On the existence and regularity of non-flat profiles for a Bernoulli free boundary problem

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Abstract

In this paper we consider a large class of Bernoulli-type free boundary problems with mixed periodic-Dirichlet boundary conditions. We show that solutions with non-flat profile can be found variationally as global minimizers of the classical Alt-Caffarelli energy functional.

Contents

1 Introduction

In the classical paper [AC81], Alt and Caffarelli studied the existence and regularity of solutions to the one-phase free boundary problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \cap \{u > 0\}, \\ u = 0 & \text{on } \Omega \cap \partial\{u > 0\}, \\ |\nabla u| = Q & \text{on } \Omega \cap \partial\{u > 0\}, \\ u = u_0 & \text{on } \Gamma, \end{cases} \tag{1.1}$$

using a variational approach. Here Ω is an open connected subset of $\mathbb{R}^N$ with locally Lipschitz continuous boundary and Q is a nonnegative measurable function. Solutions to (1.1) are critical points for the functional

$$J(u) := \int_\Omega \left(|\nabla u|^2 + \chi_{\{u>0\}} Q^2\right) d\boldsymbol{x}, \quad u \in \mathcal{K}, \tag{1.2}$$

where

$$\mathcal{K} := \{u \in H^1_{\mathrm{loc}}(\Omega) : u = u_0 \text{ on } \Gamma\}, \tag{1.3}$$

with $\Gamma \subset \partial\Omega$ a measurable set with $\mathcal{H}^{N-1}(\Gamma) > 0$ and $u_0 \in H^1_{\mathrm{loc}}(\Omega)$ a nonnegative function satisfying

$$J(u_0) < \infty. \tag{1.4}$$

The equality $u = u_0$ on Γ is in the sense of traces. Under the assumption that Q is a Hölder continuous function satisfying

$$0 < Q_{\min} \le Q(\boldsymbol{x}) \le Q_{\max} < \infty, \tag{1.5}$$

Alt and Caffarelli proved local Lipschitz regularity of local minima and showed that the free boundary $\partial\{u > 0\}$ is a $C^{1,\alpha}_{\mathrm{loc}}$ regular curve in Ω if $N = 2$, while if $N \ge 3$ they proved that the reduced free boundary is a hypersurface of class $C^{1,\alpha}_{\mathrm{loc}}$ in Ω, for some $0 < \alpha < 1$. See also [ACF84a] for the quasi-linear case and [DP05] for the case of the p-Laplace operator. We remark that while the regularity of minimizers is optimal, the regularity of the free boundary for $N \ge 3$ was improved by Weiss in [Wei99]. Weiss, following an approach closely related to the theory of minimal surfaces and by means of a monotonicity formula, proved the existence of a maximal dimension $k^* \ge 3$ such that for $N < k^*$ the free boundary is a hypersurface of class $C^{1,\alpha}_{\mathrm{loc}}$ in Ω, for $N = k^*$ the singular set consists at most of isolated points, and if $N > k^*$ then $\mathcal{H}^s(\{\text{singular set}\}) = 0$ for every $s > N - k^*$. In [CJK04], Caffarelli, Jerison and Kenig proved the full regularity of the free boundary in dimension $N = 3$, thus showing that $k^* \ge 4$. They also conjectured that $k^* \ge 7$. In a later work De Silva and Jerison exhibited an example of a global energy minimizer with non-smooth free boundary in dimension 8 (see [DSJ09]); their result implies that $k^* \le 7$. As it was remarked in [AC81], if $N = 3$ the energy functional admits a critical point with a point singularity in the free boundary. Similar results have been obtained for two-phase free boundary problems (see [ACF84b], [Caf87], [Caf89], [Caf88]). It is important to observe that the regularity of the free boundary is strongly related to the assumption $0 < Q_{\min} \le Q(\boldsymbol{x})$ in (1.5). Indeed, in a recent paper Arama and the second author showed that for $N = 2$ and in the special case in which

$$Q(x,y) = \sqrt{(h-y)_+} \quad \text{for some } h > 0, \tag{1.6}$$

if a local minimizer u has support below the line $\{y = h\}$ and if there exists a point $\boldsymbol{x}_0 = (x_0, h) \in \partial\{u > 0\}$, then

$$|\nabla u(\boldsymbol{x})| \le C r^{1/2}, \quad \text{for } \boldsymbol{x} \in B_r(\boldsymbol{x}_0) \tag{1.7}$$

(see [AL12, Remark 3.5]). On the other hand, using a monotonicity formula and a blow up method, Varvaruca and Weiss in [VW11, Theorem A] proved that for a suitable definition of solution if the constant C in (1.7) is one then the rescaled function

$$\frac{u(\boldsymbol{x}_0 + r\boldsymbol{x})}{r^{3/2}} \to \frac{\sqrt{2}}{3}\rho^{3/2}\cos\left(\frac{3}{2}\left(\min\left\{\max\left\{\theta, -\frac{5\pi}{6}\right\}, -\frac{\pi}{6}\right\} + \frac{\pi}{2}\right)\right) \quad \text{as } r \to 0^+,$$

strongly in $W^{1,2}_{\mathrm{loc}}(\mathbb{R}^2)$ and locally uniformly on $\mathbb{R}^2$, where $(x,y) = (\rho\cos\theta, \rho\sin\theta)$, and near $\boldsymbol{x}_0$ the free boundary $\partial\{u > 0\}$ is the union of two C^1 graphs with right and left tangents at $\boldsymbol{x}_0$ (see also [WZ12]). This type of singular solutions are related to Stokes' conjecture on the existence of extreme water waves (see [Sto80]). The existence of extreme waves and the corner singularity have been proved in a series of papers (see [AF87], [AFT82], [McL97], [Plo02], [Tol78]; see also [CS10], [KN78], [McL87], [PT04]) using a hodograph transformation to map the set $\{u > 0\}$ onto an annulus.

Note that for water waves of finite depth it is customary to define

$$\Omega := (-\lambda/2, \lambda/2) \times (0, \infty), \qquad \Gamma := (-\lambda/2, \lambda/2) \times \{0\}, \qquad u_0 \equiv m \tag{1.8}$$

(see (1.1), (1.3)). *The main drawback in proving the existence of regular and extreme water waves using the variational setting of* (1.2) *is that global minimizers of the energy functional J specialized to the case* (1.6), (1.8) *are one dimensional functions of the form $u = u(y)$, which correspond to flat profiles (see Theorem 3.1).* For this reason the paper [AL12] gives interesting results only for local minimizers or when the Dirichlet boundary datum u_0 is not constant on the bottom, a situation which is not compatible with water waves. Necessary and sufficient minimality conditions in terms of the second variation of J have been derived by Fonseca, Mora and the second author in [FLM]. We refer to the papers [CS04], [CSS06], [CSV16], [CWW16], [CWW18], [Fra07], [KW18], [Tol14] and the references therein for alternative approaches to water waves.

The purpose of this paper is to show that by adding an additional Dirichlet boundary condition on part of the later boundary it is possible to construct global minimizers of J in the setting (1.6), (1.8), which are not one dimensional. To be precise, we let Ω be the half infinite rectangular parallelepiped

$$\Omega := \mathcal{R} \times (0, \infty), \tag{1.9}$$

where $\mathcal{R}$ is the open cube of $\mathbb{R}^{N-1}$ with center at the origin and side-length $\lambda > 0$, that is,

$$\mathcal{R} = \left(-\frac{\lambda}{2}, \frac{\lambda}{2}\right)^{N-1}.$$

We will impose periodic boundary conditions on the lateral portion of the boundary, therefore we will require that the class of admissible functions is a subset of the Sobolev space

$$\begin{aligned} H^1_{\mathcal{R},\mathrm{loc}}(\Omega) := \{u \in H^1_{\mathrm{loc}}(\mathbb{R}^N_+) : u(\boldsymbol{x} + \lambda \boldsymbol{e}_i) = u(\boldsymbol{x}) \text{ for } \mathcal{L}^N\text{-a.e. } \boldsymbol{x} \in \mathbb{R}^N_+ \\ \text{and every } i = 1, \dots, N-1\}. \end{aligned} \tag{1.10}$$

With the choice

$$Q(\boldsymbol{x}) := (h - x_N)_+^b, \tag{1.11}$$

where $b, h > 0$, the functional J in (1.2) can be rewritten as

$$J_h(u) := \int_\Omega \left(|\nabla u|^2 + \chi_{\{u>0\}}(h - x_N)_+^{2b}\right) d\boldsymbol{x}, \qquad \text{for } u \in \mathcal{K}_\gamma, \tag{1.12}$$

where

$$\mathcal{K}_\gamma := \{u \in H^1_{\mathcal{R},\mathrm{loc}}(\Omega) : u = u_0 \text{ on } \Gamma_\gamma\}, \qquad \gamma > 0. \tag{1.13}$$

Here the Dirichlet datum u_0, defined by

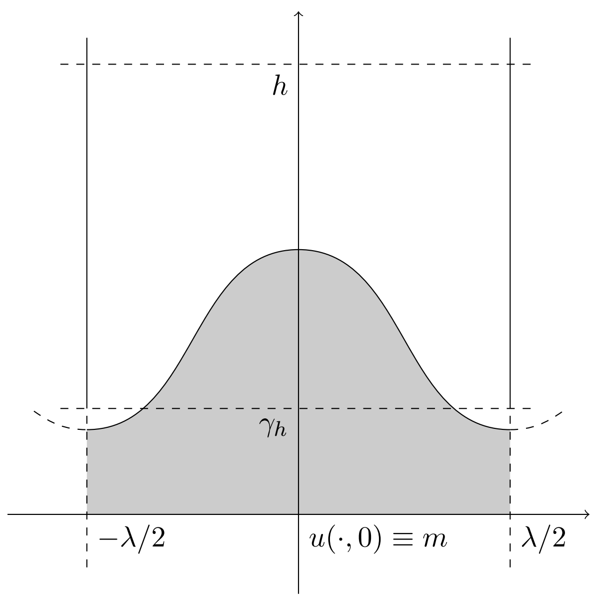
$$u_0(\boldsymbol{x}) := \frac{m}{\gamma}(\gamma - x_N)_+, \qquad m > 0, \tag{1.14}$$

is prescribed on

$$\Gamma_\gamma := (\mathcal{R} \times \{0\}) \cup (\partial\mathcal{R} \times (\gamma, \infty)). \tag{1.15}$$

In particular, notice that u_0 is constant on $\mathcal{R} \times \{0\}$ and zero on $\partial\mathcal{R} \times (\gamma, \infty)$.

One of our main results is that if γ is chosen sufficiently small (depending on the other parameters of the problem, b, m, λ, h) *then global minimizers of J_h over $\mathcal{K}_\gamma$ are not one-dimensional.*



Theorem 1.1 (Existence of non-flat minimizers)**.** *Given $b, m, \lambda, h > 0$, let $\Omega, J_h, \mathcal{K}_\gamma, u_0$ be defined as in* (1.9)*,* (1.12)*,* (1.13) *and* (1.14)*, respectively. Then there exists $\overline{\gamma} = \overline{\gamma}(b, m, \lambda, h) > 0$ such that if $0 < \gamma < \overline{\gamma}$ then every global minimizer $u \in \mathcal{K}_\gamma$ of the functional J_h is not of the form $u = u(x_N)$. Moreover, for every $h > 0$ it is possible to choose $0 < \gamma_h < \overline{\gamma}$ in such a way that the map $h \mapsto \gamma_h$ is continuous and decreasing.*

Next, we study qualitative properties of global minimizers as we vary the height h. By adapting to our setting the monotonicity techniques in [ACF82, Section 5] and [Fri88, Theorem 10.1] and the non-degeneracy lemma [AC81, Lemma 3.4], we are able to prove the following result (see also [AL12, Theorem 5.6]).

Theorem 1.2 (Existence of a critical height). *Given $b, m, \lambda > 0$, let $\Omega, J_h, \mathcal{K}_{\gamma_h}, u_0$ be defined as in* (1.9)*,* (1.12)*,* (1.13) *and* (1.14)*, respectively, where the map $h \mapsto \gamma_h$ is given as in Theorem 1.1. Then there exists a critical height $0 < h_{\rm cr} < \infty$ with the property that*

(i) if $h_{\rm cr} < h < \infty$ then every global minimizer of J_h in $\mathcal{K}_{\gamma_h}$ has support below the hyperplane $\{x_N = h\}$;

(ii) if $0 < h < h_{\rm cr}$ then every global minimizer is positive in $\mathcal{R} \times [h, \infty)$.

Note that by the regularity results of [AC81], [Wei99], [CJK04], for every $h \neq h_{\rm cr}$ the reduced free boundary $\partial^*\{u > 0\}$ of every global minimizer of J_h in $\mathcal{K}_{\gamma_h}$ is a hypersurface of class C^∞ locally in Ω and

$$\Sigma := \big(\partial\{u > 0\} \setminus \partial^*\{u > 0\}\big) \cap \Omega$$

is empty if $N = 2, 3$ and $\mathcal{H}^s(\Sigma) = 0$ for every $s > N - 4$ if $N \geq 4$ (see Corollary 4.5).

It is important to observe that the previous theorem shows that the critical height $h_{\rm cr}$ is the only value of h for which the free boundaries of global minimizers of J_h in $\mathcal{K}_{\gamma_h}$ can touch the hyperplane $\{x_N = h\}$. By the comparison principle in Theorem 4.6 and the convergence of minimizers u_h of J_h as in Theorem 4.12, it follows that by letting $h \nearrow h_{\rm cr}$ there exists a global minimizer $u^- \in \mathcal{K}_{\gamma_{h_{\rm cr}}}$ of $J_{h_{\rm cr}}$ whose support (restricted to Ω) is contained in $\mathcal{R} \times [0, h_{\rm cr}]$, while if $h \searrow h_{\rm cr}$ then there exists another global minimizer u^+ of $J_{h_{\rm cr}}$ with $u^- \leq u^+$ and whose support cannot be strictly below the hyperplane $\{x_N = h_{\rm cr}\}$ (see Theorem 4.19). We have not been able to prove that the support of any global minimizer touches the hyperplane $\{x_N = h_{\rm cr}\}$. This would follow if we had uniqueness at this level (see Theorem 4.15).

