Summer Undergraduate Applied Mathematics Institute

Characterization of Unconfinable Hexagonal Cells

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Abstract

We address an open question posed by Alfeld, Piper, and Schumaker in 1987 and again by Alfeld in 2000 regarding the characterization of unconfinable cells. For cells with 6 interior edges, we obtain a geometric characterization of confinability in terms of cross-ratios. This characterization allows us to show that a hexagonal cell in which the diagonals intersect at the interior vertex is unconfinable if and only if the lines containing opposite edges and the diagonal through the remaining points are either parallel or are concurrent.

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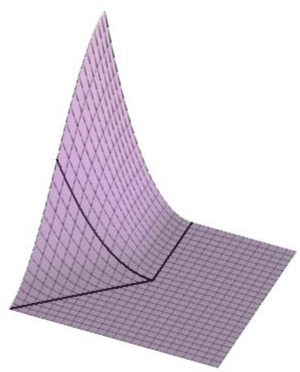
In 1987, Peter Alfeld, Bruce Piper, and Larry Schumaker addressed the problem of describing when a collection of points impose independent conditions on $\mathcal{C}^1$ quadratic splines [1]. While they were able to show a collection of different results, there are still some open questions to consider. The focus of this paper will be to answer some of these questions and to discuss methods that can be used to answer the open questions that still remain.

Definition 0.1. An $\mathcal{S}_2^1$*-spline* is a piecewise-defined bivariate quadratic function that is continuously differentiable.



Example of an $\mathcal{S}_2^1$-spline

The above figure is an example of an $\mathcal{S}_2^1$-spline defined on a particular partition of $\mathbb{R}^2$. The black lines in the figure are the boundary lines of the different regions that make up the spline. It can be seen from the figure that the spline is continuously differentiable, particulary along these boundary lines. One interesting thing to note is that a quadratic polynomial would not be able to have this behavior, since a quadratic polynomial that is identically zero on a region of the plane must be identically zero everywhere. This is one reason why splines are useful and a motivation for our investigation into the open problems involving them.

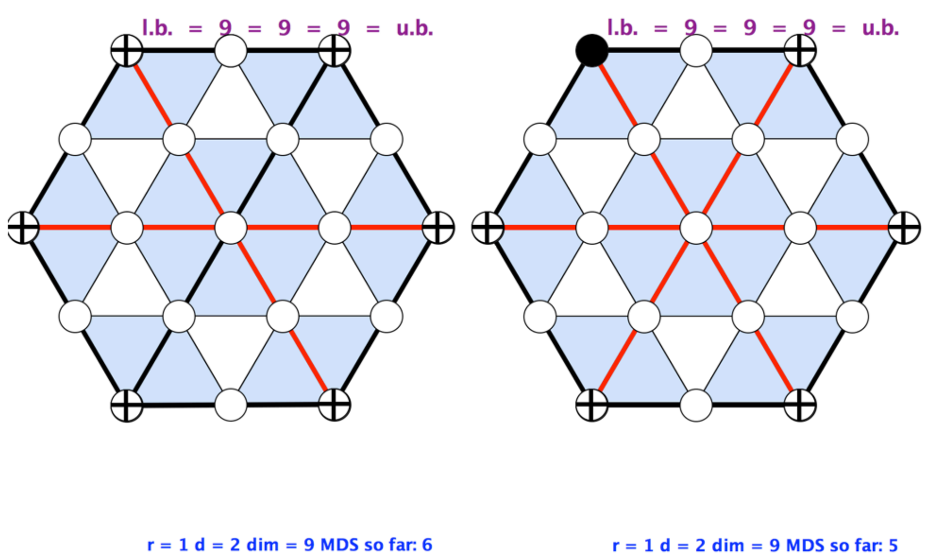
In order to address these open problems, we must know what it means for a set of points to impose independent conditions on $\mathcal{S}_2^1$-splines. Furthermore, we must know what set of points we are considering. To this end, we have the following two definitions.

Definition 0.2. A set of points $V = \{v_1, \ldots, v_n\}$ *imposes independent conditions on curves of degree* d if and only if for every set of real numbers $\{z_1, \ldots, z_n\}$, there exists a bivariate polynomial f of degree d such that $f(v_i) = z_i$ for $i = 1, \ldots, n$.

Definition 0.3 (Lai-Schumaker)**.** Suppose Δ is a triangulation consisting of a set of triangles which all share one common interior vertex v_0. Suppose every triangle in Δ has at least one neighbor with which it shares a common edge. Then we call Δ a cell.

This paper will mainly focus on when the boundary vertices of a hexagonal cell impose independent conditions on $\mathcal{S}_2^1$-splines. We introduce the following notation to refer to hexagonal cells that have this property.

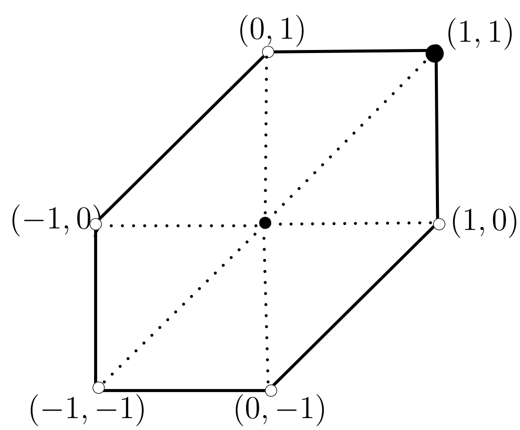
Definition 0.4. A hexagonal cell Δ_6 with boundary vertices $\{v_1, \ldots, v_6\}$ is *confinable* if and only if the set of boundary vertices imposes independent conditions on $\mathcal{S}_2^1$-splines. Otherwise, we say that Δ_6 is *unconfinable*.



An unconfinable cell and a confinable cell

The figure on the right is a regular hexagonal cell, and the figure on the left is a regular hexagonal cell with one of the boundary vertices slightly shifted. A cross represents a boundary vertex that can be assigned independently, and a solid dot represents a boundary vertex that depends on the boundary vertices that have already been assigned. Although the figures look exactly the same, we can see that in the figure on the right, we can assign all six of the boundary vertices independently, but we cannot do the same in the figure on the right. This means that the figure on the right is an unconfinable hexagonal cell, and the figure on the left is confinable. One of the purposes of this paper will be to describe properties of hexagonal cells that result in this difference.

The motivation for investigating this particular cell is that while results have proven for other types of polygonal cells, there is not a lot known about the hexagonal cell. For example, Alfeld, Piper, and Schumaker [1] showed that n-gons are always confinable when n is odd or $n = 4$; however, for hexagonal cells, there are only partial results, including an example showing that hexagonal cells are not always confinable.



The figure above was the only known example of an unconfinable hexagonal cell at the time. Although it is not a regular hexagon, it is commonly referred to as such since there exists a linear transformation that maps this hexagon to the regular hexagon.

Since not a lot is known about the hexagonal cell, we investigate the characteristics of unconfinable hexagonal cells and obtain a complete characterization of the family of hexagonal cells that we call concordant. In addition, we also present several previously unknown examples of unconfinable concordant hexagonal cells that are not linearly equivalent to the regular hexagon. Finally, we give a previously unknown example of an unconfinable discordant hexagonal cell. These ideas can be summarized into the following result.

Theorem 0.5 (The Main Result)**.** *Let Δ_6 be a concordant hexagonal cell. Then the following are equivalent:*

1. *Δ_6 is unconfinable.*
2. *There exists a projective transformation under which the image of Δ_6 is the regular hexagonal cell.*

The subsequent sections in this paper will explain the main ideas we use to prove this result. In Section 1, we derive an explicit basis for $\mathcal{S}_2^1$-splines on a hexagonal cell. This allows us to consider confinability as a linear algebra problem, which will the focus of section 2. From this, we characterize confinability in geometric terms (section 3) and use projective geometry to characterize concordant hexagonal cells (section 4).

1 A basis for the space of C^1 quadratic splines on a cell

The main goal of this section is to compute an explicit basis for the vector space of bivariate C^1 quadratic splines on a certain triangulation called cell. We start the section with the relevant definitions.

1.1 Definitions and the statement of the basis result

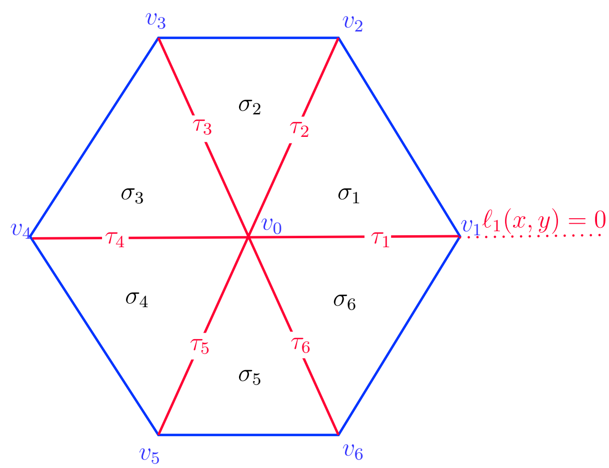
Definition 1.1. Suppose Δ is a triangulation consisting of a set of triangles which all share one common interior vertex v_0. Suppose every triangle in Δ has at least one neighbor with which it shares a common edge. Then we call Δ a *cell*.

A *polygonal cell* Δ_n is an interior cell with n boundary edges and vertices that form an n-gon.

Notation 1.2. Everywhere in this paper, Δ_n will denote a polygonal cell with n distinct interior edges emanating from the interior vertex v_0.

We label the boundary vertices $v_1, \ldots, v_n$ in a conterclockwise direction; we label the interior edges of the triangulation by τ_i, $i = 1, \ldots, n$; and the triangles containing the vertices v_0, v_i and v_{i+1} will be labeled by σ_i.

For convenience of notation, we agree that the addition on the indices of the boundary vertices is an operation modulo n; so for example, the vertex v_{n+1} is v_1, and so on.



Definition 1.3. Given a triangulation Δ consisting of triangles $\sigma_1, \ldots, \sigma_n$, a *quadratic spline on* Δ is a function s with the domain $\bigcup_i \sigma_i$ such that $s \restriction \sigma_i$ is a quadratic polynomial function for all i.

Definition 1.4. The set of all points at which a function is nonzero is called the *support* of the function.

Here is the main result of this section.

Proposition 1.5. *Let Δ_n be a polygonal cell. There is a (counterclockwise) enumeration of the boundary vertices v_i, $i = 1, \ldots, n$ and a family of C^1 quadratic splines s_i on Δ_n, for $i = 1, \ldots, n-3$, such that*

1. *every C^1 quadratic spline s on Δ_n can be uniquely expressed in the form*

$$s(x,y) = p(x,y) + \sum_{i=1}^{n-3} a_i s_i(x,y),$$

where p is a quadratic polynomial and a_i are constants;

2. *the support of s_i is contained in the set $\sigma_i \cup \sigma_{i+1} \cup \sigma_{i+2}$;*

3. *for each $i = 1, \ldots, n-3$, for $k = i+2$ or $k = i+3$ we have $s_i(v_k) = d(v_k; l_i) \cdot d(v_k; l_{i+3})$, where $d(v_k; l_i)$ is the distance from the vertex v_k to the line containing the edge τ_i.*

Definition 1.6. We call a quadratic C^1 spline s_i in the above proposition a *basic spline*.

Remark 1.7. Furthermore the union over all supports of s_i through s_{n-3} is strictly contained by the union of the regions $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ which leaves $s_i(x, y) = 0$ for all points (x, y) in σ_n and all s_i in $\mathbb{S}_n$.

