HOMOGENIZATION OF FULLY NONLINEAR FIRST & SECOND ORDER PDE IN STATIONARY ERGODIC MEDIA

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- medium with self-averaging properties

  periodic, almost periodic, random (stationary ergodic)

- phenomenon/law described by a spatially inhomogeneous equation

$$F[u, x] = 0$$

- look from far away — scale ε

$$F\left[u^\varepsilon, \frac{x}{\varepsilon}\right] = 0$$

- microscopic properties average out in the limit $\varepsilon \to 0$

$$u^\varepsilon \longrightarrow \bar{u}$$

and

$$\bar{F}[\bar{u}\,] = 0$$

spatially homogeneous, translation invariant equation

PROBABILISTIC SETTING

$\Omega = \{\text{“all equations”}\}$ P probability on Ω frequency to have certain equation

$F_\omega(\cdot,\cdot) = F(\cdot,\cdot,\omega)$ element of Ω

- at position x we observe $F(\cdot, x, \omega)$

  at position $x+y$ we observe same configuration labeled by different ω but with the same frequency

  $$F(\cdot, x+y, \omega) = F(\cdot, x, \tau_y \omega) \quad \text{and } \tau_y \text{ is measure preserving} \qquad \text{(stationarity)}$$

- at large scale “media averages”

  under translations operators “repeat each other”

  $$A \subset \Omega\ ,\ P(A) > 0 \Longrightarrow P\Big(\underbrace{\bigcup_{y\in\mathbb{R}^n}}_{\text{translation invariant}} \tau_y A\Big) = 1 \qquad \text{(ergodicity)}$$

GENERAL PROBLEM

$$(\mathrm{BVP})_\varepsilon \qquad \begin{cases} F(D^2u_\varepsilon, \varepsilon D^2u_\varepsilon, Du_\varepsilon, u_\varepsilon, x, \frac{x}{\varepsilon}, \omega) = 0 & \text{in} \ \ U \subset \mathbb{R}^N \\ u_\varepsilon = g \ \ \text{on} \ \ \partial U \end{cases}$$

ASSUMPTION

F stationary ergodic wrt $(x/\varepsilon, \omega)$

CONCLUSION

$$u_\varepsilon(\cdot, \omega) \underset{\varepsilon \to 0}{\longrightarrow} \bar{u} \qquad \text{in } C(\bar{U}) \text{ and a.s. in } \omega$$

$$\overline{(\mathrm{BVP})} \qquad \begin{cases} \bar{F}(D^2\bar{u}, D\bar{u}, \bar{u}, x) = 0 & \text{in} \ \ U \\ \bar{u} = g \ \ \text{on} \ \ \partial U \end{cases}$$

HISTORY

- periodic

  almost periodic

many references

many open problems

- stationary ergodic

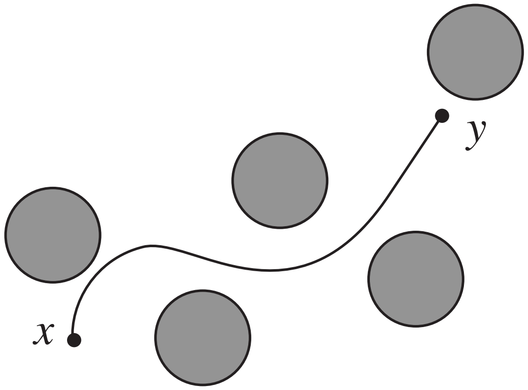
linear theory	Papanicolaou-Varadhan; Kozlov; Jikov; Yurinskii;...
variational nonlinear	Dal Masso-Modica
nonvariational nonlinear	Souganidis; Rezankhanlou-Tarver
viscous Hamilton-Jacobi	Lions-Souganidis Kosygina-Rezankhanlou-Varadhan Freidlin (N=1); Sznitman;...
fully nonlinear second-order	Caffarelli-Souganidis-Wang; Caffarelli-Souganidis

HAMILTON-JACOBI EQUATIONS

$$H\big(Du^\varepsilon, u_\varepsilon, \frac{x}{\varepsilon}\big) = 0$$

- effective geodesics

  random medium



$$|Du^\varepsilon| - V\big(\frac{x}{\varepsilon}, \omega\big) = 0 \qquad u^\varepsilon(x,\omega) = \inf\Big\{\int_0^1 V\big(\frac{x(s)}{\varepsilon}, \omega\big)\, ds : x(0) = 0,\ x(1) = x,\ |\dot{x}| \leqq 1\Big\}$$

$$u^\varepsilon(\cdot, \omega) \to \bar{u} \quad \text{as} \quad \varepsilon \to 0$$

$$\bar{H}(D\bar{u}\,) = 0 \qquad \bar{u}(x) = \inf\Big\{\int_0^1 \bar{H}^*\big(x(s)\big)\, ds : x(0) = 0,\ x(1) = x,\ |\dot{x}| \leqq 1\Big\}$$

($\bar{H}^*$ convex dual of $\bar{H}$)

recall H convex

$$H^*(q) = \sup\{q \cdot p - H(p)\}$$

$$H(p) = \sup\{p \cdot q - H^*(q)\}$$

“VISCOUS” HAMILTON-JACOBI (BELLMAN EQUATIONS)

$$-\varepsilon\,\mathrm{tr}\,A(x,\frac{x}{\varepsilon},\omega)D^2u_\varepsilon + H(Du_\varepsilon, u_\varepsilon, x, \frac{x}{\varepsilon},\omega) = 0$$

- large deviations of Brownian motions in random environments

$$\begin{cases} dX_t^\varepsilon = \sqrt{2\varepsilon}\,\sigma(X_t^\varepsilon, \varepsilon^{-1}X_t^\varepsilon,\omega)\,dB_t \\ X_0^\varepsilon = x \in U \end{cases}$$

τ_x^ε exit time from $U \subset \mathbb{R}^N$

“effective” asymptotics of $P(X^\varepsilon_{\tau_x} \in \Gamma)$ $(\Gamma \subset \partial U)$

$$u_\varepsilon = P(X^\varepsilon_{\tau^\varepsilon_x} \in \Gamma) = \exp(-\frac{1}{\varepsilon}I^\varepsilon)$$

$$-\varepsilon\,\mathrm{tr}[(\sigma\sigma^T)(x,\frac{x}{\varepsilon},\omega)D^2I^\varepsilon] + ((\sigma\sigma^T)(x,\frac{x}{\varepsilon},\omega)DI^\varepsilon, DI^\varepsilon) = 1 \text{ in } U$$

$$I^\varepsilon = \exp[-\frac{1}{\varepsilon}(\bar{I} + o(1))]$$

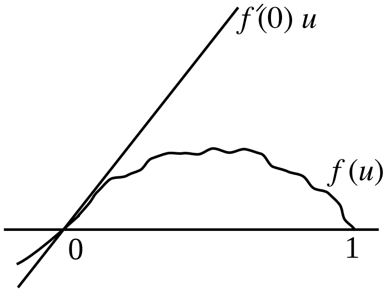
$$\begin{cases} \big((\bar{\sigma}\bar{\sigma}^T)(x)D\bar{I}, D\bar{I}\big) = 1 \text{ in } U \\ \bar{I} = \begin{cases} 0 & \text{on } \Gamma \\ +\infty & \text{on } \partial U \setminus \Gamma \end{cases} \end{cases}$$

$\bar{I}$ “effective” rate for large deviations

- premixed turbulent combustion with separated scales in random environments

$$\begin{cases} u_{\varepsilon t} - \varepsilon \Delta u_\varepsilon + a\big(x, \dfrac{x}{\varepsilon^\alpha}, \omega\big) \cdot D u_\varepsilon = \dfrac{1}{\varepsilon} f\big(u_\varepsilon^\varepsilon, x, \dfrac{x}{\varepsilon^\alpha}, \omega\big) \ \text{ in } \ \mathbb{R}^N \times (0,\infty) \\ u_\varepsilon(\cdot, 0) = 0 \ \text{ off } \ G_0 \subset \mathbb{R}^N \end{cases}$$

f KPP-type with equilibria at 0 and 1



$$u_\varepsilon \longrightarrow \begin{cases} 0 \ \text{ in } \ \{\bar v > 0\} \\ 1 \ \text{ in } \ \text{int } \{\bar v = 0\} \end{cases}$$

$$\begin{cases} \max[\bar v_t + \bar H(D\bar v, x), \bar v] = 0 \ \text{ in } \ \mathbb{R}^N \times (0,\infty) \\ \bar v(\cdot, 0) = \begin{cases} 0 & \text{in } \ G_0 \\ -\infty & \text{off } \ \overline{G_0} \end{cases} \end{cases}$$

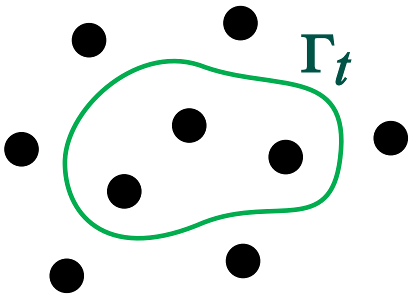
interface $\partial\{\bar v > 0\}$

effective Hamiltonian $\bar H$ is identified at the limit, as $\varepsilon \to 0$, of

$$v_{\varepsilon t} - \varepsilon^{1-\alpha} \Delta v_\varepsilon + a\big(x, \frac{x}{\varepsilon^\alpha}, \omega\big) \cdot D v_\varepsilon - |D v_\varepsilon|^2 - f_u\big(0, x, \frac{x}{\varepsilon^\alpha}, \omega\big) = 0$$

- OPEN for other f (bistable)
- random travelling waves

- front propagation



$$V = -\delta\,\mathrm{tr}\,Dn + a(n,x,\omega)$$

$$u_t - \delta M(D^2u, Du) + H(Du,x,\omega) = 0$$

$$M(X,p) = \mathrm{tr}(I - \hat{p}\otimes\hat{p})X$$
$$H(p,x,\omega) = a(-\hat{p},x,\omega)|p|$$
$$(\hat{p} = p/|p|)$$

- scale in space-time Γ_t^ε

  identify effective front $\bar{\Gamma}_t$

$$\Gamma_t^\varepsilon \xrightarrow[\varepsilon\to 0]{} \Gamma_t \quad \text{a.s.}$$