Concerning the scaling of $h_{\rm cr}$ we are able to show that for all m as in (1.14),

$$h_{\rm cr} \leq \frac{2b + 2}{(2b + 1)^{b/(b+1)}} m^{1/(b+1)},$$

where b and m are the parameters in (1.11) and (1.14), while if m is sufficiently small then there exists a constant $C_b > 0$ such that

$$h_{\rm cr} \geq C_b m^{1/(b+1)}$$

(see Lemma 4.8 and Lemma 4.9).

Finally, we remark that while the additional Dirichlet constraint $u = 0$ on $\partial\mathcal{R} \times (\gamma_h, \infty)$ allows us to construct non-flat global minimizers, it has the disadvantage of potentially destroying the regularity of minimizers and their free boundaries at the interface $\partial\mathcal{R} \times \{\gamma_h\}$, where one has Dirichlet boundary conditions on $\partial\mathcal{R} \times (\gamma_h, \infty)$ and periodic boundary conditions on $\partial\mathcal{R} \times (0, \gamma_h)$. At least in dimension $N = 2$, we do not expect a loss of regularity if the free boundary hits the fixed boundary at $y = \gamma_h$. Indeed, in a domain U with a corner (or a cut) a harmonic function u with zero Dirichlet boundary conditions near the corner can be written in polar coordinates (r, θ) as

$$u = \begin{cases} c r^{\pi/\omega} \sin(\pi\theta/\omega) + u_{\rm reg} & \text{if } \pi/\omega \notin \mathbb{N}, \\ c r^{\pi/\omega}[\log r \sin(\pi\theta/\omega) + \theta \cos(\pi\theta/\omega)] + u_{\rm reg} & \text{otherwise,} \end{cases}$$

(see [Dau88], [Gri85], [KO83], [MP75]). Here ω is the angle corresponding to the corner in U and $u_{\rm reg}$ is of class H^2 near the corner. In our setting, the Bernoulli condition $|\nabla u| = Q$ on $\partial\{u > 0\}$ (see (1.1)) should at least heuristically force the constant c to be zero, so that $u = u_{\rm reg}$. This problem is currently under investigation (see [GL]). The idea is to approximate the free boundary problem (1.1) with a family of singularly perturbed elliptic problems of the form

$$\Delta u_\varepsilon = \frac{1}{2}\beta_\varepsilon(u_\varepsilon)Q^2,$$

see Section 2 below. This approach has been used successfully in the study of the existence and regularity of solutions to free boundary problems, starting from the celebrated paper of Berestycki, Caffarelli and Nirenberg [BCN90], where they studied a problem in combustion and flame propagation theory. We refer to [Caf95], [DP05], [DPS03], [Gur99], [JP16], [Kar18], [LW98], [MT07] and the references therein for some of the recent literature on this type of singularly perturbed free boundary problems.

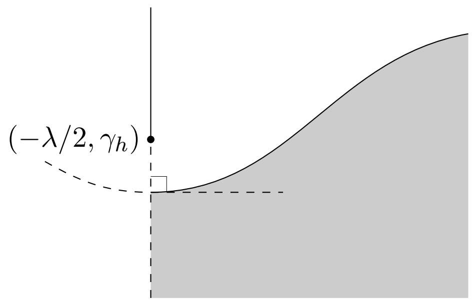


Figure A

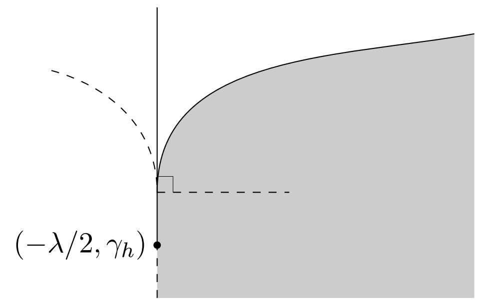


Figure B

On the other hand, in dimension $N = 2$ if the free boundary $\partial\{u > 0\}$ of a global minimizer $u \in \mathcal{K}_{\gamma_h}$ of J_h touches the fixed boundary strictly above the line $\{y = \gamma_h\}$ then we are in a position to apply the recent work of Chang-Lara and Savin [CLS17] (see also [ACF83], [ACF85], [Wei04]) in which it is shown that the free boundary of a viscosity solution of (1.1) detaches tangentially from a portion of the fixed boundary where u vanishes and is a $C^{1,1/2}$ regular hypersurface locally in a neighborhood of $\partial\Omega$ (see Figure B). The result is obtained relating the behavior of the free boundary to a Signorini-type obstacle problem. Due to the periodic boundary conditions below the line $\{y = \gamma_h\}$, if the free boundary $\partial\{u > 0\}$ of a global minimizer $u \in \mathcal{K}_{\gamma_h}$ of J_h touches the fixed boundary strictly below the line $\{y = \gamma_h\}$ (as in Figure A) then the regularity follows from interior regularity (see Corollary 2.13 and Theorem 4.1).

We refer to the work of Raynor [Ray08] for a variational proof of the Lipschitz continuity of global minimizers of J near a Neumann fixed boundary.

Our paper is organized as follows: for the convenience of the reader, in Section 2 we recover well-known results about the minimization problem for J in $\mathcal{K}$, defined as in (1.2) and (1.3) respectively. We do this by observing that solutions to a family of opportunely regularized problems can be found variationally as global minimizers of the energy functionals $\{J_\varepsilon\}_\varepsilon$, which we define in detail below. By studying the Gamma-convergence of the family of functionals $\{J_\varepsilon\}_\varepsilon$ to J, we prove the existence of a global minimizer for J in $\mathcal{K}$. This is the content of Theorem 2.4 and Corollary 2.5. In Section 2.4 we turn our attention to the main focus of this paper: the case of mixed periodic and Dirichlet boundary conditions. In Section 3 we prove Theorem 1.1.

In Theorem 4.1 we address the issue of interior regularity for the free boundary. The theorem is complemented by Corollary 4.5. In Subsection 4.3 we prove Theorem 1.2. In the last subsection, we investigate further properties of global minimizers such as symmetries and uniqueness: in Theorem 4.15 we prove that even if the functional J_h is highly non-convex, the minimization problem in $\mathcal{K}_{\gamma_h}$ has a unique solution for all but countably many values of h. In addition we show that the support of these solutions is symmetric with respect to the coordinate hyperplanes $\{x_i = 0\}$ for $i = 1, \dots, N-1$. These results are obtained by studying the convergence of sequences of the form $\{u_n\}_n$, where u_n is a global minimizer of J_{h_n} in $\mathcal{K}_{\gamma_{h_n}}$ for some given sequence of real numbers $\{h_n\}_n$. The key observation is that if $\{h_n\}_n$ is a monotone sequence, then $u_n \to u$ where u only depends on whether $\{h_n\}_n$ is increasing or decreasing.

2 Preliminary results

2.1 Basic definitions

Given a metric space X and a family of functionals $\mathcal{F}_\varepsilon \colon X \to \overline{\mathbb{R}}$, $\varepsilon > 0$, we say that $\{\mathcal{F}_\varepsilon\}_\varepsilon$ *Gamma converges* to $\mathcal{F} \colon X \to \overline{\mathbb{R}}$ as $\varepsilon \to 0^+$, and we write $\mathcal{F}_\varepsilon \xrightarrow{\Gamma} \mathcal{F}$, if for every sequence $\varepsilon_n \to 0^+$ the following two conditions hold:

(*i*) for every $x \in X$ and every sequence $\{x_n\}_n \subset X$ such that $x_n \to x$,

$$\liminf_{n\to\infty} \mathcal{F}_{\varepsilon_n}(x_n) \geq \mathcal{F}(x); \tag{2.1}$$

(*ii*) for every $x \in X$, there is a sequence $\{x_n\}_n \subset X$ such that $x_n \to x$ and

$$\limsup_{n\to\infty} \mathcal{F}_{\varepsilon_n}(x_n) \leq \mathcal{F}(x). \tag{2.2}$$

In our case $X = L^2_{\mathrm{loc}}(\Omega)$. We refer to [Bra02] and [DM93] for more details about Gamma convergence.

Throughout the paper we assume that Ω is an open connected subset of $\mathbb{R}^N$ with locally Lipschitz continuous boundary. We remark that Ω may be unbounded. Indeed in Section 2.4, and in all the subsequent sections, we will take Ω to be a half infinite rectangular parallelepiped. The main purpose of this section is to study the minimization problem for J in $\mathcal{K}$, defined as in (1.2) and (1.3) respectively. Here

$$Q \in L^2_{\mathrm{loc}}(\Omega), \quad Q \geq 0. \tag{2.3}$$

To this end, following [BCN90] we introduce the family of approximate identities β_ε, defined as

$$\beta_\varepsilon(s) := \frac{1}{\varepsilon}\beta\left(\frac{s}{\varepsilon}\right), \tag{2.4}$$

where

$$\beta \in C(\mathbb{R}; [0,\infty)), \quad \operatorname{supp}\beta \subset [0,1], \quad \int_0^\infty \beta(s)\,ds = \int_0^1 \beta(s)\,ds = 1. \tag{2.5}$$

We also define B_ε by

$$B_\varepsilon(t) := \int_0^t \beta_\varepsilon(s)\,ds. \tag{2.6}$$

It follows that B_ε is nonnegative, increasing, Lipschitz continuous, with

$$B_\varepsilon(t) = \begin{cases} 0 & \text{if } t \leq 0, \\ \displaystyle\int_0^{t/\varepsilon} \beta(s)\,ds & \text{if } 0 < t < \varepsilon, \\ 1 & \text{if } t \geq \varepsilon. \end{cases} \tag{2.7}$$

Finally, we consider the functional

$$J_\varepsilon(u) \coloneqq \int_\Omega \left(|\nabla u|^2 + B_\varepsilon(u)Q^2\right) d\boldsymbol{x} \tag{2.8}$$

defined for $u \in \mathcal{K}$.

2.2 Gamma convergence and global minimizers

The proof of the existence of a global minimizer for J_ε in the next theorem is adapted from [AC81, Theorem 3.1].

Theorem 2.1. *Let Ω be an open and connected subset of $\mathbb{R}^N$ with locally Lipschitz continuous boundary, and assume that (1.4), (2.3), (2.5) hold. Let J_ε and $\mathcal{K}$ be defined as in (2.8) and (1.3), respectively. Then there exists a global minimizer $u_\varepsilon \in \mathcal{K}$ of the functional J_ε. Furthermore, u_ε is a weak solution of the mixed Dirichlet-Neumann problem*

$$\begin{cases} \Delta u_\varepsilon = \frac{1}{2}\beta_\varepsilon(u_\varepsilon)Q^2 & \text{in } \Omega, \\ u_\varepsilon = u_0 & \text{on } \Gamma, \\ \partial_\nu u_\varepsilon = 0 & \text{on } \partial\Omega \setminus \Gamma, \end{cases} \tag{2.9}$$

where ν is the outward unit normal vector to $\partial\Omega$.

Proof. We claim that for every $u \in \mathcal{K}$,

$$J_\varepsilon(u) \le J(u), \tag{2.10}$$

where J is the functional defined in (1.2). Indeed, by (2.5) and (2.7) we have that for every $u \in L^1_{\text{loc}}(\Omega)$,

$$B_\varepsilon(u(\boldsymbol{x})) \le \chi_{\{u>0\}}(\boldsymbol{x}) \text{ for } \mathcal{L}^N\text{-a.e. } \boldsymbol{x} \in \Omega,$$

and the claim follows. In particular, we see from (1.4) and (2.10) that $J_\varepsilon(u_0) < \infty$.

Let $\alpha \coloneqq \min\{J_\varepsilon(u) : u \in \mathcal{K}\}$ and let $\{u_{k,\varepsilon}\}_k \subset \mathcal{K}$ be a minimizing sequence, that is,

$$\lim_{k\to\infty} J_\varepsilon(u_{k,\varepsilon}) = \alpha.$$

Then $\{\nabla u_{k,\varepsilon}\}_k$ is bounded in $L^2(\Omega;\mathbb{R}^N)$. Let $\Omega_r \coloneqq \Omega \cap B_r(\mathbf{0})$, where r is such that $\mathcal{H}^{N-1}(B_r(\mathbf{0}) \cap \Gamma) > 0$. Then by Poincaré's inequality we have that

$$\int_{\Omega_r} |u_{k,\varepsilon} - u_0|^2 \, d\boldsymbol{x} \le C(\Gamma, \Omega_r) \int_{\Omega_r} |\nabla u_{k,\varepsilon} - \nabla u_0|^2 \, d\boldsymbol{x}.$$

Therefore $\{u_{k,\varepsilon}\}_k$ is bounded in $H^1(\Omega_r)$ and hence, up to extraction of a subsequence (not relabeled), we can assume that $u_{k,\varepsilon} \to u_\varepsilon$ in $L^2(\Omega_r)$ and pointwise almost everywhere as $k \to \infty$ to some $u_\varepsilon \in H^1_{\text{loc}}(\Omega_r)$. By letting $r \nearrow \infty$ and by using a diagonal argument, up to extraction of a further subsequence, we have that

$$\begin{aligned} &\nabla u_{k,\varepsilon} \rightharpoonup \nabla u_\varepsilon \text{ in } L^2(\Omega, \mathbb{R}^N), \\ &u_{k,\varepsilon} \to u_\varepsilon \text{ in } L^2_{\text{loc}}(\Omega), \\ &u_{k,\varepsilon} \to u_\varepsilon \text{ pointwise almost everywhere in } \Omega. \end{aligned} \tag{2.11}$$

Moreover, since B_ε is Lipschitz continuous and nonnegative (see (2.5) and (2.6)), by the weakly lower semicontinuity of the L^2-norm and Fatou's lemma, we have that

$$\int_\Omega \left(|\nabla u_\varepsilon|^2 + B_\varepsilon(u_\varepsilon)Q^2\right) d\boldsymbol{x} \leq \liminf_{k\to\infty} \int_\Omega \left(|\nabla u_{k,\varepsilon}|^2 + B_\varepsilon(u_{k,\varepsilon})Q^2\right) d\boldsymbol{x} = \alpha.$$

To conclude, notice that $u_\varepsilon \in \mathcal{K}$ since $\mathcal{K}$ is closed with respect to the convergence in (2.11). Moreover, one can check that u_ε is a weak solution of (2.9) by considering variations of the functional J_ε. We omit the details. □

Corollary 2.2. *Let $u_\varepsilon \in \mathcal{K}$ be a global minimizer of the functional J_ε. Then, under the assumptions of Theorem 2.1,*

$$0 \leq u_\varepsilon(\boldsymbol{x}) \leq \|u_0\|_{L^\infty(\Gamma)}$$

for $\mathcal{L}^N$-a.e. $\boldsymbol{x} \in \Omega$, provided ε is small enough.