1.2 Properties of Quadratic Polynomial Functions

This section is largely expository. We give a linear algebra argument showing that an k-variable polynomial of degree n which is equal to zero on a grid of $(n+1)^k$ points must be the zero polynomial. It follows, in particular, that if a k-variable polynomial vanishes on an open subset of $\mathbb{R}^k$, then the polynomial must vanish everywhere. We use this fact in a calculation of a certain determinant via a polynomial argument.

The following fact is well known; we provide a proof to make the presentation self-contained.

Claim 1.8. *Let $p(x)$ be a polynomial of degree n that is equal to zero at $n+1$ distinct points, $a_0, \ldots, a_n$. Then $p(x)$ is a zero polynomial.*

Proof. Let polynomial $p(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_{n-1} x^{n-1} + c_n x^n$ be equal to zero at $a_0, \ldots, a_n$. Substituting a_i into $p(x)$ yields a homogeneous system of $n+1$ equations in $c_0, \ldots, c_n$:

$$\begin{cases} c_0 + c_1 a_0 + c_2 a_0^2 + \cdots + c_{n-1} a_0^{n-1} + c_n a_0^n & = 0 \\ c_0 + c_1 a_1 + c_2 a_1^2 + \cdots + c_{n-1} a_1^{n-1} + c_n a_1^n & = 0 \\ \vdots \quad \vdots \quad \vdots \quad \ddots \quad \vdots \quad \vdots & \vdots \\ c_0 + c_1 a_n + c_2 a_n^2 + \cdots + c_{n-1} a_n^{n-1} + c_n a_n^n & = 0 \end{cases}$$

Taking $c_0, \dots, c_n$ to be the variables of this system, we write the matrix,

$$\mathcal{M} = \begin{pmatrix} 1 & a_0 & a_0^2 & \cdots & a_0^{n-1} & a_0^n \\ 1 & a_1 & a_1^2 & \cdots & a_1^{n-1} & a_1^n \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & a_n & a_n^2 & \cdots & a_n^{n-1} & a_n^n \end{pmatrix}$$

and we aim to show that the determinant of $\mathcal{M}$ is not equal to zero. The matrix $\mathcal{M}$ is a Vandermonde Matrix, (for example see [5, p. 301]), which has determinant equal to $\prod\limits_{0\le i,j\le n} (a_j - a_i)$. Since all a_i are distinct by assumption, each $a_j - a_i$ is not equal to 0, therefore $\det \mathcal{M}$ is not equal to zero. This implies that there is only one solution to this system, where all coefficients c_i are equal to zero. Therefore, $p(x)$ is a zero polynomial. □

Lemma 1.9. *Let $p(x_1, x_2, \dots, x_k)$ be a polynomial in k variables such that the highest power of each variable is at most n. Let $p(x_1, x_2, \dots, x_k)$ be equal to zero at the $(n+1)^k$ points in $\mathbb{R}^k$ in the cartesian product $\{a_{1,0}, a_{1,1}, \dots, a_{1,n}\} \times \{a_{2,0}, a_{2,1}, \dots, a_{2,n}\} \times \cdots \times \{a_{k,0}, a_{k,1}, \dots, a_{k,n}\}$. If the elements in each set in the product are distinct, then $p(x_1, x_2, \dots, x_k)$ is a zero polynomial.*

Proof. We induct on the number of variables k. If $k = 1$, then $p(x_1)$ is a polynomial of degree at most n, and is equal to zero at $a_{1,0}, \dots, a_{1,n}$. By Claim 1.8, this is a zero polynomial. Assume the statement of the lemma holds for some $k \geq 1$. That is, for any $p(x_1, x_2, \dots, x_k)$ such that the highest power of each x_i is no more than n, and p is equal to zero at every point in the set $\{a_{1,0}, \dots, a_{1,n}\} \times \cdots \times \{a_{k,0}, \dots, a_{k,n}\}$, then it is a zero polynomial.

Assume $p(x_1, x_2, \dots, x_k, x_{k+1})$ is a polynomial with the highest power of each variable at most n, and $p(x_1, x_2, \dots, x_k, x_{k+1})$ vanishes on $\{a_{1,0}, \dots, a_{1,n}\} \times \cdots \times \{a_{k,0}, \dots, a_{k,n}\} \times \{a_{k+1,0}, \dots, a_{k+1,n}\}$. We can write p as follows:

$$p(x_1, x_2, \dots, x_k, x_{k+1}) = p_0(x_1, \dots, x_k) + p_1(x_1, \dots, x_k)x_{k+1} \\ + p_2(x_1, \dots, x_k)x_{k+1}^2 + \cdots + p_{n-1}(x_1, \dots, x_k)x_{k+1}^{n-1} + p_n(x_1, \dots, x_k)x_{k+1}^n.$$

For all points in $\{a_{1,0}, \ldots, a_{1,n}\}, \{a_{k,0}, \ldots, a_{k,n}\}$, the one variable polynomial $p(a_{1,0}, \ldots, a_{1,n} \ldots, a_{k,n}, x_{k+1})$ is equal to zero at each of the $n+1$ points $a_{k+1,0}, \ldots, a_{k+1,n}$. Therefore, by Claim 1.8, the coefficients of $p(a_{1,0}, \ldots, a_{k,n}, x_{k+1})$ are all equal to zero. These coefficients are given by

$$p_0(a_{1,0}, \ldots, a_{k,n}), p_1(a_{1,0}, \ldots, a_{k,n}), \ldots, p_n(a_{1,0}, \ldots, a_{k,n}).$$

Each $p_j(a_{1,0}, \ldots, a_{k,n})$, for $0 \leq j \leq n$, is equal to zero at $(n+1)^k$ distinct points, and the highest power of each is at most n. By the induction hypothesis, it follows that each p_j is a zero polynomial. This implies that $p(x_1, x_2, \ldots, x_k, x_{k+1})$ is also a zero polynomial. □

Now we use the above fact to establish the following.

Claim 1.10. *Let* $a_i, b_i \in \mathbb{R}$, $i \in \{1, 2, 3\}$*, and matrix* $\mathcal{M}$ *is given by* $\begin{bmatrix} a_1^2 & a_2^2 & a_3^2 \\ a_1b_1 & a_2b_2 & a_3b_3 \\ b_1^2 & b_2^2 & b_3^2 \end{bmatrix}$. *Then*

$$\det \mathcal{M} = (a_1b_2 - a_2b_1)(a_1b_3 - a_3b_1)(a_2b_3 - a_3b_2).$$

Proof. Of course, the equality can be verified directly, but we give a more elegant argument.

Suppose first that $a_i \neq 0$ for $i = 1, 2, 3$. Factoring out a_i^2 from the i^{th} column, we convert this matrix into the Vandermonde Matrix, where the (i, j) entry of the 3×3 matrix is given by $(\frac{b_j}{a_j})^{i-1}$ for all $1 \leq i, j \leq 3$. Using the Vandermonde determinant formula, $\det \mathcal{M}$ is given by

$$\begin{aligned} \det \mathcal{M} &= a_1^2 a_2^2 a_3^3 \begin{vmatrix} 1 & 1 & 1 \\ \frac{b_1}{a_1} & \frac{b_2}{a_2} & \frac{b_3}{a_3} \\ (\frac{b_1}{a_1})^2 & (\frac{b_2}{a_2})^2 & (\frac{b_3}{a_3})^2 \end{vmatrix} \\ &= a_1^2 a_2^2 a_3^3 \Big(\frac{b_2}{a_2} - \frac{b_1}{a_1}\Big)\Big(\frac{b_3}{a_3} - \frac{b_1}{a_1}\Big)\Big(\frac{b_3}{a_3} - \frac{b_2}{a_2}\Big) \\ &= (a_1b_2 - a_2b_1)(a_1b_3 - a_3b_1)(a_2b_3 - a_3b_2). \end{aligned}$$

Therefore, we have the needed formula in the case when none of the a_i are zero. To establish the equality in the remaining cases, we use the polynomial argument. Let $p(a_1, a_2, a_3) = (a_1b_2 - a_2b_1)(a_1b_3 - a_3b_1)(a_2b_3 - a_3b_2)$. Note that p is a polynomial in a_1, a_2, a_3 and that $\det \mathcal{M}$ is also a polynomial in these variables. Since the difference between the polynomials is zero on an open subset of $\mathbb{R}^3$, the difference is zero everywhere and so $\det \mathcal{M} = (a_1b_2 - a_2b_1)(a_1b_3 - a_3b_1)(a_2b_3 - a_3b_2)$ for all a_i, b_i. □

1.3 Basic Spline Functions

In this section, we establish that for any cell Δ_n, we can find splines $s_1 \ldots, s_{n-3}$ that satisfy the conditions (2) and (3) of Proposition 1.5. We later show that these splines, together with the 6 basis quadratic bivariate polynomials, form a basis for the space $\mathbb{S}_n$.

We begin with a technical lemma that will allow us to explicitly compute the coefficients for the splines s_i.

Lemma 1.11. *Let $\ell_1(x, y), \ldots, \ell_4(x, y)$ be linear forms such that neither ℓ_1 nor ℓ_4 is proportional to ℓ_2 and neither ℓ_1 nor ℓ_4 is proportional to ℓ_3. There exist non-zero real numbers k_1 and k_4 and real numbers k_2 and k_3 such that*

$$k_1\ell_1^2 + k_2\ell_2^2 + k_3\ell_3^2 + k_4\ell_4^2 = 0. \tag{1}$$

Moreover, if $\ell_i = a_ix + b_iy$ for unit vectors (a_i, b_i), $i = 1, \ldots, 4$, then the values of the coefficients are given by the formulas

$$k_1 = \sin\theta_{2,4} \sin\theta_{3,4} \sin\theta_{2,3} \tag{2}$$

$$k_2 = -\sin\theta_{1,4} \sin\theta_{3,4} \sin\theta_{1,3} \tag{3}$$

$$k_3 = \sin\theta_{1,4} \sin\theta_{2,4} \sin\theta_{1,2} \tag{4}$$

$$k_4 = -\sin\theta_{1,2} \sin\theta_{1,3} \sin\theta_{2,3} \tag{5}$$

where $\theta_{i,j}$ is the angle between the normal vectors to the lines $\ell_i(x, y) = 0$ and $\ell_j(x, y) = 0$.

Proof. It suffices to establish the "moreover" part. Suppose that $\ell_i(x, y) = a_ix + b_iy$, where (a_i, b_i) is a unit vector. Expanding the squares and collecting the terms with respect to x^2, xy and y^2, we obtain the following system of equations:

$$a_1^2k_1 + a_2^2k_2 + a_3^2k_3 + a_4^2k_4 = 0,$$

$$a_1b_1k_1 + a_2b_2k_2 + a_3b_3k_3 + a_4b_4k_4 = 0,$$
$$b_1^2k_1 + b_2^2k_2 + b_3^2k_3 + b_4^2k_4 = 0.$$

Converting this system of equations to a matrix expression, we have:

$$\underbrace{\begin{bmatrix} a_1^2 & a_2^2 & a_3^2 & a_4^2 \\ a_1b_1 & a_2b_2 & a_3b_3 & a_4b_4 \\ b_1^2 & b_2^2 & b_3^2 & b_4^2 \end{bmatrix}}_{\mathcal{M}} \cdot \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \vec{0}$$

Consider the matrix $\mathcal{N}$, found by appending row 1 of matrix $\mathcal{M}$ onto the matrix $\mathcal{M}$ itself:

$$\mathcal{N} = \begin{bmatrix} a_1^2 & a_2^2 & a_3^2 & a_4^2 \\ a_1^2 & a_2^2 & a_3^2 & a_4^2 \\ a_1b_1 & a_2b_2 & a_3b_3 & a_4b_4 \\ b_1^2 & b_2^2 & b_3^2 & b_4^2 \end{bmatrix}$$

Clearly the determinant of this matrix is 0. Let $\mathcal{M}_{i,j,k}$ denote the submatrix of the matrix $\mathcal{M}$ that is created by the columns $i, j,$ and k of the matrix $\mathcal{M}$. Using the first row expansion, we can find the following expression for the determinant of $\mathcal{N}$:

$$0 = \det \mathcal{N} = a_1^2 \det \mathcal{M}_{2,3,4} - a_2^2 \det \mathcal{M}_{1,3,4} + a_3^2 \det \mathcal{M}_{1,2,4} - a_4^2 \det \mathcal{M}_{1,2,3}$$

Therefore,

$$\begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} \det \mathcal{M}_{2,3,4} \\ -\det \mathcal{M}_{1,3,4} \\ \det \mathcal{M}_{1,2,4} \\ -\det \mathcal{M}_{1,2,3} \end{bmatrix}$$

gives a solution to the first equation in the system, $a_1^2k_1 + a_2^2k_2 + a_3^2k_3 + a_4^2k_4 = 0$.

