L^p estimates for a singular integral operator motivated by Calderón's second commutator

Eyvindur Ari Palsson

University of Rochester
Department of Mathematics
Rochester, NY, USA

Incompressible Fluids, Turbulence and Mixing
In honor of Peter Constantin's 60th birthday

October 15 2011

The Water Wave Problem

- Inviscid, incompressible, irrotational fluid in $\Omega(t)$, $t \geq 0$, under influence of gravity.

The Water Wave Problem

- Inviscid, incompressible, irrotational fluid in $\Omega(t)$, $t \geq 0$, under influence of gravity.
- Air above fluid and a free interface $\sum(t)$, $t \geq 0$, separates the two.

The Water Wave Problem

- Inviscid, incompressible, irrotational fluid in $\Omega(t)$, $t \geq 0$, under influence of gravity.
- Air above fluid and a free interface $\sum(t)$, $t \geq 0$, separates the two.
- Motion of fluid is described by

$$
\begin{aligned}
&\mathbf{v}_t + \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla P - \mathbf{k} \quad \text{on } \Omega(t), t \geq 0, \\
&\text{div } \mathbf{v} = 0, \ \text{curl } \mathbf{v} = 0 \quad \text{on } \Omega(t), t \geq 0, \\
&P = 0 \quad \text{on } \textstyle\sum(t), \\
&(1, \mathbf{v}) \text{ is tangent to the free surface } (t, \textstyle\sum(t))
\end{aligned}
$$

Theorem (Sijue Wu, 2009)

Let initial interface be a graph $z(\alpha, 0) = \alpha + i\epsilon f(\alpha)$*, initial velocity* $z_t(\alpha, 0) = \epsilon g(\alpha), \alpha \in \mathbb{R}$ *where* f*,* g *are smooth and decay fast at infinity.*

There exist $\epsilon_0 > 0$*,* $T > 0$*, depending only on* f *and* g *such that for* $0 < \epsilon < \epsilon_0$*, the initial value problem of the 2-D water wave system has a unique classical solution for a time period* $[0, e^{T/\epsilon}]$*. During this time period, the solution has the same regularity as the initial data and remains small, and the interface is a graph.*

- Variables related through the double layer potential operator, K.

- Variables related through the double layer potential operator, K.
- If $u = Kv$ then need to show L^p estimates of the type

$$\|\Gamma u\|_p \leq C\|\Gamma v\|_p$$

where Γ is a differentiation operator.

- Variables related through the double layer potential operator, K.
- If $u = Kv$ then need to show L^p estimates of the type

$$\|\Gamma u\|_p \leq C\|\Gamma v\|_p$$

  where Γ is a differentiation operator.
- The core problem is finding L^p estimates for the commutator $[\Gamma, K]$.

- Sijue Wu faces integrals of the type

$$p.v. \int_{\mathbb{R}} F\left(\frac{A(x)-A(y)}{x-y}\right) \frac{\Pi_{i=1}^{n}(B_i(x)-B_i(y))}{(x-y)^{n+1}} f(y)\, dy$$

- Sijue Wu faces integrals of the type

$$p.v. \int_{\mathbb{R}} F\left(\frac{A(x)-A(y)}{x-y}\right) \frac{\Pi_{i=1}^{n}(B_i(x)-B_i(y))}{(x-y)^{n+1}} f(y)\, dy$$

- L^p estimates are known if $A', B_i' \in L^\infty(\mathbb{R})$ for $i = 1, \ldots, n$ and $f \in L^2(\mathbb{R})$.

- Sijue Wu faces integrals of the type

$$p.v. \int_{\mathbb{R}} F\left(\frac{A(x)-A(y)}{x-y}\right) \frac{\Pi_{i=1}^{n}(B_i(x)-B_i(y))}{(x-y)^{n+1}} f(y)\, dy$$

- L^p estimates are known if $A', B_i' \in L^\infty(\mathbb{R})$ for $i = 1, \ldots, n$ and $f \in L^2(\mathbb{R})$.
- Novelty in Sijue Wu's paper is that she faces $B_1' \in L^2(\mathbb{R})$.

- Sijue Wu faces integrals of the type

$$p.v. \int_{\mathbb{R}} F\left(\frac{A(x)-A(y)}{x-y}\right) \frac{\Pi_{i=1}^{n}(B_i(x)-B_i(y))}{(x-y)^{n+1}} f(y)\, dy$$

- L^p estimates are known if $A', B_i' \in L^\infty(\mathbb{R})$ for $i = 1, \ldots, n$ and $f \in L^2(\mathbb{R})$.
- Novelty in Sijue Wu's paper is that she faces $B_1' \in L^2(\mathbb{R})$.
- Her setup is $A', B_i' \in L^\infty(\mathbb{R})$ for $i = 2, \ldots, n$, $B_1' \in L^2(\mathbb{R})$ and $f \in L^\infty(\mathbb{R})$.