Proof. To prove the upper bound, we can assume without loss of generality that $m := \|u_0\|_{L^\infty(\Gamma)} < \infty$, since otherwise there is nothing to prove. For every $0 < \varepsilon < m$ and for every $\eta > 0$, let $v_\varepsilon := \max\{u_\varepsilon - m, 0\}$ and consider $u_\varepsilon^\eta := u_\varepsilon - \eta v_\varepsilon$. Then $u_\varepsilon^\eta \in \mathcal{K}$ and

$$B_\varepsilon(u_\varepsilon(\boldsymbol{x})) = B_\varepsilon(u_\varepsilon^\eta(\boldsymbol{x})) \tag{2.12}$$

for $\mathcal{L}^N$-a.e. $\boldsymbol{x} \in \Omega$. Indeed, the equality holds almost everywhere in $\{v_\varepsilon = 0\}$, while for almost every $\boldsymbol{x}$ such that $v_\varepsilon(\boldsymbol{x}) > 0$ we have that

$$u_\varepsilon(\boldsymbol{x}) > u_\varepsilon^\eta(\boldsymbol{x}) = (1-\eta)u_\varepsilon(\boldsymbol{x}) + \eta m > (1-\eta)m + \eta m > \varepsilon.$$

Therefore (2.12) follows from (2.7). This, together with the minimality of u_ε, implies that

$$\int_\Omega |\nabla u_\varepsilon|^2 \, d\boldsymbol{x} \leq \int_\Omega |\nabla u_\varepsilon^\eta|^2 \, d\boldsymbol{x}.$$

Expanding the square on the right-hand side, rearranging the terms, and dividing by η in the previous inequality yields

$$2\int_\Omega \nabla u_\varepsilon \cdot \nabla v_\varepsilon \, d\boldsymbol{x} \leq \eta \int_\Omega |\nabla v_\varepsilon|^2 \, d\boldsymbol{x} = \eta \int_\Omega \nabla u_\varepsilon \cdot \nabla v_\varepsilon \, d\boldsymbol{x},$$

where in the last equality we have used the fact that $\nabla u_\varepsilon = \nabla v_\varepsilon$ a.e. in the set $\{u_\varepsilon > m\}$ while $\nabla v_\varepsilon = 0$ a.e. in the set $\{u_\varepsilon \leq m\}$. Taking $\eta < 2$, since Ω is connected, we have that $v_\varepsilon \equiv c_\varepsilon$ for some constant c_ε. In turn, its trace is c_ε, but since $u_\varepsilon = u_0 \leq m$ on Γ, necessarily $c_\varepsilon = 0$. Thus $u_\varepsilon \leq m$ as desired.

The proof that u_ε is nonnegative is similar taking $u_\varepsilon^\eta := u_\varepsilon - \eta \min\{u_\varepsilon, 0\}$ and therefore we omit it. □

Theorem 2.3 (Compactness)**.** *Let Ω be an open and connected subset of $\mathbb{R}^N$ with locally Lipschitz continuous boundary, and let J_ε and $\mathcal{K}$ be defined as in* (2.8) *and* (1.3)*, respectively. Assume that* (2.3)*,* (2.5) *hold. Given $\varepsilon_n \to 0^+$ and $\{u_n\}_n \subset \mathcal{K}$ such that*

$$\sup\{J_{\varepsilon_n}(u_n) : n \in \mathbb{N}\} < \infty, \tag{2.13}$$

there are a subsequence $\{\varepsilon_{n_k}\}_k$ of $\{\varepsilon_n\}_n$ and $u \in \mathcal{K}$ such that $u_n \to u$ in $L^2_{\text{loc}}(\Omega)$.

Proof. Since $\{\nabla u_n\}_n$ is bounded in $L^2(\Omega;\mathbb{R}^N)$ by (2.13) and $B_\varepsilon \geq 0$, the desired convergence follows as in the proof of (2.11). We omit the details. □

In view of the previous theorem, we study the Γ-convergence of the family of functionals defined as in (2.8) with respect to convergence in $L^2_{\mathrm{loc}}(\Omega)$. To be precise, we define $\mathcal{J}_\varepsilon, \mathcal{J}\colon L^2_{\mathrm{loc}}(\Omega) \to [0,\infty]$ by

$$\mathcal{J}_\varepsilon(u) := \begin{cases} J_\varepsilon(u) & \text{if } u \in \mathcal{K}, \\ \infty & \text{if } u \in L^2_{\mathrm{loc}}(\Omega) \setminus \mathcal{K}, \end{cases} \tag{2.14}$$

and

$$\mathcal{J}(u) := \begin{cases} J(u) & \text{if } u \in \mathcal{K}, \\ \infty & \text{if } u \in L^2_{\mathrm{loc}}(\Omega) \setminus \mathcal{K}. \end{cases} \tag{2.15}$$

Theorem 2.4. *Let Ω be an open and connected subset of $\mathbb{R}^N$ with locally Lipschitz continuous boundary, and let $\mathcal{J}_\varepsilon$ and $\mathcal{J}$ be defined as above. Assume that (2.3), (2.5) hold. Then $\mathcal{J}_\varepsilon \xrightarrow{\Gamma} \mathcal{J}$ with respect to L^2_{loc} convergence.*

Proof. We prove (2.1). Let $u_n \to u$ in $L^2_{\mathrm{loc}}(\Omega)$. Without loss of generality, we may assume that

$$\liminf_{n\to\infty} \mathcal{J}_{\varepsilon_n}(u_n) < \infty,$$

since otherwise there is nothing to prove. By extracting successive subsequences, we may find a subsequence $\{\varepsilon_{n_k}\}_k$ of $\{\varepsilon_n\}_n$ such that $\sup\{\mathcal{J}_{\varepsilon_{n_k}}(u_{n_k}) : k \in \mathbb{N}\} < \infty$, $u_{n_k} \to u$ pointwise $\mathcal{L}^N$-a.e. in Ω and the following limits exist and are finite

$$\lim_{k\to\infty} \int_\Omega |\nabla u_{n_k}|^2 \, d\boldsymbol{x}, \quad \lim_{k\to\infty} \int_\Omega B_{\varepsilon_{n_k}}(u_{n_k}) Q^2 \, d\boldsymbol{x}.$$

In turn,

$$\int_\Omega |\nabla u|^2 \, d\boldsymbol{x} \leq \liminf_{k\to\infty} \int_\Omega |\nabla u_{n_k}|^2 \, d\boldsymbol{x} = \lim_{k\to\infty} \int_\Omega |\nabla u_{n_k}|^2 \, d\boldsymbol{x}. \tag{2.16}$$

Now fix $\delta > 0$ and let K be any compact set contained in $\{u > \delta\}$. By Egorov's theorem, for every $\eta > 0$ there exists a compact set $K_\eta \subset K$ such that $\mathcal{L}^N(K \setminus K_\eta) \leq \eta$ and $\{u_{n_k}\}_k$ converges uniformly to u on K_η. Notice that $\{B_{\varepsilon_{n_k}}(u_{n_k})\}_k$ is bounded in $L^\infty(\Omega)$ and hence admits a further subsequence (not relabeled) that converges in the weak star topology to some function $\xi \in L^\infty(\Omega)$. By uniform convergence, we can find $\overline{k}$ such that $u_{n_k} \geq \delta/2$ on K_η for $k \geq \overline{k}$. Moreover, if $\varepsilon_{n_k} \leq \delta/2$, $B_{\varepsilon_{n_k}}(u_{n_k}(\boldsymbol{x})) = 1$ for $\mathcal{L}^N$-a.e. $\boldsymbol{x}$ in K_η by (2.7), and hence

$$0 = \int_{K_\eta} (B_{\varepsilon_{n_k}}(u_{n_k}) - 1) u_{n_k} \, d\boldsymbol{x} \to \int_{K_\eta} (\xi - 1) u \, d\boldsymbol{x}.$$

Since $u > 0$ on K_η, then necessarily $\xi = 1$ $\mathcal{L}^N$-a.e. in K_η. Letting $\eta \searrow 0$, $K \nearrow \{u > \delta\}$ and $\delta \searrow 0$ we conclude that $\xi = 1$ $\mathcal{L}^N$-a.e. in $\{u > 0\}$ and hence

$$\xi(\boldsymbol{x}) \geq \chi_{\{u>0\}}(\boldsymbol{x}) \text{ for } \mathcal{L}^N\text{-a.e. } \boldsymbol{x} \in \Omega.$$

Let now D be a compact subset of Ω. By the previous inequality, the fact that $Q^2 \in L^1(D)$ and $B_{\varepsilon_{n_k}}(u_{n_k}) \overset{*}{\rightharpoonup} \xi$ in $L^\infty(\Omega)$,

$$\int_D \chi_{\{u>0\}} Q^2 \, d\boldsymbol{x} \leq \int_D \xi Q^2 \, d\boldsymbol{x} = \lim_{k\to\infty} \int_D B_{\varepsilon_{n_k}}(u_{n_k}) Q^2 \, d\boldsymbol{x} \leq \lim_{k\to\infty} \int_\Omega B_{\varepsilon_{n_k}}(u_{n_k}) Q^2 \, d\boldsymbol{x}.$$

Finally, letting $D \nearrow \Omega$ we get

$$\int_\Omega \chi_{\{u>0\}} Q^2 \, d\boldsymbol{x} \le \lim_{k\to\infty} \int_\Omega B_{\varepsilon_{n_k}}(u_{n_k}) Q^2 \, d\boldsymbol{x},$$

which together with (2.16) proves that $\mathcal{J}(u) \le \liminf_{n\to\infty} \mathcal{J}_{\varepsilon_n}(u_n)$.

We prove (2.2). Let $u \in L^2_{\rm loc}(\Omega)$ and define $u_n \equiv u$. If $\mathcal{J}(u) = \infty$, then there is nothing to prove. Thus, assume that $\mathcal{J}(u) < \infty$. By (2.10) we have $\mathcal{J}_{\varepsilon_n}(u_n) \le \mathcal{J}(u)$ and therefore the result follows. □

Corollary 2.5. *Let Ω be an open and connected subset of $\mathbb{R}^N$ with locally Lipschitz continuous boundary, and assume that* (1.4), (2.3), (2.5) *hold. Let $\mathcal{J}$ and $\mathcal{K}$ be defined as in* (2.15) *and* (1.3) *respectively. Then there exists a global minimizer $u \in \mathcal{K}$ of the functional $\mathcal{J}$. Furthermore, every global minimizer of $\mathcal{J}$ in $\mathcal{K}$ is subharmonic, locally Lipschitz continuous in Ω and harmonic in the set where it is positive.*

Proof. Let $\varepsilon_n \to 0^+$. By Theorem 2.1, for every $n \in \mathbb{N}$ we can find u_n, a global minimizer of $\mathcal{J}_{\varepsilon_n}$. Then by (1.4) we have

$$\sup\{\mathcal{J}_{\varepsilon_n}(u_n) : n \in \mathbb{N}\} \le \mathcal{J}(u_0) < \infty.$$

Let $\{\varepsilon_{n_k}\}_k$ and $u \in \mathcal{K}$ be given as in Theorem 2.3. Then, by Theorem 2.4, u is a global minimizer of $\mathcal{J}$. The rest is classical, see [AC81, Lemma 2.2, Lemma 2.4, Corollary 3.3]. □

Remark 2.6. *In view of the previous corollary, given a global minimizer $u \in \mathcal{K}$ of the functional $\mathcal{J}$, we can work with the precise representative*

$$u(\boldsymbol{x}) = \lim_{r\to 0^+} \fint_{B_r(\boldsymbol{x})} u(\boldsymbol{y}) \, d\boldsymbol{y}, \quad \boldsymbol{x} \in \Omega.$$

2.3 Uniform gradient estimates and boundary regularity

In view of Corollary 2.2, we study uniform properties of nonnegative and uniformly bounded solutions of (2.9). In particular (see Corollary 2.10), combining the results of [BCN90] with the ones of [Gur99] and [Kar06], we show that under certain regularity conditions on $\partial\Omega$ and u_0, if u_ε is a global minimizers of $\mathcal{J}_\varepsilon$ in $\mathcal{K}$ (see (1.3), Theorem 2.1 and (2.14)), then the family $\{u_\varepsilon\}_\varepsilon$ satisfies a uniform-in-ε Lipschitz estimate away from $\partial\Gamma$, where $\partial\Gamma$ refers to the boundary of Γ as a subspace of $\partial\Omega$. In this subsection we work with sets that have the uniform C^2-regularity property.