We can construct a similar argument appending the other two rows of $\mathcal{M}$ onto itself, resulting in two new matrices:

$$\mathcal{N}' = \begin{bmatrix} a_1b_1 & a_2b_2 & a_3b_3 & a_4b_4 \\ a_1^2 & a_2^2 & a_3^2 & a_4^2 \\ a_1b_1 & a_2b_2 & a_3b_3 & a_4b_4 \\ b_1^2 & b_2^2 & b_3^2 & b_4^2 \end{bmatrix} \text{ and } \mathcal{N}'' = \begin{bmatrix} b_1^2 & b_2^2 & b_3^2 & b_4^2 \\ a_1^2 & a_2^2 & a_3^2 & a_4^2 \\ a_1b_1 & a_2b_2 & a_3b_3 & a_4b_4 \\ b_1^2 & b_2^2 & b_3^2 & b_4^2 \end{bmatrix}$$

both with determinant equal to 0. Notice that the bottom three rows in all matrices $\mathcal{N}, \mathcal{N}'$, and $\mathcal{N}''$ are the same, so taking the determinant yields that

$$0 = \det \mathcal{N}' = a_1 b_1 \det \mathcal{M}_{2,3,4} - a_2 b_2 \det \mathcal{M}_{1,3,4} + a_3 b_3 \det \mathcal{M}_{1,2,4} - a_4 b_4 \det \mathcal{M}_{1,2,3}$$

and

$$0 = \det \mathcal{N}'' = b_1^2 \det \mathcal{M}_{2,3,4} - b_2^2 \det \mathcal{M}_{1,3,4} + b_3^2 \det \mathcal{M}_{1,2,4} - b_4^2 \det \mathcal{M}_{1,2,3}.$$

And so $\begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} \det \mathcal{M}_{2,3,4} \\ -\det \mathcal{M}_{1,3,4} \\ \det \mathcal{M}_{1,2,4} \\ -\det \mathcal{M}_{1,2,3} \end{bmatrix}$ is a solution to all three equations in the system.

Notice that each of the matrices $\mathcal{M}_{j+1,j+2,j+3}$ for j from 1 to 4 (where the indices are modulo 4) are of the form required by Claim 1.10. Applying this Claim, we can say that k_1 which is equal to the determinant of $\mathcal{M}_{2,3,4}$, is equal to $(a_2b_3 - a_3b_2)(a_2b_4 - a_4b_2)(a_3b_4 - a_4b_3)$

$$\det \begin{bmatrix} a_2^2 & a_3^2 & a_4^2 \\ a_2b_2 & a_3b_3 & a_4b_4 \\ b_2^2 & b_3^2 & b_4^2 \end{bmatrix} = (a_2b_3 - a_3b_2)(a_2b_4 - a_4b_2)(a_3b_4 - a_4b_3)$$

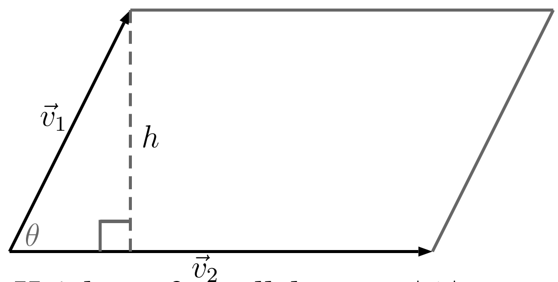
Notice that each of these differences are the determinants of 2×2 matrices:

$$(a_2b_3 - a_3b_2) = \det \begin{bmatrix} a_2 & a_3 \\ b_2 & b_3 \end{bmatrix} \text{ and so on.}$$

Therefore k_1 can now be written as the product of three determinants:

$$k_1 = \det \begin{bmatrix} a_2 & a_3 \\ b_2 & b_3 \end{bmatrix} \cdot \det \begin{bmatrix} a_2 & a_4 \\ b_2 & b_4 \end{bmatrix} \cdot \det \begin{bmatrix} a_3 & a_4 \\ b_3 & b_4 \end{bmatrix}$$

Recall that the determinant of a 2×2 matrix is the area of the parallelogram determined by the column vectors of the matrix. Another form for the area of this parallelogram is $|\vec{v}_1||\vec{v}_2| \sin\theta$ where $\vec{v}_1$ and $\vec{v}_2$ are the column vectors of the 2×2 matrix and θ is the angle between them.



Height h of parallelogram: $|\vec{v}_1| \sin \theta$
Area of parallelogram: $|\vec{v}_2||\vec{v}_1| \sin \theta$

Since we took the vectors (a_i, b_i) to be unit length, we have that

$$\det \begin{bmatrix} a_i & a_j \\ b_i & b_j \end{bmatrix} = \sin \theta_{i,j}$$

where $\theta_{i,j}$ is the angle between the vectors (a_i, b_i) and (a_j, b_j).

Now we have:

$$k_1 = \sin \theta_{2,3} \sin \theta_{3,4} \sin \theta_{2,4}$$

Similarly,

$$\begin{aligned} k_2 &= -\sin \theta_{1,4} \sin \theta_{3,4} \sin \theta_{1,3} \\ k_3 &= \sin \theta_{1,4} \sin \theta_{2,4} \sin \theta_{1,2} \\ k_4 &= -\sin \theta_{1,2} \sin \theta_{1,3} \sin \theta_{2,3}. \end{aligned}$$

□

Now we show how to construct a basic spline function using the solution to 1, assuming that certain edges are not parallel.

Proposition 1.12. *Let Δ_n be a cell with the interior vertex $v_0 = (0, 0)$. Suppose that the edge τ_i is not parallel to τ_{i+2} and τ_{i+1} is not parallel to τ_{i+3}. There exists a quadratic spline s_i such that*

1. *the support of s_i is contained in the set $\sigma_i \cup \sigma_{i+1} \cup \sigma_{i+2}$;*

2. *for each $i = 1, \ldots, n-3$, for $k = i+2$ or $k = i+3$ we have $s_i(v_k) = d(v_k; l_i) \cdot d(v_k; l_{i+3})$, where $d(v_k; l_i)$ is the distance from the vertex v_k to the line containing the edge τ_i.*

Proof. For $j = i, \dots, i+3$, let $\ell_j = a_j x + b_j y$ be linear forms such that the line $\ell_j = 0$ contains the edge τ_j and (a_j, b_j) is a unit vector oriented so that the angle between the vectors (a_j, b_j) and (a_k, b_k) is equal to the angle between the edges τ_j and τ_k. Let $k_{i,1}, \dots, k_{i,4}$ be the coefficients given by 2 in Lemma 1.11. Let

$$s_i(x,y) := \begin{cases} k_{i,1}\ell_i^2(x,y) & (x,y) \in \sigma_i \\ k_{i,2}\ell_{i+1}^2(x,y) + k_{i,1}\ell_i^2(x,y) & (x,y) \in \sigma_{i+1} \\ -k_{i,4}\ell_{i+3}^2(x,y) & (x,y) \in \sigma_{i+2} \\ 0 & otherwise \end{cases}$$

It is immediate from the construction that s_i is C^1 across the edges τ_i, τ_{i+1}, and τ_{i+3}. It remains to check that the difference

$$\left[k_{i,2}\ell_{i+1}^2(x,y) + k_{i,1}\ell_i^2(x,y)\right] - \left[-k_{i,4}\ell_{i+3}^2(x,y)\right]$$

is a multiple of ℓ_3^2. The latter follows since the coefficients $k_{i,1}, \dots, k_{i,4}$ and the linear forms $\ell_i, \dots, \ell_{i+3}$ satisfy 1 of Lemma 1.11. This establishes the first claim in the proposition.

The second claim follows via a direct computation from the form of the coefficients $k_{i,1}$ and $k_{i,4}$ obtained in Lemma 1.11, noting that in any cell consecutively numbered edges cannot be parallel. □

We finish this subsection by showing that it is always possible to enumerate the boundary vertices of Δ_n in such a way that the assumptions of Proposition 1.12 hold for $i = 1, \dots, n-3$.

Claim 1.13. *Let Δ_n be a polygonal cell. Then*

1. *there cannot be a pair of edges (τ_k, τ_{k+1}) such that $\tau_k \parallel \tau_{k+1}$;*
2. *If there is a pair (τ_i, τ_{i+2}) of parallel edges, then there can be at most one other pair of parallel edges. Moreover, if the other pair exists, it is either the pair (τ_{i-1}, τ_{i+1}) or (τ_{i+1}, τ_{i+3}).*

Proof. (1) Follows easily from the definition of a cell.

(2) Suppose there is one pair of interior edges (τ_i, τ_{i+2}) of a polygonal cell such that $\tau_i \parallel \tau_{i+2}$. The line containing these edges partitions the plane into two parts.

Suppose there is another distinct pair of edges (τ_j, τ_{j+2}), such that $\tau_j \parallel \tau_{j+2}$. The line containing this pair intersects the line containing (τ_i, τ_{i+2}) at v. This means that either τ_j or τ_{j+2} lies in the part of the plane in between τ_i and τ_{i+2} (moving counterclockwise). Since there is only one edge between τ_i and τ_{i+2}, (namely, τ_{i+1}) either $\tau_j = \tau_{i+1}$ or $\tau_{j+2} = \tau_{i+1}$.

If $\tau_j = \tau_{i+1}$ then we have $j = i + 1$ which gives $j + 2 = i + 3$ so $\tau_{j+2} = \tau_{i+3}$ and the pair $(\tau_j, \tau_{j+2}) = (\tau_i, \tau_{i+3})$. Else, if $\tau_{j+2} = \tau_{i+1}$ then we have $j + 2 = i + 1$ which gives $j = i - 1$ so $\tau_j = \tau_{i-1}$ and the pair $(\tau_j, \tau_{j+2}) = (\tau_{i-1}, \tau_{i+1})$.

And so, given the pair (τ_i, τ_{i+2}) exists such that $\tau_i \parallel \tau_{i+2}$, if there is another such pair, then it is either the pair (τ_{i-1}, τ_{i+1}) or (τ_{i+1}, τ_{i+3}). □

Lemma 1.14. *There exists an enumeration of the interior edges of Δ_n, $n \geq 5$, such that τ_i is not parallel to τ_{i+2} and τ_{i+1} is not parallel to τ_{i+3} for i from 1 to $n-3$.*

Proof. We start with an arbitrary enumeration of the edges in the counterclockwise direction. If there are no pairs (τ_k, τ_{k+2}) of parallel edges, then we are done. If there is exactly one pair of parallel edges (τ_k, τ_{k+2}), then, re-enumerating the edges if nesessary, we may assume that $k = n - 1$. (Recalling our convention to that the addition on indices is modulo n, this means that $\tau_{k+2} = \tau_1$ in this case.) It remains to note that no pair of edges of the form (τ_i, τ_{i+2}), $i = 1, \ldots, n-2$, will contain the pair (τ_{n-1}, τ_1).

