Lecture Notes of Carlos Kenig

Part 1: Elliptic Quantitative Unique Continuation, with an Application to Anderson Localization

The classical unique continuation theorem, which originates in the work of Carleman, in its simplest form is the following:

Proposition: *Assume that $\Delta u = Vu$ in $\{|x| < 10\}$, with $|u| \le C_0$ and $||V||_{L^\infty} \le M$. If $|u(x)| \le C_N |x|^N$, for all $N \ge 0$, then $u \equiv 0$.*

In order to establish this Proposition, Carleman developed a method, the "method of Carleman estimates", which still permeates the subject. An example of such an estimate is the following one due to Hörmander (1983).

Lemma: *There exist C_1, C_2, C_3, depending only on the dimension n, and an increasing function $w(r), 0 < r < 10$, so that $\frac{1}{C_1} \le \frac{w(r)}{r} \le C_1$ and such that, for all $f \in C_0^\infty(B(0,10)\backslash\{0\})$, $\alpha > C_2$, we have*

$$\alpha^3 \int w^{(-1-2\alpha)} f^2 \le C_3 \int w^{(2-2\alpha)} |\Delta f|^2.$$

I will give the proof of this Lemma later on, but let's illustrate Carleman's method by showing how it yields the Proposition.

Proof of Proposition: Let $\varphi \in C_0^\infty(B(0,10)), \varphi \equiv 1$ on $B(0,2), 0 \le \varphi \le 1$, $\psi \in C^\infty(\mathbb{R}^n)$, $\psi \equiv 1$ for $|x| \ge 1, \psi \equiv 0$ for $|x| < 1/2$. For $\epsilon > 0$, small, let

$$f_\epsilon(x) = \psi\left(\frac{x}{\epsilon}\right)\varphi(x)u(x),$$

and apply the Lemma to f_ϵ. We obtain (with $\eta_\epsilon(x) = \psi\left(\frac{x}{\epsilon}\right)\varphi(x)$):

$$\alpha^3 \int w^{-1-2\alpha} f_\epsilon^2 \le C_3 \int w^{2-2\alpha} (\Delta f_\epsilon)^2 \le C_3 \int w^{2-2\alpha} \left[\eta_\epsilon(x)\Delta u + 2\nabla\eta_\epsilon \nabla u + \Delta\eta_\epsilon u\right]^2.$$

Note that $\nabla\eta_\epsilon = \frac{1}{\epsilon}\nabla\psi\left(\frac{x}{\epsilon}\right)\varphi + \psi\left(\frac{x}{\epsilon}\right)\nabla\varphi$, while

$$\Delta\eta_\epsilon = \frac{1}{\epsilon^2}\Delta\psi\left(\frac{x}{\epsilon}\right)\varphi + \psi\left(\frac{x}{\epsilon}\right)\Delta\varphi + \frac{2}{\epsilon}\nabla\psi\left(\frac{x}{\epsilon}\right)\nabla\varphi.$$

In order to control the term involving ∇u, we use the Caccioppoli inequality

$$(\dagger) \qquad \int_{B(x_0,r)} |\nabla u|^2 \leq 2M \int_{B(x_0,2r)} |u|^2 + \frac{C}{r^2} \int_{B(x_0 2r)} |u|^2$$

(The inequality is obtained by multiplying $\Delta u + Vu = 0$ with $\theta^2 u, \theta \equiv 1$ on $B(x_0, r)$, supp $\theta \subset B(x_0, 2r)$ and integrating by parts). Now, our assumption that $|u(x)| \leq C_N |x|^N$ and (†) easily give that all the terms in which ψ is differentiated on $r.h.s. \to 0$ as $\epsilon \to 0$. Hence, letting $\epsilon \to 0$, we obtain:

$$\alpha^3 \int w^{-1-2\alpha} |u\varphi|^2 \leq C_3 \int w^{2-2\alpha} \left[\varphi(x)V(x)u(x) + u(x)\Delta\varphi(x) + 2\nabla u \cdot \nabla\varphi\right]^2.$$

We now use $||V||_\infty \leq M$,

$$w^{2-2\alpha}(x) \leq w^3(x) w^{-1-2\alpha}(x) \leq w(10)^3 w^{-1-2\alpha}(x)$$

and choose α so large that $4w(10)^3 M^3 C_3 \leq \alpha^3/2$, to obtain:

$$\frac{\alpha^3}{2} \int w^{-1-2\alpha} |u\varphi|^2 \leq C_3 \int w^{2-2\alpha} \left[u(x)\Delta\varphi(x) + 2\nabla u \nabla\varphi\right]^2.$$

Now, supp $(\Delta\varphi, \nabla\varphi) \subset B(0, 10) \backslash B(0, 2)$, so, by the monotonicity of w we have that the right hand side is smaller than

$$C_3 C_n w^{2-2\alpha}(2) \int\limits_{\text{supp}(\nabla\varphi, \Delta\varphi)} |u|^2 + |\nabla u|^2 \leq C_0^2 C_3 C_n w^{2-2\alpha}(2),$$

where we use (†) once more. Hence, $\frac{\alpha^3}{2} \int\limits_{|x|<2} \left[\frac{w(x)}{w(2)}\right]^{-1-2\alpha} |u|^2 \leq C_3 C_n C_0^2$. Note that $w(2)/w(x) > 1$ on $|x| < 2$, so that $u \equiv 0$ on $|x| < 2$, by letting $\alpha \to \infty$. A chain of balls argument finishes the proof.

A natural "quantitative" question might be: How large can N be and still have $u \not\equiv 0$? Clearly, some normalization on u is needed, because, when $n = 2$ $u(z) = Re(z^N)$ is harmonic for each N. This question was studied by H. Donnelly and C. Fefferman (1988) for eigenfunctions on compact manifolds, i.e. solutions to $-\Delta_g u = \lambda u$, $||u||_{L^2(M)} = 1$, who showed that the possible order of vanishing is $O(\lambda^{1/2})$, which is sharp. We will return to this question later on.

Another "quantitative unique continuation" problem arose in my work with J. Bourgain on Anderson localization for the Bernoulli model, to which I will now turn to. The problem of Anderson localization for the Bernoulli model is a well-known problem in the theory of disordered media. The problem originates in a seminal 1958 paper by Anderson, who argued that, for a simple Schrödinger operator in a disordered medium, "at sufficiently low densities, transport does not take place; the exact wave functions are localized in a small region of space." In our work with Bourgain (2004) we concentrated on continuous models; the corresponding issues for discrete problems remain open. Thus, consider a random Schrödinger operator on $\mathbb{R}^n$ of the form

$$H_\epsilon = -\Delta + V_\epsilon$$

where the potential $V_\epsilon(x) = \sum_{j\in\mathbb{Z}^n} \epsilon_j \varphi(x-j)$, where $\epsilon_j \in \{0,1\}$ are independent, $0 \leq \varphi \leq 1$, $\varphi \in C_0^\infty(B(0,1/10)$. It is not hard to see that, under these assumptions,

$$\inf \operatorname{spec} H_\epsilon = 0 \qquad\qquad \text{a.s.}$$

In this context, Anderson localization means that, near the bottom of the spectrum (i.e. for energies $E > 0$, $E < \delta$, $\delta = \delta(n)$ small) H_ϵ has pure point spectrum, with exponentially decaying eigenfunctions, a.s. This phenomenon is by now well-understood in the case when the random potential V_ϵ has a continuous site distribution (i.e. the ϵ_j take their values in $[0,1]$). When $n = 1$, this was first proved, for all energies, for potentials with a continuous site distribution by Goldsh'ein-Molchanov-Pastur (1977). The extensions to $n > 1$, for the same potentials, were achieved by the method of "multi-scale analysis", developed by Fröhlich-Spencer ($\sim$ 1983). When the random variables are discrete valued (i.e. the Anderson-Bernoulli model), the result was established for $n = 1$, by Carmona-Klein-Martinelli (1987) and by Shubin-Vakilian-Wolff (1987). Neither one of their methods extends to $n > 1$. We now have

Theorem (Bourgain-Kenig 2004). *For energies near the bottom of the spectrum* $(0 < E < \delta)$, H_ϵ *displays Anderson localization a.s., for* $n \geq 1$.

The only previous result when $n > 1$ was due to Bourgain (2003), who considered instead $V_\epsilon(x) = \sum_{j\in\mathbb{Z}^n} \epsilon_j \varphi(x-j)$, where $\varphi(x) \sim \exp(-|x|)$ instead of $\varphi \in C_0^\infty$. The non vanishing of the tail of φ as $|x| \to \infty$ was essential in Bourgain's argument (which also applied to the corresponding discrete problem on $\mathbb{Z}^n$). In our work on the true Bernoulli model, we overcome this by the use of a quantitative unique continuation result. The proof of the above Theorem proceeds by an "induction on scales" argument. Thus, we consider restrictions of H_ϵ to cubes $\Lambda \subset \mathbb{R}^n$, of size-length l. We establish our estimates by induction on l. The estimates that we establish are weak versions of the so-called "Wegner estimates". Thus, let $H_\Lambda =$ the restriction of H_ϵ to Λ, with Dirichlet boundary conditions and let $R_\Lambda(z) = (H_\Lambda - z)^{-1}$ be the resolvent in $L^2(\Lambda)$. We fix an E (the energy) and set $R_\Lambda = R_\Lambda(E + i0)$. We also let $\chi_x = \chi_{B(x;1)}$, for $x \in \mathbb{R}^n$. Our basic estimate, for $0 < E < \delta, \delta$ small, is:

Proposition A: $\exists \Omega_l \subset \{0,1\}^{\Lambda \cap \mathbb{Z}^n}$ *s.t.*

$(a_1)|\Omega_\ell| > 1 - l^{-\rho}$ *(ρ is any number $< 3n/8$), such that, for ϵ belonging to Ω_l, the resolvent satisfies*

(a)$_2$: $||R_\Lambda|| \leq \exp\,(l^-)$

(a)$_3$: $||\chi_x R_\Lambda \chi_{x'}|| \leq \exp\,(-Cl)\quad,\quad |x - x'| \geq l/10.$

Such estimates (with exponentially small exceptional sets in (a)$_1$) are what in the literature are called "Wegner estimates". The difficulty in proving such estimates in the Bernoulli case, as opposed to the case in which we have a continuous site distribution, is that we cannot obtain the estimate by varying a single j at a time. Here, "rare event" bounds must be obtained by considering the dependence of eigenvalues on a large collection of variables $\{\epsilon_j\}_{j\in S}$. The proof of the Proposition A is obtained by induction on l. An added difficulty is the fact that the bounds on the exceptional set are weaker from the standard ones, but we show that they still suffice to obtain Anderson localization.