- Sijue Wu faces integrals of the type

$$p.v. \int_{\mathbb{R}} F\left(\frac{A(x)-A(y)}{x-y}\right) \frac{\Pi_{i=1}^{n}(B_i(x)-B_i(y))}{(x-y)^{n+1}} f(y)\, dy$$

- L^p estimates are known if $A', B_i' \in L^\infty(\mathbb{R})$ for $i = 1, \ldots, n$ and $f \in L^2(\mathbb{R})$.
- Novelty in Sijue Wu's paper is that she faces $B_1' \in L^2(\mathbb{R})$.
- Her setup is $A', B_i' \in L^\infty(\mathbb{R})$ for $i = 2, \ldots, n$, $B_1' \in L^2(\mathbb{R})$ and $f \in L^\infty(\mathbb{R})$.
- Bounds can be obtained using $T(b)$ theorem.

Calderón Commutators

Definition

The k-th Calderón commutator, $k \in \{1, 2, 3, \ldots\}$, is given by

$$\mathcal{C}_A^{(k)} f(x) = \int_{\mathbb{R}} \frac{1}{x-y} \left(\frac{A(x) - A(y)}{x-y} \right)^k f(y) dy$$

where A is Lipschitz and $A' \in L^\infty(\mathbb{R})$.

Calderón Commutators

Definition

The k-th Calderón commutator, $k \in \{1, 2, 3, \ldots\}$, is given by

$$\mathcal{C}_A^{(k)} f(x) = \int_{\mathbb{R}} \frac{1}{x-y} \left(\frac{A(x) - A(y)}{x-y} \right)^k f(y) dy$$

where A is Lipschitz and $A' \in L^\infty(\mathbb{R})$.

- Coifman and Meyer showed in 1975

  $$\mathcal{C}_A^{(k)} : L^p \to L^p \text{ for } 1 < p < \infty$$

  for $k = 1, 2, \ldots$

Bilinear Hilbert Transform

- $\dfrac{A(x)-A(y)}{x-y} = \displaystyle\int_0^1 A'(x+\alpha(y-x))d\alpha$

Bilinear Hilbert Transform

- $\dfrac{A(x)-A(y)}{x-y} = \displaystyle\int_0^1 A'(x+\alpha(y-x))d\alpha$
- Calderón wrote

$$\begin{aligned}
\mathcal{C}_A^{(1)} f(x) &= \int_{\mathbb{R}} \left(\frac{A(x)-A(y)}{x-y}\right)(x-y)^{-1} f(y)dy \\
&= \int_0^1 \int_{\mathbb{R}} A'(x+\alpha(y-x))(x-y)^{-1} f(y)dyd\alpha \\
&= \int_0^1 \int_{\mathbb{R}} A'(x+\alpha t) f(x+t)\frac{1}{t}dtd\alpha
\end{aligned}$$

- The Bilinear Hilbert Transform is defined as

$$BHT_\alpha(f_1, f_2)(x) = p.v. \int_{\mathbb{R}} f_1(x + \alpha t) f_2(x + t) \frac{1}{t} dt$$

- The Bilinear Hilbert Transform is defined as

$$BHT_{\alpha}(f_1, f_2)(x) = p.v. \int_{\mathbb{R}} f_1(x + \alpha t) f_2(x + t) \frac{1}{t} dt$$

Theorem (Lacey and Thiele from 1997 and 1999)

Let $\alpha \notin \{0, 1\}$, $1 < p_1, p_2 \leq \infty$ and $\frac{2}{3} < p := \frac{p_1 p_2}{p_1 + p_2} < \infty$. Then there exists a constant C_{α,p_1,p_2} such that

$$\|BHT_{\alpha}(f_1, f_2)\|_p \leq C_{\alpha,p_1,p_2} \|f_1\|_{p_1} \|f_2\|_{p_2}$$

for all f_1 and f_2 in $\mathcal{S}(\mathbb{R})$.

- The Bilinear Hilbert Transform is defined as

$$BHT_\alpha(f_1, f_2)(x) = p.v. \int_{\mathbb{R}} f_1(x + \alpha t) f_2(x + t) \frac{1}{t} dt$$

Theorem (Lacey and Thiele from 1997 and 1999)

Let $\alpha \notin \{0, 1\}$, $1 < p_1, p_2 \leq \infty$ and $\frac{2}{3} < p := \frac{p_1 p_2}{p_1 + p_2} < \infty$. Then there exists a constant C_{α,p_1,p_2} such that

$$\|BHT_\alpha(f_1, f_2)\|_p \leq C_{\alpha,p_1,p_2} \|f_1\|_{p_1} \|f_2\|_{p_2}$$

for all f_1 and f_2 in $\mathcal{S}(\mathbb{R})$.