If there are two pairs of parallel edges, indexed in the counterclockwise order as $\{\tau_k, \tau_{k+1}, \tau_{k+2}, \tau_{k+3}\}$, then, re-enumerating the edges if nesessary, we may assume that $k = n - 1$ (so the two pairs of parallel edges are (τ_{n-1}, τ_1) and (τ_n, τ_2)). In this case it is also the case that no pair of edges of the form (τ_i, τ_{i+2}), $i = 1, \ldots, n-2$, will contain either of the two the pairs. □

1.4 Proof of Proposition 1.5

Let $\mathbb{S}_n$ be the vector space of all $\mathcal{S}_2^1$-splines defined on Δ_n, $n \geq 5$. Since any vector can be written as a linear combination of the elements in the basis of its vector space, every $\mathcal{S}_2^1$-spline defined on a polygonal cell Δ_n can be written as the linear combination of the elements of a basis of $\mathbb{S}_n$. Therefore it will be sufficient to find an appropriate basis for $\mathbb{S}_n$. We first use the following theorem to find the dimension of $\mathbb{S}_n$.

Theorem 1.15 (Lai-Schumaker [4]). *Suppose Δ_n is an interior cell associated with an interior vertex v where n edges meet with m different slopes. Then for any $0 \le r \le d$,*

$$\dim \mathcal{S}_d^r(\Delta_n) = \binom{r+2}{2} + n\binom{d-r+1}{2} + \gamma,$$

where

$$\gamma := \sum_{j=1}^{d-r} (r + j + 1 - jm)_+$$

Specializing the above theorem to our situation, we obtain the following.

Claim 1.16. *The dimension of $\mathbb{S}_n$ is equal to $n+3$ in all cases except for one: the case in which $n = 4$ and the interior edges of Δ_n associated with the quadrilateral on which all splines in $\mathbb{S}_n$ are defined have exactly 2 distinct slopes.*

Proof. It follows from Theorem 1.15 that the dimension of the space $\mathbb{S}_n$ is given by the following:

$$\dim \mathcal{S}_2^1(\Delta_n) = \dim(\mathbb{S}_n) = \binom{1+2}{2} + n\binom{2-1+1}{2} + \gamma$$
$$\dim(\mathbb{S}_n) = 3 + n + \gamma$$

where

$$\gamma = \sum_{j=1}^{2-1} (1 + j + 1 - jm)_+$$
$$= (1 + 1 + 1 - m)_+$$
$$\gamma = (3 - m)_+$$

It is clear that there must be at least three distinct slopes for the interior edges of a polygonal cell Δ_n whose number of boundary vertices, n, is not equal to 4, and at least two distinct slopes for the interior edges of Δ_4. Therefore

$$m \geq \begin{cases} 3 & \text{for } \Delta_n, n \neq 4 \\ 2 & \text{for } \Delta_4 \end{cases}$$

It follows that $\gamma = 0$ for all Δ_n such that $n \neq 4$. Therefore, the dimension of $\mathbb{S}_n$ such that $n \neq 4$ is $n+3$. In the case that $n = 4$, γ will be either 1 or 0. The only polygon for which $\gamma = 1$ is the quadrilateral with a singular interior vertex, v. So we can say that in this case $\dim \mathbb{S}_4 = 8$ and in all other cases:

$$\dim \mathbb{S}_n = n + 3. \tag{6}$$

□

Claim 1.17. *The set of functions* $\mathcal{B} = \{1, x, y, x^2, xy, y^2, s_1, s_2, \ldots, s_{n-3}\}$ *is a basis for* $\mathbb{S}_n$*, for* $n \geq 5$.

Proof. Since the dimension of $\mathbb{S}_n$ is equal to $n + 3$ by Claim 1.16 and since $\mathcal{B}$ contains $n + 3$ elements, it suffices to show that $\mathcal{B}$ is linearly independent.

Suppose

$$c_1 + c_2 x + c_3 y + c_4 x^2 + c_5 xy + c_6 y^2 + d_1 s_1 + d_2 s_2 + \cdots + d_{n-3} s_{n-3} = 0$$

for every $(x, y) \in \Delta_n$. We show that $c_i = d_j = 0$ for $i = 1, \ldots, 6$ and $j = 1, \ldots, n - 3$.

Recall that by the definition of the basic spline, for every $(x, y) \in \sigma_n$ and every $i = 1, \ldots, n-3$, we have $s_i(x, y) = 0$. Therefore $c_1 + c_2 x + c_3 y + c_4 x^2 + c_5 xy + c_6 y^2 = 0$ for all $(x, y) \in \sigma_n$. This implies by Lemma 1.9 that $c_i = 0$ for $i = 1, \ldots, 6$.

Now we have $d_1 s_1 + d_2 s_2 + \cdots + d_{n-3} s_{n-3} = 0$ on Δ_n. We induct on j from 1 to $n - 3$ to show that $d_j = 0$. The spline s_1 is non-zero at v_2. Since v_2 is not contained in the support of s_i for all $i \neq 1$, all other basic splines are zero at this point. It follows that $d_1 s_1(v_2) = 0$. Since $s_1(v_2) \neq 0$ we conclude $d_1 = 0$.

Now suppose that $d_m = 0$ for all $m = 1, \ldots, k$, and show $d_{k+1} = 0$. We have

$$\underbrace{d_1 s_1(v_{k+2}) + d_2 s_2(v_{k+2}) + \cdots + d_k s_k(v_{k+2})}_{\text{0 by the Induction Hypothesis}} \quad + d_{k+1} s_{k+1}(v_{k+2})$$
$$+ \underbrace{d_{k+2} s_{k+2}(v_{k+2}) + \cdots + d_{n-3} s_{n-3}(v_{k+2})}_{\substack{\text{Since } v_{k+2} \text{ is not in the} \\ \text{support of } s_m \text{ for } m > k+1, \text{ the values of} \\ s_{k+2} \text{ through } s_{n-3} \text{ are 0 at } v_{k+2}}} = d_{k+1} s_{k+1}(v_{k+2}) = 0.$$

Since $s_{k+1}(v_{k+2}) \neq 0$, it follows that $d_{k+1} = 0$ as desired. Therefore $d_j = 0$ for $j = 1, \ldots, n-3$ and $\mathcal{B}$ is a basis for $\mathbb{S}_n$. □

This completes the proof of Proposition 1.5.

2 Linear Algebraic discussion of unconfinability of hexagonal cells

2.1 Introduction

As a direct result from the last section, we have that if $s(x, y)$ is an $\mathcal{S}_2^1$-spline on Δ_6, then it can be expressed as:

$$s(x, y) = p(x, y) + \sum_{i=1}^{3} a_i s_i(x, y), \tag{7}$$

where $p(x, y)$ is a quadratic polynomial and $s_i(x, y)$ is a basic spline.

This section focuses on using this form and results from linear algebra to address the problem of unconfinability on certain hexagonal cells. We begin by looking at a result from [3], which determines when a set of points will impose independent conditions on polynomials of degree d. This result allows us to determine what restricitions we must place on hexagonal cells, and we use these results to define the special type of hexagonal cell that will be the focus of the next two sections. From this, we state a prove a series of linear algebra results that will be useful in the proof of the main result of this section. We conclude this section with a discussion of properties of quadratic polynomials that vanish on a certain number of the boundary vertices of the hexagonal cell, which will lead into the geometric descriptions we consider in the next section.

By looking at (7), we can see that there are two main parts to the form of a spline: the quadratic polynomial and the sum of the basic splines. When examining systems of equations of splines, it will be useful to have matrices that represent these parts. To this end, we define the following matrices.

$$\mathcal{Q} := \begin{pmatrix} x_1^2 & x_1y_1 & y_1^2 & x_1 & y_1 & 1 \\ x_2^2 & x_2y_2 & y_2^2 & x_2 & y_2 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_6^2 & x_6y_6 & y_6^2 & x_6 & y_6 & 1 \end{pmatrix}, \quad \mathcal{S}_i := \begin{pmatrix} s_i(v_1) \\ s_i(v_2) \\ \vdots \\ s_i(v_6) \end{pmatrix}, i = 1, \ldots, 3$$

$$\mathcal{M} := \begin{pmatrix} \mathcal{Q} & | & \mathcal{S}_1 & | & \mathcal{S}_2 & | & \mathcal{S}_3 \end{pmatrix}$$

Our first goal is to establish some results that will allow us to know how to handle the matrix $\mathcal{Q}$, specifically in terms of calculating its rank. Most of the results that follow involve points that lie on a conic, and so the following definition has been provided as a convenient reference.

Definition 2.1. A *conic* is a collection of points that satisfy the equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$.

The following fact appears in [3] and describes conditions that deterimine when a set of points will impose independent conditions on polynomials of degree d.

Fact 2.2. *[3, Proposition 1] Let d be the degree of a plane curve and let $V = \{v_1, \ldots, v_n\} \subset \mathbb{P}$ be any collection of $n \leq 2d + 2$ distinct points. The points of V fail to impose independent conditions on curves of degree d if and only if either $d + 2$ of the points of V are collinear or $n = 2d + 2$ and V is contained in a conic.*

It follows from this fact that if $V = \{v_1, \ldots, v_6\}$, then the points of V fail to impose independent conditions on quadratic polynomials if and only if either 4 of the points of V are collinear or V is contained in a conic. For this reason, we only want to consider hexagonal cells whose boundary vertices meet these conditions, which gives us the following definition.

Definition 2.3. A *conic cell* is a hexagonal cell such that no four of the boundary vertices are collinear and all six of the boundary vertices lie on a conic.

2.2 Linear Algebra

Remark 2.4. If no five boundary vertices of a conic cell are collienar, then the rank of $\mathcal{M}$ is greater than or equal to 5.

The following claim and two lemmas are useful linear algebra results that will help us in proving the characteristics that determine when a hexagonal cell is unconfinable.

Lemma 2.5. *If A is a matrix such that $A^T\vec{c} = \vec{\mathbf{0}}$ and $\vec{s} \in \text{Col}(A)$, where $\text{Col}(A)$ is the column space of A, then $\vec{c} \cdot \vec{s} = 0$*

Proof. Let $A = (a_{ij})$ and $\vec{s} \in \text{Col}(A)$. Then $A^T\vec{c} = \vec{\mathbf{0}}$ means that for all i, $\sum\limits_j a_{ij}c_i = 0$. Since $\vec{s} \in \text{Col}(A)$, for all i, $s_i = \sum\limits_j \lambda_j a_{ij}$, where $\lambda_j \in \mathbb{R}$. This means that $\vec{c} \cdot \vec{s} = \sum\limits_i c_i s_i = \sum\limits_i c_i \sum\limits_j \lambda_j a_{ij} = \sum\limits_i \sum\limits_j c_i \lambda_j a_{ij} = \sum\limits_j \sum\limits_i c_i \lambda_j a_{ij} = \sum\limits_j \lambda_j \sum\limits_i c_i a_{ij} = 0$.