For instance, to obtain the bound (a)$_2$, we write $H_\Lambda - E + i0 = H^0_\Lambda + 1 + V_\epsilon - 1 - E + i0$ and if we let $\Gamma_\Lambda = \Gamma_\Lambda(E,\epsilon) = (H^0_\Lambda + 1)^{-1/2}(-V_\epsilon + 1 + E)(H^0_\Lambda + 1)^{-1/2}$, which is a compact operator on $L^2(\Lambda)$, it is easy to see that $||R_\Lambda|| \simeq ||(1 - \Gamma_\Gamma)^{-1}||$. Thus, the issue is to obtain a lower bound for

$$\mathrm{dist}(1,\ \mathrm{spec}\ \Gamma_\Lambda).$$

In carrying this out by induction in l, one of our key tools is a probabilistic lemma on Boolean functions, introduced by Bourgain in his 2003 work.

Lemma: Let $f = f(\epsilon_1, \ldots \epsilon_d)$ be a bounded function on $\{0,1\}^d$ and denote $I_j = f|_{\epsilon_j=1} - f|_{\epsilon_j=0}$, the j^{th} influence, which is a function of $\epsilon_{j'}, j' \neq j$. Let $J \subset \{1,\ldots,d\}$ be a subset with $|J| \leq \delta^{-1/4}$, so that $k < |I_j| < \delta$, for all $j \in J$. Then, for all E,

$$\mathrm{meas}\ \{|f - E| < k/4\} \leq |J|^{-1/2}$$

(here meas refers to normalized counting measure on $\{0,1\}^d$.) The proof of this Lemma relies on Sperner's Lemma in the theory of partially ordered sets. The function to which this Lemma is applied is the eigenvalue. It then becomes crucial to find bounds for the j^{th} influence of eigenvalues. To calculate it, note that V_ϵ and H_ϵ, defined as functions of $\epsilon \in \{0,1\}^{\mathbb{Z}^n}$, admit obvious extensions $V_t, H_t, t \in [0,1]^{\mathbb{Z}^n}$, namely $V_t(x) = \sum_{j\in\mathbb{Z}^n} t_j \varphi(x - j)$ and we also have the analogues of $R_\Lambda, \Gamma_\Lambda$. Let $E_\tau(t)$ be an eigenvalue parametrization of spec $\Gamma_\Lambda(t)$, near 1. Upper estimates on I_j (for $f = E_\tau$ as a function on $\{0,1\}^{\mathbb{Z}^n}$) are standard and the crucial issue is lower bounds for I_j

$$I_j = \int_0^1 \frac{\partial E_\tau}{\partial t_j}\,(\epsilon_{j'}, j' \in \mathbb{Z}^n\backslash\{j\}; t_j)\,dt_j.$$

A calculation shows that $\frac{\partial E_\tau}{\partial t_j} = \left\langle \frac{\partial}{\partial t_j}\Gamma_\Lambda \xi, \xi \right\rangle = -\int_\Lambda \varphi(x-j)|(H^0_\Lambda+1)^{-1/2}\xi|^2$, where $\xi = \xi_\tau(t)$ is the corresponding normalized eigenfunction of Γ_Λ. If $\zeta = (H^0_\Lambda + 1)^{-1/2}\xi$ we see that

$$-I_j = \int_0^1 \int_\Lambda \varphi(x-j)|\zeta(\epsilon_{j'}, t_j)|^2 dt_j dx$$

From the fact that ξ is a normalized eigenfunction of Γ_Λ one obtains $H^0_\Lambda \zeta = E_\tau^{-1}(1+E-V_\epsilon)\zeta$, so that

$$|\Delta\zeta| \leq C|\zeta|.$$

Moreover, $1 \leq ||\xi||_{L^2} \leq C||\zeta||_{L^2}$ and from interior estimates $|\zeta| \leq C$. We see then that what we need to estimate from below is $\displaystyle\int_{B(j,1)} |\zeta|^2$, where $j \in \Lambda$. This then leads us to the following quantitative unique continuation problem at infinity. Suppose that u is a solution to

$$\Delta u + Vu = 0 \quad \text{in} \quad \mathbb{R}^n,$$

where $|V| \leq 1,\ |u| \leq C_0$ and $u(0) = 1$.

For R large, define

$$M(R) = \inf_{|x_0|=R} \sup_{B(x_0,1)} |u(x)|.$$

Q: How small can $M(R)$ be? Note that, by unique continuation $\displaystyle\sup_{B(x_0,1)} |u(x)| \not\equiv 0$.

Theorem: (Bourgain - K 2004)

$$M(R) \geq C \exp\left(-R^{4/3} \log R\right)$$

Remark: In order for the induction on scales argument to prove Proposition A to work, if we have an estimate of the form $M(R) \geq C \exp(-CR^\beta)$, one needs $\beta < \frac{1+\sqrt{3}}{2} \simeq 1.35$. Note that $4/3 \simeq 1.33$.

It turns out that the estimate described in the Theorem is a quantitative version of a conjecture of E.M. Landis (~ 65). Landis conjectured that if $\Delta u + Vu = 0$ in $\mathbb{R}^n$, with $||V||_\infty \leq 1, ||u||_\infty \leq C_0$, and $|u(x)| \leq C \exp\left(-C|x|^{1+}\right)$, then $u \equiv 0$. This conjecture was disproved by Meshkov (1992) who constructed such a $V, u, u \not\equiv 0$, with $|u(x)| \leq C \exp\left(-C|x|^{4/3}\right)$. (Meshkov also showed that if

$|u(x)| \leq C \exp\,(-C|x|^{4/3+})$, then $u \equiv 0$.) Meshkov's example clearly shows the sharpness of our lower bound on $M(R)$.

It turns out that one can give a unified proof of the above Theorem and of the quantitative version of Carleman's Proposition that we mentioned earlier. We first formulate precisely the latter:

Suppose that we are in the following normalized situation: Let $(\Delta + V)u = 0$ in $B(0, 10)$, with $||V||_{L^\infty} \leq M$, $||u||_{L^\infty(B(0,10))} \leq C_0$. Assume that $\sup_{|x|\leq 1}|u(x)| \geq 1$. What is the sharp lower bound for

$$m(r) = \sup_{|x|\leq r}|u| \geq a_1 r^{a_2 \beta}, \text{ as } r \to 0$$

where a_1, a_2 depend only on n, C_0 and $\beta = \beta(M), M \gg 1$. As I mentioned earlier, when $V \equiv M$, H. Donnelly and C. Fefferman (1988) showed that $\beta = M^{1/2}$.

Theorem: (Bourgain -K 2004) *For general $V, \beta = M^{2/3}$. Moreover, this is the sharp rate.*

The most efficient way to prove both Theorems is through "3-ball" inequalities. For harmonic functions, such inequalities were first proved by Hadamard.

3-Ball Inequalities. Fix $R_1 = 6, r_1 = 2$ and $2r_0 \ll r_1$. Let $\zeta \in C_0^\infty(B_{R_1}), \zeta \equiv 1$ on $\left[\frac{3}{2}r_0, R_1/2\right]$, $\zeta \equiv 0$ on $[0, r_0] \cup \left[\frac{3}{4}R_1, R_1\right]$, with $|\nabla\zeta| + r_0|\nabla^2\zeta| \leq C/r_0$ on $\left[0, \frac{3}{2}r_0\right]$ and $|\nabla\zeta| + R_1|\nabla^2\zeta| \leq C/R_1$ on $\left[R_1/2, \frac{3}{4}R_1\right]$. Let $f = \zeta u$ in Carleman estimate, where we assume now that $|\Delta u| \leq M|u|$ in B_{R_1}. Then:

$$\alpha^3 \int w^{-1-2\alpha}(\zeta u)^2 \leq C_3 \int w^{2-2\alpha}|\Delta(\zeta u)|^2, \ \alpha > C_2.$$

Let $K_1 = \left\{\frac{3}{2}r_0 \leq |x| \leq R_1/2\right\}, \ K_2 = \left\{r_0 \leq |x| \leq \frac{3}{2}r_0\right\}, K_3 = \left\{R_1/2 \leq |x| \leq \frac{3R_1}{4}\right\}$. Then:

$$\alpha^3 \int_{K_1} w^{-1-2\alpha}|u|^2 \leq C_3 M^2 \int_{K_1} w^{2-2\alpha}|u|^2 + J,$$

where $J = C_3 \int_{K_2 \cup K_3} |\Delta(\zeta u)|^2 w^{2-2\alpha}$. Thus,

$$\alpha^3 \int_{K_1} w^{-1-2\alpha}|u|^2 \leq C_3 M^2 w^3(R_1/2) \int_{K_1} |u|^2 w^{-1-2\alpha} + J.$$

Thus, if $\alpha^3 > 2C_3 M^2 w^3(R_1/2)$, we obtain:

$$(\ddagger) \qquad \frac{\alpha^3}{2} \int_{K_1} w^{-1-2\alpha}|u|^2 \leq J.$$

We next estimate J. We have $|\Delta(u\zeta)| \le M|u| + 2|\nabla\zeta||\nabla u| + |\Delta\zeta||u|$. Thus,

$$|\Delta(u\zeta)| \le M|u| + \frac{C}{r_0}|\nabla u| + \frac{C}{r_0^2}|u| \text{ on } K_2,\ |\Delta(u\zeta)| \le M|u| + \frac{C}{R_1}|\nabla u| + \frac{C}{R_1^2}|u| \text{ on } K_3,$$

so

$$J \le C_3\left[M^2 + \frac{C}{r_0^4}\right]\int_{K_2} w^{2-2\alpha}|u|^2 + \frac{CC_3}{r_0^2}\int_{K_2} w^{2-2\alpha}|\nabla u|^2 + C_3\left[M^2 + \frac{C}{R_1^4}\right]\int_{K_3} w^{2-2\alpha}|u|^2$$

$$+\frac{CC_3}{R_1^2}\int_{K_3} w^{2-2\alpha}|\nabla u|^2$$

$$\le C_3\left[M^2 + \frac{C}{r_0^4}\right]w(r_0)^{2-2\alpha}\int_{K_2}|u|^2 + \frac{CC_3}{r_0^2}w(r_0)^{2-2\alpha}\int_{K_2}|\nabla u|^2$$

$$+C_3\left[M^2 + \frac{C}{R_1^4}\right]w(R_1/2)^{2-2\alpha}\int_{K_3}|u|^2 + \frac{CC_3}{R_1^2}w(R_1/2)^{2-2\alpha}\int_{K_3}|\nabla u|^2.$$