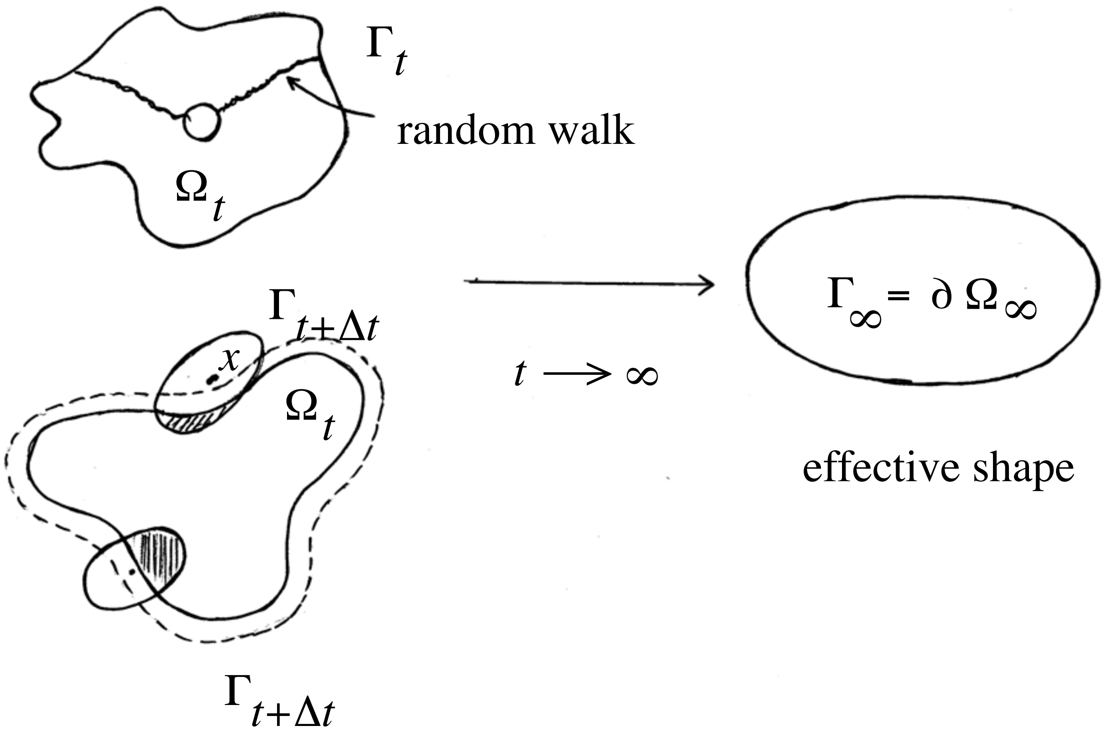
- long time behavior (percolation)

  identify effective shape

$$\frac{1}{t}\Gamma_t \xrightarrow[t\to\infty]{} \bar{\Gamma} \quad \text{a.s.}$$

percolation

voter model



$$V = -\delta \operatorname{tr} Dn - a(n, x, \omega) \qquad u_t - \delta M(D^2 u, Du) + H(Du, x, \omega) = 0$$

different scallings – different effects of heterogeneity

- $(x,t) \to \left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}\right)$

  $$V = -\varepsilon\delta \operatorname{tr} Dn - a\Big(n, \frac{x}{\varepsilon}, \omega\Big) \qquad u_t^\varepsilon - \varepsilon\delta M(D^2 u^\varepsilon, Du^\varepsilon) + H\Big(Du^\varepsilon, \frac{x}{\varepsilon}, \omega\Big) = 0$$

  OPEN ⟨ $\delta = 0$ and H not convex unless medium is almost periodic

  $\delta \neq 0$ unless medium is almost periodic

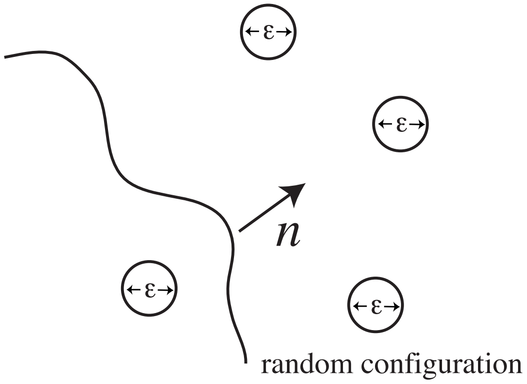
- $(x,t) \to \left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right)$

  $$V = -\delta \operatorname{tr} Dn - \frac{1}{\varepsilon} a\Big(a, \frac{x}{\varepsilon}, \omega\Big) \qquad u_t^\varepsilon - \delta M(D^2 u^\varepsilon, Du^\varepsilon) + \frac{1}{\varepsilon} H\Big(Du^\varepsilon, \frac{x}{\varepsilon}, \omega\Big) = 0$$

  OPEN in all settings (including periodic)

FULLY NONLINEAR, SECOND-ORDER, UNIFORMLY ELLIPTIC EQUATIONS

$$F(D^2u_\varepsilon, Du_\varepsilon, u_\varepsilon, x, \frac{x}{\varepsilon}, \omega) = 0$$

- Monge-Ampère $$\det D^2 u_\varepsilon = f(\frac{x}{\varepsilon}, \omega)$$
- minimal surface
$$-\operatorname{tr}\left[A(\frac{x}{\varepsilon},\omega)\left(I - \frac{Du_\varepsilon \otimes Du_\varepsilon}{1+|Du_\varepsilon|^2}\right) D^2u_\varepsilon\right] = f(\frac{x}{\varepsilon},\omega)$$
- 

$$V = -\operatorname{tr} A\left(\frac{x}{\varepsilon},\omega\right) Dn_\varepsilon$$
$$A\left(\frac{x}{\varepsilon}\right) Dn_\varepsilon Dn_\varepsilon = 1$$

OPEN

$$u_{\varepsilon t} - \operatorname{tr}\{A(\frac{x}{\varepsilon},\omega)[I - (A(\frac{x}{\varepsilon},\omega)Du_\varepsilon Du_\varepsilon)^{-1} A(\frac{x}{\varepsilon},\omega)Du_\varepsilon Du_\varepsilon] D^2u_\varepsilon\}$$
$$+ \frac{2}{\varepsilon}(A(\frac{x}{\varepsilon},\omega)Du_\varepsilon Du_\varepsilon)^{-1} \operatorname{tr} A(\frac{x}{\varepsilon},\omega)Du^\varepsilon \otimes D_y A(\frac{x}{\varepsilon},\omega)Du_\varepsilon Du_\varepsilon = 0$$

GENERAL PROBLEM

$$(\mathrm{BVP})_\varepsilon \qquad \begin{cases} F(D^2u_\varepsilon, \varepsilon D^2u_\varepsilon, Du_\varepsilon, u_\varepsilon, x, \frac{x}{\varepsilon}, \omega) = 0 \ \text{ in } \ U \subset \mathbb{R}^N \\ u_\varepsilon = g \ \text{ on } \ \partial U \end{cases}$$

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PROBABILISTIC SETTING

$(\Omega, \mathcal{F}, P)$ probability space

- $f : \mathbb{R}^N \times \Omega \to \mathbb{R}$ stationary iff $P(\{\omega : f(x,\omega) > \alpha\})$ independent of x

$\tau_x : \Omega \to \Omega$ measure preserving group

$f : \mathbb{R}^N \times \Omega \to \mathbb{R}$ stationary iff $f(x+y,\omega) = f(x, \tau_y \omega)$

$$f(x,\omega) = f(0, \tau_x \omega) = \tilde{f}(\tau_x \omega) \qquad \tilde{f} : \Omega \to \mathbb{R}$$

- τ_x ergodic iff $\tau_x A = A$ for all $x \in \mathbb{R}^N \Rightarrow P(A) = 0$ or 1

τ_x ergodic $\Longrightarrow$ all translation invariant quantities are non-random

EXAMPLES

- periodic functions

$\tilde{f} : \mathbb{R} \to \mathbb{R}$ 1-periodic; $\Omega \equiv S^1$, $\mathcal{F} \equiv$ Borel sets, $P \equiv$ Lebesgue measure, $\tau_x \omega = x + \omega \bmod 1$

$$f(x, \omega) = \tilde{f}(x + \omega) \qquad \text{stationary}$$

- quasi-periodic functions

$\Omega \equiv$ torus T^N, $\quad \mathcal{F} \equiv$ Borel sets, $\quad P \equiv$ Lebesgue measure

$\tilde{f}(x) = F(\alpha_1 x, \ldots, \alpha_N x)$; $F : \mathbb{R}^N \to \mathbb{R}$ $\quad T^N$-periodic, $\alpha_1, \ldots, \alpha_N$ rationally independent

$$f(x, \omega) = F(\alpha_1 x + \omega_1, \ldots, \alpha_N x + \omega_N) \quad \text{stationary}$$

- almost periodic functions

Ω compact subset of $C(\mathbb{R}^N)$ wrt $\| \; \|_\infty$-topology, P normalized Haar measure

- regular random chessboard structure

$(\Omega, \mathcal{F}, P)$ probability space

$(X_k)_{k\in\mathbb{Z}^N}$ independent rv taking values λ and $\Lambda > 0$ with prob r and $1 - r$

$Q_k = \{x \in \mathbb{R}^N : k_i \leq x_i \leq k_i + 1,\ i = 1, \dots, N\}$ cube in $\mathbb{R}^N$

$$V(x, \omega) = \sum_{k\in\mathbb{Z}^N} X_k(\omega)\mathbf{1}_{Q_k}(x) \quad \text{stationary}$$

cubic lattice with cells occupied by two different materials with densities λ and Λ chosen independently by Bernoulli's law

λ	λ	Λ	Λ
λ	λ	Λ	Λ
λ	λ	λ	λ

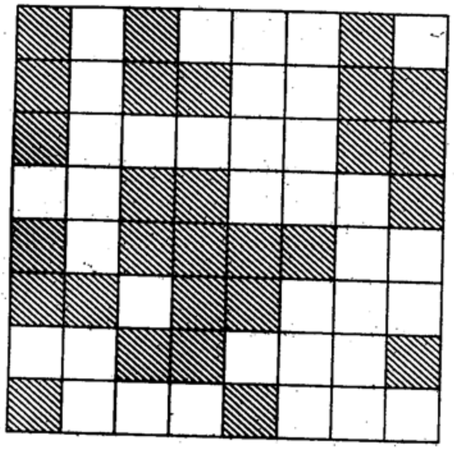
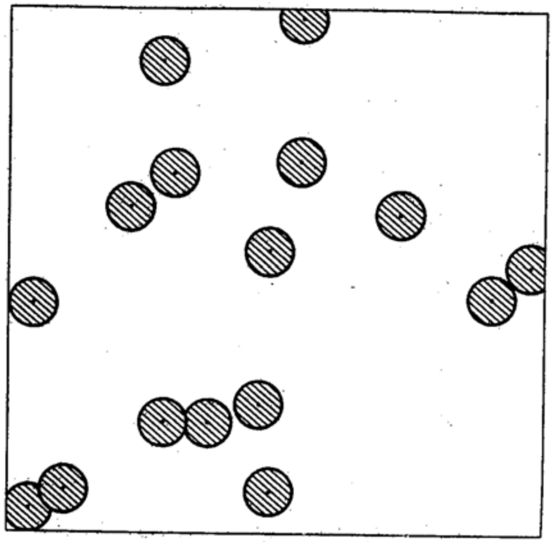


Fig. 1. [illegible]



- material with random spherical inclusions

X_A Poisson process counting process for a uniform random distribution of points

$V(x,\omega) = c_1 + (c_2 - c_1)\min\{1, X_{B(x,r)}(\omega)\}$

$V \equiv c_2$ in union of balls of radius r associated to (X_A) and c_1 elsewhere

$\mathcal{B}_N$ bdd Borel subsets of $\mathbb{R}^N$

$(X_A)_{A\in\mathcal{B}_N}$ Poisson process with parameter λ : family of rv on $(\Omega, \mathcal{F}, P)$

with nonnegative integer values such that

(a) $X_{A\cup B}(\omega) = X_A(\omega) + X_B(\omega), \quad \text{if } A\cap B = \emptyset$