□

Claim 2.6. *Let $\mathcal{Q}$ be an $n \times n$ matrix with rank $n-1$ and let $\mathcal{S}_1, \dots, \mathcal{S}_k \in \text{Col}(\mathcal{Q})$. Then the matrix $\mathcal{Q}$ augmented by the vectors $\mathcal{S}_1, \dots, \mathcal{S}_k$ has rank $n-1$.*

Proof. Since the rank of $\mathcal{Q}$ is $n-1$, then the rank of $\mathcal{Q}^T$ is also $n-1$ (see, for example, [6]). This means that there is a non-trivial solution to $\mathcal{Q}^T\vec{c} = \vec{\mathbf{0}}$. Since each $\mathcal{S}_i$ is in the column space of $\mathcal{Q}$, by Lemma 2.5, $\vec{c} \cdot \mathcal{S}_i = 0$ for each $\mathcal{S}_i$. Therefore, $\vec{c}$ is a non-trivial solution to $(\mathcal{Q} \mid \mathcal{S}_1 \mid \cdots \mid \mathcal{S}_k)^T = \vec{\mathbf{0}}$. This means that the rank of $(\mathcal{Q} \mid \mathcal{S}_1 \mid \cdots \mid \mathcal{S}_k)^T$ is $n-1$, which means that the rank of $(\mathcal{Q} \mid \mathcal{S}_1 \mid \cdots \mid \mathcal{S}_k) = ((\mathcal{Q} \mid \mathcal{S}_1 \mid \cdots \mid \mathcal{S}_k)^T)^T$ is $n-1$. □

Lemma 2.7. *There exists a bivariate quadratic polynomial $q(x, y)$ such that $q(v_i) = \mathcal{S}_j(v_i)$ if and only if $\mathcal{S}_j$ is in the column space of $\mathcal{Q}$.*

Proof. There exists a bivariate quadratic polynomial $q(x, y)$ such that $q(v_i) = \mathcal{S}_j(v_i) \iff \mathcal{S}_i$ can be written as a linear combination of the columns of $\mathcal{Q} \iff \mathcal{S}_i$ is in the column space of $\mathcal{Q}$. □

The following theorem is the main result of this section and is a characterization of unconfinability for conic cells.

Theorem 2.8. *Let Δ_6 be a conic cell. Then Δ_6 is unconfinable if and only if the rank of $\mathcal{M}$ is 5.*

Proof. ($\Rightarrow$) Suppose that Δ_6 is unconfinable. By Remark 2.4, rank$(\mathcal{M}) \geq 5$. If rank$(\mathcal{M}) = 6$, then the boundary vertices would impose independent conditions on $\mathcal{S}_2^1$-splines, and so by Definition 0.4, Δ_6 would be confinable, which is a contradiction. Therefore the rank of $\mathcal{M}$ is 5.
($\Leftarrow$) Suppose the rank of $\mathcal{M}$ is 5. Since the dimension of the column space is less than 6, the boundary vertices fail to impose independent conditions on $\mathcal{S}_2^1$-splines. This means that Δ_6 is unconfinable by Definition 0.4. □

We conclude this section with a discussion of properties of quadratic polynomials that vanish on a number of the boundary vertices of a conic cell.

2.3 Quadratic Polynomials

Claim 2.9. *Let the points in $V = \{v_1, \ldots, v_6\}$ be the boundary vertices of a conic cell. If a quadratic function vanishes at five of the six points in V, then it also vanishes at the sixth point of V.*

Proof. Let the points in $V = \{v_1, \ldots, v_6\}$ be contained on the conic described by $q(x, y) = 0$ and let $\tilde{q}(x, y)$ be a quadratic function that vanishes at five of the six points in V. Without loss of generality, we may suppose that $\tilde{q}(v_i) = 0$ for $i = 1, \ldots, 5$. Evaluating $\tilde{q}(v_i)$ for $i = 1, \ldots, 5$, we obtain a system of equations in the coefficients of $\tilde{q}$ described by the matrix

$$\mathcal{Q} = \begin{pmatrix} x_1^2 & x_1y_1 & y_1^2 & x_1 & y_1 & 1 \\ x_2^2 & x_2y_2 & y_2^2 & x_2 & y_2 & 1 \\ x_3^2 & x_3y_3 & y_3^2 & x_3 & y_3 & 1 \\ x_4^2 & x_4y_4 & y_4^2 & x_4 & y_4 & 1 \\ x_5^2 & x_5y_5 & y_5^2 & x_5 & y_5 & 1 \end{pmatrix}.$$

By Fact 2.2, five points, no four of which are collinear, impose independent conditions on polynomials of degree two. Therefore, the rows of

$\mathcal{Q}$ are linearly independent, and so the rank of $\mathcal{Q}$ is 5. By the Rank Theorem (see, for example, [6, p. 211]), the dimension of the null space is then 1. Because V is contained on the conic $q(x,y)=0$, then $q(v_i)=0$ for $i=1,\dots,5$. Therefore, the vector composed of the coefficients of q form a basis of the null space of the matrix $\mathcal{Q}$, and therefore every polynomial that is zero at $v_1,\dots,v_5$ can be written as a scalar multiple of q. Therefore we have $\tilde{q}=kq$. Since $q(v_6)=0$, we have $\tilde{q}(v_6)=kq(v_6)=0$. □

Remark 2.10. If a quadratic function vanishes on four of the boundary vertices of a conic cell and does not vanish at one of the other boundary vertices, then it does not vanish at the remaining boundary vertex.

Claim 2.11. *For any four noncollinear points $v_1,\dots,v_4$, there exists a quadratic polynomial that vanishes at each v_i for $i=1,\dots,4$ and is nonzero at v_6.*

Proof. Let $v_i=(x_i,y_i)$, $i=1,\dots,4$. The line through any v_i,v_j, $i\neq j$, is given by $\ell_{ij}=\begin{vmatrix} x-x_i & x_i-x_j \\ y-y_i & y_i-y_j \end{vmatrix}=0$. Then the bivariate quadratic polynomial

$$Q_{1423}(x,y)=\ell_{14}\cdot\ell_{23}=\begin{vmatrix} x-x_1 & x_1-x_4 \\ y-y_1 & y_1-y_4 \end{vmatrix}\cdot\begin{vmatrix} x-x_2 & x_2-x_3 \\ y-y_2 & y_2-y_3 \end{vmatrix}$$

will necessarily be zero at the four points $v_1,\dots,v_4$ used to determine the linear forms ℓ_{ij} through v_i and v_j and nonzero at v_6. □

Corollary 2.12. *Let the points in $V=\{v_1,\dots,v_6\}$ be the boundary vertices of a conic cell. Then for every bivariate quadratic polynomial that vanishes on $\{v_1,\dots,v_4\}$ and is nonzero at v_6, the ratio between the values of the quadratic at v_5 and v_6 is unique.*

Proof. Let p and $\tilde{p}$ be distinct bivariate quadratic polynomials such that $p(v_i)=\tilde{p}(v_i)=0$ for $i=1,\dots,4$, $p(v_6)\neq 0$, and $\tilde{p}(v_6)\neq 0$. By Remark 2.10, we then have that $p(v_5)\neq 0$ and $\tilde{p}(v_5)\neq 0$. This means that $\frac{p(v_5)}{\tilde{p}(v_5)}$ is defined, and so we can say that $p(v_5)=k\tilde{p}(v_5)$ for some $k\in\mathbb{R}$. Then define $P=p-k\tilde{p}$, and note that $P(v_i)=0$ for $i=1,\dots,5$. Since P is a quadratic polynomial vanishing on $\{v_1,\dots,v_5\}$ and the points in V are the boundary vertices of a conic section, $P(v_6)=0$ by Claim 2.9. Thus $p(v_6)-k\tilde{p}(v_6)=0$, and so $p(v_6)=k\tilde{p}(v_6)$. Therefore $\frac{p(v_5)}{p(v_6)}=\frac{\tilde{p}(v_5)}{\tilde{p}(v_6)}$.

□

Remark 2.13. Let the points $V = \{v_1, \ldots, v_6\}$ be the boundary vertices of a conic cell. Then there is a unique (up to a constant multiple) bivariate quadratic polynomial that vanishes on $\{v_1, \ldots, v_4\}$ and whose value is nonzero at v_5 and v_6.

3 Geometric characterization of unconfinability

In this section, we offer a geometric criterion for the unconfinability of hexagonal cells. We begin by offering a geometric interpretation for the ratio between the nonzero values of a basic spline at the boundary vertices, and follow with a similar description of bivariate quadratic polynomials that vanish on all but two boundary vertices. These interpretations, in conjunction with the linear algebra considerations of Section 2, enable characterization of unconfinability in terms of the cross-ratio.

3.1 Ratios of Nonzero Spline Values

We show that the ratio of nonzero spline values at two boundary vertices can be expressed in terms of distances along the line through the edge between these boundary vertices.

Claim 3.1. *Let Δ_6 be a conic cell, let $s(x, y)$ be a basic spline on Δ_6 such that s is identically zero on σ_1, σ_2, and σ_3, and let $d(v_i, \ell_j)$ represent the distance from the point v_i to the line $\ell_j = 0$. Then the unique ratio between the values of s at v_5 and v_6 is given by*

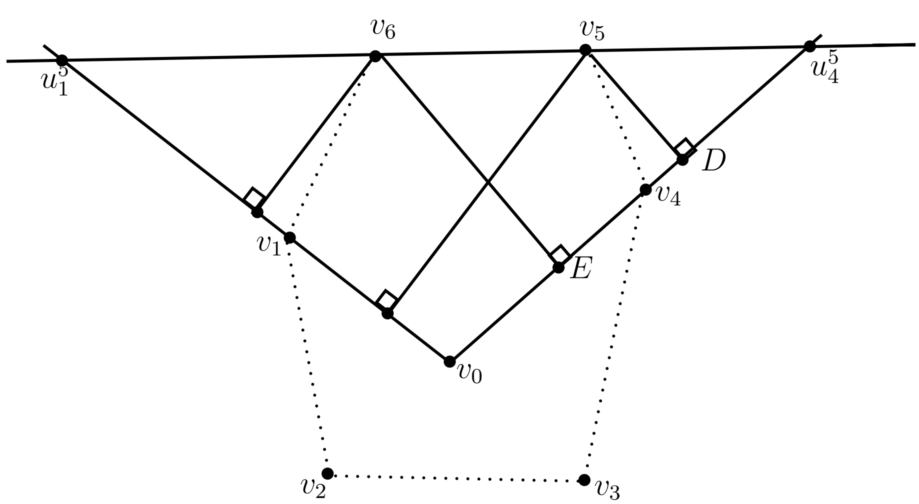
$$\frac{s(v_5)}{s(v_6)} = \frac{d(v_5, \ell_4) \cdot d(v_5, \ell_1)}{d(v_6, \ell_4) \cdot d(v_6, \ell_1)}.$$

Proof. Let p_4, p_6 denote the restriction of s to σ_4, σ_6, respectively; that is, $p_4(x, y) = c_4\ell_4^2$ and $p_6(x, y) = -c_1\ell_1^2$. Since s is an $\mathcal{S}_2^1$-spline, $s(v_5) = p_4(v_5)$ and $s(v_6) = p_6(v_6)$. Therefore, $s(v_5) = c_4(\ell_4(v_5))^2$ and $s(v_6) = -c_1(\ell_1(v_6))^2$. Because ℓ_4 is a linear form, the value of $\ell_4(v_5)$ is the distance from the point v_5 to the line $\ell_4 = 0$, which means that $\ell_4(v_5) = d(v_5, \ell_4)$. Similarly, $\ell_1(v_6) = d(v_6, \ell_1)$. If $\vec{v}_5$ represents the vector from the interior vertex of the conic cell to the point v_5, then $d(v_5, \ell_4) = \|\vec{v}_5\| \sin\theta_{45}$. In the same way, $d(v_6, \ell_1) = \|\vec{v}_6\| \sin\theta_{61}$. Therefore, we have that $s(v_5) = c_4(\ell_4(v_5))^2 = \|\vec{v}_5\|^2 \sin^2\theta_{45} \sin\theta_{56} \sin\theta_{51} \sin\theta_{61}$ and $s(v_6) = -c_1(\ell_1(v_6))^2 = \|\vec{v}_6\|^2 \sin^2\theta_{61} \sin\theta_{45} \sin\theta_{46} \sin\theta_{56}$. This means that the ratio of the values of s at v_5 and v_6 is

$$\frac{s(v_5)}{s(v_6)} = \frac{\|\vec{v}_5\|^2 \sin^2\theta_{45} \sin\theta_{56} \sin\theta_{51} \sin\theta_{61}}{\|\vec{v}_6\|^2 \sin^2\theta_{61} \sin\theta_{45} \sin\theta_{46} \sin\theta_{56}} = \frac{d(v_5, \ell_4) \cdot d(v_5, \ell_1)}{d(v_6, \ell_4) \cdot d(v_6, \ell_1)}.$$

□

We introduce the following notation in order to be able to discuss the interpolation of splines by quadratic functions in general. Let ℓ_{ij} be the linear form of the line that contains v_i and v_j. Let $v_k^j := v_{(j+k) \pmod 6)+1}$ for $j = 1, \ldots, 6$, and $k = 1, \ldots, 4$. Suppose u_k^j is the point of intersection of the line containing the interior edge through v_k^j and the line through v_j and v_{j+1}. Also let w_{mn}^j be the intersection of the line through v_j and v_{j+1} and the line through v_m^j and v_n^j.