Let $K_4 = \{x \in K_1 : \ |x| \le r_1\}$, and still assume that $\alpha^3 > 2C_3M^2w^2(R_1/2)$ and insert this into (‡). We get

$$\int_{K_4} u^2 \le w(r_1)^{2\alpha+1}\int_{K_4} w^{-2\alpha-1}|u|^2 \le C_3 w(r_1)^{2\alpha+1}\left\{\left[M^2 + \frac{C}{r_0^4}\right]\frac{w^2(r_0)}{w(r_0)^{2\alpha}}\int_{K_2}|u|^2\right.$$

$$\left.+\frac{C}{r_0^2}\,\frac{w(r_0)^2}{w(r_0)^{2\alpha}}\int_{K_2}|\nabla u|^2\right\} + C_3 w(r_1)^{2\alpha+1}\left\{\left[M^2 + \frac{C}{R_1^4}\right]\frac{w^2(R_1/2)}{w(R_1/2)^{\alpha}}\int_{K_3}|u|^2 + \frac{C}{R_1^2}\frac{w^2(R_1/2)}{w(R_1/2)^{2\alpha}}\int_{K_3}|\nabla u|^2\right\}.$$

We now use the interior regularity bounds (Caccippoli)

$$\int_{K_2}|\nabla u|^2 \le \left[M + \frac{C}{r_0^2}\right]\int\limits_{B_{2r_0}\setminus B_{r_0/2}}|u|^2$$

and

$$\int_{K_3}|\nabla u|^2 \le \left[M + \frac{C}{R_1^2}\right]\int\limits_{B_{R_1}\setminus B_{R_1/4}}|u|^2.$$

Finally, add $\int\limits_{|x|\le\frac{3}{2}r_0}|u|^2$ to both sides of the previous inequality, together with the Caccioppoli estimates, to obtain:

$$\int\limits_{|x|<r_1} |u|^2 \le CC_3 \left[\frac{w(r_1)}{w(r_0)}\right]^{2\alpha} \eta^2 w(r_1) \left[M^2 r_0^2 + \frac{1}{r_0^2}\right] + CC_3 \left[\frac{w(r_1)}{w(R_1/2)}\right]^{2\alpha} \mu^2 w(r_1) \left[M^2 R_1^2 + \frac{1}{R_1^2}\right],$$

for $\alpha^3 \ge 2C_3 M^2 w^3(R_1/2)$, where $\eta = ||u||_{L^2(B_{2r_0})}$ and $\mu = ||u||_{L^2(B_{R_1})}$. If we now define

$$\eta_1^2 = \eta^2 w(r_1) \left[M^2 r_0^2 + \frac{1}{r_0^2}\right],\ \mu_1^2 = \mu^2 w(r_1) \left[M^2 R_1^2 + \frac{1}{R_1^2}\right],$$

$A^2 = CC_3$, we have, for $\alpha \ge 2C_3 M^2 w^3(R_1/2)$,

$$\int\limits_{|x|<r_1} |u|^2 \le A^2 \left[\frac{w(r_1)}{w(r_0)}\right]^{2\alpha} \eta_1^2 + A^2 \left[\frac{w(r_1)}{w(R_1/2)}\right]^{2\alpha} \mu_1^2.$$

Let now K_0 be defined by $\frac{1}{K_0} = 1 + \frac{\log[w(r_1)/w(r_0)]}{\log\left[\frac{\omega(R_1/2)}{w(r_1)}\right]}$, so that $\frac{1}{K_0} \simeq \log 1/r_0$. Let $\alpha_1 = \frac{K_0}{2\log\left[\frac{w(R_1/2)}{w(r_1)}\right]} \log((\mu_1/\eta_1)^2)$. If $\alpha_1^3 \ge 2C_3 M^2 w^3(R_1/2)$, we can insert it above. With these choices, a computation gives

$$||u||_{L^2(B_{r_1})} \le \sqrt{2} A \eta_1^{K_0} \mu_1^{1-K_0} = \sqrt{2} A \quad \left\{||u||_{L^2(B_{2r_0})} w(r_1)^{1/2} \left[M^2 r_0^2 + \tfrac{1}{r_0^2}\right]^{1/2}\right\}^{K_0} \cdot$$

$$\cdot \left\{||u||_{L^2(B_{R_1})} w(r_1)^{1/2} \left[M^2 R_1^2 + \tfrac{1}{R_1^2}\right]^{1/2}\right\}^{1-K_0}.$$

If, on the other hand, $\alpha_1 \le 2^{1/3} C_3^{1/3} w(R_{1/2}) M^{2/3}$, since $||u||_{L^2(B_{r_1})} \le \mu$ and

$$\frac{K_0}{2\log\left[\frac{w(R_1/2)}{w(r_1)}\right]} \log((\mu_1/\eta_1)^2) \le 2^{1/3} C_3^{1/3} w(R_1/2) M^{2/3},\ \mu_1^2 \le \eta_1^2 \exp\left(\tilde{C} M^{2/3}/K_0\right)$$

and so

$$||u||_{L^2(B_{r_1})} \le \frac{[M^2 r_0^2 + \frac{1}{r_0^2}]^{1/2}}{[M^2 R_1^2 + \frac{1}{R_1^2}]^{1/2}} ||u||_{L^2(B_{2r_0})} \exp\left(\frac{\tilde{C} M^{2/3}}{K_0}\right).$$

Combining both estimates, we obtain:

$$
\begin{aligned}
||u||_{L^2(B_{r_1})} \;&\leq \sqrt{2}A\left\{||u||_{L^2(B_{2r_0})}w(r_1)^{1/2}\left[M^2r_0^2+\frac{1}{r_0^2}\right]^{1/2}\right\}^{K_0}\cdot \\
&\cdot\left\{||u||_{L^2(B_{r_1})}w(r_1)^{1/2}\left[M^2R_1^2+\frac{1}{R_1^2}\right]^{1/2}\right\}^{1-K_0} \\
&+\left[\frac{M^2r_0^2+1/r_0^2}{M^2R_1^2+1/R_1^2}\right]^{1/2}||u||_{L^2(B_2r_0)}\ \exp\,(\tilde{C}M^{2/3}/K_0),
\end{aligned}
$$

with $1/K_0 \simeq \log 1/r_0$. This is our "3-ball inequality". We can now combine it with the elliptic regularity estimate $||u||_{L^\infty(B_1)} \leq C_n\left\{M^{n/2}+1\right\}||u|_{L^2(B_2)} \leq C_nM^{n/2}||u||_{L^2(B_2)}$ (for $M>1$). Hence,

$$
||u||_{L^\infty(B_1)} \leq \mathrm{I}+\mathrm{II},
$$

where

$$
\begin{aligned}
I \quad &= \quad \sqrt{2}AC_nM^{n/2}\left\{||u||_{L^2(B_{2r_0})}w(r_1)^{1/2}\left[M^2r_0^2+\frac{1}{r_0^2}\right]^{1/2}\right\}^{K_0}\cdot \\
&\cdot\left\{||u||_{L^2(B_{R_1})}w(r_1)^{1/2}\left[M^2R_1^2+1/R_1^2\right]\right\}^{1-K_0}, \\
II \quad &= \quad C_nM^{n/2}\left\{\frac{M^2r_0^2+1/r_0^2}{M^2R_1^2+1/R_1^2}\right\}^{1/2}||u||_{L^2(B_{2r_0})}\exp\left(\tilde{C}M^{2/3}/K_0\right).
\end{aligned}
$$

Recall that in our quantitative unique continuation question, we assume that $||u||_{L^\infty(B_1)} \geq 1$. If I $\leq II$, we obtain

$$
\begin{aligned}
&\mathrm{I} \leq 2\ \mathrm{II}\ \leq 2C_nM^{n/2}\left\{\tfrac{M^2r_0^2+1/r_0^2}{M^2R_1^2+1/R_1^2}\right\}^{1/2}||u||_{L^2(B_2r_0)}\cdot\exp\,(\tilde{C}M^{2/3}/K_0) \\
&\leq \tilde{C}_nM^{n/2}(M+1/r_0)r_0^{n/2}\ \exp\,(\tilde{C}M^{2/3}/K_0)\max_{|x|\leq 2r_0}|u| \\
&\leq \tilde{C}_n\ \exp\,(2\tilde{C}M^{2/3}/K_0)\max_{|x|\leq 2r_0}|u| \leq \tilde{C}_nr_0^{-CM^{2/3}}\max_{|x|\leq 2r_0}|u|,
\end{aligned}
$$

which gives the desired lower bound with $\beta = M^{2/3}$ (recall that $\frac{1}{K_0} \simeq \log 1/r_0$). If, on the other hand II $\leq$ I, we have

$$1 \leq 2C_n M^{n/2} \left\{ ||u||_{L^2(B2r_0)} w(r_1)^{1/2} \left[M^2 r_0^2 + \frac{1}{r_0^2} \right]^{1/2} \right\}^{K_0}$$

$$\cdot \left\{ ||u||_{L^2(B_{R_1})} w(R_1)^{1/2} \left[M^2 R_1^2 + 1/R_1^2 \right]^{1/2} \right\}^{1-K_0}$$

Raising both sides to $1/K_0$ and using the bound $||u||_{L^\infty} \leq C_0$, we obtain

$$1 \leq (2C_n M^{n/2})^{1/K_0} M ||u||_{L^\infty(B2r_0)} \cdot (C_0 C_n R_1^{n/2})^{\frac{1}{K_0}-1} . M^{1/K_0 - 1}$$

$$\leq C_n^{1/K_0} C_0^{1/K_0} (M^{n/2+1})^{1/K_0} ||u||_{L^\infty(B_{2r_0})}.$$

Since $\frac{1}{K_0} \simeq \log 1/r_0$, the right hand side is bounded by

$$\left(\frac{1}{r_0}\right)^{C_n} \left(\frac{1}{r_0}\right)^{C \log C_0} \left(\frac{1}{r_0}\right)^{C \log M} ||u||_{L^\infty(B_{2r_0})},$$

which gives a better bound. Thus the Theorem follows.