Definition 2.7. ([Fra79, Definition 4.1]) *Let Ω be an open subset of $\mathbb{R}^N$. We say that Ω has the* uniform C^2-regularity property *if there exist a locally finite open cover $\{U_s\}_s$ of $\partial\Omega$, and corresponding C^2 homeomorphisms $\boldsymbol{\phi}_s$, such that:*

(i) *for each s, $\boldsymbol{\phi}_s(U_s) = B_1(\boldsymbol{0})$ and $\boldsymbol{\phi}_s(\Omega \cap U_s) = B_1^+(\boldsymbol{0})$;*

(ii) $\bigcup_s \boldsymbol{\phi}_s^{-1}(B_{1/2}(\boldsymbol{0})) \supset \{\boldsymbol{x} \in \overline{\Omega} : \operatorname{dist}(\boldsymbol{x}, \partial\Omega) \le \tau\}$, *for some $\tau > 0$;*

(iii) *there exists an integer R such that any $R+1$ distinct sets U_s have empty intersection;*

(iv) *for some sequence of points $\{c_s\}_s \subset \mathbb{R}^N$,*

$$\|\boldsymbol{\phi}_s\|_{C^2(\overline{U_s};\mathbb{R}^N)}, \|\boldsymbol{\phi}_s^{-1} - c_s\|_{C^2(\overline{B_1(\boldsymbol{0})};\mathbb{R}^N)} \le M,$$

for some M independent of s.

Remark 2.8. (*i*) *Definition 2.7 is standard in the treatment of regularity results for PDEs in unbounded domains. We remark that it is equivalent to the definition of boundary uniformly of class C^2 (see* [Fra79, Definition 3.4] *and* [Fra79, Theorem 4.2]*). Moreover, it is also equivalent to* [BCN90, Property P]*.*

(*ii*) *For any given $d > 0$, eventually replacing R with a larger number, we can assume without loss of generality that* $\operatorname{diam} U_s \le d$*.*

Theorem 2.9. *Let Ω be an open connected subset of $\mathbb{R}^N$ with boundary $\partial\Omega$ uniformly of class C^2, and let $u_0 \in C^{1,\alpha}(\overline{\Omega})$, $0 < \alpha < 1$. Let $\{u_\varepsilon\}_\varepsilon \subset W^{2,p}_{\mathrm{loc}}(\Omega)$, $N < p < \infty$, be a family of nonnegative uniformly bounded solutions of* (2.9) *where Q, in addition to* (2.3)*, is assumed to be locally bounded in $\overline{\Omega}$. Then, for every K compactly contained in $\overline{\Omega} \setminus \partial\Gamma$, there exists a constant C such that*

$$|\nabla u_\varepsilon(\boldsymbol{x})| \le C, \quad \boldsymbol{x} \in K, \tag{2.17}$$

where C only depends on $N, p, K, \|Q\|_{L^\infty(K)}, \|\beta\|_{L^\infty(\mathbb{R})}, \|u_0\|_{C^{1,\alpha}(\overline{\Omega})}, \sup_\varepsilon \|u_\varepsilon\|_{L^\infty(\Omega)}$ and $\partial\Omega$ through τ, R and M as in Definition 2.7.

Proof. Let K be a compact subset of $\overline{\Omega} \setminus \partial\Gamma$. If $K \subset \Omega$ the desired result follows directly from [BCN90, Theorem 3.1 (a)]. Thus, assume that $K \cap \partial\Omega$ is non-empty and let $d_K \coloneqq \operatorname{dist}(K, \partial\Gamma)$. Let $\{U_s\}_s$ be as in Definition 2.7 with $\operatorname{diam} U_s \le d_K/2$ (see Remark 2.8 (*ii*)). By a compactness argument, we can find an integer S such that $K \cap U_s$ is empty for every $s > S$. Then there are $\mathcal{D}, \mathcal{N} \subset \mathbb{N}$ such that:

(*i*) $\mathcal{D}, \mathcal{N}$ are disjoint and $\mathcal{D} \cup \mathcal{N} = \{1, \dots, S\}$;

(*ii*) $U_i \cap \partial\Omega \subset \Gamma$ for every $i \in \mathcal{D}$ and $U_j \cap \partial\Omega \subset \partial\Omega \setminus \Gamma$ for every $j \in \mathcal{N}$;

(*iii*) $\displaystyle\bigcup_{s \in \mathcal{D}\cup\mathcal{N}} \boldsymbol{\phi}_s^{-1}(B_{1/2}(\mathbf{0})) \supset K \cap \{\boldsymbol{x} \in \overline{\Omega} : \operatorname{dist}(\boldsymbol{x}, \partial\Omega) \le \tau\}$, where τ is as in Definition 2.7;

(*iv*) $\displaystyle\bigcup_{s \in \mathcal{D}\cup\mathcal{N}} U_s \cap \overline{\Omega} \subset \overline{\Omega} \setminus \{\boldsymbol{x} \in \overline{\Omega} : \operatorname{dist}(\boldsymbol{x}, \partial\Gamma) < d_K/2\}$.

Notice that we are in a position to apply [Kar06, Theorem 3.1] in $U_i \cap \Omega$, $i \in \mathcal{D}$, and [BCN90, Theorem 3.1 (b)] in $U_j \cap \Omega$, $j \in \mathcal{N}$. Therefore, there exists a constant C (depending on the other parameters of the problem, but independent of ε) such that

$$|\nabla u_\varepsilon(\boldsymbol{x})| \le C, \quad \boldsymbol{x} \in \bigcup_{s \in \mathcal{D}\cup\mathcal{N}} \boldsymbol{\phi}_s^{-1}(B_{1/2}(\mathbf{0})).$$

Moreover, again by [BCN90, Theorem 3.1 (a)], a similar estimate holds in $K \cap \{\boldsymbol{x} \in \Omega : \operatorname{dist}(\boldsymbol{x}, \partial\Omega) \ge \tau/2\}$ and hence, by (*iii*), everywhere in K. □

Corollary 2.10. *Let Ω be an open and connected subset of $\mathbb{R}^N$ with boundary $\partial\Omega$ uniformly of class C^2, and assume that* (1.4), (2.3), (2.5) *hold. In addition, we assume that $u_0 \in C^{1,\alpha}(\overline{\Omega})$, $0 < \alpha < 1$, and that Q, in addition to* (2.3)*, is locally bounded in $\overline{\Omega}$. Let $\mathcal{J}_\varepsilon, \mathcal{J}$ and $\mathcal{K}$ be defined as in* (2.14), (2.15) *and* (1.3) *respectively. Then, given $\varepsilon_n \to 0^+$ and $\{u_n\}_n \subset \mathcal{K}$ such that u_n is a global minimizer of $\mathcal{J}_{\varepsilon_n}$ for every $n \in \mathbb{N}$, we have that $\{u_n\}_n \subset W^{2,p}_{\mathrm{loc}}(\Omega)$, $N < p < \infty$, and moreover there exists a subsequence $\{\varepsilon_{n_k}\}_k$ such that $\{u_{n_k}\}_k$ converges locally uniformly in $\overline{\Omega} \setminus \partial\Gamma$ to a function u that is a global minimizer of $\mathcal{J}$ in $\mathcal{K}$. In particular, u is locally Lipschitz continuous in $\overline{\Omega} \setminus \partial\Gamma$.*

Proof. By Theorem 2.1, for every $n \in \mathbb{N}$, u_n is a weak solution of (2.9) with $\varepsilon = \varepsilon_n$. Moreover, by Corollary 2.2, the sequence $\{u_n\}_n$ is nonnegative and uniformly bounded from above by $\|u_0\|_{L^\infty(\Gamma)}$, which is finite by assumption. By standard elliptic regularity theory, $\{u_n\}_n \subset W^{2,p}_{\mathrm{loc}}(\Omega)$, $N < p < \infty$ (see, e.g., [GT84] and [Nar14]). Let $\{\varepsilon_{n_k}\}_k$, u be given as in Theorem 2.3. Then, reasoning as in the proof of Corollary 2.5, we obtain that u is a global minimizer of $\mathcal{J}$ in $\mathcal{K}$. Notice that by Theorem 2.9, we are in a position to apply the Ascoli-Arzelà Theorem to $\{u_{n_k}\}_k$. This proves the existence of a further subsequence (which we don't relabel) that converges uniformly to u on every compact subsets of $\overline{\Omega}\setminus\partial\Gamma$. To conclude, it is enough to notice that u inherits the gradient estimates on every compact subset of $\overline{\Omega} \setminus \partial\Gamma$ from the weak star convergence in L^∞ of (a subsequence of) $\{\nabla u_{n_k}\}_k$. □

Remark 2.11. (*i*) *Under the slightly more restrictive assumptions that $\partial\Omega$ is smooth and $u_0 \in C^{2,\alpha}(\overline{\Omega})$, an estimate up to the boundary near the Dirichlet fixed boundary can be obtained as in* [Gur99, Section 2.3].

(*ii*) *As we remarked in the introduction, the behavior of global minimizers near points where two different boundary conditions meet is under study* [GL].

2.4 Mixed periodic and Dirichlet boundary conditions

In this subsection we adapt the previous results to the case in which Neumann boundary conditions are replaced by periodic boundary conditions. To be precise, we take Ω to be the half infinite rectangular parallelepiped $\Omega = \mathcal{R} \times (0, \infty)$ in (1.9), where we recall that $\mathcal{R} = (-\lambda/2, \lambda/2)^{N-1}$, and we take $\Gamma = \Gamma_\gamma$ to be the bottom of the parallelepiped $\mathcal{R} \times \{0\}$ and a part of the lateral boundary, to be precise

$$\Gamma_\gamma := (\mathcal{R} \times \{0\}) \cup (\partial\mathcal{R} \times (\gamma, \infty)),$$

where $\gamma > 0$. We will assume that the Dirichlet datum u_0 is zero on $\partial\mathcal{R} \times (\gamma, \infty)$ and a $\mathcal{R}$-periodic function in $\mathcal{R} \times \{0\}$, that is,

$$u_0(\boldsymbol{x}) := \begin{cases} 0 & \text{if } \boldsymbol{x} \in \partial\mathcal{R} \times (\gamma, \infty), \\ v_0(\boldsymbol{x}') & \text{if } \boldsymbol{x} = (\boldsymbol{x}', 0) \in \mathcal{R} \times \{0\}, \end{cases} \tag{2.18}$$

where $v_0 \in C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^{N-1})$ is nonnegative and $\mathcal{R}$-periodic. Define

$$\mathcal{K}_\gamma := \{u \in H^1_{\mathcal{R},\mathrm{loc}}(\Omega) : u = u_0 \text{ on } \Gamma_\gamma\}.$$

Here $H^1_{\mathcal{R},\mathrm{loc}}(\Omega)$ is the Sobolev space defined in (1.10). Furthermore, in addition to (2.3), we assume that

$$Q \in L^\infty_{\mathrm{loc}}(\overline{\Omega}), \quad Q(\boldsymbol{x} + \lambda \boldsymbol{e}_i) = Q(\boldsymbol{x}) \text{ for } \mathcal{L}^N\text{-a.e. } \boldsymbol{x} \in \mathbb{R}^N_+ \text{ and every } i = 1, \dots, N-1. \tag{2.19}$$

We remark that since u_0 as in (2.18) can be extended to a function in $\mathcal{K}_\gamma$ such that $J(u_0) < \infty$, the existence of global minimizers in $\mathcal{K}_\gamma$ for J_ε, J can be adapted from the results of the previous subsections, essentially without change. We omit the details.

Theorem 2.12. *Let $\Omega, J_\varepsilon, \mathcal{K}_\gamma, u_0$ be defined as in* (1.9), (2.8), (1.13) *and* (2.18) *respectively, where u_0 is given as in* (2.18), *Q as in* (2.3) *and* (2.19), *and β as in* (2.5). *Let $u_\varepsilon \in \mathcal{K}_\gamma$ be a global minimizer of the functional J_ε. Then, for every K compactly contained in $\overline{\Omega} \setminus (\partial\mathcal{R} \times \{\gamma\})$, there exists a constant C as in Theorem 2.9 such that*

$$|\nabla u_\varepsilon(\boldsymbol{x})| \le C, \quad \boldsymbol{x} \in K. \tag{2.20}$$

Proof. For simplicity we only give the proof for $N = 3$. The proof is analogous in the other cases, and simpler if $N = 2$.