Distances related to ratios of basic spline values.

Corollary 3.2. *Let Δ_6 be a conic cell, let $s(x,y)$ be a basic spline on Δ_6 such that s is identically zero on σ_1, σ_2, and σ_3, and let $d(v_i, \ell_j)$ represent the distance from the point v_i to the line $\ell_j = 0$. Then the unique ratio between the values of s at v_5 and v_6 is given by*

$$\frac{s(v_5)}{s(v_6)} = \frac{|v_5u_4^5| \cdot |v_5u_1^5|}{|v_6u_4^5| \cdot |v_6u_1^5|}.$$

Proof. Let D denote the point of intersection of ℓ_4 and the perpendicular from v_5 to ℓ_4 and let E denote the point of intersection of ℓ_4 and the perpendicular from v_6 to ℓ_4. Notice that the triangle with vertices D, v_5, u_4^5 is similar to the triangle with vertices E, v_6, u_4^5 since they both contain a right angle and they share $\angle Du_4^5v_5$. Therefore, $\frac{|v_5u_4^5|}{|v_6u_4^5|} = \frac{d(v_5,\ell_4)}{d(v_6,\ell_4)}$. Using the same argument involving the similarity of triangles, we can get that $\frac{|v_5u_1^5|}{|v_6u_1^5|} = \frac{d(v_5,\ell_1)}{d(v_6,\ell_1)}$. Therefore, from Claim 3.1 we have that

$$\frac{s(v_5)}{s(v_6)} = \frac{d(v_5,\ell_4) \cdot d(v_5,\ell_1)}{d(v_6,\ell_4) \cdot d(v_6,\ell_1)} = \frac{|v_5u_4^5| \cdot |v_5u_1^5|}{|v_6u_4^5| \cdot |v_6u_1^5|}.$$

□

3.2 Ratios of Bivariate Quadratic Polynomials

In Section 2, we proved that there exists a unique (up to scalar multiplication) bivariate quadratic polynomial that is nonzero at two boundary vertices and vanishes at the other boundary vertices. We now discuss a geometric expression for the ratio between the values of such quadratic polynomials at the boundary vertices.

Claim 3.3. *Suppose we have a set of six points $V = \{v_1, \ldots, v_6\}$ that are the boundary vertices of a conic cell. Then for all bivariate quadratic polynomials $q(x,y)$ that vanish on $\{v_1, \ldots, v_4\}$ but not on v_6, the ratio between the values of q at v_5 and v_6 is given by*

$$\frac{q(v_5)}{q(v_6)} = \frac{d(v_5,\ell_{ab}) \cdot d(v_5,\ell_{cd})}{d(v_6,\ell_{ab}) \cdot d(v_6,\ell_{cd})}$$

for all permutations a, b, c, d of the indices $\{1, \ldots, 4\}$, and where $d(v_i, \ell_{jk})$ the distance from v_i to the line $\ell_{jk} = 0$.

Proof. By Corollary 2.12, for any two bivariate quadratic polynomials that vanish on $\{v_1, \ldots, v_4\}$ but not on v_6, the ratio between the values of the quadratic function at v_5 and v_6 is unique. Therefore the quadratic polynomial can be constructed using two lines through any two pairs of the vertices on which we desire that q vanish, so the bivariate quadratic polynomial

$$Q_{abcd}(x,y) = \begin{vmatrix} x-x_a & x_a-x_b \\ y-y_a & y_a-y_b \end{vmatrix} \cdot \begin{vmatrix} x-x_c & x_c-x_d \\ y-y_c & y_c-y_d \end{vmatrix} \tag{8}$$

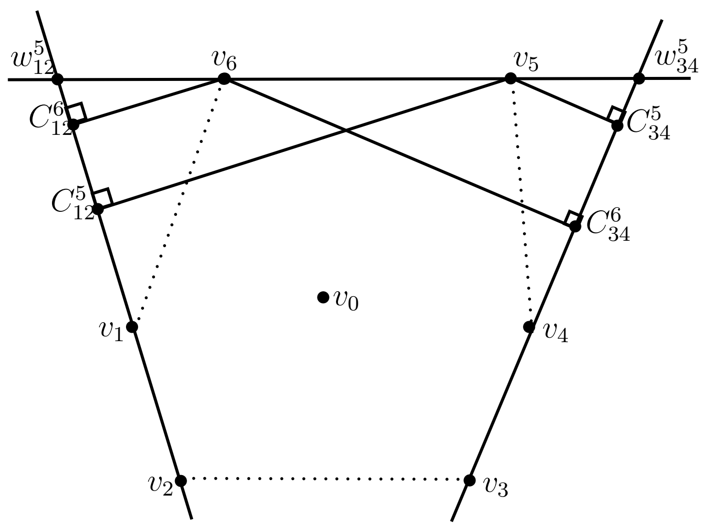
vanishes on $\{v_a, v_b, v_c, v_d\}$ but not on v_6. Evaluating Q_{abcd} at v_5 and v_6, we see that the first determinant in (8) gives the area of the parallelogram with sides $\vec{v}_{ab}$ and $\vec{v}_{ia}$, where $i = 5, 6$ and $\vec{v}_{jk} = \vec{v}_j - \vec{v}_k$. This area can also be expressed as $\|\vec{v}_{ab}\| \cdot \|\vec{v}_{ia}\| \sin\theta_{b,a,i}$, where $\theta_{b,a,i}$ is the angle between $\vec{v}_{ab}$ and $\vec{v}_{ia}$. Similarly interpreting the second determinant, we can express $Q_{abcd}(x_i, y_i)$ as

$$Q_{abcd}(x_i,y_i) = \|\vec{v}_{ab}\| \cdot \|\vec{v}_{ia}\| \sin\theta_{b,a,i} \cdot \|\vec{v}_{cd}\| \cdot \|\vec{v}_{ic}\| \sin\theta_{d,c,i}.$$

The ratio between $Q_{abcd}(v_5)$ and $Q_{abcd}(v_6)$ can then be determined by substitution into this formula. Cancellation of like terms gives

$$\frac{Q_{abcd}(v_5)}{Q_{abcd}(v_6)} = \frac{\|\vec{v}_{5a}\| \sin\theta_{b,a,5} \cdot \|\vec{v}_{5c}\| \sin\theta_{d,c,5}}{\|\vec{v}_{6a}\| \sin\theta_{b,a,6} \cdot \|\vec{v}_{6c}\| \sin\theta_{d,c,6}} = \frac{d(v_5,\ell_{ab}) \cdot d(v_5,\ell_{cd})}{d(v_6,\ell_{ab}) \cdot d(v_6,\ell_{cd})}.$$

By Corollary 2.12, we know that this ratio is the same for any quadratic polynomial that vanishes on $\{v_1, \ldots, v_4\}$ but not on v_5, v_6. Therefore $\frac{q(v_5)}{q(v_6)} = \frac{Q_{abcd}(v_5)}{Q_{abcd}(v_6)} = \frac{d(v_5,\ell_{ab}) \cdot d(v_5,\ell_{cd})}{d(v_6,\ell_{ab}) \cdot d(v_6,\ell_{cd})}$. □



Distances related to ratios of quadratic polynomial values.

Corollary 3.4. *Suppose the set $V = \{v_1, \ldots, v_6\}$ gives the boundary vertices of a conic cell. Then for any bivariate quadratic polynomial $q(x, y)$ that vanishes on $\{v_1, \ldots, v_4\}$ but not on v_6, the unique ratio between the values of q at v_5 and v_6 is given by*

$$\frac{q(v_5)}{q(v_6)} = \frac{|v_5 w_{ab}^5| \cdot |v_5 w_{cd}^5|}{|v_6 w_{ab}^5| \cdot |v_6 w_{cd}^5|} \tag{9}$$

where $|CD|$ is the length of line segment with endpoints C and D and a, b, c, d are all distinct indices in $\{1, \ldots, 4\}$.

Proof. By Claim 3.3, we have $\frac{q(v_5)}{q(v_6)} = \frac{d(v_5,\ell_{ab}) \cdot d(v_5,\ell_{cd})}{d(v_6,\ell_{ab}) \cdot d(v_6,\ell_{cd})}$. Let C_{ij}^k denote the point of intersection of the perpendicular line segment from ℓ_{ij} through v_k. The triangle with vertices v_5, C_{ab}^5, w_{ab}^5 is similar to the triangle with vertices v_6, C_{ab}^6, w_{ab}^5 because the two triangles share $\angle v_5 w_{ab}^5 v_a$ and both contain a right angle. Therefore there is a constant ratio between corresponding sides of the two triangles. Hence $\frac{d(v_5,\ell_{ab})}{d(v_6,\ell_{ab})} = \frac{|v_5 C_{ab}^5|}{|v_6 C_{ab}^6|} = \frac{|v_5 w_{ab}^5|}{|v_6 w_{ab}^5|}$. By the same argument, $\frac{d(v_5,\ell_{cd})}{d(v_6,\ell_{cd})} = \frac{|v_5 C_{cd}^5|}{|v_6 C_{cd}^6|} = \frac{|v_5 w_{cd}^5|}{|v_6 w_{cd}^5|}$. Therefore we can substitute into (9) for $\frac{q(v_5)}{q(v_6)}$ and obtain $\frac{q(v_5)}{q(v_6)} = \frac{|v_5 w_{ab}^5| \cdot |v_5 w_{cd}^5|}{|v_6 w_{ab}^5| \cdot |v_6 w_{cd}^5|}$. □

3.3 Geometric Characterization of Unconfinability

Using these ratios of distances, we are able describe when the values of a basic spline at the boundary vertices of a hexagonal cell can be interpolated by a bivariate quadratic polynomial, and thus provide a criterion for unconfinability.

Claim 3.5. *Suppose we have a conic cell Δ_6 with boundary vertices $\{v_1, \ldots, v_6\}$. Suppose also that we have an S_2^1-spline $s(x, y)$ that vanishes on the four boundary vertices other than v_j and v_{j+1} for some $j \in \{1, \ldots, 6\}$. Then the values of s at the six boundary vertices can be interpolated by a bivariate quadratic polynomial if and only if*

$$\frac{q(v_j)}{q(v_{j+1})} = \frac{s(v_j)}{s(v_{j+1})},$$

where $q(x, y)$ is the bivariate quadratic polynomial, unique up to a scalar multiple, that also vanishes on the same four vertices and is nonzero at v_j, v_{j+1}.