We next turn to the proof of the first Theorem. Fix $x_0, |x_0| = R$ so that $M(R) = \inf_{|x_0|=R} \sup_{B(x_0,1)} |u(x)| = \sup_{B(x_0,1)} |u(x)|$. Set $u_R(x) = u(R(x + x_0/R))$, so that $||u_R||_\infty \leq C_0$, $|\Delta u_R| \leq R^2 |u_R|$, so that, using our previous notation, $M = R^2$. Note also that if $\tilde{x}_0 = -x_0/R$, $|\tilde{x}_0| = 1$ and $u_R(\tilde{x}_0) = u(0) = 1$, so that $||u_R||_{L^\infty(B_1)} \geq 1$. Also, $\sup_{B(x_0,1)} |u(x)| = \sup_{B(0,2r_0)} |u_R(y)|$, where $2r_0 = 1/R$. Our previous estimate gives

$$M(R) = \sup_{B_{2r_0}} |u_R(y)| \geq C(2r_0)^{M^{2/3}} = C(1/2R)^{R^{4/3}} = C \exp\left(-CR^{4/3} \log R\right)$$

as claimed. The example of Meshkov shows that this is sharp. To show that it is also sharp for the "rate of vanishing" theorem in its uniform form, we will find $r_j \to 0$, potentials V_j, $||V_j||_\infty = M_j$ and solutions to $\Delta u_j + V_j u_j = 0$, with $||u_j||_\infty \leq C_0, ||u_j||_{L^\infty(B_1)} \geq 1$ and $\max_{|x|<r_j} |u_j(x)| \leq C r_j^{CM_j^{2/3-}}$. In fact, let u be the Meshkov solution, normalized by $u(0) = 1$, $||u||_\infty \leq C_0$, $|\Delta u| \leq |u|$ and $|u(x)| \leq C \exp\left(-C|x|^{4/3}\right)$. For $R_j \to \infty$, let $M_j = R_j^2$, fix $x_{0,j}$, $|x_{0,j}| = R_j$, and let $u_j(x) = u(R_j(x + x_{0,j}/R_j))$. Clearly the required conditions on u_j are verified. Let $r_j = \frac{2}{2R_j}$. Then,

$$\max_{|x|<r_j} |u_j(x)| \leq \max_{|x| \leq 1/R_j 2} |u(R_j(x + x_{0,j}/R_j))| \leq C \exp\left(-CR_j^{4/3}\right) \leq C \left(\frac{1}{2R_j}\right)^{CR_j^{4/3-}},$$

since $\log(2R_j) \le CR_j^{0+}$ for R_j large. Before ending this topic by proving the Lemma concerning the Carleman estimate, I would like to point out a few questions that these results suggest.

Question 1: In my work with Bourgain, we establish Anderson localization for the continuous Bernoulli model $-\Delta + V_\epsilon$, where Δ is the usual Laplacian in $\mathbb{R}^n$. It is also of importance to study the corresponding discrete Bernoulli model, where Δ is now the Laplacian on $\mathbb{Z}^n$. Here there are no results for $n > 1$. The reason why our approach does not apply in this setting (at least without modifications) is because unique continuation fails and there could be solutions vanishing in a ball which are not identically 0.

Question 2: In Meshkov's example mentioned earlier, u and V are complex valued. Can the exponent $4/3$ be improved to $1+$ in our theorem, for real valued u and V? Is Landis' conjecture true for real valued u and V?

Question 3: Can one improve the lower bound on the "quantitative order of vanishing theorem" to $\beta = M^{1/2}$, for real valued u and V, thus giving the same order as in the Donnelly-Fefferman work?

Remark: The Carleman lemma admits various extensions. One can add to the left hand of the inequality a term of the form $\alpha \int w^{1-2\alpha}|\nabla f|^2$. One can also substitute Δ by Δ_g given by $\sum \partial_{x_i} g_{ij}(x) \partial_{x_j}$, with $g_{ij}(x)\zeta_i\zeta_j \ge \lambda|\zeta|^2$, $|g_{ij}(x)| \le \lambda^{-1}$ and $|g_{ij}(x) - g_{ij}(x')| \le \lambda^{-1}|x - x'|$. (In view of the well-known Pliš-Miller examples, the Lipschitz condition is best possible.) The order of vanishing theorem can be extended to solutions of $\sum \partial_{x_i} g_{ij}(x) \partial_{x_j} u + \sum b_i(x) \frac{\partial u}{\partial x_i} + V(x)u = 0$, with $|b_i| \le N$, $|V| \le M$.

We finish this part of the course by giving a proof of the Carleman estimate i.e.: $\exists w$, $\frac{1}{C_1} \le \frac{w(r)}{r} \le C_1$, $0 < r < 6$, $w \uparrow$, so that, for $\alpha < C_2$, $f \in C_0^\infty(B_6 \backslash \{0\})$,

$$\int \alpha^3 w^{-1-2\alpha}|f|^2 \le C_3 \int w^{2-2\alpha}|\Delta f|^2.$$

Let $g = w^{-\alpha} f$ and compute:

$$(*) \qquad w^{-\alpha}\Delta f = \Delta g + \frac{\alpha^2|\nabla w|^2}{w^2} g + 2\alpha \frac{|\nabla w|^2}{w^2} A(g),$$

where $A(g) = \frac{w\nabla w \nabla g}{|\nabla w|^2} + \frac{1}{2}F_w g$, and $F_w = \frac{w\Delta w - |\nabla w|^2}{|\nabla w|^2}$. For future use, we set $M_w = \frac{1}{2}(M_{ij} + M_{ji})$, where

$$M_{ij} = \frac{1}{2} \operatorname{div}\left(\frac{w\nabla w}{|\nabla w|^2}\right)\delta_{ij} - \partial_{x_j}\left(\frac{w\partial_{x_i} w}{|\nabla w|^2}\right) - \frac{1}{2}F_w\delta_{ij}.$$

Step 1: We have

$$\int \frac{w^2}{|\nabla w|^2}(w^{-\alpha}\Delta f)^2 \geq 4\alpha \int M_w \nabla g \cdot \nabla g + \alpha \int F_w \Delta(g^2) + 4\alpha^2 \int \frac{|\nabla w|^2}{w^2} A(g)^2.$$

Proof: Let

$$J_1 = \int \left(2\alpha \frac{|\nabla w|}{w} A(g)\right)^2, \; J_2 = 2\int \left[2\alpha A(g)\left\{\Delta g + \alpha^2 \frac{|\nabla w|^2}{w^2} g\right\}\right].$$

Then, (*) gives that $\int \frac{w^2}{|\nabla w|^2}\{w^{-\alpha}\Delta f\}^2 \geq J_1 + J_2$. Consider J_2 and note that $\int \frac{|\nabla w|^2}{w^2} A(g) \cdot g = 0$, so that

$$J_2 = 4\alpha \int A(g)\Delta g = 2\alpha \int \left(\frac{2w\nabla w \nabla g}{|\nabla w|^2}\Delta g - F_w |\nabla g|^2\right) + \alpha \int F_w \Delta(g^2).$$

We now use the Rellich identity:

$$2(\vec{\beta}\cdot\nabla g)\Delta g = 2 \text{ div } ((\vec{\beta}\cdot\nabla g)\nabla g) - \text{ div } (\vec{\beta}|\nabla g|^2) + \text{ div } (\vec{\beta})|\nabla g|^2 - 2\delta_{ij}\partial_i\beta_k\partial_k g \cdot \partial_j g$$

and choose $\vec{\beta} = \frac{w\nabla w}{|\nabla w|^2}$, use the formula above and the divergence theorem to conclude:

$$4\alpha \int A(g)\Delta g = 4\alpha \int M_w \nabla g \cdot \nabla g + \alpha \int F_w \Delta(g^2).$$

This gives Step 1.

We will now choose w. Let $\sigma(x) = |x|$, $\varphi(s) = s \exp\left(\int_0^s \frac{e^{-t}-1}{t} dt\right)$, $\phi(s) = \frac{\varphi(s)}{s\varphi'(s)} = e^s$. We define $\omega(x) = \varphi(\sigma(x)) = \varphi(r)$, $r = |x|$. Notice that for $0 < r < 6$, $\varphi(r) \uparrow$, $\varphi(r) \simeq r$, $\nabla w(x) = \varphi'(r)\frac{x}{r}$, so that $|\nabla w(x)| \simeq 1$. With this definition, $F_w = (n-2)\phi(\sigma) - \sigma\phi'(\sigma)$ and $M_w = \sigma\phi'(\sigma)\left[I - \frac{\nabla\sigma\otimes\nabla\sigma}{|\nabla\sigma|^2}\right]$, so that $M_w \nabla g \cdot \nabla g \geq \sigma\phi'(\sigma)$. $|\tilde{\nabla} g|^2$ and hence $\int M_w \nabla g \cdot \nabla g \geq \int \sigma\phi'(\sigma)|\tilde{\nabla} g|^2$, where $\tilde{\nabla} g = \nabla g - \frac{\nabla\sigma\cdot\nabla g}{|\nabla\sigma|^2}\cdot\nabla\sigma$.