Step 1: Subdivide $\mathbb{R}^2$ into cubes of length λ and centers at $\lambda\mathbb{Z}^2$ and let $\mathcal{R}_1, \ldots, \mathcal{R}_8$ be the cubes adjacent to $\mathcal{R}$. Define $\mathcal{R}_0 \coloneqq \mathcal{R}$, $U_n \coloneqq \mathcal{R}_n \times (0, \gamma)$. Consider the open set

$$U \coloneqq \left(\bigcup_{n=0}^{8} \overline{U_n} \right)^{\circ}.$$

For every open set $V \subset \mathbb{R}^3$ consider the functional

$$J_\varepsilon(v; V) \coloneqq \int_V \left(|\nabla v|^2 + B_\varepsilon(v) Q^2 \right) d\boldsymbol{x},$$

where $\boldsymbol{x} = (x, y, z) \in \mathbb{R}^3$. We claim that u_ε restricted to U is a local minimizer of the functional $J_\varepsilon(\cdot, U)$ in the sense that

$$J_\varepsilon(u_\varepsilon, U) \leq J_\varepsilon(v, U)$$

for all $v \in H^1_{\mathrm{loc}}(U)$ such that $\operatorname{supp}(v - u_\varepsilon) \Subset \overline{B_r(\boldsymbol{x}_0)} \subset U$ for some $\boldsymbol{x}_0 \in U_0$ and $0 < r < \lambda$. To see this, let v be any such function and write

$$J_\varepsilon(v, U) = \sum_{n=0}^{8} J_\varepsilon(v, U_n).$$

Given $n = 0, \ldots, 8$ write $\mathcal{R}_n = \lambda \boldsymbol{\tau}_n + \mathcal{R}$, where $\boldsymbol{\tau}_n = (\boldsymbol{\tau}_n', 0) \in \mathbb{Z}^3$. Note that $\boldsymbol{\tau}_0 = \mathbf{0}$. Define

$$v_\varepsilon(\boldsymbol{x}) \coloneqq \begin{cases} v(\boldsymbol{x} + \lambda \boldsymbol{\tau}_n) & \text{if } \boldsymbol{x} \in -\lambda \boldsymbol{\tau}_n + (B_r(\boldsymbol{x}_0) \cap U_n) \text{ for } n = 0, \ldots, 8, \\ u_\varepsilon(\boldsymbol{x}) & \text{otherwise in } \mathbb{R}^3. \end{cases}$$

We claim that v_ε is well-defined and belongs to $\mathcal{K}_\gamma$. Indeed, since $\boldsymbol{x}_0 \in \Omega$ and $r < \lambda$, we have that

$$\big(-\lambda \boldsymbol{\tau}_n + (B_r(\boldsymbol{x}_0) \cap U_n) \big) \bigcap \big(-\lambda \boldsymbol{\tau}_m + (B_r(\boldsymbol{x}_0) \cap U_m) \big) = \emptyset$$

if $n \neq m$; hence v_ε is well defined. Moreover, since $u_\varepsilon(\boldsymbol{x} + \lambda \boldsymbol{\tau}_n) = u_\varepsilon(\boldsymbol{x})$ and $u_\varepsilon = v$ on $\partial B_r(\boldsymbol{x}_0)$, it follows that $v_\varepsilon \in H^1_{\mathrm{loc}}(\mathbb{R}^3)$ and in particular $v_\varepsilon \in \mathcal{K}_\gamma$ as desired. Let

$$W \coloneqq \bigcup_{n=0}^{8} \big(-\lambda \boldsymbol{\tau}_n + (B_r(\boldsymbol{x}_0) \cap U_n) \big).$$

Since u_ε is $\mathcal{R}$-periodic in (x, y) and a global minimizer of J_ε, and since by construction $v_\varepsilon = u_\varepsilon$ in $\Omega \setminus W$ (see Figure 1), we have that

$$\begin{aligned} J_\varepsilon(u_\varepsilon, B_r(\boldsymbol{x}_0)) &= J_\varepsilon(u_\varepsilon, W) = J_\varepsilon(u_\varepsilon) - J_\varepsilon(u_\varepsilon, \Omega \setminus W) \\ &\leq J_\varepsilon(v_\varepsilon) - J_\varepsilon(u_\varepsilon, \mathbb{R}^3 \setminus W) = J_\varepsilon(v_\varepsilon, W) \\ &= J_\varepsilon(v, B_r(\boldsymbol{x}_0)). \end{aligned}$$

This proves the claim. Reasoning as in Theorem 2.9 and Corollary 2.10, we can conclude that (2.20) holds for every $K \Subset \overline{R} \times [0, \gamma)$.

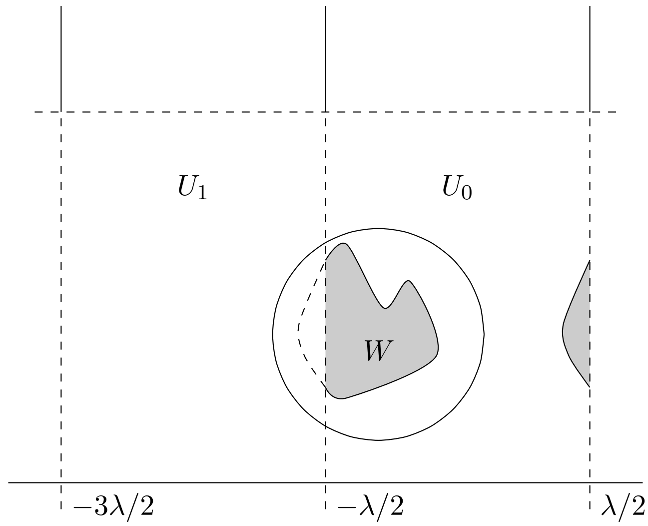


Figure 1: The function v_ε is set to be equal to u_ε outside of the shaded region W.

Step 2: Let $\boldsymbol{z} \in \partial\mathcal{R} \times (\gamma, \infty)$ be an extreme point for the convex set $\overline{\Omega}$. Without loss of generality we assume that $\boldsymbol{z} = (-\lambda/2, -\lambda/2, z_3)$. Let $0 < r < \min\{\lambda, z_3 - \gamma\}$. Let w be a solution of

$$\begin{cases} \Delta w = 0 & \text{in } B_r(\boldsymbol{z}) \cap \Omega, \\ w = u_\varepsilon & \text{on } \partial(B_r(\boldsymbol{z}) \cap \Omega). \end{cases}$$

Then w is nonnegative and by the maximum principle, together with Corollary 2.2,

$$w(\boldsymbol{x}) \leq \|v_0\|_{L^\infty(\mathcal{R})} \tag{2.21}$$

for every $\boldsymbol{x} \in B_r(\boldsymbol{z}) \cap \Omega$. For every $0 < \rho \leq r$, define

$$V_\rho := B_\rho(\boldsymbol{z}) \cap (-\lambda/2, \lambda/2) \times (-3\lambda/2, \lambda/2) \times (0, \infty).$$

Let $\widetilde{w}$ be the function obtained extending w to the half ball V_r by an odd reflection with respect to the plane $\{y = -\lambda/2\}$. Since $u_\varepsilon = 0$ on $\partial\mathcal{R} \times (\gamma, \infty)$, the extended function is still harmonic. By [GT84, Theorem 9.13], Morrey's inequality and (2.21) there is a constant $C_1 > 0$ such that

$$|\nabla \widetilde{w}(\boldsymbol{x})| \leq C_1, \quad \text{for } \boldsymbol{x} \in V_{r/2}.$$

Let $0 < \varepsilon < r/2$; then for every $\boldsymbol{x} \in B_\varepsilon(\boldsymbol{z}) \cap \Omega$, recalling that u_ε is subharmonic in Ω,

$$u_\varepsilon(\boldsymbol{x}) \leq w(\boldsymbol{x}) \leq C_1|\boldsymbol{x} - \boldsymbol{x}_0| \leq C_1\varepsilon. \tag{2.22}$$

Let $\widetilde{u}_\varepsilon$ be the function obtained extending u_ε by an odd reflection about the hyperplane $\{y = -\lambda/2\}$ and let $\widetilde{\beta}$ be the function obtained from β by an odd reflection. Then $\widetilde{u}_\varepsilon$ is a weak solution of

$$\Delta \widetilde{u}_\varepsilon = \frac{1}{2}\widetilde{\beta}_\varepsilon(\widetilde{u}_\varepsilon)Q^2$$

in V_r. Consider the rescaled function

$$U_\varepsilon(\boldsymbol{x}) := \frac{\widetilde{u}_\varepsilon(\boldsymbol{z} + \varepsilon\boldsymbol{x})}{\varepsilon}, \quad \boldsymbol{x} \in \{B_1(\boldsymbol{0}) : \boldsymbol{z} + \varepsilon\boldsymbol{x} \in V_\varepsilon\} =: B. \tag{2.23}$$

Then, by [GT84, Theorem 9.13], there exists a constant $C_2 > 0$ such that

$$|\nabla u_\varepsilon(\boldsymbol{z})| = |\nabla U_\varepsilon(\boldsymbol{x})| \leq C_2 \left(\frac{\|U_\varepsilon\|_{L^\infty(B)}}{\varepsilon} + \varepsilon \|\widetilde{\beta}_\varepsilon\|_{L^\infty(\mathbb{R})} \right) \leq C,$$

where in the last inequality we have used (2.5), (2.22) and (2.23). By the previous inequality, Theorem 2.9 and Corollary 2.10, (2.20) is satisfied for every $K \Subset \overline{R} \times (\gamma, \infty)$. This concludes the proof. □

Corollary 2.13. *Under the assumptions of Theorem 2.12, given $\varepsilon_n \to 0^+$ and $\{u_n\}_n \subset \mathcal{K}_\gamma$ such that u_n is a global minimizer of J_{ε_n} for every $n \in \mathbb{N}$, we have that $\{u_n\}_n \subset W^{2,p}_{\text{loc}}(\Omega)$, $N < p < \infty$, and moreover there exists a subsequence $\{\varepsilon_{n_k}\}_k$ such that $\{u_{n_k}\}_k$ converges locally uniformly in $\overline{\Omega} \setminus (\partial\mathcal{R} \times \{\gamma\})$ to a function $u \in \mathcal{K}_\gamma$ that is a global minimizer of the functional J, defined as in (1.2). In particular, u is locally Lipschitz continuous in $\overline{\Omega} \setminus (\partial\mathcal{R} \times \{\gamma\})$.*

Proof. The proof is analogous to the proof of Corollary 2.10, therefore we omit it. □

3 Existence of nontrivial minimizers

The purpose of this section is to prove Theorem 1.1. To be precise, we consider Ω to be a half infinite rectangular parallelepiped as in (1.9) and study the minimization problem for J_h in $\mathcal{K}_\gamma$, defined as in (1.12) and (1.13) respectively. The main difference with respect to the setting of Section 2.4 is that in this section we assume that Q is of the form (1.11) and the Dirichlet datum u_0, which we prescribe on Γ_γ (see (1.15)), is given by (1.14).

We observe that if $u \in \mathcal{K}_\gamma$ is of the form $u = u(x_N)$, then $u(0) = m$, $u(\gamma) = 0$, and by Tonelli's theorem

$$J_h(u) = \int_{\mathcal{R}} \int_0^\infty \left(|u'(x_N)|^2 + \chi_{\{u>0\}}(\boldsymbol{x}', x_N)(h - x_N)_+^{2b} \right) dx_N d\boldsymbol{x}' = \lambda^{N-1} \mathcal{I}_h(u), \tag{3.1}$$

where the functional $\mathcal{I}_h$ is defined via

$$\mathcal{I}_h(v) := \int_0^\infty \left(v'(t) + \chi_{\{v>0\}}(t)(h - t)_+^{2b} \right) dt \tag{3.2}$$

in the class

$$\mathcal{K}_{\gamma,\text{1-d}} := \{ v \in H^1_{\text{loc}}((0,\infty)) : v(0) = m \text{ and } v(\gamma) = 0 \},$$

where $H^1_{\text{loc}}((0,\infty))$ is the space of all functions $v \in L^2_{\text{loc}}((0,\infty))$ such that $v \in H^1((0,r))$ for every $r > 0$. Thus

$$\inf\{J_h(u) : u \in \mathcal{K}_\gamma\} \leq \lambda^{N-1} \inf\{\mathcal{I}_h(v) : v \in K_{\gamma,\text{1-d}}\}.$$

To prove Theorem 1.1 we will show that there exists a constant $\overline{\gamma}$ for which the inequality above is a strict inequality for all $0 < \gamma < \overline{\gamma}$. We begin by studying the one-dimensional minimization problem for $\mathcal{I}_h$ in the class $\mathcal{K}_{\gamma,\text{1-d}}$. Given $b, m, h, \gamma > 0$, let

$$h^\# := \frac{b+1}{b^{b/(b+1)}} m^{1/(b+1)}, \quad h^* := \frac{2b+2}{(2b+1)^{b/(b+1)}} m^{1/(b+1)}, \tag{3.3}$$

define $g_h \colon \mathbb{R}^+ \to \mathbb{R}$ by

$$g_h(t) := \frac{m^2}{t} + \frac{h^{2b+1} - (h - \min\{h,t\})^{2b+1}}{2b+1} \tag{3.4}$$

and, for $t > 0$, let $v_t \colon \mathbb{R}^+ \to \mathbb{R}$ be defined by

$$v_t(s) = \frac{m}{t}(t-s)_+. \tag{3.5}$$

Observe that $g_h \in C^1(\mathbb{R}^+)$.

Theorem 3.1. *Given $b, m, h, \gamma > 0$, let $\mathcal{I}_h$ be the functional defined in (3.2) and let $h^\#, h^*, g_h$ and v_t be given as above. Then*

$$\inf\{\mathcal{I}_h(v) : v \in \mathcal{K}_{\gamma,1\text{-}d}\} = \inf\{g_h(t) : 0 < t < \gamma\}, \tag{3.6}$$

and the following hold:

(i) if $h \le h^\#$ then g_h is decreasing and v_γ is the only global minimizer of $\mathcal{I}_h$ in the class $\mathcal{K}_{\gamma,1\text{-}d}$,

(ii) if $h^\# < h < h^*$ then g_h has two critical points, t_h, T_h,

$$0 < t_h < \frac{h}{b+1} < T_h < h, \tag{3.7}$$

which correspond to a point of local minimum and a point of local maximum of g_h respectively. Moreover, there exists a unique $\tau_h > T_h$ such that $g_h(t_h) = g_h(\tau_h)$. In this case we have that

(a) if $0 < \gamma \le t_h$ then g_h is decreasing in $(0, \gamma)$ and v_γ is the only global minimizer of $\mathcal{I}_h$ in the class $\mathcal{K}_{\gamma,1\text{-}d}$;

(b) if $t_h < \gamma < \tau_h$ then $\inf\{\mathcal{I}_h(v) : v \in \mathcal{K}_{\gamma,1\text{-}d}\} = g_h(t_h)$ and v_{t_h} is the only global minimizer of $\mathcal{I}_h$ in the class $\mathcal{K}_{\gamma,1\text{-}d}$;

(c) if $\gamma = \tau_h$ then $\inf\{\mathcal{I}_h(v) : v \in \mathcal{K}_{\gamma,1\text{-}d}\} = g_h(t_h) = g_h(\tau_h)$ and v_{t_h}, v_{τ_h} are the only minimizers of $\mathcal{I}_h$ in the class $\mathcal{K}_{\gamma,1\text{-}d}$;

(d) if $\gamma > \tau_h$ then $\inf\{\mathcal{I}_h(v) : v \in \mathcal{K}_{\gamma,1\text{-}d}\} = g_h(\gamma)$ and v_γ is the only global minimizer of $\mathcal{I}_h$ in the class $\mathcal{K}_{\gamma,1\text{-}d}$;

(iii) if $h \ge h^*$ then t_h is a point of absolute minimum for g_h. Moreover, v_γ is the only global minimizer of $\mathcal{I}_h$ in the class $\mathcal{K}_{\gamma,1\text{-}d}$ if $0 < \gamma \le t_h$, while if $t_h < \gamma$ then the only global minimizer is given by v_{t_h}.