(b) (A_k) finite, disjoint family $\Longrightarrow$ X_{A_k} are independent

(c) $P(X_A = m) = e^{-\lambda|A|}\dfrac{\lambda^m|A|^m}{m!}$

- random chessboard structures with cells of arbitrary size

$N = 2$

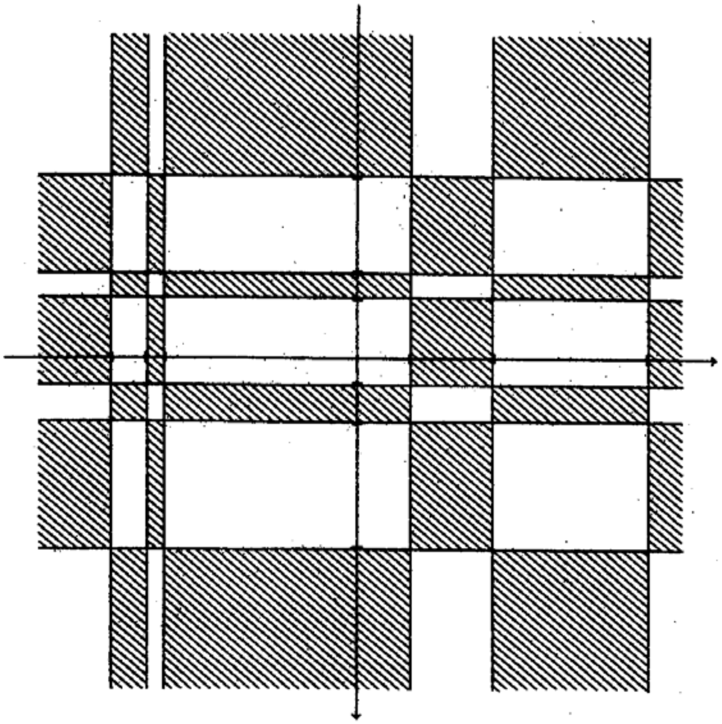
X_A, Y_A independent one-dimensional Poisson processes

V rv independent of X_A and Y_A taking the values c_1, c_2 with prob. $1/2$

$$A(t) = \begin{cases} [0,t) & t > 0 \\ [t,0) & t < 0 \\ \emptyset & t = 0 \end{cases}$$

$$V(x,\omega) = \begin{cases} V(\omega) & \text{if} \quad X_{A(x_1)}(\omega) + Y_{A(x_2)}(\omega) \text{ is even} \\ c_1 + c_2 - V(\omega) & \text{if} \quad X_{A(x_1)}(\omega) + Y_{A(x_2)}(\omega) \text{ is odd} \end{cases}$$

For fixed $\omega \in \Omega$, $V(x,\omega)$ is constant on each cell of the chessboard with alternated values c_1 and c_2 (the value at the origin is given by $V(\omega)$)



Review (formal) of linear homogenization
for uniformly elliptic equations

- nondivergence form

$$-a_{ij}\Big(\frac{x}{\varepsilon}\Big)u^{\varepsilon}_{x_i x_j} = f\Big(x,\frac{x}{\varepsilon}\Big) \qquad a_{ij} \text{ uniformly elliptic}$$

- formal expansion

$$u^{\varepsilon}(x) = u(x) + \varepsilon w\Big(x,\frac{x}{\varepsilon}\Big) + \varepsilon^2 v\Big(x,\frac{x}{\varepsilon}\Big) + h.o.t. \qquad (y = x/\varepsilon)$$

$$Du^{\varepsilon} = Du + D_y w + \varepsilon D_x w + \varepsilon D_y v + \varepsilon^2 D_x v + h.o.t.$$

$$D^2u^{\varepsilon} = D^2u + \frac{1}{\varepsilon}D^2_y w + 2D^2_{x,y}w + \varepsilon D^2_x w + D^2_y v + \varepsilon D^2_{x,y}v + \varepsilon^2 D^2_x v + h.o.t.$$

- substitute in equation

$$\varepsilon^{-1} \qquad a_{ij}w_{y_i y_j} = 0 \quad \overset{\text{ellipticity}}{\Longrightarrow} \quad w(y,x) \equiv w(x) \quad (\equiv 0)$$

$$\varepsilon^{0} \qquad -a_{ij}(u_{x_i x_j} + v_{y_i y_j}) = f(x,y)$$

$$Lv = -a_{ij}(y)v_{y_i y_j} = f(x,y) + a_{ij}(y)u_{x_i x_j}$$

- $m > 0$ st $L^*m = -(a_{ij}m)_{y_iy_j} = 0$ (invariant measure)

Fredholm's alternative $\Longrightarrow$

$$Lv = f(x,y) + a_{ij}u_{x_ix_j} \quad \text{has a solution} \qquad \text{(corrector)}$$

iff

$$f(x,\cdot) + a_{ij}(\cdot)u_{x_ix_j} \perp m$$

$$-\bar{a}_{ij}u_{x_ix_j} = \bar{f}$$

$$\bar{a}_{ij} = -\Big(\int a_{ij}(y)m(y)dy\Big) \qquad \text{and} \qquad \bar{f} = \int f(x,y)m(y)dy$$

- divergence form

$$L^\varepsilon u^\varepsilon = -\partial_{x_i}\Big(a_{ij}\Big(\frac{x}{\varepsilon}\Big)\partial_{x_j}u^\varepsilon\Big) = f\Big(x,\frac{x}{\varepsilon}\Big) \qquad\qquad a_{ij}\ \text{uniformly elliptic}$$

- notation $\qquad \Phi(x) = \phi\Big(x,\frac{x}{\varepsilon}\Big) \qquad \partial_{x_i}\Phi = \partial_{x_i}\phi + \frac{1}{\varepsilon}\partial_{y_i}\phi$

$$L^\varepsilon\Phi = -\big(\partial_{x_i} + \frac{1}{\varepsilon}\partial_{y_i}\big)\big(a_{ij}\big(\partial_{x_j} + \frac{1}{\varepsilon}\partial_{y_j}\big)\phi\big) = \big(\varepsilon^{-2}L_1 + \varepsilon^{-1}L_1 + \varepsilon^0 L_0\big)\phi$$

$$L_1 = -\partial_{y_i}(a_{ij}(y)\partial_{y_j})$$

$$L_2 = -\partial_{y_i}(a_{ij}(y)\partial_{x_j}) - \partial_{x_i}(a_{ij}(y)\partial_{y_j})$$

$$L_3 = -\partial_{x_i}(a_{ij}(y)\partial_{x_j})$$

- formal expansion

$$u^\varepsilon(x) = u\Big(x,\frac{x}{\varepsilon}\Big) + \varepsilon w\Big(x,\frac{x}{\varepsilon}\Big) + \varepsilon^2 v\Big(x,\frac{x}{\varepsilon}\Big) + h.o.t.$$

- $L^\varepsilon u^\varepsilon = 0 \quad \Longrightarrow$

$$L_1 u = 0$$

$$L_1 w + L_2 u = 0$$

$$L_1 v + L_2 w + L_3 u = f$$

- $L_1 u = 0 \quad \overset{\text{ellipticity}}{\Longrightarrow} \quad u(x,y) \equiv u(x)$

- $L_1 w + L_2 u = 0 \quad \Longrightarrow \quad L_1 w = \partial_{y_i} a_{ij}(y) \partial_{x_j} u$

  $w(x,y) = -X(y) \cdot Du(x)$

  $L_1 X^i = -\partial_{y_i} a_{ij}$

  Fredholm's alternative $\Longrightarrow \quad \exists \;\; X$

- $L_1 v + L_2 w + L_3 u = f \quad \Longrightarrow \quad L_1 v = f - L_2 w + L_3 u$

  Fredholm's alternative $\Longrightarrow$ v exists iff rhs $\perp 1$

  $$\int (L_2 w + L_3 u)\, dy = \int f\, dy$$

  $$\int L_2 w = -\int \partial_{y_i}(a_{ij}(y)\partial_{x_j} w) + \partial_{x_i}(a_{ij}(y)\partial_{y_j} w) = -\partial_{x_j} \int a_{ij} \partial_{y_j} w$$

  $$= \cdots = \partial_{x_i}\left[\Big(\int a_{ij}\partial_{y_j} X^i\Big)\partial_{x_j} u\right] \qquad \Longrightarrow$$

  $$-\bar{a}_{ij} u_{x_i x_j} = \bar{f}$$

  $$\bar{a}_{ij} = \int (a_{ij} - a_{ik}\partial_{y_k} X^j) dy \quad \text{and} \quad \bar{f} = \int f(x,y) dy$$

$$H(Du_\varepsilon, \frac{x}{\varepsilon}, \omega) = 0$$

formal expansion

$$u_\varepsilon(x) = \bar{u}(x) + \varepsilon \underbrace{v(\frac{x}{\varepsilon}, \omega)}_{\text{corrector}} + O(\varepsilon^2)$$

$$H(D\bar{u} + D_y v, \frac{x}{\varepsilon}, \omega) = 0$$

macroscopic (cell) problem
(nonlinear eigenvalue)

for each p $\exists!$ $\bar{H}(p)$ such that

$$H(Dv + p, y, \omega) = \bar{H}(p) \text{ in } \mathbb{R}^N$$

has a solution (corrector) v satisfing

$$|y|^{-1} v(y, \omega) \to 0 \text{ as } |y| \to \infty \text{ a.s. in } \omega$$

why the decay?