Also,

$$\begin{aligned}
\int F_w \Delta(g^2) &= (n-2)\int \phi(\sigma)\Delta(g^2) - \int \sigma\phi'(\sigma)\Delta(g^2) \\
&= (n-2)\int \Delta\phi g^2 - 2\int \sigma\phi'(\sigma) g\Delta g - 2\int \sigma\phi'|\nabla g|^2 \\
&= (n-2)\int \Delta\phi g^2 - 2\int \sigma\phi' g\Delta g - 2\int \sigma\phi'|\tilde{\nabla} g|^2 - 2\int \sigma\phi' \frac{(\nabla\sigma\cdot\nabla g)^2}{|\nabla\sigma|^2}.
\end{aligned}$$

We next turn to $-2\int \sigma\phi' g\Delta g$, using (*), to get

$$-2\int \sigma\phi' g\Delta g = -2\int \sigma\phi' g\Delta f w^{-\alpha} + 2\alpha^2\int \sigma\phi'\frac{|\nabla w|^2}{w^2}g^2 + 2\int 2\alpha\frac{|\nabla w|^2}{w^2}A(g)g\sigma\phi'.$$

Thus,

$$4\alpha\int M_w\nabla g\cdot\nabla g + \alpha\int F_w\Delta(g^2) \geq 4\alpha\int \sigma\phi'|\tilde{\nabla} g|^2 + \alpha(n-2)\int \Delta\phi g^2 - 2\alpha\int \sigma\phi' g\Delta f w^{-\alpha}$$

$$+2\alpha^3\int \sigma\phi'\frac{|\nabla w|^2}{w^2}g^2 + 4\alpha^2\int \frac{|\nabla w|^2}{w^2}A(g)g\sigma\phi' - 2\alpha\int \sigma\phi'|\tilde{\nabla} g|^2 - 2\alpha\int \sigma\phi'\frac{(\nabla\sigma\cdot\nabla g)^2}{|\nabla\sigma|^2}$$

$$= 2\alpha\int \sigma\phi'|\tilde{\nabla} g|^2 + 2\alpha^3\int \sigma\phi'\frac{|\nabla w|^2}{w^2}g^2 - R_1$$

where

$$R_1 = -\alpha(n-2)\int \Delta\phi g^2 + 2\alpha\int \sigma\phi' g\Delta f w^{-\alpha} - 4\alpha^2\int \frac{|\nabla w|^2}{w^2}A(g)g\sigma\phi' + 2\alpha\int \sigma\phi'\frac{(\nabla\sigma\cdot\nabla g)^2}{|\nabla\sigma|^2}.$$

Recall that $\phi(s) = e^s$, so that $\Delta\phi(\sigma) = \phi''(\sigma)|\nabla\sigma|^2 + \phi'(\sigma)\Delta\sigma = \phi''(\sigma)|\nabla\sigma|^2 + \frac{(n-1)}{\sigma}\phi'(\sigma)$ so that for $0 \leq \sigma \leq 6$, $|\Delta\phi(\sigma)| \leq C/\sigma$. Hence,

$$R_1 \leq C\left\{\alpha\int w^{-1}g^2 + \alpha\int w^{1-\alpha}|g||\Delta f| + \alpha^2\int w^{-1}|A(g)||g| + \alpha\int w\frac{|\nabla\sigma\cdot\nabla g|^2|}{|\nabla\sigma|^2}\right\}.$$

Once we combine this with Step 1, we obtain:

Step 2:

$$4\alpha^2\int \frac{|\nabla w|^2}{w^2}A(g)^2 + 2\alpha\int \sigma\phi'|\tilde{\nabla} g|^2 + 2\alpha^3\int \sigma\phi'\frac{|\nabla w|^2}{w^2}g^2 \leq \int \frac{w^2}{|\nabla w|^2}(w^{-\alpha}\Delta f)^2 +$$

$$+C\left\{\alpha\int w^{-1}g^2 + \alpha\int w^{1-\alpha}|g||\Delta f| + \alpha^2\int w^{-1}|A(g)||g| + \alpha\int w\frac{|\nabla\sigma\cdot\nabla g|^2}{|\nabla\sigma|^2}\right\}.$$

To conclude the proof, recall that $A(g) = \frac{w\nabla w\cdot\nabla g}{|\nabla w|^2} + \frac{1}{2}F_w g$ and that $F_w = (n-2)\phi - \sigma\phi'(\sigma)$.

Hence, $|F_w| \leq C$ in B_6. Thus,

$$\frac{w}{|\nabla w|^2}\frac{|\nabla w\cdot\nabla g|^2}{|\nabla w|^2} \leq Cw^{-1}|Ag|^2 + \frac{C|g|^2}{w},$$

so that

$$\alpha\int w\frac{|\nabla\sigma\cdot\nabla g|^2}{|\nabla\sigma|^2}=\alpha\int w\frac{|\nabla w\cdot\nabla g|}{|\nabla w|^2}\leq C\alpha\int w^{-1}|Ag|^2+C\alpha\int |g|^2w^{-1}$$

$$\leq\tilde{C}\alpha\int\frac{|\nabla w|^2}{w^2}|Ag|^2+C\alpha\int |g|^2w^{-1}.$$

Also,

$$C\alpha^2\int w^{-1}|Ag||g|\leq\frac{1}{2}\alpha^2\int\frac{|\nabla w|^2}{w^2}|Ag|^2+\tilde{C}\alpha^2\int g^2w^{-1}$$

and

$$\alpha\int w^{(1-\alpha)}|g||\Delta f|\leq\int w^2(w^{-\alpha}|\Delta f|)^2+C\alpha^2\int g^2w^{-1}$$

and the Lemma follows choosing α large enough.

Part 2. Parabolic Unique Continuation and Backward Uniqueness

We start out with a version of the Carleman lemma, valid for parabolic equations. We will state it for the heat equation $\partial_t u-\Delta u$. For t_0, r fixed, define $Q_r^{t_0}=B_r\times(-t_0,t_0)$, $\tilde{Q}_r^{t_0}=B_r\setminus\{0\}\times(-t_0,t_0)$.

Lemma: (Escauriaza-Vessella 2003) *(In this Lemma, $R_0\simeq 1, T\simeq 1$). Let $f\in C_0^\infty\left(\tilde{Q}_{R_0}^T\right)$, $\alpha>C_2, w$ as in the elliptic Lemma. Then:*

$$\int\limits_{Q_{R_0}^T}\left[\alpha^3|f|^2w^{-1-2\alpha}+\alpha\left|\nabla f\right|^2w^{1-2\alpha}\right]\leq C_3\int\limits_{Q_{R_0}^T}\left|\partial_t f-\Delta f\right|^2w^{2-2\alpha}.$$

With this Lemma in hand, arguing similarly to the elliptic case, we obtain:

Corollary 1: ("3 cylinder inequality")

Let r_0, r_1, R be as in the elliptic case. Assume that

$$|\partial_t u-\Delta u|\leq M\left(|\nabla u|+|u|\right)\text{ in }Q_R^T.$$

Then, with K_0 so that $\dfrac{1}{K_0} \simeq \log \dfrac{1}{r_0}$ and for $0 < t_0 < T$, we have:

$$\|u\|_{L^2(Q_{r_1}^{T-t_0})} \leq C\left\{\left(\left[\frac{1}{r_0}\right]^{n/2}\|u\|_{L^2(Q_{2r_0}^T)}\right)^{K_0}\cdot\left(\|u\|_{L^2(Q_{R_1}^T)}\right)^{1-K_0} + \left(\frac{1}{r_0}\right)^{n/2}\|u\|_{L^2(Q_{2r_0}^T)}\cdot\exp(CM/K_0)\right\}.$$

Corollary 2: *If u is as above and (say) bounded on $Q_{R_1}^T$ and $\|u\|_{L^2(Q_s^T)} = O(s^N)$ as $s \to 0$ for each N, then $u \equiv 0$.*

A typical application of Corollary 2 is to the following "unique continuation through spatial boundaries" theorem.

Corollary 3: *Assume that $|\partial_t w - \Delta w| \leq M\left(|w| + |\nabla w|\right)$ in $B_{4R} \times [t_0, t_1]$, $|w| \leq M_1$ and $w \equiv 0$ on $(B_{4R}\backslash B_R) \times [t_0, t_1]$. Then $w \equiv 0$ in $B_{4R} \times [t_0, t_1]$.*

A brief Digression on the Navier-Stokes Equation

Given a smooth divergence free vector field in $\mathbb{R}^3$, decaying fast as $x \to \infty$, we seek a v solving

$$(NS)\begin{cases} \partial_t v + \text{ div } (v\otimes v) - \Delta v = -\nabla p \\ \\ \text{div } v = 0,\ v(x,0) = a(x) \end{cases}$$

where $v : \mathbb{R}^3 \times \mathbb{R}_+ \longrightarrow \mathbb{R}^3$, $p : \mathbb{R}^3 \longrightarrow \mathbb{R}$ are smooth and $\text{div}(v\otimes v) = \partial_{x_i}(v_i v_j)$. Later on we will consider the same problem for more general a.

In 1934, Leray showed:

i) $\exists T^* > 0$ s.t. (NS) has a unique solution, well-behaved as $x \to +\infty$, in $\mathbb{R}^3 \times (0, T^*)$).

ii) (NS) has at least one 'weak solution' satisfying an 'energy inequality'. Moreover, this 'weak solution' coincides with the smooth solution for $0 < t < T^*$.

iii) If $[0, T^*]$ is the maximal interval of existence of a smooth solution, then, for each $p > 3$, there exists $\varepsilon_p > 0$ such that $\left(\displaystyle\int_{\mathbb{R}^3} |u(x,t)|^p dx\right)^{1/p} \geq \frac{\varepsilon_p}{(T_* - t)^{\frac{1}{2}(1-3/p)}}$ as $t \to T^*$.

Let me clarify the definition of 'weak solution'.

$\dot{C}_0^\infty =$ all C_0^∞ vector fields, with divergence 0 in $\mathbb{R}^3$.

$\overset{o}{J} =$ closure of $\dot{C}_0^\infty$ in L^2; $\overset{o}{J}{}_2^1 =$ closure in W_2^1.

$Q_T = \mathbb{R}^3 \times (0,T)$. We say that v is a weak solution to (NS) in Q_T if $v : Q_T \to \mathbb{R}^3$, with $v \in L^\infty\left((0,T);\ \overset{o}{J}\right) \cap L^2\left((0,T);\ \overset{o}{J}{}_2^1\right)$, $t \longmapsto \int v(x,t)w(x)dx$ can be extended continuously to $[0,T]$, for any $w \in L^2$, and

$$\int\limits_{Q_T} \left[-v\partial_t\theta - v\otimes v : \nabla\theta + \nabla v : \nabla\theta\right] dx\, dt = 0,\ \forall \theta \in \dot{C}_0^\infty(Q_T), \|v(-,t)-a\|_{L^2} \underset{t\to 0}{\to} 0$$

and

$$\frac{1}{2}\int |v(x,t_0)|^2\, dx + \int\limits_{\mathbb{R}^3\times[0,t_0]} |\nabla v|^2\, dx\, dt \le \frac{1}{2}\int\limits_{\mathbb{R}^3} |a(x)|^2\, dx,$$

for all $t_0 \in (0,T)$.

Theorem: *(Leray 34) For $a \in \overset{o}{J}$, $\exists$ a weak solution v on $\mathbb{R}^3 \times [0,\infty)$.*

Theorem: (Prodi-Serrin-Ladyzhenskaya 60's) If v_1, v are two weak solutions, $a \in \overset{o}{J}$, and for some $T>0$, $v \in L_{s,\ell}(Q_T) = \left\{ f : \left(\int_0^T \|f(-,t)\|_s^\ell dt\right)^{\frac{1}{\ell}} < \infty \right\}$, with $\frac{3}{s}+\frac{2}{\ell} = 1$, $s \in (3,+\infty]$, then $v = v_1$ in Q_T and v is smooth in $\mathbb{R}^3 \times (0,T]$.