In particular, in all the previous cases we have that

$$\inf\{\mathcal{I}_h(v) : v \in \mathcal{K}_{\gamma,1\text{-}d}\} = g_h(\gamma), \tag{3.8}$$

provided that $\gamma \le t_h$ when $h \ge h^\#$.

Proof. We divide the proof into several steps.
Step 1: By Corollary 2.5 we have that there exists a global minimizer v of $\mathcal{I}_h$ in $\mathcal{K}_{\gamma,1\text{-d}}$. We claim that v is linear on $\{v > 0\}$. Indeed, the minimality of v implies that the set $\{v > 0\}$ is connected; the claim follows recalling that v is harmonic in $\{v > 0\}$ (see Corollary 2.5). Thus, v is of the form $v = v_t$ for some $0 < t < \gamma$ and so (3.6) follows by noticing that

$$\mathcal{I}_h(v_t) = g_h(t). \tag{3.9}$$

Thus it remains to study $\inf\{g_h(t) : 0 < t < \gamma\}$.
Step 2: Since

$$g_h'(t) = \begin{cases} -\frac{m^2}{t^2} + (h-t)^{2b} & \text{if } t \leq h, \\ -\frac{m^2}{t^2} & \text{if } t > h, \end{cases}$$

we have that $g_h'(t) < 0$ if $t \geq h$. Moreover, $g_h'(t) \leq 0$ for $t < h$ if and only if

$$\psi_h(t) \coloneqq -m^2 + t^2(h-t)^{2b} \leq 0. \tag{3.10}$$

Since ψ_h has a global maximum in $(0, h)$ at the point $t = h/(b+1)$, it follows that

$$\psi_h(h/(b+1)) = -m^2 + \frac{b^{2b}}{(b+1)^{2b+2}} h^{2b+2} \leq 0 \tag{3.11}$$

if and only if $h \leq h^\#$, where $h^\#$ is the number given in $(3.3)_1$. Consequently, if $h \leq h^\#$ then g_h is decreasing and so

$$\inf\{g_h(t) : 0 < t < \gamma\} = g_h(\gamma),$$

which, together with (3.6) and (3.9), shows that v_γ is the only global minimizer of $\mathcal{I}_h$ in the class $\mathcal{K}_{\gamma,\text{1-d}}$.
Step 3: If $h > h^\#$, then in view of (3.10) and (3.11) there exist

$$0 < t_h < \frac{h}{b+1} < T_h < h$$

such that g_h strictly decreases in $(0, t_h)$ and in (T_h, ∞), and strictly increases in (t_h, T_h). It follows that

$$\inf\{g_h(t) : 0 < t < \gamma\} = \begin{cases} g_h(\gamma) & \text{if } 0 < \gamma \leq t_h, \\ g_h(t_h) & \text{if } t_h < \gamma \leq T_h, \\ \min\{g_h(t_h), g_h(\gamma)\} & \text{if } \gamma > T_h. \end{cases} \tag{3.12}$$

Hence, in what follows, it remains to treat the case $\gamma > T_h$. Notice that

$$\inf\{g_h(t) : 0 < t < \gamma\} = g_h(t_h) \leq \lim_{t\to\infty} g_h(t) = \frac{h^{2b+1}}{2b+1} \tag{3.13}$$

if and only if

$$m^2(2b+1) \leq \sup\{f_h(t) : 0 < t < h\},$$

where $f_h(t) \coloneqq t(h-t)^{2b+1}$. The function f_h has a maximum at $t = h/(2b+2)$, and so,

$$m^2(2b+1) \leq f_h(h/(2b+2)),$$

or equivalently $h \geq h^*$, where h^* is the number given in $(3.3)_2$. Hence by (3.13) if $h \geq h^*$ then $g_h(t_h) < g_h(\gamma)$, which, by (3.6), (3.9), and (3.12), proves (iii), while if $h < h^*$ then by (3.13) there exists $T_h < \tau_h$ such that $g_h(t_h) = g_h(\tau_h)$.

Properties $(a), (b), (c), (d)$ now follow again by (3.6),(3.9), and (3.12). □

The content of Theorem 3.1 is visually summarized in the following figure.

We are now ready to prove Theorem 1.1.

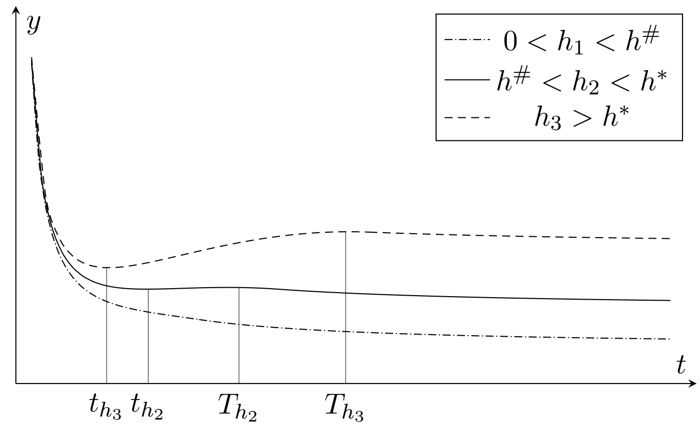


Figure 2: We let $b = 1/2$, $m = 1$ and plot g_h for different values of h.

Proof of Theorem 1.1. We will present the proof in full detail only for the case $N = 2$.
Step 1: Let $\delta, \gamma > 0$ and consider the function

$$w(x,y) = m\left(1 - \frac{y\lambda}{(\gamma+\delta)\lambda - 2x\delta}\right)_+,$$

defined in $(0, \lambda/2) \times (0,\infty)$, and extend it to $\mathcal{R} \times (0,\infty)$ with an even reflection about $\{0\} \times (0,\infty)$ and then to $\mathbb{R}^2_+$ by periodicity. With a slight abuse of notation we keep referring to the newly obtained function as w. We remark that by construction w is supported in the polygonal region with vertices in $\{(\pm\lambda/2, 0), (\pm\lambda/2, \gamma), (0, \gamma+\delta)\}$, and hence belongs to $\mathcal{K}_\gamma$. We will show that for every h we can find $\overline{\gamma}$ such that if $\gamma < \overline{\gamma}$, then

$$J_h(w) < \min\{J_h(v) : v \in \mathcal{K}_\gamma \text{ and } v = v(y)\}. \tag{3.14}$$

Notice that by (3.1), (3.6) and (3.8), if $\gamma \leq t_h$ when $h > h^\#$, we have that

$$\min\{J_h(v) : v \in \mathcal{K}_\gamma \text{ and } v = v(y)\} = \lambda g_h(\gamma),$$

and therefore to prove (3.14) one has to show that

$$J_h(w) < \lambda g_h(\gamma). \tag{3.15}$$

To this end, we explicitly compute $J_h(w)$: since for $(x, y) \in (0, \lambda/2) \times (0, \infty)$,

$$\frac{\partial w}{\partial x}(x,y) = \begin{cases} \frac{2m\delta y\lambda}{((\gamma+\delta)\lambda - 2x\delta)^2} & \text{if } (\gamma+\delta)\lambda - 2x\delta > y\lambda, \\ 0 & \text{othwerwise,} \end{cases}$$

and

$$\frac{\partial w}{\partial y}(x,y) = \begin{cases} -\frac{m\lambda}{(\gamma+\delta)\lambda - 2x\delta} & \text{if } (\gamma+\delta)\lambda - 2x\delta > y\lambda, \\ 0 & \text{othwerwise,} \end{cases}$$

by means of a direct computation we obtain

$$\int_\Omega |\nabla w|^2 \, d\boldsymbol{x} = m^2 \left(\frac{4\delta}{3\lambda} + \frac{\lambda}{\delta}\right) \log\left(1 + \frac{\delta}{\gamma}\right) \tag{3.16}$$

and

$$\int_\Omega \chi_{\{w>0\}}(h-y)_+^{2b}\,d\boldsymbol{x} = \begin{cases} \frac{\lambda h^{2b+1}}{2b+1} & \text{if } h \le \gamma, \\ \frac{\lambda h^{2b+1}}{2b+1} - \frac{\lambda(h-\gamma)^{2b+2}}{2\delta(2b+1)(2b+2)} & \text{if } \gamma < h \le \gamma+\delta, \\ \frac{\lambda h^{2b+1}}{2b+1} - \frac{\lambda(h-\gamma)^{2b+2} - \lambda(h-\gamma-\delta)^{2b+2}}{2\delta(2b+1)(2b+2)} & \text{if } \gamma+\delta < h. \end{cases} \tag{3.17}$$

Case $h \le \gamma$: Since (3.15) is equivalent to

$$\left(\frac{4\delta}{3\lambda} + \frac{\lambda}{\delta}\right) \log\left(1 + \frac{\delta}{\gamma}\right) < \frac{\lambda}{\gamma}, \tag{3.18}$$

as one can check from (3.4), (3.16) and (3.17), if we let $q = q(\lambda, \delta)$ be the unique value such that

$$\left(\frac{4\delta}{3\lambda} + \frac{\lambda}{\delta}\right) \log\left(1 + \frac{\delta}{q}\right) = \frac{\lambda}{q}, \tag{3.19}$$

then (3.18) is satisfied for every $\gamma < q$. To see that such a number q exists and is unique one can consider the function

$$\varphi(\gamma) := \left(\frac{4\delta}{3\lambda} + \frac{\lambda}{\delta}\right) \log\left(1 + \frac{\delta}{\gamma}\right) - \frac{\lambda}{\gamma}$$

and observe that it satisfies

$$\lim_{\gamma\to 0^+} \varphi(\gamma) = -\infty, \quad \lim_{\gamma\to\infty} \varphi(\gamma) = 0; \tag{3.20}$$

moreover, since $\varphi'(\gamma) = 0$ if and only if $\gamma = \overline{q}$, where

$$\overline{q} := \frac{3\lambda^2}{4\delta},$$

we conclude that s is increasing in $(0, \overline{q})$ and decreasing in $(\overline{q}, \infty)$. By (3.20) there exists a unique $q \in (0, \overline{q})$ such that $\varphi(q) = 0$, which is equivalent to (3.19). This also shows that if $h \le \min\{q, h^\#\}$ we can choose $\overline{\gamma}$ to be q. On the other hand, if $h > \min\{q, h^\#\}$ then either $h > q$ or $h > h^\#$ and in both cases we are forced to consider values of γ that are below h.
Case $h > \gamma$: for simplicity, instead of (3.15), we show that $\overline{\gamma}$ can be chosen in such a way that for every $\gamma < \overline{\gamma}$ the following more restrictive inequality is satisfied:

$$m^2\left(\frac{4\delta}{3\lambda} + \frac{\lambda}{\delta}\right) \log\left(1 + \frac{\delta}{\gamma}\right) < \frac{\lambda m^2}{\gamma} - \frac{\lambda h^{2b+1}}{2b+1}. \tag{3.21}$$

For φ defined as above, let $q_h = q_h(\delta, \lambda, m)$ be the unique value for which

$$m^2\varphi(q_h) + \frac{\lambda h^{2b+1}}{2b+1} = 0. \tag{3.22}$$

Notice that $0 < q_h < q$ and

$$m^2\left(\frac{4\delta}{3\lambda} + \frac{\lambda}{\delta}\right) \log\left(1 + \frac{\delta}{q_h}\right) = \frac{\lambda m^2}{q_h} - \frac{\lambda h^{2b+1}}{2b+1}.$$

Then (3.21) is satisfied for every $\gamma < q_h$. To summarize, if we set

$$\Theta(h) := \begin{cases} q & \text{if } h \leq \min\{q, h^{\#}\}, \\ q_h & \text{if } q < h^{\#} \text{ and } q < h \leq h^{\#}, \\ \min\{q_h, t_h\} & \text{if } h > h^{\#}, \end{cases} \tag{3.23}$$

then for $0 < \gamma < \Theta(h)$ we have that (3.14) holds and so the first part of Theorem 1.1 is proved.

Step 2: In this step we prove that the map Θ is decreasing. Since s is increasing in $(0, q)$, it follows from (3.22) that the function $h \mapsto q_h$ is decreasing. Therefore, we are left to prove that the map $h \mapsto t_h$ is also decreasing. To see this, we observe that since t_h is given implicitly by $m^2 = t_h^2(h - t_h)^{2b}$ (see (3.10)), by the implicit function theorem, this map is differentiable and as one can check

$$\frac{dt_h}{dh} = -\frac{bt_h}{h - (b+1)t_h} < 0 \quad \text{ for } h > h^{\#}. \tag{3.24}$$

Step 3: Notice that Θ is lower semicontinuous and decreasing. Let $\theta \colon \mathbb{R}^+ \to \mathbb{R}^+$ be a lower semicontinuous and decreasing function such that $\theta(h) < \Theta(h)$ for every h, and consider θ_n, the Yosida transform of θ, defined as

$$\theta_n(x) := \inf_y \{\theta(y) + n|x - y|\}.$$

Recall that θ_n is Lipschitz continuous with Lipschitz constant at most n and that $\theta_n(x) \nearrow \theta(x)$ as $n \to \infty$. We claim that θ_n is decreasing. To see this, consider $x < z$ and let $\epsilon > 0$ be given. By definition, we can find x_ϵ such that

$$\theta_n(x) \geq \theta(x_\epsilon) + n|x - x_\epsilon| - \epsilon.$$

Notice that if $x_\epsilon \leq z$ we have

$$\theta_n(x) \geq \theta(x_\epsilon) - \epsilon \geq \theta(z) - \epsilon \geq \theta_n(z) - \epsilon,$$

while if $x_\epsilon > z$ then $n|x - x_\epsilon| > n|z - x_\epsilon|$ so that

$$\theta_n(x) \geq \theta(x_\epsilon) + n|x - x_\epsilon| - \epsilon > \theta(x_\epsilon) + n|z - x_\epsilon| - \epsilon \geq \theta_n(z) - \epsilon.$$

By the arbitrariness of ϵ, this proves the claim. Thus, θ_n has all the desired properties for every $n \in \mathbb{N}$. This concludes the proof for the case $N = 2$.