- expansion
- uniqueness of $\bar{H}$

macroscopic problem

$$H(q, y, \omega) = H(q)$$

$$H(Dv + p) = \bar{H}(p)$$

$$v = q \cdot y$$

$\bar{H}(p) = H(q + p)$ not unique !

proof of homogenization when correctors exist

$$u_\varepsilon + H(Du_\varepsilon, \frac{x}{\varepsilon}) = 0 \ \text{ in } \ \mathbb{R}^N$$

- assume $\quad u_\varepsilon \to \bar{u}$ in $C(\mathbb{R}^N)$
- need to show $\quad \begin{cases} \text{if } \phi \text{ smooth and } x_0 \text{ local max of } u - \phi, \text{ then} \\ \bar{u}(x_0) + \bar{H}(D\phi(x_0)) \leqq 0 \end{cases}$
- ψ corrector for $p = D\phi(x_0)$

$$H(D\phi(x_0) + D\psi, y) = \bar{H}(D\phi(x_0))$$

- $u_\varepsilon - (\phi + \underbrace{\varepsilon\psi(\frac{x}{\varepsilon})}_{\longrightarrow 0 \text{ as } \varepsilon \to 0})$ attains a max at x_ε and $x_\varepsilon \to x_0$

$$u_\varepsilon(x_\varepsilon) + \underbrace{H(D\phi(x_\varepsilon) + D\psi(\frac{x_\varepsilon}{\varepsilon}), \frac{x_\varepsilon}{\varepsilon})}_{= \bar{H}(D\phi(x_0)) + o(1)} \leqq 0$$

$$\vdots$$

"proof of existence of correctors"

$$H(Dv + p, y, \omega) = \bar{H}(p)$$

H coercive $\qquad H(q, y, \omega) \to +\infty \quad \text{as} \quad |q| \to \infty$

H bounded for fixed q $\qquad \sup_y H(q, y, \omega) < \infty$

- approximate $\qquad \alpha v_\alpha + H(Dv_\alpha + p, y, \omega) = 0 \;\text{ in }\; \mathbb{R}^N$
- estimate $\qquad \sup_{\alpha>0}(\|\alpha v_\alpha\|_\infty + \|Dv_\alpha\|_\infty) < \infty$
- normalize $\qquad \hat{v}_\alpha(y) = v_\alpha(y) - v_\alpha(0)\ ; \qquad \|\,|y|^{-1}\hat{v}_\alpha(y)\| + \|D\hat{v}_\alpha\| \le C$

  $$\alpha\hat{v}_\alpha + H(D\hat{v}_\alpha + p, y, \omega) = -\alpha v_\alpha(0)$$
- pass to the limit $\alpha \to 0$ $\qquad \alpha_n \to 0\ , \quad \hat{v}_{\alpha_n} \to \hat{v}$

  $$H(D\hat{v} + p, y) = \lim_{\alpha_n \to 0}(-\alpha_n v_{\alpha_n}(0))$$

Difficulties

- measurability
- growth of $\hat{v}$ at infinity

CANNOT BE RESOLVED USING EXISTING ESTIMATES

NO CORRECTORS

$$|v'| = f(x,\omega) + \bar{H}(0)$$

$$\inf_y f(y,\omega) = 0 \quad \Longrightarrow \quad \bar{H}(0) = 0$$

$$|v'| = \underbrace{2 - \cos(2\pi x + \omega_1) - \cos(2\pi\alpha x + \omega_2)}_{f(x,\omega)} \qquad \begin{cases} (\omega_1, \omega_2) \in [0,1] \times [0,1] \\ \alpha \text{ irrational} \end{cases}$$

$$f(x,\omega) > 0 \ \text{ a.s. in } \ \omega = (\omega_1, \omega_2)$$

$$\begin{array}{r} |w'| = F(x) \\ F > 0 \text{ for } |x| \gg 1 \end{array} \quad \Longrightarrow \quad w(x) = \pm \int_{\bar{x}}^{x} F(y)dy$$

$$v(x,\omega) = 2|x| - \frac{\sin(2\pi x + \omega_1)}{2\pi} - \frac{\sin(2\pi\alpha x + \omega_2)}{2\pi\alpha} \qquad (|x| \gg 1)$$

$$\Longrightarrow \quad \frac{v(x,\omega)}{|x|} \to 2 \text{ as } |x| \to \infty \ \text{ a.s. in } \omega = (\omega_1, \omega_2)$$

Lions-Souganidis (2003)

MAIN TOOL

(sub-additive) ergodic theorem identifies

uniquely a.s. a (limiting) quantity controlling the behavior of u_ε as $\varepsilon \to 0$

SUB-ADDITIVE ERGODIC THEOREM

$(\Omega, \mathcal{F}, P)$ probability space, $\mu : \mathcal{P} \times \mathbb{R} \to \mathbb{R}$ such that

- $0 \leq \mu(A, \omega) \leqq c|A|$
- sub-additive $A = \cup A_k,\ A_k$ mutually disjoint $\Longrightarrow \mu(A,\omega) \leqq \sum \mu(A_k, \omega)$
- stationary $\mu(z + A, \omega)$ same distribution as $\mu(A, \omega)$

$$\Longrightarrow \quad \frac{\mu(tQ,\omega)}{|tQ|} \xrightarrow[t\to\infty]{} \bar{\mu}(\omega) \quad \text{a.s.} \qquad (Q \text{ ``nice'' subset of } \mathbb{R}^N)$$

$$\text{ergodicity} \quad \Longrightarrow \quad \bar{\mu}(\omega) = \bar{\mu} \quad \text{a.s.}$$

HAMILTON-JACOBI EQUATIONS

$$\begin{cases} H(Du_\varepsilon, u_\varepsilon, x, \frac{x}{\varepsilon}, \omega) = 0 \text{ in } U \\ u_\varepsilon = g \text{ on } \partial U \end{cases} \qquad u_\varepsilon(\cdot,\omega) \to \bar{u} \text{ in } C(\bar{U}) \text{ a.s. in } \omega \qquad \begin{cases} \bar{H}(D\bar{u}, \bar{u}, x) = 0 \text{ in } U \\ \bar{u} = g \text{ on } \partial U \end{cases}$$

$H(p, r, x, y, \omega)$ coercive, convex wrt p

stationary ergodic wrt (y, ω)

- $\bar{H}(p, r, x) = \inf_{\Phi \in S} \sup_{y \in \mathbb{R}^N} H(p + D\Phi, r, x, y, \omega)$ ($\bar{H}$ independent of ω)

  $S = \{\Phi : \mathbb{R}^N \times \Omega \to \mathbb{R} \ : \ \text{Lip}, \ D\Phi \text{ stationary}, \ E(D\Phi) = 0\}$

  ergodic thm $\Longrightarrow$ if $\Phi \in S$, then $|y|^{-1}\Phi(y, \omega) \underset{|y| \to \infty}{\longrightarrow} 0$ a.s. in ω

- $\bar{H}^*(q, r, x) = \lim_{t \to \infty} \frac{1}{t} \inf \left[\int_0^t H^*(-\dot{\gamma}(s), r, x, \gamma(s), \omega) ds : \gamma(0) = 0, \ \gamma(t) = tq \right]$

- no correctors (in general)

- H not convex OPEN

Souganidis (1999)
Rezankhanlou-Tarver (2000)
Lions-Souganidis (2003)
Schwab (2006) (time dependent)

VISCOUS HAMILTON-JACOBI (BELLMAN) EQUATIONS

$$\begin{cases} -\varepsilon \operatorname{tr} A(x, \frac{x}{\varepsilon}, \omega) D^2 u_\varepsilon + H(Du_\varepsilon, u_\varepsilon, x, \frac{x}{\varepsilon}, \omega) = 0 \text{ in } U \\ u_\varepsilon = g \text{ on } \partial U \end{cases} \qquad \begin{cases} \bar{H}(D\bar{u}, \bar{u}, x) = 0 \text{ in } U \\ \bar{u} = g \text{ on } \partial U \end{cases}$$

$$u_\varepsilon(\cdot, \omega) \underset{\varepsilon \to 0}{\longrightarrow} \bar{u} \quad \text{in} \quad C(\bar{U}) \text{ a.s. in } \omega$$

$A(x, y, \omega)$ degenerate elliptic, stationary ergodic

$H(p, r, x, y, \omega)$ coercive, convex in p, stationary ergodic

- $\bar{H}(p, r, x) = \inf_{\Phi \in S} \sup_{y \in \mathbb{R}^N} \left[-\operatorname{tr} A(x, y, \omega) D^2 \Phi + H(p + D\Phi, r, x, y, \omega) \right]$ (independent of ω)
- $\bar{H}(p, r, x) = \sup_{(b,\Phi) \in \mathcal{E}} E\Big[(\theta \cdot b(\omega) - H^*(b(\omega), x, \omega)) \Phi(\omega) \Big]$ $(A = I)$

  $\mathcal{E} = \{(b, \Phi) : \operatorname{div}(b\Phi) = \Delta \Phi\}$
- $\bar{H}^*(p, r, x) = \lim_{t \to \infty} \frac{1}{t} \inf E\left\{ \int_0^t H^*(\alpha(s), r, x, \gamma(s), \omega) ds : \gamma(\cdot) \in \mathcal{B}(0, t, 0, tp, \alpha) \right\}$

  $\mathcal{B}(a, b, x, y, \alpha) \equiv$ "Brownian bridges" in (a, b) with drift α connecting x and y
- no correctors (in general)
- H not convex

  A depending on p

OPEN

Lions-Souganidis (2004)

Kosygina-Rezankhanlou-Varadhan (2005)

Kosygina-Varadhan (2006) (time dependent, uniformly elliptic)

SKETCH OF PROOF # 1 (A deg. elliptic)

$\exists\,!\ \ L^\varepsilon(y,\hat y,s,\sigma,\omega)$ st

- $$\begin{cases} L^\varepsilon_\sigma - \varepsilon\,\mathrm{tr}\,A(\varepsilon^{-1}y,\omega)D^2L^\varepsilon + H(DL^\varepsilon,\varepsilon^{-1}y,\omega)=0 \ \text{ in } \ \mathbb{R}^N\times(s,t) \\ L^\varepsilon(y,\hat y,s,s,\omega) = \begin{cases} \infty & \text{if } y\neq\hat y \\ 0 & \text{if } y=\hat y \end{cases} \end{cases}$$

- L^ε loc Lip in $y,\hat y$, Hölder in σ for $y-\hat y$ bdd, $L^\varepsilon(y,\hat y,s,\sigma,\omega)=L^\varepsilon(0,\hat y-y,s,\sigma,\tau_y\omega)$

uniqueness / comparison $\Longrightarrow$

$$L^\varepsilon(y,\hat y,s,t,\omega)\leqq L^\varepsilon(y,z,s,\sigma,\omega)+L^\varepsilon(z,\hat y,\sigma,t,\omega) \qquad (s<\sigma<t)$$

subadditive ergodic thm $\Longrightarrow$

$$L^\varepsilon(y,\hat y,0,t,\omega)\underset{\varepsilon\to0}{\longrightarrow} t\bar L\Big(\frac{y-\hat y}{t}\Big) \quad \text{a.s. in } \omega \quad \text{for all } y,\hat y\in\mathbb{R}^N \text{ with } \bar L \text{ convex}$$

$$u_\varepsilon(x,t,\omega)\underset{\substack{\uparrow\\ \text{needs uniform estimates}}}{\approx}\inf[g(y)+L^\varepsilon(x,y,0,t,\omega)] \ \underset{\varepsilon\to0}{\longrightarrow}\ \bar u(x,t)=\inf[g(y)+t\bar L(\frac{x-y}{t})]$$

$$\begin{cases} \bar u_t+\bar H(D\bar u)=0 \\ \bar u = g \ \text{ on } \ \mathbb{R}^N\times\{0\} \end{cases} \qquad\qquad (\bar H=(\bar L)^*)$$

- $A \equiv 0$

$$L^\varepsilon(y, \hat{y}, s, t, \omega) = \inf \Big\{ \int_s^t H^*\big(-\dot{\gamma}(s), r, x, \frac{\gamma(s)}{\varepsilon}, \omega\big)\, ds : \ \gamma(s) = y, \ \gamma(t) = \hat{y} \Big\}$$

- A nondegenerate, $\quad H(p, r, x, y, \omega) = O(|p|^k) \qquad$ with $\ k > 2$

$$L^\varepsilon(y, \hat{y}, s, t, \omega) = \inf E \ \Big\{ \int_s^t H^*\big(\alpha(s), r, x, \frac{\gamma(s)}{\varepsilon}, \omega\big)\, ds : \gamma(\cdot) \in \mathcal{B}(s, t, y, \hat{y}, \alpha) \Big\}$$