Remark: Standard Sobolev embeddings show that if v is a weak solution, $v \in L_{s,\ell}(Q_T)$ with $\frac{3}{s}+\frac{2}{\ell} = \frac{3}{2}$, $s \in [2,6]$. The significance of the condition in the second Theorem comes from scaling: if u is a weak solution to (NS), $u_\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t)$, $p_\lambda(x,t) = \lambda^2 p(\lambda x, \lambda^2 t)$ is another weak solution and the norms in $L_{s,\ell}$ with $\frac{3}{s}+\frac{2}{\ell} = 1$ are invariant under the scaling.

Escauriaza-Seregin-Sverak (2002) provided the end-point in iii) of Leray's work and the case $s=3$ in the above theorem.

Theorem: (ESS-2002) *If $]\,0,T^*\,[$ is the maximal interval of existence, $T^* < \infty$, then,*

$$\lim_{t\uparrow T^*} \int |v(x,t)|^3\, dx = +\infty.$$

Also, if $v \in L_{3,\infty}(Q_T)$ is a weak solution, then $v \in L_{5,5}(Q_T)$ and hence, by second Theorem $\left(\frac{3}{5}+\frac{2}{5}=1\right)$ *v is smooth and unique in Q_T.*

The main interest in this result comes from the fact that it seems out of the reach of "standard methods". These are methods which give regularity of solutions, provided that some scale invariant quantity becomes small, when restricted to a small set. For example, a norm of the type $\|f\|_{s,\ell} < \infty$, with $s, \ell < \infty$ implies this kind of smallness in small sets, automatically. This is not the case for the $L_{3,\infty}$ norm and possible concentration effects may be an obstacle to regularity. A typical result one can achieve by "standard methods" is:

Theorem: *(Caffarelli-Kohn-Nirenberg, '82) There exists $\varepsilon_0 > 0, C_{0,k}, k = 0, 1, \ldots,$ such that if (U, P) is a "suitable weak solution" to (NS) in Q and $\int\limits_Q |U|^3 + |P|^{\frac{3}{2}} dx\, dt \le \varepsilon_0$, then $\nabla^k U$ is Hölder continuous on $\overline{Q}\left(\frac{1}{2}\right)$, $\max\limits_{Q(\frac{1}{2})} \left|\nabla^k U\right| \le C_{o,k}$.*

To prove their Theorem, ESS argue by contradiction. They say that if $z_0 = (x_0, t_0)$ is a "bad point" (where regularity breaks down), by using a variant of the C-K-N result mentioned above, $\exists R_k \downarrow 0$ s.t.,

$$\sup_{t_0 - R_k^2 \le t \le t_0} \frac{1}{R_k} \int\limits_{B(x_0,R_k)} |v|^2 dx \ge \varepsilon_* > 0.$$

(This is a scale-invariant quantity). Then, they perform a blow-up:

Let

$$\begin{aligned} v^{R_k}(x,t) &= R_k v(x_0 + R_k x,\ t_0 + R_k^2 t), \\ p^{R_k}(x,t) &= R_k^2 p(x_0 + R_k x,\ t_0 + R_k^2 t). \end{aligned}$$

They then show that, after passing to a subsequence,

$v^{R_k} \overset{*}{\longrightarrow} u$ in $L_{3,\infty}(\mathbb{R}^{3+1}),\ \operatorname{div} u = 0,\ q \in L_{\frac{3}{2},\infty}(\mathbb{R}^{3+1}),$

$$\forall Q \subset\subset \mathbb{R}^{3+1},\ \int\limits_Q |u|^4 + |\nabla u|^2 + |\partial_t u|^{\frac{4}{3}} + |\nabla^2 u|^{\frac{4}{3}} + |\nabla q|^{\frac{4}{3}} < \infty,$$

$v^{R_k} \longrightarrow u$ in $L^3(Q)$ and (u, q) is a "suitable weak solution" to (NS) in Q. They also show that

$$\sup_{-1\le t\le 0}\int\limits_{B(0,1)}|u(x,t)|^2dx\ge\varepsilon_*$$

and that $\forall x^*\in\mathbb{R}^3$,

$$\int\limits_{B(x^*,1)}|u(x,0)|^2dx=0.$$

(Here they use that $v(-,t)\in L^3$ for all t.) One then should have, by "backward uniqueness for $(N-S)$" that $u\equiv 0$ for $t<0$, which would be a contradiction. To actually carry out this, they introduce the vorticity $w=\nabla\wedge u=(\partial_{x_j}u_k-\partial_{x_k}u_j)\ 1\le j\le 3,\ 1\le k\le 3$, which verifies $\partial_t w=\Delta w+\nabla\wedge(wu)$, i.e., a 'standard' parabolic equation, without the pressure. Then, for R large, $\forall T>0,\ \forall k=0,1,\ldots,\ \nabla^k u$ is Hölder continuous, bounded in the set

$$\left(\mathbb{R}^3\backslash B\left(0,\frac{R}{2}\right)\right)\times(-2T,0]$$

by CKN. Inserting this information in the equation for vorticity, we have, in $(\mathbb{R}^3\backslash B(0,R))\times(-T,0]$ that $|\partial_t w-\Delta w|\le M\left(|\nabla w|+|w|\right),\ |w|\le C$ and $w(x,0)\equiv 0$. From a "backward uniqueness" for standard parabolic equations (to be described momentarily) one would conclude that

$$w\equiv 0\text{ on }\mathbb{R}^3\backslash B(0,R)\times(-T,0]\,.$$

From this one concludes further regularity on u in $B(0,4R)\times(-T,0]$, which then gives us that $|\partial_t w-\Delta w|\le M\left(|w|+|\nabla w|\right),\ |w|\le C$ on $B(0,4R)\times(-T,0]$, with $w\equiv 0$ on $B(0,4R))\backslash B(0,R)\times(-T,0]$. By our uniqueness through "spatial boundaries", $w\equiv 0$ on $B(0,4R)\times(-T,0]$ and hence $w\equiv 0$ in $\mathbb{R}^3\,(-T,0]$. But then, $u\in L_{3,\infty}\left(\mathbb{R}^3\times(-T,0]\right)$, $\Delta u\equiv$ div $w\equiv 0$, so that $u\equiv 0$, a contradiction. The "backward uniqueness" theorem for parabolic equations that is used above is the following:

Theorem: *Assume $|\partial_t u-\Delta u|\le M\left(|u|+|\nabla u|\right)$ in $\mathbb{R}^n_+\times[0,1],\ |u(x,t)|\le Ce^{B|x|^2}$ and assume that $u(x,1)\equiv 0$. Then $u\equiv 0$ in $\mathbb{R}^n_+\times[0,1]$ (here $\mathbb{R}^n_+=\{x=(x^1,x_n):x_n>0\}$).*

Note that there is no assumption made on $u\Big|_{(x_n=0)\times[0,1]}$. The standard "backward unique continuation" theorem for parabolic equations is the same as the result mentioned before, but for u defined in $\mathbb{R}^n\times[0,1]$.

Let us now reexamine backward unique continuation theorems from the perspective of the Landis elliptic conjecture we discussed earlier. Thus, let us consider solutions to

$$\partial_t u - \Delta u + W(x,t)\nabla u + V(x,t)u \equiv 0$$

in $\mathbb{R}^n \times (0,1]$, with $\|V\|_\infty \leq M,\ \|W\|_\infty \leq M$ and let us restrict ourselves to bounded solutions, i.e., $\|u\|_{L^\infty} \leq C_0$.

In 1974, in connection with backward uniqueness, and in parallel with Landis' elliptic conjecture, Landis and Oleinik posed the following:

Conjecture: *Assume that u is as above and at $t = 1$ we don't have $u(x,1) \equiv 0$ but*

$$|u(x,1)| \leq C \exp\left(-C|x|^{2+\varepsilon}\right) \text{ for some } \varepsilon > 0.$$

Is $u \equiv 0$?

Note that the growth rate is clearly optimal, but for real and complex solutions. We now have:

Theorem: *(Escauriaza-Kenig-Ponce-Vega, 2005) In the situation above, if $|u(x,1)| \leq C_k \exp(-k|x|^2)$ for each k, then $u \equiv 0$. Moreover, if $\|u(-,1)\|_{L^2(B_1)} \geq \delta > 0$, there exists N s.t., for $|x| > N$ we have*

$$\|u(-,1)\|_{L^2(B(x,1))} \geq \exp(-N|x|^2 \log|x|).$$

Corresponding uniqueness results and quantitative results hold also in the case of $\mathbb{R}^n_+ \times (0,1)$. The proof is inspired by the elliptic one I discussed earlier. The main points are a rescaling argument and a quantification of the size of the constants involved in the "two sphere and one cylinder" inequalities satisfied by solutions of parabolic equations, in terms of the L^∞ norm of the lower order coefficients and the time of existence of solutions.

We end this part of the course with:

Question 4: Consider variable coefficient parabolic equations, i.e., $\partial_t u - \sum \partial_{x_i}\ a_{ij}(x,t)\ \partial_{x_j}\ u + W(x,t).\ \nabla u + V(x,t)u = 0$ in $\mathbb{R}^n \times (0,1]$, where $\{a_{ij}(x,t)\}$ is uniformly elliptic and symmetric. What conditions on the local smoothness and the behavior of the coefficients at infinity are needed for the previous theorem to hold? For example, we conjecture that $\left|\nabla_{(x,t)} a_{ij}(x,t)\right| \leq \dfrac{C}{(1+|x|)^{1+\varepsilon}}$ suffices for this. The elliptic result we discussed before can be proved under the corresponding assumption. (Escauriaza, unpublished).

Part 3: Unique Continuation for Dispersive Equations

We now turn to the possible existence of results in the spirit of the parabolic ones, for dispersive equations.

Let us consider for example, non-linear Schrödinger equations, i.e. equations of the form

$$i\partial_t u + \Delta u + F(u, \overline{u})u = 0 \ \text{ in } \mathbb{R}^n \times [0, 1],$$

where F is a suitable non-linearity.

The first thing we would like to discuss is what is the analog of the backward uniqueness result for parabolic equations which we have just discussed.

The first obstacle in doing this is that Schrödinger equations are time reversible and so "backward in time" makes no sense.

As is usual in the study of uniqueness questions, we consider first linear Schrödinger equations of the form

$$i\partial_t u + \Delta u + V(x, t)u = 0$$

in $\mathbb{R}^n \times [0, 1]$, for suitable $V(x, t)$, so that in the end we can let $V(x, t) = F(u(x, t))$.