Proof. (⇒) Suppose the values of s at the boundary vertices can be interpolated by a bivariate quadratic polynomial. Then there exists some $q(x, y)$ such that $q(v_i) = s(v_i)$ for $i = 1, \ldots, 6$. We then have $q(v_j) = s(v_j)$ and $q(v_{j+1}) = s(v_{j+1})$, and therefore $\frac{q(v_j)}{q(v_{j+1})} = \frac{s(v_j)}{s(v_{j+1})}$. Because $s(v_i) = 0$ for $i \neq j, j+1$, then $q(v_i) = 0$ for $i \neq j, j+1$ and so by Corollary 2.12, this bivariate quadratic polynomial is unique up to a scalar multiple and $q(v_j)$ and $q(v_{j+1})$ are both nonzero.
(⇐) Suppose $\frac{q(v_j)}{q(v_{j+1})} = \frac{s(v_j)}{s(v_{j+1})}$, where q is the unique bivariate quadratic polynomial that vanishes on all boundary vertices except v_j and v_{j+1}. Then there is some k such that $s(v_j) = kq(v_j)$ and $s(v_{j+1}) = kq(v_{j+1})$. Trivially, $kq(v_i) = s(v_i) = 0$ for $i \neq j, j+1$. Hence $kq(v_i) = s(v_i)$ for $i = 1, \ldots, 6$, and the values of s at the six boundary vertices are interpolated by kq. □

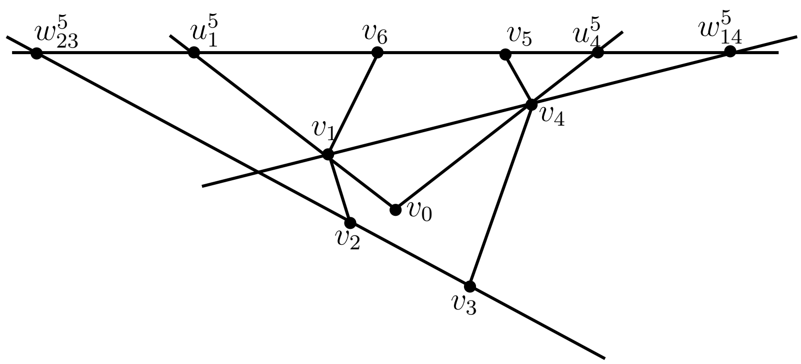
We now introduce the cross-ratio, a concept in projective geometry, which provides a concise method of comparing the ratios of values that we have discussed thus far.

Definition 3.6. Suppose x_1, x_2, x_3, and x_4 are points on a projective line L, with x_1, x_2, x_3 distinct. The *cross-ratio* of these four points, denoted $[x_1, x_2, x_3, x_4]$, is given by $\mathbf{P}_{x_1x_2x_3}(x_4)$, where $\mathbf{P}_{x_1x_2x_3}$ is the unique projective transformation such that $\mathbf{P}(x_1) = \infty$, $\mathbf{P}(x_2) = 0$, and $\mathbf{P}(x_3) = 1$.

Fact 3.7 (Theorem 21 of [7]). *Let x_1, x_2, x_3, x_4 be elements of $\mathbb{R} \cup \infty$, the first three of which are distinct. With the usual conventions about operations with 0 and ∞, the cross-ratio $[x_1, x_2, x_3, x_4]$ is given by*

$$[x_1, x_2, x_3, x_4] = \frac{(x_3 - x_1)(x_4 - x_2)}{(x_3 - x_2)(x_4 - x_1)}.$$

Claim 3.8. *Suppose Δ_6 is a conic cell with boundary vertices $\{v_1, \ldots, v_6\}$, and let s be a basic spline that is identically zero on σ_1, σ_2, and σ_3. Then for all bivariate quadratic polynomials that vanish on $\{v_a, v_b, v_c, v_d\}$ and whose values at v_j and v_{j+1} are nonzero, we have $\frac{q(v_j)}{q(v_{j+1})} = \frac{s(v_j)}{s(v_{j+1})}$ if and only if the two cross-ratios $[v_j, v_{j+1}, w^j_{ab}, u^j_1]$ and $[v_j, v_{j+1}, u^j_4, w^j_{cd}]$ are equal, where a, b, c, d are all distinct indices in $\{1, \ldots, 4\}$.*



Points used in calculation of cross-ratios for $j = 5$.

Proof. By Corollary 3.4 and Corollary 3.2, the ratio of the values of the quadratic equals the ratio of the values of the basic spline at v_5 and v_6 if and only if

$$\frac{q(v_j)}{q(v_{j+1})} = \frac{|v_j w^j_{ab}| \cdot |v_j w^j_{cd}|}{|v_{j+1} w^j_{ab}| \cdot |v_{j+1} w^j_{cd}|} = \frac{|v_j u^j_1| \cdot |v_j u^j_4|}{|v_{j+1} u^j_1| \cdot |v_{j+1} u^j_4|} = \frac{s(v_j)}{s(v_{j+1})}.$$

We can then rewrite the ratios as follows:

$$\frac{(w^j_{ab} - v_j)(w^j_{cd} - v_j)}{(w^j_{ab} - v_{j+1})(w^j_{cd} - v_{j+1})} = \frac{(u^j_1 - v_j)(u^j_4 - v_j)}{(u^j_1 - v_{j+1})(u^j_4 - v_{j+1})}.$$

Rearranging the terms gives

$$\frac{(w^j_{ab} - v_j)(u^j_1 - v_{j+1})}{(w^j_{ab} - v_{j+1})(u^j_1 - v_j)} = \frac{(u^j_4 - v_j)(w^j_{cd} - v_{j+1})}{(u^j_4 - v_{j+1})(w^j_{cd} - v_j)}$$

which, by Fact 3.7 is exactly the equality of the cross-ratios

$$[v_j, v_{j+1}, w^j_{ab}, u^j_1] = [v_j, v_{j+1}, u^j_4, w^j_{cd}].$$

□

Theorem 3.9. *Let Δ_6 be a conic cell with boundary vertices $\{v_1, \ldots, v_6\}$. Then the following are equivalent:*

1. *The function values at the six boundary vertices of Δ_6 of a basic spline s that is identically zero on $\sigma_1, \sigma_2,$ and σ_3 can be interpolated by a bivariate quadratic polynomial.*

2. *There exists a bivariate quadratic polynomial* $q(x, y)$ *that vanishes on* $\{v_1, \ldots, v_4\}$ *and whose value is nonzero at* v_5 *and* v_6 *such that*

$$\frac{s(v_5)}{s(v_6)} = \frac{q(v_5)}{q(v_6)}.$$

3. *For all bivariate quadratic polynomials* $q(x, y)$ *that vanish on* $\{v_1, \ldots, v_4\}$ *and whose value is nonzero at* v_5 *and* v_6*, we have the equality*

$$\frac{s(v_5)}{s(v_6)} = \frac{q(v_5)}{q(v_6)}.$$

4. *The two cross ratios* $[v_5, v_6, w^5_{14}, u^5_1]$ *and* $[v_5, v_6, u^5_4, w^j_{23}]$ *are equal.*

Proof. (1⇔2) This follows from Claim 3.5 with $j = 5$.
(2⇔3) (⇒) Suppose such a $q(x, y)$ exists. Then by Remark 2.13, this bivariate quadratic polynomial is unique up to a constant multiple, and so for all bivariate quadratic polynomials that vanish on $\{v_1, \ldots, v_4\}$ and whose value is nonzero at v_5 and v_6, $\frac{s(v_5)}{s(v_6)} = \frac{q(v_5)}{q(v_6)}$.
(⇐) Trivially, if all bivariate quadratic polynomials $q(x, y)$ that vanish on
$\{v_1, \ldots, v_4\}$ and whose values are nonzero at v_5 and v_6 are such that $\frac{s(v_5)}{s(v_6)} = \frac{q(v_5)}{q(v_6)}$, then there exists a $q(x, y)$ such that the equality of ratios holds.
(3⇔4) This follows from Claim 3.8, with $j = 5$, $a = 1$, $b = 4$, $c = 2$, and $d = 3$. □

Theorem 3.10. *Let* Δ_6 *be a conic cell with boundary vertices* $\{v_1, \ldots, v_6\}$*. Then the following are equivalent:*

1. *The conic cell* Δ_6 *is unconfinable.*

2. *The rank of* $\mathcal{M}$ *is 5.*

3. *The values of each of the column vectors* $\mathcal{S}_1, \mathcal{S}_2$*, and* $\mathcal{S}_3$ *can be interpolated by a bivariate quadratic polynomial.*

4. *The cross-ratios* $[v_j, v_{j+1}, w^j_{ab}, u^j_1]$ *and* $[v_j, v_{j+1}, u^j_4, w^j_{cd}]$ *are equal for* $j = 2, 3, 4$.

Proof. (1⇔2) This follows from Theorem 2.8.
(2⇔3) This follows from Claim 2.6 and Lemma 2.7.
(3⇔4) This follows from Theorem 3.9, taking $j = 2, 3, 4$ and the splines that are identically zero on $\sigma_4, \sigma_5,$ and σ_6, on σ_5, σ_6, and σ_1, and on σ_6, σ_1,and σ_2. □

4 Unconfinability of concordant hexagonal cells

The central aim of this section is to completely characterize the unconfinability of the class of hexagonal cells that we call concordant, and show that the property is preserved under projective transformations.

4.1 Definitions and Projective Geometry

Definition 4.1. A hexagonal cell where the lines connecting v_i and v_{i+3} for $i = 1, 2, 3$ all intersect at the interior vertex is called *concordant*. Otherwise, it is called *discordant*.

In order to continue our discussion of the characteristics of unconfinable cells, we briefly recall the most important definitions regarding projective geometry. For more information, seek a text on the subject, such as [7].

Definition 4.2. Let E be an $(n+1)$-dimensional vector space. A *projective point* is a one-dimensional vector subspace of E. An n-dimensional *projective space* $P^n(E)$ is the set of all the projective points.

Definition 4.3. Let F be a $(k+1)$-dimensional subspace of E. Then the k-dimensional projective space $P^k(F)$ is called the *projective subspace* of $P^n(E)$. Let L be a 2-dimensional vector subspace of $\mathbb{R}^3$, then $P^1(L)$ is a *projective line*.

Definition 4.4. Let g be a bijective linear map from a vector space E to a vector space E'. Then g induces a function $\mathbf{P}(g)$ between the projective spaces $P^n(E)$ and $P^n(E')$. The map $\mathbf{P}(g) : P^n(E) \to P^n(E')$ is called a *projective transformation*.

The following hold for projective transformations:

i. The image of a projective line is a projective line,

ii. The image of the intersection of two projective lines is the intersection of the images of the projective lines.

In order to prove the result, the following facts will be needed.