- homogenization $\Longrightarrow$ formula

- formula $$u^\varepsilon - \Delta u^\varepsilon + H\Big(Du^\varepsilon, \frac{x}{\varepsilon}, \omega\Big) = 0 \text{ in } \mathbb{R}^N$$

- homogenization $\Longrightarrow \begin{cases} u^\varepsilon \to \bar{u} \\ \bar{u} + \bar{H}(D\bar{u}) = 0 \text{ in } \mathbb{R}^N \end{cases} \quad \Longrightarrow \quad \bar{u} = -\bar{H}(0)$

- viscosity proof $\Longrightarrow \ \bar{u} + \hat{H}(D\bar{u}) \geqq 0 \text{ in } \mathbb{R}^N \quad \Longrightarrow \quad -\bar{H}(0) = \bar{u} \geqq -\hat{H}(0)$

$$(\hat{H}(p) = \inf_{\phi \in S} \sup_y [-\Delta\phi + H(D\phi + p, y, \omega)])$$

- choose ϕ st $$-\Delta\Phi + H(D\Phi, y, \omega) \leqq \bar{H}(0)$$

$$\begin{cases} u_t - \Delta u + H(Du, y, \omega) = 0 \text{ in } \mathbb{R}^N \times (0, \infty) \\ u = \phi \text{ on } \mathbb{R}^N \times (0, \infty) \end{cases}$$

- comparison $\Longrightarrow \quad u(y,t) \geqq \phi(y) - \hat{H}(0)t$

  homogenization $\Longrightarrow \quad \dfrac{u(0,t)}{t} \to -\bar{H}(0)$

  $\Longrightarrow \qquad -\bar{H}(0) \geqq -\hat{H}(0)$

- estimate

$$\begin{cases} u_t^\varepsilon - \varepsilon\Delta u^\varepsilon + H\Big(Du^\varepsilon, \frac{x}{\varepsilon}\Big) = 0 \text{ in } \mathbb{R}^N \times (0,\infty) \\ u^\varepsilon = u_0 \text{ on } \mathbb{R}^N \times \{0\} \end{cases}$$

$$H(p,y) \geqq C_1|p|^{1/2} - C_2\ ,\ C_1 > \|D_yH\|$$

$u_0 \in \mathrm{Lip}$ $\Longrightarrow$ $|u_t^\varepsilon| \leqq C_0$

$w = |Du^\varepsilon|^2$

$$w_t - \varepsilon\Delta w + 2\varepsilon|D^2u|^2 + D_pH \cdot Dw + \frac{2}{\varepsilon}D_yH \cdot Du^\varepsilon = 0$$

at interior max

$$|D^2u|^2 + \frac{1}{\varepsilon}D_yH \cdot Du^\varepsilon \leqq 0$$

$$\text{eqn} \quad \Longrightarrow \quad H - C_0 \leqq \varepsilon\Delta u \quad \Longrightarrow \quad \varepsilon^2(\Delta u)^2 \geqq (H - C_0)_+^2$$

$$(\Delta u)^2 \leqq |D^2u|^2 \quad \Longrightarrow \quad (H - C_0)_+^2 \leqq \|D_yH\|\,|Du|$$

- no correctors

$$\begin{cases} -\Delta u + |Du|^2 = V(y,\omega) + \bar{H}(0) \\ |y|^{-1}u(y,\omega) \to 0 \ \text{ as } \ |y| \to \infty \end{cases}$$

V potential of the Russian School (Carmona-Lacrois)

$$V(y,\omega) = F(W_y) \qquad \begin{cases} F \in C^\infty(\mathbb{T}), \ \inf F(y) = 0, \ \sup F(y) = 1, \quad \text{nondegenerate} \\ (Y_t)_{t\in\mathbb{R}} \ \text{ Brownian motion on torus } \ \mathbb{T} \end{cases}$$

$$L = -\Delta + V \qquad\qquad \sigma(L) = [0,\infty)$$

$$w = e^{-u}$$

$$-\Delta w + Vw = -\bar{H}(0)w \qquad \Longrightarrow \qquad \bar{H}(0) = 0$$

if corrector exists, it is an eigenvector

but $\nexists$ no eigenvector for 0

SKETCH OF PROOF #2 (A uniformly elliptic)

$$u^\varepsilon - \varepsilon\Delta u^\varepsilon + H\big(Du^\varepsilon, \frac{x}{\varepsilon}, \omega\big) = 0 \qquad \text{in} \ \ \mathbb{R}^N$$

recall

$$H(p,y,\omega) = \sup_q [p\cdot q - H^*(q,y,\omega)]$$

assume

$$u^\varepsilon \to \bar{u} \quad \text{as} \quad \varepsilon \to 0$$

- $\mathcal{E} = \{(b, \Phi : b \text{ bdd}, \ \Phi \geqq 0, \ E\Phi = 1, \ \Delta\Phi = \operatorname{div}(b\Phi)\}$ $\qquad \bar{H}_2(p) = \sup\limits_{(b,\Phi)\in\mathcal{E}} E[(p\cdot b - H^*(b,0,\omega))\Phi] = 0$

Fix $(b,\Phi)\in\mathcal{E}$, write $\tilde{b}(x,\omega) = b(\tau_x\omega), \ \tilde{\Phi}(x,\omega) = \Phi(\tau_x\omega)$

$$u^\varepsilon - \varepsilon\Delta u^\varepsilon + \tilde{b}\big(\frac{x}{\varepsilon},\omega\big)\cdot Du^\varepsilon - H^*\big(\tilde{b}\big(\frac{x}{\varepsilon},\omega\big), \frac{x}{\varepsilon}, \omega\big) \leq 0$$

$\Phi \geqq 0 \quad \Longrightarrow$

$$u^\varepsilon - \varepsilon\Delta u^\varepsilon + \tilde{b}\big(\frac{x}{\varepsilon},\omega\big)\cdot Du^\varepsilon - H^*\big(\tilde{b}\big(\frac{x}{\varepsilon},\omega\big), \frac{x}{\varepsilon}, \omega\big)\tilde{\Phi}\big(\frac{x}{\varepsilon},\omega\big) \leqq 0$$

$$E\big[\big(u^\varepsilon - \varepsilon\Delta u^\varepsilon + \tilde{b}\big(\frac{x}{\varepsilon},\omega\big)\cdot Du^\varepsilon - H^*\big(\tilde{b}\big(\frac{x}{\varepsilon},\omega\big), \frac{x}{\varepsilon}, \omega\big)\big)\tilde{\Phi}\big(\frac{x}{\varepsilon},\omega\big)\big] \leqq 0$$

$$E\big[u^\varepsilon\tilde{\Phi} + \frac{1}{\varepsilon}(-\Delta\tilde{\Phi} + \operatorname{div}(\tilde{b}\tilde{\Phi}))u^\varepsilon - H^*\big(\tilde{b}\big(\frac{x}{\varepsilon},\omega\big), \frac{x}{\varepsilon}, \omega\big)\tilde{\Phi}\big] \leqq 0$$

$u^\varepsilon \to \bar{u}$

$-\Delta\Phi + \operatorname{div}(b\Phi) = 0 \qquad \Longrightarrow \qquad \bar{u} + E[-H^*(b(\omega),0,\omega)\Phi(\omega)] \leqq 0$

$E\Phi = 1$, stationarity

(b,Φ) arbitrary $\qquad \Longrightarrow \qquad \bar{u} + \bar{H}_2(0) \leqq 0$

$\qquad \Longrightarrow \qquad \bar{u} \leqq -\bar{H}_2(0)$

- $S = \{\psi : D\psi \text{ stationary}, \ E(D\psi) = 0\}$ $\qquad \bar{H}_1(p) = \inf_{\Psi \in S} \sup_y (-\Delta\psi + H(D\psi + p, y, \omega))$

ϕ smooth $\qquad \bar{u} - \phi$ strict min at x_0

choose $\Psi \in S$ st $\qquad \bar{H}_1(D\phi(x_0)) = \sup_y (-\Delta\psi + H(D\psi + D\phi(x_0), y, \omega))$

$u^\varepsilon \to \bar{u}$

$\psi \in S$ $\qquad \Longrightarrow \qquad u^\varepsilon \big(\phi + \varepsilon \Psi\big(\frac{\cdot}{\varepsilon}\big)\big)$ min at x_0 and $x_\varepsilon \to x_0$

$$u^\varepsilon \underbrace{-\Delta\Psi + H\big(D\phi(x_\varepsilon) + \varepsilon\psi\big(\frac{x_\varepsilon}{\varepsilon}\big), \frac{x_\varepsilon}{\varepsilon}, \omega\big)}_{\leqq \bar{H}_1(D\phi(x_0)) + o(1)} \geqq 0$$

$u^\varepsilon \to \bar{u} \qquad \Longrightarrow \qquad \bar{u}(x_0) + \bar{H}_1(D\phi(x_0)) \geqq 0$

$\Longrightarrow \qquad \bar{u} \geqq -\bar{H}_1(0)$

- $\bar{H}_1 = \bar{H}_2$

$$\bar{H}_1(p) = \sup_{(b,\Phi)\in\mathcal{E}} [E(p \cdot b(\omega) - H^*(b(\omega),\omega))\Phi(\omega)]$$

$$= \sup_{(b,\Phi)\in\mathcal{E}} \inf_U E[(p \cdot b(\omega) - H^*(b(\omega),\omega) + A_b U]\Phi(\omega)$$

$$= \sup_\Phi \sup_b \inf_U E[p \cdot b(\omega) - H^*(b(\omega),\omega) - A_b U]\Phi(\omega)$$

DU stationary

$A_b U = -\Delta U + b \cdot DU$

$\inf_U E[\varphi A_b U] = -\infty$

unless φ is invariant for A_b

(uniform ellipticity)

min max

$$= \sup_\Phi \inf_U \sup_b E([p \cdot b(\omega) - H^*(b(\omega),\omega) + A_b U]\Phi(\omega))$$

$$= \sup_\Phi \inf_U \sup_b E[(p + DU) \cdot b(\omega) - H^*(b(\omega),\omega)) - \Delta U]\Phi$$

$$= \sup_\Phi \inf_U E(H(p + DU,\omega) - \Delta U)\Phi$$

min max

$$= \inf_U \sup_\Phi E(H(p + DU,\omega) - \Delta U)\Phi$$

duality

$$= \inf_U \operatorname*{ess\,sup}_\omega (H(p + DU,\omega) - \Delta U)$$

CORRECTORS

- periodic setting H coercive, A deg elliptic — correctors

$$\exists!\ \bar{H}(p) \text{ st } -\operatorname{tr} A(Du+p,y)D^2u + H(Du+p,y) = \bar{H}(p)$$

Papanicolaou-Lions-Varadhan
Evans (unif. elliptic)
Lions-Souganidis (deg elliptic)

- almost periodic setting, H coercive, A deg elliptic — approximate correctors

$$\exists! \quad \bar{H}(p) \quad \text{st} \quad \forall\ \delta > 0\ \exists\ v_\delta^{\pm} \text{ st}$$

$$-\operatorname{tr} A(Dv_\delta^{\pm}+p,y)D^2v_\delta^{\pm} + H(Dv_\delta^{\pm}+p,y) \begin{matrix} \leqq \\ \geqq \end{matrix} \bar{H}(p) \begin{matrix} +\delta \\ -\delta \end{matrix}$$