We first recall the following well-known version of the uncertainty principle, due to Hardy:

Let $f : \mathbb{R} \to \mathbb{C}$ be such that $f(x) = O(\exp(-\pi Ax^2))$ and such that its Fourier transform is $\hat{f}(\xi) = O(\exp(-\pi B\xi^2))$ with $A, B > 0$. Then, if $A \cdot B > 1$, we must have $f \equiv 0$.

For instance, if

$$|f(x)| \leq C_\epsilon \exp\left(-C_\epsilon |x|^{2+\epsilon}\right)$$

$$\text{and} \quad |\hat{f}(\xi)| \leq C_\epsilon \exp\left(-C_\epsilon |\xi|^{2+\epsilon}\right),$$

for some $\epsilon > 0$, then $f \equiv 0$.

(The usual proof of this result uses the theory of analytic functions of exponential type.)

It turns out that this version of the uncertainty principle can be easily translated into an equivalent formulation for the free Schrödinger equation.

If v solves $i\partial_t v + \partial_x^2 u = 0$ in $\mathbb{R} \times [0, 1]$, with $v(x, 0) = v_0(x)$, then

$$v(x, t) = \frac{c}{\sqrt{t}} \int e^{i|x-y|^2/4t} v_0(y)dy$$

so that

$$v(x,1) = ce^{i|y|^2/4} \int e^{-ixy/2} e^{\frac{|y|^2}{4}} v_0(y)\ dy$$

If we then apply the corollary to Hardy's uncertainty principle to $f(y) = e^{i|y|^2/4} v_0(y)$, we see that if

$$|v(x,0)| \le C_\epsilon \exp(-C_\epsilon |x|^{2+\epsilon}) \quad \text{and}$$

$$|v(x,1)| \le C_\epsilon \exp(-C_\epsilon |x|^{2+\epsilon}) \quad \text{for some } \epsilon > 0,$$

we must have $v(x,t) \equiv 0$.

Thus, for time-reversible dispersive equations, the analog of "backward in time uniqueness" should be "uniqueness from behavior at two different times".

We are thus interested in such results with "data which is 0 at infinity" or with "rapidly decaying data" and even in quantitative versions, where we obtain "lower bounds for all non-zero solutions".

It turns out that, for the case of "data which is 0 at infinity", this question has been studied for some time.

For the one-dimensional cubic Schrödinger equation,

$$i\partial_t u + \partial_x^2 u \mp |u|^2 u = 0 \quad \text{in } \mathbb{R} \times [0,1],$$

B.Y. Zhang (1997) showed that if $u \equiv 0$ on $(-\infty, a] \times \{0,1\}$, or on $[a, +\infty) \times \{0,1\}$, for some $a \in \mathbb{R}$, then $u \equiv 0$ on $\mathbb{R} \times [0,1]$. His proof used inverse scattering (thus making it only applicable to the one-dimensional cubic Schrödinger equation), exploiting a non-linear Fourier transform and analyticity.

In 2002, Kenig-Ponce-Vega introduced a general method which allowed them to prove the corresponding results for solutions to

$$i\partial_t u + \Delta u + V(x,t) u = 0$$

in $\mathbb{R}^n \times [0,1]$, $n \ge 1$, for a large class of potentials V. We thus have:

Theorem. *(Kenig-Ponce-Vega 2000). If $V \in L_t^1 L_x^\infty \cap L_{\text{loc}}^\infty$ and $\lim_{R\to\infty} ||V||_{L_t^1 L^\infty(|x|>R)} = 0$ and there exists a strictly convex cone $\Gamma \subset \mathbb{R}^n$ and a $y_0 \in \mathbb{R}^n$ so that*

$$\text{supp } u(-,0) \subset y_0 + \Gamma$$

$$\text{supp } u(-,1) \subset y_0 + \Gamma,$$

then we must have $u \equiv 0$ on $\mathbb{R}^n \times [0,1]$.

This work was extended by Ionescu-Kenig (2004) who considered more general potentials V and the case when Γ is a half-space. For instance, if $V \in L_{xt}^{\frac{n+2}{2}}(\mathbb{R}^n \times \mathbb{R})$ or more generally, $V \in L_t^p L_x^q(\mathbb{R}^n \times [0,1])$ with $\frac{2}{p} + \frac{n}{p} \le 2,\ 1 < p < \infty$ (for $n = 1, 1 < p \le 2$) or $V \in C([0,1];\ L^{n/2}(\mathbb{R}^n)),\ n \ge 3$, the result holds, with Γ a half-plane. This work involves some delicate constructions of parametrices and is quite involved technically.

We next turn to our extension of Hardy's uncertainty principle to this context, i.e. the case of "rapidly decaying data". Here there seems to be no previous literature on the problem.

Theorem (*) *(Escauriaza-Kenig-Ponce-Vega 2005). Let u be a solution to $i\partial_t u + \Delta u + V(x,t)u = 0$ in $\mathbb{R}^n \times [0,1]$ with $u \in C([0,1];\ H^2(\mathbb{R}^n))$. Assume that $V \in L^\infty(\mathbb{R}^n \times [0,1]),\ \nabla_x V \in L^1([0,1];\ L^\infty(\mathbb{R}^n))$ and $\lim_{R\to\infty} ||V||_{L_t^1 L^\infty(|x|>R)} = 0$. If $u_0 = u(x,0)$ and $u_1 = u(x,1)$ belong to $H^1(e^{k|x|^2}dx)$, for each $k > 1$, then $u \equiv 0$.*

As we will see soon, there actually even is a quantitative version of this result. The rest of this lecture will be devoted to a sketch of the proof of Theorem (*). Our starting point is:

Lemma *(Kenig-Ponce-Vega 2002). Suppose that $u \in C([0,1];\ L^2(\mathbb{R}^n)),\ H \in L_t^1 L_x^2$ and $||V||_{L_t^1 L_x^\infty} \le \epsilon$ where $\epsilon = \epsilon(n)$ is small enough. Suppose that $u_0(x) = u(x,0),\ u_1(x) = u(x,1)$ both belong to $L^2(\mathbb{R}^n; e^{2\beta x_1}dx)$ and $H \in L^1([0,1];\ L^2(e^{2\beta x_1}dx))$. Then $u \in C([0,1];\ L^2(e^{2\beta x_1}dx))$ and*

$$\begin{aligned}\sup_{0\le t\le 1} ||u(-,t)||_{L^2(e^{2\beta x_1}dx)} &\le C\left\{||u_0||_{L^2(e^{2\beta x_1}\ dx)} + ||u_1||_{L^2(e^{2\beta x_1}\ dx)}\right.\\ &\quad \left. + ||H||_{L^1([0,1]);L^2(e^{2\beta x_1}\ dx))}\right\},\end{aligned}$$

with C independent of β.

The proof of this lemma is quite subtle. If we know *a priori* that $u \in C([0,1];\, L^2(e^{2\beta x_1}dx))$, the proof could be carried out by a variant of the energy method (after conjugation with the weight $e^{2\beta x_1}$) where we split into frequencies $\xi_1 > 0$ and $\xi_1 < 0$, performing the time integral from 0 to t or from t to 1, according to each case.

However, since we are not free to prescribe both u_0 and u_1, we cannot use *apriori* estimates.

We thus introduce a fixed smooth function φ, with $\varphi(0) = 0,\ \varphi'$ non-increasing, $\varphi'(r) \equiv 1$ for $r \le 1$, $\varphi'(r) = 0$ for $r \ge 2$. We then let, for λ large, $\varphi_\lambda(r) = \lambda\varphi(r/\lambda)$, so that $\varphi_\lambda(r) \uparrow r$ as $\lambda \to \infty$. We replace the weight $e^{2\beta x_1}(\beta > 0)$ with $e^{2\beta\varphi_\lambda(x_1)}$ and prove the analogous estimate for these weights, uniformly in λ, for $\lambda \ge C(1+\beta^6)$. The point is that all the quantities involved are now *apriori* finite.

The price one pays is that, after conjugation with the weight $e^{2\beta\varphi_\lambda(x_1)}$, the resulting operators are no longer constant coefficient (as is the case for $e^{2\beta x_1}$) and their study presents complications.

At this point there are two approaches: in KPV 2002 one adapts the use of the energy estimates, combined with commutator estimates and the standard pseudo-differential calculus.

The second approach, in IK 2004, constructs parametrices for the resulting operators and proves bounds for them.

With this Lemma as our point of departure, our first step is to deduce from it further weighted estimates.

Corollary *(EKPV 2005). If we are under the hypothesis of the previous Lemma and in addition for some* $a > 0, \alpha > 1,\ u_0,\ u_1 \in L^2(e^{a|x|^\alpha}dx),\ H \in L^1([0,1];\ L^2(e^{a|x|^\alpha}dx))$*, then there exist* $C_{\alpha,n}$*,* $C_n > 0$ *such that*

$$\sup_{0<t<1} \int_{|x|>C_{\alpha,n}} |u(x,t)|^2 e^{C_n a|x|^\alpha} dx < \infty.$$

The idea used for the proof of the corollary is as follows: let $u_R(x,t) = u(x,t)\eta_R(x)$, where $\eta_R(x) = \eta(x/R)$ and $\eta \equiv 0$ for $|x| \leq 1,\ \eta \equiv 1$ for $|x| \geq 2$. We apply the Lemma to u_R and a choice of $\beta = bR^{\alpha-1}$, for suitable b, in each direction $x_1, x_2, \ldots, x_n$. The corollary then follows readily.

The next step in the proof of the theorem is to deduce lower bounds for L^2 space-time integrals, in analogy with the elliptic and parabolic situations. These are our "quantitative lower bounds".

Theorem. *Let* $u \in C([0,1];\ H^2(\mathbb{R}^n))$ *solve* $i\partial_t u + \Delta u + Vu = 0$ *in* $\mathbb{R}^n \times [0,1]$*. Assume that*

$$\int_0^1 \int_{\mathbb{R}^n} |u|^2 + |\nabla u|^2 dx\ dt \leq A^2$$

and that

$$\int_{\frac{1}{2}-\frac{1}{8}}^{\frac{1}{2}+\frac{1}{8}} \int_{|x|<1} |u|^2 dx\ dt \geq 1,$$

with $||V||_{L^\infty} \leq L$*. Then there exists* $C_n > 0$ *and* $R_0 = R_0(n, A, L) > 0$ *such that if* $R \geq R_0$*, we have* $\delta(R) \geq C_n\ exp(-C_n R^2)$*, where*

$$\delta(R) = \left(\int_0^1 \int_{R-1 \leq |x| \leq R} (|u|^2 + |\nabla u|^2) dx\ dt \right)^{1/2}.$$

Once the Theorem is proved, applying the Corollary to u and ∇u (which verifies a similar equation to the one u does) we see that the Theorem yields a contradiction, which proves our theorem.