- Applications to AKNS systems.

Trilinear Hilbert Transform

$$\mathcal{C}_A^{(2)} f(x) = \int_{\mathbb{R}} \left(\frac{A(x) - A(y)}{x - y} \right)^2 (x-y)^{-1} f(y) dy$$
$$= \int_0^1 \int_0^1 \int_{\mathbb{R}} A'(x + \alpha_1 t) A'(x + \alpha_2 t) f(x+t) \frac{1}{t} dt\, d\alpha_1\, d\alpha_2$$

Trilinear Hilbert Transform

$$\mathcal{C}_A^{(2)} f(x) = \int_{\mathbb{R}} \left(\frac{A(x) - A(y)}{x - y} \right)^2 (x-y)^{-1} f(y) dy$$
$$= \int_0^1 \int_0^1 \int_{\mathbb{R}} A'(x + \alpha_1 t) A'(x + \alpha_2 t) f(x+t) \frac{1}{t} dt d\alpha_1 d\alpha_2$$

- The Trilinear Hilbert Transform is defined as

$$THT_{\vec{\alpha}}(f_1, f_2, f_3)(x) = p.v. \int_{\mathbb{R}} f_1(x+\alpha_1 t) f_2(x+\alpha_2 t) f_3(x+t) \frac{1}{t} dt$$

Trilinear Hilbert Transform

$$\mathcal{C}_A^{(2)} f(x) = \int_{\mathbb{R}} \left(\frac{A(x) - A(y)}{x - y} \right)^2 (x-y)^{-1} f(y) dy$$
$$= \int_0^1 \int_0^1 \int_{\mathbb{R}} A'(x + \alpha_1 t) A'(x + \alpha_2 t) f(x + t) \frac{1}{t} dt d\alpha_1 d\alpha_2$$

- The Trilinear Hilbert Transform is defined as

$$THT_{\vec{\alpha}}(f_1, f_2, f_3)(x) = p.v. \int_{\mathbb{R}} f_1(x+\alpha_1 t) f_2(x+\alpha_2 t) f_3(x+t) \frac{1}{t} dt$$

- Open question: Find L^p estimates for $THT_{\vec{\alpha}}$.

Main Theorem

Define

$$T_\beta(f_1, f_2, f_3)(x) = \\ p.v. \int_{\mathbb{R}} \left(\int_0^1 f_1(x + \alpha t) d\alpha \right) f_2(x + \beta t) f_3(x + t) \frac{1}{t} dt$$

Theorem (P.)

Let $\beta \notin \{0, 1\}$, $1 < p_1, p_2, p_3 \leq \infty$,

$$\frac{1}{2} < p := \frac{p_1 p_2 p_3}{p_1 p_2 + p_1 p_3 + p_2 p_3} < \infty \quad \textit{and} \quad \frac{2}{3} < \frac{p_2 p_3}{p_2 + p_3} \leq \infty.$$

Then there exists a constant C_{β, p_1, p_2, p_3} *such that*

$$\| T_\beta(f_1, f_2, f_3) \|_p \leq C_{\beta, p_1, p_2, p_3} \| f_1 \|_{p_1} \| f_2 \|_{p_2} \| f_3 \|_{p_3}$$

for all f_1, f_2 *and* f_3 *in* $\mathcal{S}(\mathbb{R})$.

Future Applications?

- Motivated by Sijue Wu's operator it would be interesting to obtain L^p estimates for

$$p.v. \int_{\mathbb{R}} F\left(\frac{A(x+t)-A(x)}{t}\right) b(x+\beta t) f(x+t) \frac{1}{t}\, dt$$

Future Applications?

- Motivated by Sijue Wu's operator it would be interesting to obtain L^p estimates for

$$p.v. \int_{\mathbb{R}} F\left(\frac{A(x+t)-A(x)}{t}\right) b(x+\beta t) f(x+t) \frac{1}{t}\, dt$$

- First need L^p estimates for

$$p.v. \int_{\mathbb{R}} \left(\frac{A(x+t)-A(x)}{t}\right)^m b(x+\beta t) f(x+t) \frac{1}{t}\, dt$$

with polynomial bounds in m.

Future Applications?

- Motivated by Sijue Wu's operator it would be interesting to obtain L^p estimates for

$$p.v. \int_{\mathbb{R}} F\left(\frac{A(x+t)-A(x)}{t}\right) b(x+\beta t) f(x+t) \frac{1}{t}\, dt$$

- First need L^p estimates for

$$p.v. \int_{\mathbb{R}} \left(\frac{A(x+t)-A(x)}{t}\right)^m b(x+\beta t) f(x+t) \frac{1}{t}\, dt$$

  with polynomial bounds in m.

- Result on operator on previous slide is the first step, showing a wide range of L^p estimates for the case $m=1$.