Ishii ($A{\equiv}0$)

Lions-Souganidis ($A{\neq}0$)

$$\Updownarrow$$

$$\alpha v_\alpha - \operatorname{tr} A(Dv_\alpha+p)D^2v_\alpha + H(Dv^\alpha+p,y) = 0$$

$$\alpha v_\alpha \quad \text{converges, as } \alpha \to 0, \text{ uniformly in } \ \mathbb{R}^N$$

- random, H coercive, convex, A deg elliptic — local approximate correctors

$$\exists!\ \bar{H}(p) \quad \text{st} \quad \forall\ R > 0\ \exists\ v_R^{\pm} : \mathbb{R}^N \times \Omega \to \mathbb{R} \quad \text{st}$$

$$Du_R^{\pm} \text{ stationary with mean } 0,\ |Dv_R^{\pm}| \leqq K \text{ and } \forall\ k\ \exists\ \varepsilon_R^k \to 0, \text{ as } R \to \infty, \text{ st}$$

$$-\operatorname{tr} A(y,\omega)D^2v_R^- + H(Dv_R^- + p,y,\omega) \leqq \bar{H}(p) + \varepsilon_R^k$$

$$\text{in} \quad B_{kR}$$

$$-\operatorname{tr} A(y,\omega)D^2v_R^+ + H(Dv_R^+ + p,y,\omega) \geqq \bar{H}(p) - \varepsilon_R^k$$

$$\begin{cases} u_t + H(Du, x, \omega) = 0 \ \text{ in } \ \mathbb{R}^N \times (0, \infty) \\ u = u_0 \text{ on } \mathbb{R}^N \times \{0\} \end{cases}$$

H coercive convex, stationary ergodic

$S(t)$ solution operator

Recall

$$S = \{\Phi \in C^{0,1}(\mathbb{R}^N) \text{ a.s. in } \omega,\ |y|^{-1}\Phi(y,\omega) \to 0 \text{ a.s. in } \omega\}$$

$$\exists \text{ corrector } v \in S$$

iff

for some $\Phi \in S$ st $H(p + D\phi, x) \leqq \bar{H}(p)$

$$\sup_{t \geqq 0}((S(t)\Phi)(x) + t\bar{H}) < \infty$$

Lions-Souganidis

Examples

$$(*) \qquad \text{(i)} \quad f : \mathbb{R}^N \times \Omega \to \mathbb{R} \text{ stationary}, \qquad \text{(ii)} \quad \inf_x f(x,\omega) = 0, \qquad \text{(iii)} \quad f(x,\omega) > 0$$

- $$|Du^\varepsilon| + \varepsilon u^\varepsilon = f(x,\omega) \qquad \Longrightarrow \qquad \varepsilon u^\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} 0$$

f satisfies $(*)_{\text{(i)}}$, $(*)_{\text{(ii)}}$

$$K(\omega) = \{x : f(x,\omega) = 0\}$$

$$\text{ergodicity} \quad \Longrightarrow \quad P(K(\omega) \neq \emptyset) = 0 \text{ or } 1$$

$$K(\omega) = \emptyset \qquad \Longrightarrow \qquad \text{no correctors}$$

$$K(\omega) \neq \emptyset \quad \Longrightarrow \quad u(x) = \inf_{\bar{x}\in K(\omega)} L(x,\bar{x},\omega) \text{ corrector}$$

$$L(x,y,\omega) = \inf\left\{ \int_0^T f(x(s),\omega)\, ds : x(0) = x,\ x(T) = y,\ |\dot{x}| \leqq 1,\ T > 0\right\}$$

- f satisfies $(*)_{(\text{i})}$, $(*)_{(\text{ii})}$ and $N = 1$

$$|u' + P|^2 = f(y,\omega) + \bar{H}(p) \quad \text{in} \quad \mathbb{R}$$

$$\bar{H}(p) = \begin{cases} \inf_y f(y,\omega) & \text{if} \quad |P| \leqq E\sqrt{f(y,\omega) - \inf_y f(y,\omega)} \\ \alpha & \text{if} \quad |P| = E\sqrt{(f(x,\omega) - \inf_y f(y,\omega) + \alpha} \end{cases}$$

- $H(p) = bp_+ - ap_- \qquad\qquad (0 < a < b)$

$$H(Du + p) = f(x,\omega) + \bar{H}(p)$$

$\bar{H}(p)$ unique constant st $E\frac{1}{b}(f + \lambda)_+ = E\frac{1}{a}(f - \lambda)_-$

$$u_x = \frac{1}{b}(f + \lambda)_+ - \frac{1}{a}(f - \lambda)_-$$

- $H(p_1, p_2) = |p_1 + P_1|^2 + |p_2|^2 - (f_1(x,\omega) + f_2(x_2,\omega))$

$$\begin{matrix} \min f_1(\cdot,\omega) = 0 \\ \inf f_2(\cdot,\omega) = 0 \end{matrix} \quad ; \quad \text{and} \quad |P_1| \geqq E\sqrt{f_1(x,\omega)} \quad \Longrightarrow \quad \text{no corrector}$$

$$\bar{H}(P_1, 0) = \alpha \text{ where } |P_1| = E\sqrt{f_1(x,\omega) + \alpha}$$

general result

f satisfies $(*)_{\text{(i)}}$ and $(*)_{\text{(ii)}}$

$$\varepsilon u^{\varepsilon} \text{ converges uniformly in } \mathbb{R}^N \text{ to } 0$$

iff

$$\sup_{x\in\mathbb{R}^N} \inf_{|y-x|\leqq R} f(y,\omega) \to \infty \quad \text{as} \quad R\to\infty$$

- true for periodic, almost periodic settings

- not true (in general) in random settings

  $\xi(\cdot,\omega)\in C(\mathbb{R})$ stationary ergodic st supp of law of $\inf\limits_{|y|\leqq R} \xi(y,\omega)$ unbounded from above

  $V \in C^{0,1}(\mathbb{R})$ st $V(-\infty)=0\ ,\quad V(+\infty)=1\ ,\quad V'\geqq 0$

$$f(x,\omega)=V(\xi(x,\omega))$$
$$\Downarrow$$
$$\text{for all } R>0 \qquad \sup_{x} \inf_{|y-x|\leqq R} f(y,\omega)=1 \quad \forall\ R>0 \qquad \text{a.s. in } \omega$$

UNIFORMLY ELLIPTIC SECOND-ORDER PDE

$$\begin{cases} F\left(D^2u_\varepsilon, Du_\varepsilon, u_\varepsilon, x, \frac{x}{\varepsilon}, \omega\right) = 0 \text{ in } U \\ u_\varepsilon = g \quad \text{on} \quad \partial U \end{cases} \qquad \begin{matrix} u_\varepsilon(\cdot, \omega) \underset{\varepsilon \to 0}{\longrightarrow} \bar{u}(\cdot) \\ \text{in } C(\bar{U}\,) \text{ a.s. in } \omega \end{matrix} \qquad \begin{cases} \bar{F}(D^2\bar{u}, D\bar{u}, \bar{u}, x) = 0 \text{ in } U \\ \bar{u} = g \quad \text{on} \quad \partial U \end{cases}$$

F uniformly elliptic, stationary ergodic — Caffarelli-Souganidis-Wang

F uniformly elliptic, strongly mixing with power decay

$$\|u_\varepsilon(\cdot, \omega) - \bar{u}(\cdot)\|_\infty = O(\varepsilon^{-\log \varepsilon})$$

Caffarelli-Souganidis

existence of correctors   OPEN   even in the linear case!

$$F(D^2u_\varepsilon, Du_\varepsilon, \frac{x}{\varepsilon}, \omega) = 0$$

formal expansion $$u_\varepsilon(x) = \bar{u}(x) + \varepsilon w(\frac{x}{\varepsilon}, \omega) + \varepsilon^2 v(\frac{x}{\varepsilon}, \omega) + O(\varepsilon^3)$$

$$F(D^2_y v + D^2_x \bar{u}, D_x\bar{u}, \frac{x}{\varepsilon}, \omega) = 0$$

microscopic (cell)
problem
(nonlinear eigenvalue)

$$\begin{cases} \text{for each } (X,p) \quad \exists!\ \bar{F}(X,p) \text{ such that} \\ F(D^2v + X, p, y, \omega) = \bar{F}(X,p) \text{ in } \mathbb{R}^N \\ \text{has a solution (corrector) } v \text{ satisfying} \\ |y|^{-2}v(y,\omega) \to 0 \text{ as } |y| \to \infty \quad \text{a.s. in } \omega \end{cases}$$

Why the decay?

- expansion
- uniqueness of $\bar{F}$

$$F(X,p,y,\omega) = F(X)$$

cell problem $$F(D^2v + X) = \bar{F}(X)$$

$$v = \frac{1}{2}(Qx, x) \qquad \bar{F}(X) = F(X+Q) \text{ not unique!}$$

$$F(D^2 v + X, y, \omega) = \bar{F}(X)$$

F uniformly elliptic

- approximate $\qquad \alpha v_\alpha + F(D^2 v_\alpha + X, y, \omega) = 0 \;$ in $\; \mathbb{R}^N$
- estimate $\qquad \sup_{\alpha>0}(\alpha \|v_\alpha\|_\infty + [v_\alpha]_{C^{0,\gamma}_{\text{loc}}}) < \infty$
- normalize $\qquad \hat{v}_\alpha(y) = v_\alpha(y) - v_\alpha(0) \; ; \; \| \, |y|^{-\gamma} \hat{v}(y) \|_{L^\infty_{\text{loc}}} \leqq C \; , \quad [\hat{v}_\alpha]_{0,\gamma} \leqq C$

  $$\alpha \hat{v}_\alpha + F(D^2_y \hat{v}_\alpha + X, y) = -\alpha v_\alpha(0)$$
- pass to the limit $\alpha \to 0$ $\qquad \alpha_n \to 0 \; , \; \hat{v}_{\alpha_n} \to \hat{v}$

  $$F(D^2 \hat{v} + X, y) = \lim_{\alpha_n \to 0}(-\alpha_n v_{\alpha_n}(0))$$

Difficulties

- measurability
- growth at infinity for $\hat{v}$

CANNOT BE RESOLVED USING EXISTING ESTIMATES

$D^2 v$ stationary, $E(D^2 v) = 0 \Rightarrow |y|^{-2} v(y, \omega) \to 0$ as $|y| \to \infty$ a.s. in ω

$D^2 v_\alpha$ stationary, $E(D^2 v_\alpha) = 0$ but $\nexists$ estimates implying $D^2 v_\alpha \to D^2 v$

MOTIVATION OF NEW APPROACH

formal asymptotics

$$F\left(D^2 u_\varepsilon, \frac{x}{\varepsilon}, \omega\right) = 0$$

$$u_\varepsilon(x) = \bar{u}(x) + \varepsilon^2 v\left(\frac{x}{\varepsilon}, \omega\right) + O(\varepsilon^3)$$

microscopic problem

$$\begin{cases} F(Q + D_y^2 v, y, \omega) = \bar{F}(X) \\ |y|^{-2} v(y, \omega) \to 0 \text{ as } |y| \to \infty \text{ a.s. in } \omega \end{cases}$$

- no need
  - (i) v to be independent of ε as long as $\varepsilon^2 v^\varepsilon\left(\frac{x}{\varepsilon}, \omega\right) \to 0$
  - (ii) solve cell problem in all of $\mathbb{R}^N$
- it suffices to find, for each Q, a unique constant $\bar{F}(Q)$ such that

if $v^\varepsilon(\cdot, \omega)$ solves $\begin{cases} F(D_y^2 v^\varepsilon, y, \omega) = \bar{F}(Q) \text{ in } B_{1/\varepsilon} \\ v^\varepsilon = Q \text{ on } \partial B_{1/\varepsilon} \end{cases}$ then $\|\varepsilon^2 v^\varepsilon(\cdot, \omega) - Q\|_{C(\bar{B}_{1/\varepsilon})} \xrightarrow[\varepsilon \to 0]{} 0$ a.s.

equivalently $\left(v_\varepsilon(y) = \varepsilon^2 v^\varepsilon\left(\frac{y}{\varepsilon}\right)\right)$

if $v_\varepsilon(\cdot, \omega)$ solves $\begin{cases} F\left(D^2 v_\varepsilon, \frac{y}{\varepsilon}\right) = \bar{F}(Q) \text{ in } B_1 \\ v_\varepsilon = Q \text{ on } \partial B_1 \end{cases}$ then $\|v_\varepsilon - Q\|_{C(\bar{B}_1)} \xrightarrow[\varepsilon \to 0]{} 0$ a.s.