Step 4: If $N \geq 3$, let S be the convex hull of $\mathcal{R} \times \{\gamma\}$ and the point $\{(\mathbf{0}', \gamma + 1)\}$. Define $f \colon \mathcal{R} \to \mathbb{R}^+$ via

$$f(\boldsymbol{x}') := \sup\{t : (\boldsymbol{x}', t) \in S\}.$$

Then we can set

$$w(\boldsymbol{x}', x_N) := \frac{m}{f(\boldsymbol{x}')}(f(\boldsymbol{x}') - x_N)_+,$$

and the rest follows similarly. □

Remark 3.2. *We report here the explicit values of t_h and T_h for the case $b = 1/2$. As previously mentioned in the introduction to this paper, this case is of particular interest since it corresponds to Bernoulli-type free boundary problems related to water waves. For $0 < t < h$,*

$$g_h'(t) = -\frac{m^2}{t^2} + h - t.$$

If $h > h^{\#}$, the cubic equation $t^3 - ht^2 + m^2 = 0$ has three real solutions, two of which are positive. Setting

$$\theta := \arccos\left(1 - \frac{3^3}{2}\frac{m^2}{h^3}\right)$$

so that $0 < \theta < \pi$, the two positive solutions are given by

$$t_h := \frac{2h}{3}\cos\frac{\theta + 4\pi}{3} + \frac{h}{3} \in \left(0, \frac{2h}{3}\right),$$
$$T_h := \frac{2h}{3}\cos\frac{\theta}{3} + \frac{h}{3} \in \left(\frac{2h}{3}, h\right).$$

We also know that

$$t_h < 2^{1/3}m^{2/3} < T_h.$$

Indeed, for every $\eta \in (0, h - h^{\#})$, by (3.7) and (3.24) we have

$$t_h < t_{h-\eta} < \frac{2}{3}(h - \eta) < T_{h-\eta} < T_h.$$

To conclude, let $\eta \to h - h^{\#}$.

4 Properties of global minimizers

In this section we carry out the study of properties of global minimizers of the functional J_h.

4.1 Interior regularity

In this subsection we study the regularity of solutions and of their free boundaries inside Ω. The next theorem shows that the reduced free boundary of a global minimizer u is regular except for the case in which $\operatorname{supp} u \subset \mathcal{R} \times [0, h]$ and $\operatorname{supp} u \not\subset \mathcal{R} \times [0, h)$. Part (ii) significantly improves the understanding of the so-called *non-physical solutions* in [AL12].

Theorem 4.1. *Let $\Omega, J_h, \mathcal{K}_\gamma, u_0$ be defined as in (1.9), (1.12), (1.13) and (1.14), respectively. Let $u \in \mathcal{K}_\gamma$ be a global minimizer of the functional J_h and assume that one of the following holds*

(i) $\operatorname{supp} u \subset \mathcal{R} \times [0, h)$;

(ii) there is $\boldsymbol{x} \in \Omega$ with $x_N \geq h$ such that $u(\boldsymbol{x}) > 0$.

Then the reduced free boundary $\partial^*\{u > 0\}$ is a hypersurface of class C^∞ locally in Ω and

$$\Sigma := \big(\partial\{u > 0\} \setminus \partial^*\{u > 0\}\big) \cap \Omega$$

is empty if $N = 2, 3$ and $\mathcal{H}^s(\Sigma) = 0$ for every $s > N - 4$ if $N \geq 4$.

Proof. (i) Assume that $\operatorname{supp} u \subset \mathcal{R} \times [0, h)$ and let $\overline{x}_N$ be such that $\operatorname{supp} u \subset \mathcal{R} \times [0, \overline{x}_N)$. Observe that for every $\boldsymbol{x} \in \operatorname{supp} u$

$$Q(\boldsymbol{x}', x_N) = (h - x_N)_+^b \geq (h - \overline{x}_N)^b > 0,$$

and so the regularity of the free boundary follows from [AC81, Theorem 8.4], [Wei99, Theorem 4.5] and [CJK04].

(ii) Since $u(\boldsymbol{x}) > 0$ for some $\boldsymbol{x} \in \Omega$ with $x_N \geq h$, and u is locally Lipschitz continuous by Corollary 2.10, we have that there exists $\boldsymbol{y} \in \Omega$ with $y_N > h$ such that $u(\boldsymbol{y}) > 0$. Since $Q \equiv 0$ in $\mathcal{R} \times (h, \infty)$ we have that u is harmonic in $\mathcal{R} \times (h, \infty)$ and so by the maximum principle $u > 0$ in $\mathcal{R} \times (h, \infty)$. It remains to show that the free boundary cannot intersect the hyperplane $\{x_N = h\}$. Indeed, let u be a solution with unbounded support and assume by contradiction that there exists

$$\boldsymbol{x}_0 \in \partial\{u > 0\} \cap \{x_N = h\}.$$

Then, by [AL12, Remark 3.5], there exists a constant C such that

$$|\nabla u(\boldsymbol{y})| \leq C r^b \tag{4.1}$$

for all $\boldsymbol{y} \in B_r(\boldsymbol{x}_0)$, where $r > 0$ is sufficiently small. Let B_ρ be any ball in $\mathcal{R} \times (h, \infty)$ such that $\boldsymbol{x} \in \partial B_\rho$. Since $u(\boldsymbol{y}) > u(\boldsymbol{x}_0) = 0$ for every $\boldsymbol{y} \in B_\rho$, we have that (4.1) is in contradiction with Hopf's Lemma. This concludes the proof. □

4.2 Existence of solutions with bounded support

The next proposition is a classical result due to Alt and Caffarelli (see [AC81, Lemma 3.4] and [AC81, Remark 3.5]). For the convenience of the reader we adapt here the statement to our framework (see also [AL12, Theorem 3.6]).

Proposition 4.2. *Given $b, m, \lambda, h, \gamma > 0$, let $\Omega, J_h, \mathcal{K}_\gamma, u_0$ be defined as in* (1.9), (1.12), (1.13) *and* (1.14)*, respectively. Then for every $k \in (0,1)$ there exists a positive constant $C(k)$ such that for every minimizer u of J_h in $\mathcal{K}_\gamma$ and for every ball $B_r(\boldsymbol{x}) \subset \Omega$, if*

$$\frac{1}{r} \fint_{\partial B_r(\boldsymbol{x})} u \, d\mathcal{H}^{N-1} \leq C(k)(h - x_N - kr)_+^b,$$

then $u \equiv 0$ in $B_{kr}(\boldsymbol{x})$. Moreover, the result is still valid for balls not contained in Ω if $u = 0$ on $B_r(\boldsymbol{x}) \cap \partial\Omega$. In particular, this holds if $B_r(\boldsymbol{x}) \cap \partial\Omega \subset \partial\mathcal{R} \times (\gamma, \infty)$.

Theorem 4.3. *Given $b, m, \lambda > 0$, let $\Omega, J_h, \mathcal{K}_{\gamma_h}, u_0$ be defined as in* (1.9), (1.12), (1.13) *and* (1.14)*, respectively, where the map $h \mapsto \gamma_h$ is given as in Theorem 1.1. Then for every $\overline{x}_N > 0$, if $C(1/2)$ is as in Proposition 4.2 and*

$$h \geq \max\left\{\frac{5}{4}\overline{x}_N + \left(\frac{2m}{\overline{x}_N C(1/2)}\right)^{1/b}, h_0\right\},$$

where $h_0 = 3\overline{x}_N/2$, the support (restricted to Ω) of every global minimizer $u \in \mathcal{K}_{\gamma_h}$ of J_h is contained in the set $\mathcal{R} \times [0, h)$. In particular, u is a regular solution with bounded support in the sense of Theorem 4.1.

Proof. Let $\overline{x}_N > 0$ be given. Let h_0 be such that $\gamma_{h_0} = 3\overline{x}_N/2$ and let $r := \overline{x}_N/2$. Then for every $\boldsymbol{x}' \in \mathcal{R}$,

$$B_r(\boldsymbol{x}) \subset \mathbb{R}^{N-1} \times (\gamma_h, \infty)$$

for every $h \geq h_0$. Moreover, if $h \geq \frac{5}{4}\overline{x}_N + \left(\frac{2m}{\overline{x}_N C(1/2)}\right)^{1/b}$ we have

$$\frac{1}{r(h - \overline{x}_N - r/2)_+^b} \fint_{\partial B_r(\boldsymbol{x}', \overline{x}_N)} u \, d\mathcal{H}^{N-1} \leq \frac{m}{r(h - \overline{x}_N - r/2)_+^b} = \frac{2m}{\overline{x}_N (h - \frac{5}{4}\overline{x}_N)^b} \leq C(1/2),$$

where the first inequality follows from Corollary 2.2. Thus we are in a position to apply Proposition 4.2, which shows that $u \equiv 0$ in $\mathcal{R} \times [\frac{3}{4}\overline{x}_N, \frac{5}{4}\overline{x}_N]$. Since by minimality the support of u is connected, this proves that $u \equiv 0$ in $\mathcal{R} \times [\overline{x}_N, \infty)$. The regularity of u follows from Theorem 4.1 (i). □

4.3 Existence of a critical height

The following result is inspired by [Fri88, Theorem 10.1] (see also [AL12, Theorem 5.5]).

Theorem 4.4. (Monotonicity). *Given $b, m, \lambda > 0$, let $\Omega, J_h, \mathcal{K}_{\gamma_h}, u_0$ be defined as in* (1.9), (1.12), (1.13) *and* (1.14), *respectively, where the map $h \mapsto \gamma_h$ is given as in Theorem 1.1. Consider $0 < d < h$ and let u_d, u_h be global minimizers of J_d and J_h in $\mathcal{K}_{\gamma_d}$ and $\mathcal{K}_{\gamma_h}$ respectively. Then*

$$\{\boldsymbol{x} \in \Omega : u_h(\boldsymbol{x}) > 0\} \subset \{\boldsymbol{x} \in \Omega : u_d(\boldsymbol{x}) > 0\} \tag{4.2}$$

and

$$u_h \leq u_d. \tag{4.3}$$

Moreover, if there exists $\boldsymbol{x} \in \partial\{u_h > 0\}$ such that $\partial^*\{u_h > 0\} = \partial\{u_h > 0\}$ in a neighborhood of $\boldsymbol{x}$ then $u_h < u_d$ in $\{\boldsymbol{x} \in \Omega : u_d(\boldsymbol{x}) > 0\}$.

Proof. **Step 1:** Define $v_1 := \min\{u_d, u_h\}$ and $v_2 := \max\{u_d, u_h\}$. Since $h \mapsto \gamma_h$ is decreasing, we have that $v_1 \in K_{\gamma_h}$ and $v_2 \in \mathcal{K}_{\gamma_d}$, and so

$$J_d(u_d) + J_h(u_h) \leq J_d(v_2) + J_h(v_1). \tag{4.4}$$

Notice that

$$\begin{aligned}
\int_\Omega |\nabla v_1|^2 \, d\boldsymbol{x} + \int_\Omega |\nabla v_2|^2 \, d\boldsymbol{x} &= \int_{\{u_h > u_d\}} \left(|\nabla v_1|^2 + |\nabla v_2|^2\right) d\boldsymbol{x} + \int_{\{u_h \leq u_d\}} \left(|\nabla v_1|^2 + |\nabla v_2|^2\right) d\boldsymbol{x} \\
&= \int_{\{u_h > u_d\}} \left(|\nabla u_d|^2 + |\nabla u_h|^2\right) d\boldsymbol{x} + \int_{\{u_h \leq u_d\}} \left(|\nabla u_h|^2 + |\nabla u_d|^2\right) d\boldsymbol{x} \\
&= \int_\Omega |\nabla u_d|^2 \, d\boldsymbol{x} + \int_\Omega |\nabla u_h|^2 \, d\boldsymbol{x}.
\end{aligned}$$

Therefore we can rewrite (4.4) canceling out the gradient terms and by rearranging the remaining terms we obtain

$$\int_{\{u_h > u_d\}} \left(\chi_{\{u_h > 0\}}(\boldsymbol{x}) - \chi_{\{u_d > 0\}}(\boldsymbol{x})\right)\left((h - x_N)_+^{2b} - (d - x_N)_+^{2b}\right) d\boldsymbol{x} \leq 0. \tag{4.5}$$

Since the integrand is nonnegative in the set $\{u_h > u_d\}$, and recalling that u_d and u_h are continuous in Ω, we have that

$$\{u_h > 0\} \cap \{x_N < h\} \cap \{u_h > u_d\} \subset \{u_d > 0\} \cap \{x_N < h\} \cap \{u_h > u_d\},$$

which together with the fact that

$$\{u_h > 0\} \cap \{u_h \leq u_d\} \subset \{u_d > 0\} \cap \{u_h \leq u_d\}$$

yields

$$\{u_h > 0\} \cap \{x_N < h\} \subset \{u_d > 0\} \cap \{x_N < h\}. \tag{4.6}$$

We now notice that if $\operatorname{supp} u_h \subset \mathcal{R} \times [0, d]$ then (4.2) follows from (4.6), while if it is not the case, again by (4.6) we get that there is $\boldsymbol{x} \in \mathcal{R} \times (d, \infty)$ such that $u_d(\boldsymbol{x}) > 0$. Reasoning as in the proof of Theorem 4.1, we can conclude that $u_d > 0$ in $\mathcal{R} \times (d, \infty)$ and so the desired inclusion is also satisfied in $\mathcal{R} \times [h, \infty)$. This conclude the proof of (4.2).

