CNA Working Group, Fall 2008
Systems of Conservation Laws and Viscosity Solutions
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The concept of hyperbolic systems of balance laws was introduced in the works of natural philosophers of the eighteenth century; the equations which are studied in this field arise in Continuum Physics as reformulations of basic laws governing the motion of physical bodies such as fluids, gases etc. A particular feature of non-linear hyperbolic systems is the loss of regularity and appearance of shock waves in their solutions. The global well-posedness can hence be expected only in the space of discontinuous functions. Due to this fact, most modern mathematical techniques are not applicable, and new specific methods must have been devised.

The purpose of this working group is to provide an introduction to the mathematical theory of hyperbolic systems of conservation laws, with the scope of understanding the major recent advances on uniqueness, stability and convergence of the viscosity approximations.

The outlay of the program is as follows:

(1) We start by a self-contained introduction to the general theory of systems of conservation laws in 1 space dimension; preliminaries on hyperbolicity, weak solutions, Rankine-Hugoniot conditions, entropy conditions, the Riemann problem can be taken from [Evans, Bressan].

(2) The global existence of solutions (with small Total Variation) via the Glimm scheme, the deterministic version of it, or the purely deterministic wave front tracking algorithm will be discussed.

(3) The fact that these global solutions can be seen as trajectories of the unique Lipschitz continuous semigroup S was first proven by Bressan, Crasta and Piccoli, but a much simpler proof of the same fact was later worked out by Bressan, Liu and Yang (1999). The key step there is the construction of a Lyapunov functional, equivalent to the L^1 distance and almost decreasing in time along each pair of approximate solutions constructed by means of the wave front algorithm.

(4) The stability result refers to a special class of weak solutions: those obtained as limits of the Glimm or the wave front tracking algorithms. One would want for a criterion, characterizing those entropy weak solutions that coincide with a trajectory of the unique solution flow S. An example of such sufficient condition was obtained by Bressan and Lewicka (2000).

(5) We shall then concentrate on the celebrated work [Bressan-Bianchini], establishing convergence of solutions to parabolic regularization of systems of conservation laws, given by adding the small viscosity term, to the inviscid solutions of S. This is a highly nontrivial result, which has persisted as a major open problem for decades. The heart of the problem is to establish uniform bounds on the total variation of viscous solutions, valid for arbitrarily large times, The ingenious construction of [Bressan-Bianchini] relays on decomposition of gradients of solutions along a basis of unit vectors, which are gradients of those traveling wave profiles, which lie on a suitable center manifold. The construction of this center manifold and the related wave decomposition will be considered.

(6) All the above mentioned results relate to 1 space dimension. For multi-dimensional systems of conservation laws, very little is known at any sufficient level of generality, and the few existing well-posedness results are still based on the 1-dimensional ideas. For example, for systems whose fluxes exhibit certain rotational symmetry, as studied in [Ambrosio-DeLellis], the norm of the solution obeys a scalar multidimensional conservation law and hence is given by construction in [Kruzkhov]. The evolution of the angular component is, in turn, governed by a transport problem with BV coefficients. The proof of well-posedness of the original system hence relies on results in [Ambrosio], which extend the DiPerna-Lions theory of ODE with discontinuous coefficients to BV vector fields satisfying natural L^∞ bounds on the distributional divergence.

References

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See also: L. Ambrosio, *Transport equation and Cauchy problem for non-smooth vector fields. Calculus of variations and nonlinear partial differential equations*, 1–41, Lecture Notes in Math., 1927, Springer, Berlin, 2008.