REVIEW OF KEY FACTS ABOUT UNIFORMLY ELLIPTIC PDE

- linearization

$$\begin{matrix} F(D^2u,x)=0 \\ F(D^2v,x)=0 \end{matrix} \quad \underset{w=u-v}{\Longrightarrow} \quad -a_{ij}(x)w_{x_ix_j} - 0 \qquad a_{ij} \text{ bdd meas}$$

- Alexandrov-Bakelman-Pucci (ABP)-estimate

$$-a_{ij}w_{x_ix_j} = f \text{ in } B_1 \quad \Longrightarrow \quad \sup_{B_1} w^+ \leqq \sup_{\partial B_1} w^+ + C\|f^+\|_{L^N} \qquad (C \text{ universal})$$

- Fabes-Stroock (FS)-estimate

$$\begin{cases} -a_{ij}w_{x_ix_j} = f \text{ in } B_1 \\ w = 0 \text{ on } \partial B_1 \end{cases} \quad \Longrightarrow \quad w_{|B_{1/2}} \geqq C\|f\|_{L^M} \qquad (C, M \text{ universal, } M \text{ large})$$

$$0 < f < 1$$

- obstacle problem

u smallest st

$$\begin{cases} F(D^2u,x) \geqq 0 \text{ in } B_1 \\ u \geqq Q \text{ in } B_1 \end{cases} \quad \Longrightarrow$$

$$\min(F(D^2u,x), u-Q) = 0 \text{ in } B_1$$

- $u = Q$ on ∂B_1
- $F(D^2u,x) = 0$ in $\{u > Q\}$
- $0 \leqq (u-Q)(y) \leqq C|x-y|^2 \quad (u(x) = Q(x))$ $(C$ universal$)$
- $F(D^2u,x) = \mathbf{1}_{\{u=Q\}}(F(Q,x))^+$

NEED TO MEASURE “DISTANCE” BETWEEN SOLUTIONS AND QUADRATICS

$$\begin{cases} F\big(D^2 v_\varepsilon, \dfrac{x}{\varepsilon}\big) = \ell \text{ in } B_1 \\ v_\varepsilon = Q \text{ on } \partial B_1 \end{cases}$$

$$\begin{cases} F\big(Q, \dfrac{x}{\varepsilon}\big) = \ell + F\big(Q, \dfrac{x}{\varepsilon}\big) - \ell \text{ in } B_1 \\ Q = Q \text{ on } \partial B_1 \end{cases}$$

$$\overset{\text{(ABP)}}{\Longrightarrow}$$

$$\begin{aligned} \|v_\varepsilon - Q\| &\leqq C\Big\|\Big(F\big(Q, \frac{\cdot}{\varepsilon}\big) - \ell\Big)_+\Big\|_{L^N} \\ &= C\Big[\fint_{B_{1/\varepsilon}} \Big(F\big(Q, \frac{y}{\varepsilon}\big) - \ell\Big)_+^N dy\Big]^{1/N} \end{aligned}$$

$$\Big\downarrow \varepsilon \to 0$$

$$C[E(F(Q,\omega) - \ell)_+^N]^{1/N}$$

$$\|$$

$$O(1)$$

compare with solution of obstacle problem

$$\begin{cases} F\big(D^2 u_\varepsilon, \dfrac{x}{\varepsilon}\big) = \ell + \Big(F\big(Q, \dfrac{x}{\varepsilon}\big) - \ell\Big)_+ \mathbf{1}_{\{u_\varepsilon = Q\}} \text{ in } B_1 \\ u_\varepsilon = Q \text{ on } \partial B_1 \end{cases}$$

$$\overset{\text{(ABP)}}{\Longrightarrow} \qquad \|(u_\varepsilon - v_\varepsilon)_+\| \leqq C\Big\|\mathbf{1}_{\{u_\varepsilon = Q\}}\Big(F\big(Q, \frac{\cdot}{\varepsilon}\big) - \ell\Big)_+\Big\|_{L^N} \leqq C|\{u_\varepsilon = Q\}|$$

after rescalling

$$h^\varepsilon(\ell, \omega) = |\{u_\varepsilon = Q\}|$$

satisfies the assumptions of the subadditive ergodic thm (subadditive wrt to domain)

$$h^\varepsilon(\ell, \omega) \underset{\varepsilon \to 0}{\longrightarrow} h(\ell) \quad \text{a.s.}$$

measure of contact set is sub-additive

↓

apply ergodic theorem

↓

if measure of contact set is

- asymptotically 0, then v_ε and u_ε stay asymptotically close and above Q
- positive, then the contact set spreads uniformly on B_1, u_ε converges to Q and v_ε stays asymptotically below Q

SKETCH OF THE PROOF

$$\begin{cases} F\big(D^2 v_\varepsilon, \dfrac{x}{\varepsilon}, \omega\big) = \ell \ \text{ in } B_1 \\ v_\varepsilon = Q \ \text{ on } \ \partial B_1 \end{cases} \qquad \begin{cases} \min\big(F\big(D^2 u_\varepsilon, \dfrac{x}{\varepsilon}, \omega\big) - \ell, u_\varepsilon - Q\big) = 0 \ \text{ in } \ B_1 \\ u_\varepsilon = Q \ \text{ on } \ \partial B_1 \end{cases}$$

- $u_\varepsilon \geqq v_\varepsilon$
- $|\{u_\varepsilon = Q\}|$ subadditive, ergodic

$$\text{ergodic thm} \quad \Longrightarrow \quad |\{u_\varepsilon = Q\}| \xrightarrow[\varepsilon \to 0]{} h(\ell) \text{ a.s. in } \omega$$

- $h(\ell)$ monotone (nonincreasing) in ℓ and $h(\ell) = \begin{cases} 1 & \text{if } \ell \ll -1 \\ 0 & \text{if } \ell \gg 1 \end{cases}$

if $h(\ell) > 0$, then the contact set spreads uniformly in B_1

$$u_\varepsilon - Q = o(1) \quad \text{and} \quad v_\varepsilon - Q \leqq u_\varepsilon - Q = o(1) \ \text{ in } B_1$$

if $h(\ell) = 0$, then u_ε and v_ε stay close and above Q

$$u_\varepsilon - v_\varepsilon = o(1) \quad \text{and} \quad 0 \leqq u_\varepsilon - Q \leqq v_\varepsilon - Q + o(1) \ \text{ in } B_1$$

$$\bar{F}(Q) = \inf\{\ell \in \mathbb{R} : h(\ell) = 0\}$$

BACK TO FRONT PROPAGATION

$$u_t^\varepsilon + H\Big(Du^\varepsilon, \frac{x}{\varepsilon}, \omega\Big) = 0 \qquad\qquad H(p,y,\omega) = a(-\hat{p}, y, \omega)|p|$$

- H periodic/almost periodic and coercive $(|a| > 0)$ Lions-Papanicolaou-Varadhan; Ishii

$\Longrightarrow$ homogenization

- H stationary ergodic, coercive, convex (concave) Souganidis; Rezankhanlou-Tarver

general coercive H OPEN

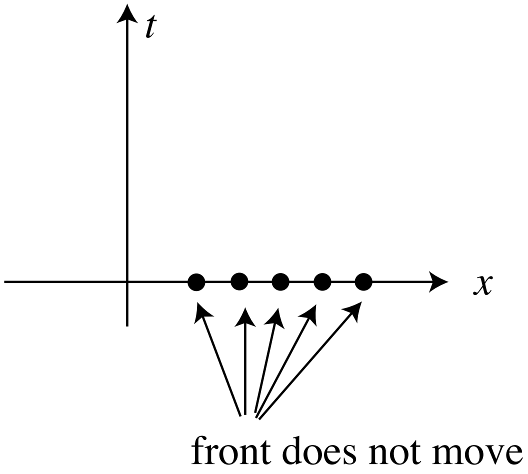
what happens if a vanishes somewhere?

H not coersive?

- trapping

$$\begin{cases} u^\varepsilon_t + \cos\left(\frac{x}{\varepsilon}\right)|u^\varepsilon_x| = 0 & \text{in } \mathbb{R}\times(0,\infty) \\ u^\varepsilon = u_0 & \text{on } \mathbb{R}\times\{0\} \end{cases}$$