Claim 4.5. *Let $P^1(L)$ be a projective line in $\mathbb{R}^3$. Let $\mathbf{P}(g) : P^2(\mathbb{R}^3) \to P^2(\mathbb{R}^3)$ be a projective transformation, and let L' be the image of L under g. Then $\mathbf{P}(g \restriction L) : P^1(L) \to P^1(L')$ is a projective transformation of the projective line.*

Proof. The linear bijection $g \restriction L$ maps linear subspaces contained in L to linear subspaces in L', and so the projective transformation $\mathbf{P}(g \restriction L) : P^1(L) \to P^1(L')$ is well defined. □

Fact 4.6 (Theorem 20 in [7])**.** *Let $P^1(L)$ and $P^1(L')$ be projective lines. If there exists a projective transformation $\mathbf{P}(u) : P^1(L) \to P^1(L')$ taking projective points a, b, c, d in $P^1(L)$ to projective points a', b', c', d' in $P^1(L')$, then the cross-ratio $[a, b, c, d]$ equals $[a', b', c', d']$.*

Lemma 4.7. *Let $\mathbf{P}(g) : P^2(\mathbb{R}^3) \to P^2(\mathbb{R}^3)$ be a projective transformation, and L a 2-dimensional vector subspace of $\mathbb{R}^3$ containing the projective points a, b, c, d. If the images of a, b, c, d under $\mathbf{P}(g)$ are a', b', c', d' respectively, then the cross-ratios $[a, b, c, d]$ and $[a', b', c', d']$ are equal.*

Proof. Let $P^1(L')$ be the image of projective line $P^1(L)$ under $\mathbf{P}(g)$. By Claim 4.5 there exists a projective transformation $\mathbf{P}(g \restriction L) : P^1(L) \to P^1(L')$ that is the projective transformation of the restriction of g to plane L. Since the images of a, b, c, d under $\mathbf{P}(g)$ are given by a', b', c', d', by Fact4.6 the cross-ratio of points a, b, c, d equals that of a', b', c', d'. □

4.2 Key Properties of Projective Transformations

We will now introduce additional definitions in order to describe concordant hexagonal cells in the projective plane. Everywhere below, we will abuse the notation by identifying an element v of the vector space E with the corresponding projective point, i.e. the linear subspace spanned by v.

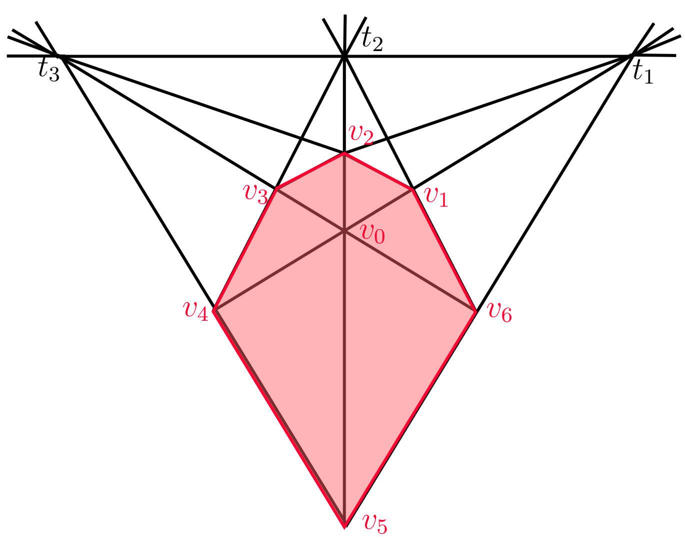
Definition 4.8. Let $\{e_0, \ldots, e_n\}$ be a basis in an $(n+1)$-dimensional vector space E, and $\mathbf{P}^n(E)$ an n-dimensional projective space. A *projective frame* is an ordered set of $n+2$ projective points relative to the basis, given by $\{e_0, \ldots, e_n, e_0 + \cdots + e_n\}$.

We let the associated projective frame of a hexagonal cell to be the projective points $\{t_1, v_0, t_3, v_5\}$.

Fact 4.9 (Theorem 4 in [7]). *Let $P^n(E)$ and $P^n(E')$ be projective spaces, with projective frames $\{p_0, \ldots, p_n, p_{n+1}\}$ and $\{p'_0, \ldots, p'_n, p'_{n+1}\}$ respectively. There exists a unique projective transformation $\mathbf{P}(g) : P^n(E) \to P^n(E')$ such that $\mathbf{P}(g(p_i)) = p'_i$ for all $i = 0, 1, \ldots, n, n+1$.*

Definition 4.10. Let a hexagonal cell Δ_6 have vertices v_i for $i = 1, \ldots, 6$ with interior vertex v_0. If the lines $\overline{v_i v_{i+3}}$, $\overline{v_{i+1} v_{i+2}}$, and $\overline{v_{i+4} v_{i+5}}$ are concurrent, then we call the collection of these three lines the i^{th} *pencil* of Δ_6, and denote the set of lines by the symbol T_i. The *pencil point* t_i is the common point of intersection of the lines in T_i.

The image below is an example of a concordant hexagonal cell with the pencils drawn and pencil points labeled t_i for $i = 1, 2, 3$.



Proposition 4.11. *A concordant hexagon is unconfinable if and only if it has three pencils T_i for $i = 1, 2, 3$.*

Proof. (⇒) Suppose a concordant hexagonal cell Δ_6 is unconfinable. Then the function values of each basic spline at the six vertices can be interpolated by the quadratic function which vanishes on $v_1^j, v_2^j, v_3^j, v_4^j$ and so we have $[v_j, v_{j+1}, w_{14}^j, u_1^j] = [v_j, v_{j+1}, u_4^j, w_{23}^j]$ by Claim 3.8. Because Δ is concordant, the diagonals intersect at the interior vertex, and therefore the line containing v_1^j and v_4^j is the same as the interior edges through both v_1^k and v_4^j, and so these lines all intersect the line containing v_j and v_{j+1} at the same point. Therefore $w_{14}^j = u_1^j$. By Definition 3.6, if $w_{14}^j = u_1^j$, then the cross-ratio is given by $[v_j, v_{j+1}, w_{14}^j, u_1^j] = \mathbf{P}_{v_j v_{j+1} w_{14}}(u_1^j) = \mathbf{P}_{v_j v_{j+1} w_{14}}(w_{14}) = 1$. Because the two cross ratios on the line through v_j and v_{j+1} are equal then also $[v_j, v_{j+1}, u_4^j, w_{23}^j] = 1$, and Definition 3.6 similarly implies that $u_4^j w_{23}^j$, so the line through v_2^j and v_3^j intersects the line containing v_j and v_{j+1} at the same point as the line through v_1^j and v_4^j, so the lines through opposite sides and the diagonal are concurrent, forming a pencil.
(⇐) Now suppose that the lines containing opposite edges of the hexagonal cell Δ and the diagonal are concurrent. Then $u_1^j = w_{14}^j$ and $w_{23}^j = u_4^j$, for $j = 1, 2, 3$, so by Definition 3.6, we have $[v_j, v_{j+1}, u_1^j, w_{14}^j] = 1$ and $[v_j, v_{j+1}, w_{23}^j, u_4^j] = 1$, so the two cross-ratios on the line through v_j and v_{j+1} are equal for $j = 1, 2, 3$, and Δ is unconfinable by Claim 3.8. □

Example 4.12. Let the *regular hexagonal cell* Δ^{reg} have vertices $(v_1, \ldots, v_6)$ given by the projective points $[\cos\frac{k\pi}{3} : \sin\frac{k\pi}{3} : 1]$ for $k = 0, \ldots, 5$. This unconfinable, concordant hexagonal cell centered at $[0 : 0 : 1]$ will have pencil points $t_i, i = 1, 2, 3$ at points at infinity, given by $t_1 = [1 : 0 : 0], t_2 = [\frac{1}{2} : \frac{\sqrt{3}}{2} : 0], t_3 = [\frac{-1}{2} : \frac{\sqrt{3}}{2} : 0]$. The associated projective frame of Δ^{reg} is $\{[0 : 0 : 2], [-2 : 0 : 0], [1 : -\sqrt{3} : 0], [-1 : -\sqrt{3} : 2]\}$.

Lemma 4.13. *Given any collection of four projective points* a, b, c, d*, there exists a unique hexagonal cell* Δ_6 *such that the pencil points* t_1, t_3 *of* Δ_6 *are* a *and* c*, the interior vertex of* Δ_6 *is* b*, and* v_5 *of* Δ_6 *is* d*.*

Proof. Let a, b, c, d be labeled t_1, v_0, t_3, v_5, and $\{t_1, v_0, t_3, v_5\}$ be the projective frame. We will construct the unique hexagonal cell. Draw a line through each pair of points in the frame. The intersection of $\overline{t_1 v_5}$ and $\overline{t_3 v_0}$ is the point v_6, and the intersection of $\overline{t_1 v_0}$ and $\overline{t_3 v_5}$ is v_4. The pencil point t_2 is the intersection of the line $\overline{t_1 t_3}$ with $\overline{v_0 v_5}$. The line $\overline{t_2 v_6}$ intersects with $\overline{t_1 v_0}$ at vertex v_1, and $\overline{t_2 v_4}$ intersects with $\overline{t_3 v_0}$ at vertex v_3. The final vertex, v_2 is the intersection of $\overline{t_1 v_3}$ and $\overline{v_0 v_5}$. Thus all vertices are uniquely determined, and the hexagonal cell can be constructed using these vertices. □

4.3 Proof of Theorem 4.14

Theorem 4.14. *Let Δ_6 be a concordant hexagonal cell. Then the following are equivalent:*

1. *The cell Δ_6 is unconfinable.*

2. *The image of Δ_6 under some projective transformation is unconfinable.*

3. *There exists a projective transformation under which the image of Δ_6 is the regular hexagonal cell Δ^{reg}.*

Proof. $(1 \iff 2)$ Given the unconfinable cell Δ_6, by Theorem 3.10 the cross-ratios of Δ_6 $[v_j, v_{j+1}, w^j_{ab}, u^j_1]$ and $[v_j, v_{j+1}, u^j_4, w^j_{cd}]$ are equal for $j = 2, 3, 4$. By Lemma 4.7 , each pair of the cross-ratios are preserved over some projective transformation, and so the cross-ratios $[v_j, v_{j+1}, w^j_{ab}, u^j_1]$ and $[v_j, v_{j+1}, u^j_4, w^j_{cd}]$ are equal for $j = 2, 3, 4$ of Δ'_6. Therefore by Theorem 3.10 , Δ'_6 is unconfinable. The converse can be seen by taking the inverse projective transformation of Δ'_6, and noting that the cross-ratios are preserved.
$(1 \Rightarrow 3)$ Let $P^2(\mathbb{R}^3)$ be the projective plane containing Δ_6, and $P^2(\mathbb{R}^{3'})$ the projective plane containing regular hexagonal cell Δ^{reg}. Let the vertices of Δ_6 be vectors in $\mathbb{R}^3$ given by v_i for $i = 1, \ldots, 6$. Since Δ_6 is unconfinable and concordant, by Proposition 4.11 each pencil of Δ_6 must have an pencil point. The three pencil points can be given as projective points t_1, t_2, t_3. By the assumption that no 4 points on Δ_6 are collinear, the vectors t_1, v_0, t_3 form a basis in $\mathbb{R}^3$. We define a projective frame in $P^2(\mathbb{R}^3)$ as $\{\alpha t_1, \beta v_0, \gamma t_3, v_5\}$. As the vectors corresponding to t_1, v_0, t_3 form a basis in $\mathbb{R}^3$, there exists scalars α, β, γ such that $\alpha t_1 + \beta v_0 + \gamma t_3 = v_5$, thus satisfying the conditions for a projective frame. By Fact 4.9, there exists a projective transformation that uniquely sends this projective frame of Δ_6 to the projective frame of Δ^{reg} in $P^2(\mathbb{R}^{3'})$. Therefore, Lemma 4.13 implies that the remaining vertices of the image Δ_6 are uniquely determined, and must be the vertices of Δ^{reg}. Thus, the image of Δ_6 under some projective transformation is the regular hexagonal cell Δ^{reg}.

(1 $\Leftarrow$ 3) We know that Δ_6 must be concordant, and want to show that it is unconfinable. We take the projective transformation of the pencils of Δ^{reg}, and by the properties of projective transformations, each $t_i, i = 1, 2, 3$ will map to the intersections of the images of the pencils. Since Δ_6 is projectively equivalent to Δ^{reg}, the images of the pencils must coincide with the edges of Δ_6, and so Δ_6 has three pencil points. Therefore, Δ_6 is unconfinable. $\square$

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