In order to prove the Theorem a key tool is the following Carleman estimate, which is a variant of the one due to V. Isakov (1993).

Lemma. *Assume that $R > 0$ and $\varphi : [0,1] \to \mathbb{R}$ is a smooth compactly supported function. Then there exists $C = C(n, ||\varphi'||_\infty, ||\varphi''||_\infty) > 0$ such that, for all $g \in C_0^\infty(\mathbb{R}^{n+1})$ with* supp $g \subset \left\{(x,t) : \left|\frac{x}{R} + \varphi(t)e_1\right| \geq 1\right\}$ *and $\alpha \geq CR^2$, we have*

$$\frac{\alpha^{3/2}}{R^2}\left|\left|e^{\alpha|\frac{x}{R}+\varphi(t)e_1|^2|}g\right|\right|_{L^2} \leq C\left|\left|e^{\alpha|\frac{x}{R}+\varphi(t)e_1|^2}(i\partial_t + \Delta)(g)\right|\right|_{L^2}.$$

(Here $e_1 = (1,0,\ldots,0)$.)

Proof. We conjugate $(i\partial_t + \Delta)$ by the weight $e^{\alpha|\frac{x}{R}+\varphi(t)e_1|^2}$ and split the resulting operator into its Hermitian and its anti-Hermitian parts. Thus, let $f = e^{\alpha|\frac{x}{R}+\varphi(t)e_1|^2}g$, so that $e^{\alpha|\frac{x}{R}+\varphi(t)e_1|^2}(i\partial_t+\Delta)g = S_\alpha f - 4\alpha A_\alpha f$, where $S_\alpha = i\partial_t + \Delta + \frac{4\alpha^2}{R^2}\left|\frac{x}{R}+\varphi(t)e_1\right|^2$ and $A_\alpha = \frac{1}{R}\left(\frac{x}{R}+\varphi e_1\right)\cdot\nabla + \frac{n}{2R^2} + i\frac{\varphi'}{2}\left(\frac{x_1}{R}+\varphi\right)$. Thus, $S_\alpha^* = S_\alpha$, $A_\alpha^* = -A_\alpha$ and

$$\begin{aligned}\left|\left|e^{\alpha|\frac{x}{R}+\varphi(t)e_1|^2}(i\partial_t + \Delta)(g)\right|\right|_{L^2}^2 &= \langle S_\alpha f - 4\alpha A_\alpha f, S_\alpha f - 4\alpha A_\alpha f\rangle \\ &\geq -4\alpha\left\langle (S_\alpha A_\alpha - A_\alpha S_\alpha) f, f\right\rangle = -4\alpha\left\langle [S_\alpha, A_\alpha] f, f\right\rangle.\end{aligned}$$

A calculation shows that

$$[S_\alpha, A_\alpha] = \frac{2}{R^2}\Delta - \frac{4\alpha^2}{R^4}\left|\frac{x}{R}+\varphi e_1\right|^2 - \frac{1}{2}\left[\left(\frac{x_1}{R_\varphi}\right)\varphi'' + (\varphi')^2\right] + 2i\frac{\varphi'}{R}\partial_{x_1}$$

and

$$\begin{aligned}\left|\left|e^{\alpha|\frac{x}{R}+\varphi(t)e_1|^2}(i\partial t + \Delta)g\right|\right|_{L^2}^2 &\geq \tfrac{16\alpha^3}{R^4}\int\left|\tfrac{x}{R}+\varphi(t)e_1\right|^2|f|^2 \\ &+\tfrac{8\alpha}{R^2}\int|\nabla f|^2 + 2\alpha\int\left[\left(\tfrac{x_1}{R}+\varphi\right)\varphi'' + (\varphi')^2\right]|f|^2 - \tfrac{8\alpha i}{R}\int\varphi'\partial_{x_1}f\overline{f}.\end{aligned}$$

Using our support hypothesis on g, and taking $\alpha > CR^2$, with $C = C(n, ||\varphi'||_{\infty,}||\varphi''||_\infty)$ yields our estimate.

In order to use the Lemma to prove our Theorem; we choose $\theta_R,\ \theta \in C_0^\infty(\mathbb{R}^n),\ \varphi \in C_0^\infty((0,1))$, so that

$$\theta_R(x) = \begin{cases} 1 & \text{if} \quad |x| < R-1 \\ 0 & \text{if} \quad |x| \geq R, \end{cases}$$

$$\theta(x) = \begin{cases} 0 & \text{if} \quad |x| \le 1 \\ \\ 1 & \text{if} \quad |x| \ge 2, \end{cases}$$

$0 \le \varphi \le 3$, with

$$\varphi = \begin{cases} 1 & \text{on} \quad \left[\frac{1}{2} - \frac{1}{8}, \frac{1}{2} + \frac{1}{8}\right] \\ \\ 0 & \text{on} \quad \left[0, \frac{1}{4}\right] \cup \left[\frac{3}{4}, 1\right]. \end{cases}$$

We let $g(x,t) = \theta_R(x)\theta\left(\frac{x}{R} + \varphi(t)e_1\right) u(x,t)$. Note that supp $g \subset \left\{\left|\frac{x}{R} + \varphi(t)e_1\right| \ge 1\right\}$, $g(x,t) \equiv 0$ if $|x| > R$ and if $t \in \left[0, \frac{1}{4}\right] \cup \left[\frac{3}{4}, 1\right]$, $|x| \le R$, $g(x,t) \equiv 0$, so that the Lemma applies. Note that $g \equiv u$ in $B_{R-1} \times \left[\frac{1}{2} - \frac{1}{8}, \frac{1}{2} + \frac{1}{8}\right]$ where $\left|\frac{x}{R} + \varphi(t)e_1\right| \ge 3 - 1 = 2$. We have:

$$(i\partial_t + \Delta + V)(g) = \theta\left(\frac{x}{R} + \varphi e_1\right)\{2\nabla\theta_R \cdot \nabla u + u\Delta\theta_R\}$$

$$+\theta_R(x)\left\{2R^{-1}\nabla\theta\left(\frac{x}{R} + \varphi e_1\right) \cdot \nabla u + R^{-2}u\Delta\theta\left(\frac{x}{R} + \varphi e_1\right) + i\varphi'(t)\partial_{x_1}\theta\left(\frac{x}{R} + \varphi e_1\right)u\right\}.$$

The first term on the right-hand side is supported in $(B_R \backslash B_{R-1}) \times [0,1]$, where $\left|\frac{x}{R} + \varphi e_1\right| \le 4$. The second one is supported in $\left\{(x,t) \ : \ 1 \le \left|\frac{x}{R} + \varphi e_1\right| \le 2\right\}$. Thus

$$\left|\left|e^{\left|\frac{x}{R} + \varphi(t)e_1\right|^2} g\right|\right|^2_{L^2(dxdt)} \ge e^{4\alpha}||u||_{L^2\left(B_1 \times \left[\frac{1}{2} - \frac{1}{8}, \frac{1}{2} + \frac{1}{8}\right]\right)} \ge e^{4\alpha},$$

and by the Lemma,

$$\frac{\alpha^{3/2}}{R^2}\left|\left|e^{\alpha\left|\frac{x}{R} + \varphi(t)e_1\right|^2} g\right|\right|_{L^2(dx\ dt)} \le C_n L\left|\left|e^{\alpha\left|\frac{x}{R} + \varphi(t)e_1\right|^2} g\right|\right|_{L^2(dx\ dt)} + C_n e^{16\alpha}\delta(R) + C_n e^{4\alpha}A,$$

provided $\alpha \ge C_n R^2$. If we choose $\alpha = C_n R^2$, for R large we can hide the first term on the right-hand side in the left-hand side to obtain

$$Re^{4\alpha} \le \tilde{C}_n e^{16\alpha}\delta(R) + \tilde{C}_n e^{4\alpha}A,$$

so that $R \le \tilde{C}_n e^{12\alpha}\delta(R) + \tilde{C}_n A$, and for R large, depending on A, we obtain $R \le 2\tilde{C}_n e^{12\alpha}\delta(R)$, which, since $\alpha = C_n R^2$, is the desired result. ■

Question 5: Can one obtain sharper versions of Theorem (*) in the spirit of the uncertainty principle of Hardy? For instance, assume

$u_0 \in H^1(e^{-a_0|x|^2}dx)$ for a fixed $a_0 > 0$ and
$u_1 \in H^1(e^{-k|x|^2}dx)$ for all $k > 0$.

Prove, for the class of V as in Theorem (*) that $u \equiv 0$.

References

1. J. Bourgain and C. Kenig, On localization in the Anderson-Bernoulli model in higher dimensions, *Invent. Math.* **161** (2005), 389–426.

2. L. Escauriaza, C. Kenig, G. Ponce, and L. Vega, Decay at infinity of caloric functions within characteristic hyperplanes, preprint,
http://arkiv.org/abs/math.AP/0509436, Sept. 19, 2005.

3. L. Escaouriaza, C. Kenig, G. Ponce, and L. Vega, On unique continuation of solutions of Schrödinger equations, preprint,
http://arkiv.org/abs/math.AP/0509119, Sept. 8, 2005.

4. L. Escauriaza, G. Seregin, and V. Sverak, $L^{3,\infty}$ solutions to the Navier-Stokes equations and backward uniqueness, *Russian Math. Surv.* **58.2** (2003) 211–250.

5. A. Ionescu and C. Kenig, L^p Carleman inequalities and uniqueness of solutions of nonlinear Schrödinger equations, *Acta Math.* **193** (2004), 193-239.

6. E.M. Landis and O.A. Oleinik, Generalized analyticity and some related properties of solutions of elliptic and parabolic equations, *Russian Math. Surv.* **29** (1974), 195–212.

7. V.Z. Meshkov, On the possible rate of decay at infinity of solutions of second order partial differential equations, *Math. USSR Sbornik* **72** (1992), 343–360.

8. C. Kenig, G. Ponce, and L. Vega, On unique continuation for nonlinear Schrödinger equations, *CPAM LV* (2002), 1247-1262.