$$u^\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} u_0$$

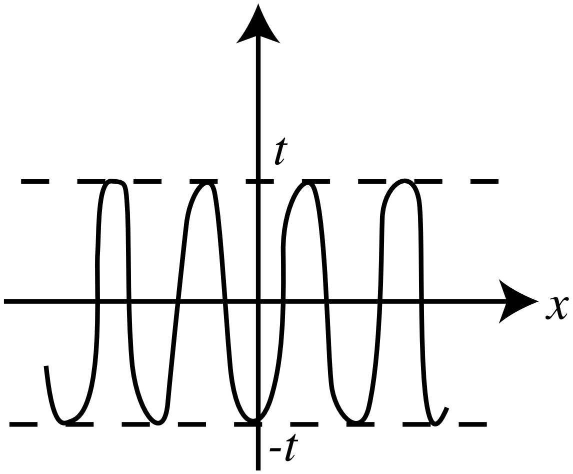


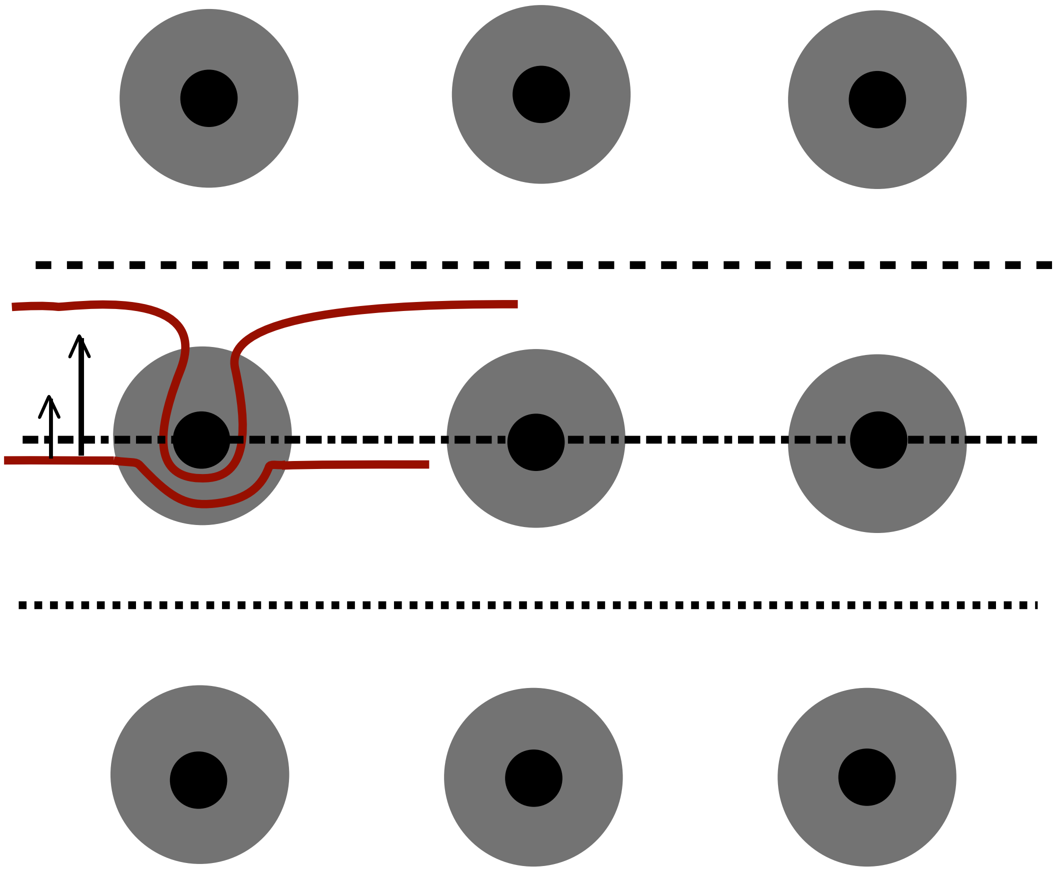
- no homogenization

$$\begin{cases} u^\varepsilon_t + \cos\left(\frac{x}{\varepsilon}\right)|Du^\varepsilon| = 0 & \text{in } \mathbb{R}^2\times(0,\infty) \\ u^\varepsilon(x,y,0) = y \end{cases}$$

$$\underline{\lim_{\varepsilon\to 0}}\, u^\varepsilon(x,y,t) = y - t$$

$$\overline{\lim_{\varepsilon\to 0}}\, u^\varepsilon(x,y,t) = y + t$$





- general result

$$\begin{cases} u_t^\varepsilon + a\big(\dfrac{x}{\varepsilon}, \omega\big)|Du^\varepsilon| = 0 & \text{in } \mathbb{R}^N \times (0,\infty) \\ u^\varepsilon = u_0 \quad \text{on } \mathbb{R}^N \times \{0\} \end{cases}$$

$$a(y,\omega) = \tilde{a}(\tau_y \omega) \quad \text{(stationary)} \quad P(\{\tilde{a} \leqq 0\}) = \theta$$

then

$$u^\varepsilon \rightharpoonup \theta u_0 + (1-\theta)\bar{u} \;\; \text{in} \;\; L^\infty(B_R \times (0,T)) \text{ weak* a.s.}$$

where

$$\begin{cases} \bar{u}_t + \bar{H}(D\bar{u}) = 0 & \text{in } \mathbb{R}^N \times (0,\infty) \\ \bar{u} = u_0 \;\; \text{on } \mathbb{R}^N \times \{0\} \end{cases}$$

$$\bar{H}(p) = \lim_{\delta \downarrow 0} \bar{H}_\delta(p)$$

$$\bar{H}_\delta \quad \text{effective Hamiltonian for} \quad \max(a^+, \delta)$$

Cardaliaguet-Lions-Souganidis

$$V = -\varepsilon\delta \operatorname{div} Dn - a\Big(n, \frac{x}{\varepsilon}\Big)$$

- periodic / almost periodic Lions, Souganidis

$$\min(a^2 - \delta(N-1)|D_y a|) > 0 \quad \Longrightarrow \quad \exists \ \text{effective velocity} \ \bar{v}$$

- stationary ergodic OPEN

$$u_t^\varepsilon - \varepsilon\delta \operatorname{tr}(I - \widehat{Du^\varepsilon} \otimes \widehat{Du^\varepsilon}) D^2 u^\varepsilon + a\Big(\widehat{Du^\varepsilon}, \frac{x}{\varepsilon}\Big)|Du^\varepsilon| = 0 \ \text{ in } \ \mathbb{R}^N \times (0, \infty)$$

$$u^\varepsilon \longrightarrow \bar{u}$$

$$\bar{u}_t + \bar{v}(\widehat{Du})|D\bar{u}| = 0 \ \text{ in } \ \mathbb{R}^N \times (0, \infty)$$

is condition necessary? YES for Lip estimates

general v OPEN

CELL PROBLEM

for each $p \in \mathbb{R}^N$ $\exists!$ $\bar{v}$ st $\exists$ bdd (periodic) solution w of

$$-\mathcal{M}(Dw + p, D^2 w) + v(y)|Dw + p| = \bar{v}(\hat{p})|p| \quad \text{in} \quad \mathbb{R}^N$$

$$\mathcal{M}(p, X) = \operatorname{tr}(I - \hat{p} \otimes \hat{p})X$$

- approximate

$$\alpha w^\alpha - \mathcal{M}(Dw^\alpha + p, D^2 w^\alpha) + v(y)|Dw^\alpha + p| = 0 \quad \text{in} \quad \mathbb{R}^N$$

- estimate

$$\alpha|w^\alpha| + |Dw^\alpha| \leqq C$$

- renormalize

$$\tilde{w}^\alpha(y) = w^\alpha(y) - w^\alpha(0)$$

estimates $\Longrightarrow$ $|\tilde{w}^\alpha| + |D\tilde{w}^\alpha| \leqq C$

- pass to the limit $\alpha \to 0$

$\tilde{w}^\alpha \to w$ bdd and $-\mathcal{M}(Dw + p, D^2 w) + v(y)|Dw + p| = \lim_{\alpha \to 0}(-\alpha w_\alpha(0))$

w bdd $\Longrightarrow$ rhs unique

- $\min(u^2 - \delta_{(N-1)}|D_y v|) > 0$ sharp for Lip estimates

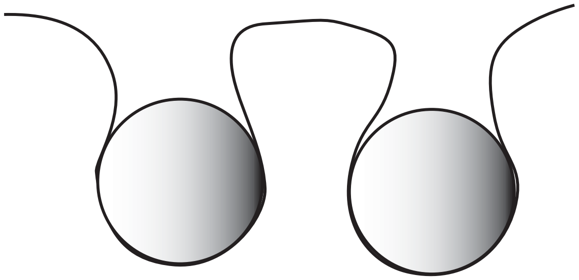
$$v(y) = \begin{cases} \frac{N-1}{\beta'} & |x| \geqq \beta' \\ \frac{N-1}{|x|} & \alpha' < |x| < \beta' \qquad\qquad v^2 = {\scriptstyle (N-1)}|D_y v| \quad \text{in} \quad \alpha' < |x| < \beta' \\ \frac{N-1}{\alpha'} & |x| \leqq \alpha' \qquad\qquad\qquad\qquad\qquad\qquad (0 < \alpha' < \alpha < \beta < \beta' < 1) \end{cases}$$

$w(x) = \left(\frac{1}{|x|} \wedge \beta\right) \vee \alpha$ solves

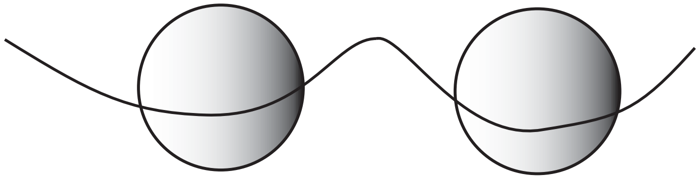
$$-\operatorname{tr}(I - \widehat{Dw} \otimes \widehat{Dw})D^2 w + v(y)|Dw| = 0 \quad \text{in} \quad \mathbb{R}^N$$

$\Longrightarrow$ no Lip estimates for w_α in general

$$V = v - \kappa$$



v_1 large



v_2 small

$$v(x_1, x_2) = \begin{cases} v_1 & \text{outside} \\ & \\ v_2 & \text{inside} \end{cases} \qquad\qquad - v_2 > v_1 > 0$$

Craciun-Bhattacharya

Special case

$$\begin{cases} u_t^\varepsilon + \left(\theta \sin\left(\frac{x}{2\pi\varepsilon}\right) + \delta\right)|Du^\varepsilon| - \varepsilon\, \mathrm{tr}(I - \widehat{Du^\varepsilon} \otimes \widehat{Du^\varepsilon}) D^2 u^\varepsilon = 0 \ \text{ in } \ \mathbb{R}^2 \times (0,\infty) \\ u^\varepsilon = u_0 \quad \text{in} \quad \mathbb{R}^2 \times \{0\} \end{cases} \qquad (x,y) \in \mathbb{R}^2$$

- trapping $\delta = 0$ and $2\pi\theta < 1 \implies u^\varepsilon \to u_0$
- homogenization $\theta < 1 \implies u^\varepsilon \to \bar{u}$

$$\begin{cases} \bar{u}_t + \bar{H}(D\bar{u}) = 0 & \text{in } \mathbb{R}^2 \times (0,\infty) \\ \bar{u} = u_0 & \text{on } \mathbb{R}^2 \times \{0\} \end{cases} \quad \text{and} \quad \delta|p_y| \leqq \bar{H}(p,q) \leqq (\theta+\delta)|p_y|$$

- no homogenization $\theta > 2\pi$ and $u_0(x,y) = -y$

$$\Downarrow$$

$$\exists\, \alpha > 0 \ \text{ st } \quad \begin{aligned} \overline{\lim}\, u^\varepsilon(x,y,t) &\geqq -y + \alpha t \\ \underline{\lim}\, u^\varepsilon(x,y,t) &\leqq -y - \alpha t \end{aligned}$$

Cardaliaguet-Lions-Souganidis

OPEN PROBLEMS

- Hamilton-Jacobi, viscous Hamilton-Jacobi

  non convex Hamiltonians, quasilinear equations, nonlinear equations

- uniformly elliptic

  existence of correctors

- degenerate elliptic

- free boundary problems in random media

- boundary conditions

  Neuman, Dirichlet, perforated domains

- error estimates

- applications

  random traveling waves, mean curvature motion, minimal surfaces