Step 2: We observe that since the equality holds in (4.5), then the equality necessarily holds in

(4.4) as well, and so v_1 and v_2 are global minimizers of J_h and J_d in $\mathcal{K}_{\gamma_h}$ and $\mathcal{K}_{\gamma_d}$ respectively. We now claim that if there is $\boldsymbol{x}_0 \in \Omega$ such that $u_d(\boldsymbol{x}_0) = u_h(\boldsymbol{x}_0) > 0$, then $u_d = u_h$ everywhere in Ω. To see this, we notice that in a neighborhood of $\boldsymbol{x}_0$ the functions $u_d - v_2$ and $u_h - v_2$ are harmonic, nonpositive and attain a maximum at an interior point. Then, by the maximum principle, $u_d - v_2 = u_h - v_2 \equiv 0$ in the connected component of $\{u_h > 0\}$ that contains $\boldsymbol{x}_0$; since $\{u_h > 0\}$ is connected by minimality, this proves the claim.

To prove (4.3), assume by contradiction that there is $\boldsymbol{x} \in \Omega$ such that $u_h(\boldsymbol{x}) > u_d(\boldsymbol{x})$. If there is $\boldsymbol{y} \in \{u_h > 0\}$ such that $u_d(\boldsymbol{y}) > u_h(\boldsymbol{y})$, then by the connectedness of $\{u_h > 0\}$, together with the fact that u_h and u_d are continuous, we have that there is $\boldsymbol{z} \in \Omega$ such that $u_h(\boldsymbol{z}) = u_d(\boldsymbol{z}) > 0$. By the claim we just proved, this would imply that $u_h = u_d$, a contradiction. Hence $u_d \le u_h$ in $\{u_h > 0\}$, which together with (4.2) implies that

$$\{u_h > 0\} = \{u_d > 0\}. \tag{4.7}$$

In turn,

$$\begin{aligned}\int_\Omega \chi_{\{u_h>0\}}(h - x_N)_+^{2b}\, d\boldsymbol{x} &= \int_\Omega \chi_{\{u_d>0\}}(h - x_N)_+^{2b}\, d\boldsymbol{x},\\ \int_\Omega \chi_{\{u_d>0\}}(d - x_N)_+^{2b}\, d\boldsymbol{x} &= \int_\Omega \chi_{\{u_h>0\}}(d - x_N)_+^{2b}\, d\boldsymbol{x}.\end{aligned} \tag{4.8}$$

From (4.7) we also see that $u_d \in \mathcal{K}_{\gamma_h}$. Since $h \mapsto \gamma_h$ is decreasing, we also have that $u_h \in \mathcal{K}_{\gamma_d}$ and hence we can conclude that $J_h(u_h) \le J_h(u_d)$ and $J_d(u_d) \le J_d(u_h)$, which, together with (4.8), implies that

$$\int_\Omega |\nabla u_h|^2\, d\boldsymbol{x} = \int_\Omega |\nabla u_d|^2\, d\boldsymbol{x}.$$

Consider $v := \frac{1}{2}u_h + \frac{1}{2}u_d \in \mathcal{K}_{\gamma_h}$. By the strict convexity of the Dirichlet energy, we have

$$J_h(v) < \int_\Omega \left(\frac{1}{2}|\nabla u_h|^2 + \frac{1}{2}|\nabla u_d|^2 + \chi_{\{v>0\}}(h - x_N)_+^{2b}\right) d\boldsymbol{x} = J_h(u_h),$$

a contradiction to the minimality of u_h, and (4.3) is hence proved.

Step 3: Finally, assume by contradiction that there is $\boldsymbol{x}_0 \in \{u_d > 0\}$ such that $u_h(\boldsymbol{x}_0) = u_d(\boldsymbol{x}_0)$, so that $u_h = u_d$ in Ω. Then for $\boldsymbol{x}$ as in the statement we have

$$(h - x_N)^b = \frac{\partial u_h}{\partial \nu}(\boldsymbol{x}) = \frac{\partial u_d}{\partial \nu}(\boldsymbol{x}) = (d - x_N)^b.$$

This is a contradiction since by assumption $d < h$. Hence $u_h < u_d$ in $\{u_d > 0\}$ and the proof is complete. □

Proof of Theorem 1.2. Let

$$h_{\mathrm{cr}} := \inf\{h > 0 : \text{there is a global minimizer } u_h \in \mathcal{K}_{\gamma_h} \text{ of } J_h \text{ s.t. } \operatorname{supp} u_h \subset \mathcal{R} \times [0, h]\}. \tag{4.9}$$

By Theorem 4.3 we have that $h_{\mathrm{cr}} < \infty$. The rest of the proof follows from the monotonicity result Theorem 4.4 as in [AL12, Theorem 5.6], we omit the details. □

Corollary 4.5. *Under the assumptions of Theorem 1.2, let $h \neq h_{\mathrm{cr}}$. Then every global minimizer of J_h in $\mathcal{K}_{\gamma_h}$ is regular in the sense of Theorem 4.1.*

Proof. The proof is a direct consequence of Theorem 1.2 and Theorem 4.1. □

4.4 Scaling of the critical height

Theorem 4.6. (Comparison principle). *Given* $b, m, \lambda, \delta, \gamma, h > 0$*, let* Ω, J_h, u_0 *be defined as in* (1.9), (1.12) *and* (1.14)*, respectively. Let* u *and* w *be a global minimizers of* J_h *in* $\mathcal{K}_\delta$ *and* $\mathcal{K}_\gamma$ *respectively, where* $\mathcal{K}_\delta, \mathcal{K}_\gamma$ *are defined as in* (1.13)*. Then either*

$$\{u > 0\} \subset \{w > 0\} \text{ and } u \leq w$$

or

$$\{w > 0\} \subset \{u > 0\} \text{ and } w \leq u.$$

Proof. Assume without loss of generality that $\delta \leq \gamma$. As in the proof of Theorem 4.4, we consider $v_1 := \min\{u, w\}$ and $v_2 := \max\{u, w\}$. Then $v_1 \in \mathcal{K}_\delta$, $v_2 \in \mathcal{K}_\gamma$ and in particular we have

$$J_h(u) + J_h(w) = J_h(v_1) + J_h(v_2).$$

Therefore v_1 and v_2 are global minimizers of J_h in $\mathcal{K}_\delta$ and $\mathcal{K}_\gamma$ respectively. Reasoning as in the proof of Theorem 4.4, we have that if there exists a point $\boldsymbol{x}_0$ such that $u(\boldsymbol{x}_0) = w(\boldsymbol{x}_0) > 0$ then $u = w$ everywhere in Ω. Next, we assume by contradiction that the supports of u and w do not satisfy the inclusions as in the statement, i.e., there exist $\boldsymbol{x}, \boldsymbol{y} \in \Omega$ such that $u(\boldsymbol{x}) > 0$, $w(\boldsymbol{y}) > 0$ and $u(\boldsymbol{y}) = w(\boldsymbol{x}) = 0$. Let $\boldsymbol{z} \in \Omega$ be such that $u(\boldsymbol{z}) > 0$ and $w(\boldsymbol{z}) > 0$ (such a point $\boldsymbol{z}$ exists since by minimality we have that $J_h(u)$ and $J_h(w)$ are both finite). We assume first that $w(\boldsymbol{z}) > u(\boldsymbol{z})$. Then, since by minimality $\{u > 0\}$ is open and connected and thus path-wise connected, we can find a continuous curve $\boldsymbol{\varphi}\colon [0, 1] \to \Omega$ joining $\boldsymbol{z}$ to $\boldsymbol{x}$, with support contained in Ω. Define

$$v(t) := w(\boldsymbol{\varphi}(t)) - u(\boldsymbol{\varphi}(t)).$$

Notice that by construction $v(0) = w(\boldsymbol{z}) - u(\boldsymbol{z}) > 0$ and $v(1) = w(\boldsymbol{x}) - u(\boldsymbol{x}) < 0$, and so there exists $t_0 \in (0, 1)$ such that $v(t_0) = 0$. Thus $0 < u(\boldsymbol{\varphi}(t)) = w(\boldsymbol{\varphi}(t))$, which in turn implies that $u = w$, a contradiction. Similarly, if $u(\boldsymbol{z}) > w(\boldsymbol{z})$, we arrive to a contradiction by considering a continuous curve $\boldsymbol{\psi}\colon [0, 1] \to \Omega$ that joins $\boldsymbol{z}$ with $\boldsymbol{y}$ and with support contained in $\{w > 0\}$. The rest of the proof is analogous to the proof of (4.3). □

Remark 4.7. *Notice that in Theorem 4.6 we also allow for the case where* $\delta = \gamma$*.*

In this lemma we show that h_{cr} in Theorem 1.2 is less than the value h^* given in $(3.3)_2$.

Lemma 4.8. *Under the assumptions of Theorem 1.2, we have that*

$$h_{\mathrm{cr}} \leq h^* = \frac{2b + 2}{(2b + 1)^{b/(b+1)}} m^{1/(b+1)}.$$

Proof. Assume by contradiction that $h_{\mathrm{cr}} > h^*$, and let $h^* < h < h_{\mathrm{cr}}$. By Tonelli's theorem and Theorem 3.1 (*iii*) we have that $w\colon \mathbb{R}^N_+ \to \mathbb{R}$ defined by

$$w(\cdot, x_N) := v_{t_h}(x_N)$$

is the unique global minimizer of J_h in $\mathcal{K}_{t_h}$. Let $u \in \mathcal{K}_{\gamma_h}$ be a global minimizer of J_h. Since $h > h^* \geq h^{\#}$ (see (3.3)), by (3.23) we have that $\gamma_h < t_h$, and hence $u(\boldsymbol{x}) = 0$ for $\boldsymbol{x} = (\boldsymbol{x}', x_N) \in \partial\mathcal{R} \times (\gamma_h, \infty)$. By continuity, we can find $\boldsymbol{x}'_0 \in \mathcal{R}$ close to $\partial\mathcal{R}$ such that

$$u(\boldsymbol{x}'_0, \gamma_h) < \frac{m}{t_h}(t_h - \gamma_h) = w(\boldsymbol{x}'_0, \gamma_h).$$

Then by Theorem 4.6, $u \le w$ and

$$\{u > 0\} \subset \{w > 0\} = \{x_N < t_h\}.$$

Thus, by (3.7), u is a solution with bounded support. This is in contradiction with the definition of $h_{\rm cr}$. $\square$

In this lemma we show that if the Dirichlet datum m is small then the critical height $h_{\rm cr}$ in Theorem 1.2 is greater than a constant multiple of $h^\#$, where $h^\#$ is given as in $(3.3)_1$.

Lemma 4.9. *Given $b, m, \lambda > 0$, let $h_{\rm cr}$ be as in Theorem 1.2 and assume that the Dirichlet datum m in* (1.14) *satisfies*

$$m < \frac{b^b}{(b+1)^{b+1}} \cdot q^{b+1} \tag{4.10}$$

for q as in (3.19)*. Then the function $h \mapsto \gamma_h$ given by Theorem 1.1 can be constructed with the property that*

$$\gamma_h \ge q/2, \qquad \text{for } h \le h^\# = \frac{b+1}{b^{b/(b+1)}} m^{1/(b+1)}. \tag{4.11}$$

In turn, there exists a constant $k_b > 0$ such that

$$h_{\rm cr} \ge k_b \frac{b+1}{b^{b/(b+1)}} m^{1/(b+1)}.$$

In particular, if m is small enough, $h_{\rm cr} \gtrsim m^{1/(b+1)}$.

Proof. We present the proof in full detail only for the case $N = 2$. We show that there exists $C_b > 0$ such that if $h \le C_b h^\#$ then every global minimizer of J_h in $\mathcal{K}_{\gamma_h}$ is a solution with unbounded support. Roughly speaking, this is due to the fact that for h small enough (with respect to other parameters in the problem), the Dirichlet energy plays a predominant role in the competition with the (weighted) area-penalizing term in J_h.

We begin by showing that γ_h can be chosen to satisfy (4.11). Let $\Theta \colon \mathbb{R}^+ \to \mathbb{R}^+$ be as in the proof of Theorem 1.1 (see (3.23)). As it was previously observed, Θ is lower semicontinuous and decreasing. Furthermore, we observe that condition (4.10) is equivalent to $h^\# < q$ and hence it follows from (3.23) that $\Theta(h) = q$ for every $h \in (0, h^\#]$. Let θ_n be the Yosida transform of $2\Theta/3$. Then, as in the proof of Theorem 1.1, $\{\theta_n\}_n$ is a sequence of continuous functions that converges monotonically to the constant $2q/3$ in any compact subset of $(0, h^\#]$. By Dini's convergence theorem, the convergence is then uniform and consequently θ_n satisfies (4.11) for n sufficiently large.

Let k_b be the unique solution of

$$2t + \frac{t^{2b+2}(b+1)^{2b+2}}{(2b+1)b^{2b}} = 1. \tag{4.12}$$

We claim that every global minimizer of $J_{k_b h^\#}$ in $\mathcal{K}_{\gamma(k_b h^\#)}$ is a solution with unbounded support. Notice that in view of (4.9) the claim implies that $h_{\rm cr} \ge k_b h^\#$. To prove the claim, we assume for the sake of contradiction that every global minimizer of $J_{k_b h^\#}$ in $\mathcal{K}_{\gamma(k_b h^\#)}$ is a solution with bounded support. Hence, as in the proof of Theorem 4.1, the support of every global minimizer is contained in $\{y \le k_b h^\#\}$ and therefore

$$\min\{J_{k_b h^\#}(u) : u \in \mathcal{K}_{\gamma(k_b h^\#)}\} \ge \min_{\mathcal{A}} \int_\Omega |\nabla v|^2 \, d\boldsymbol{x}, \tag{4.13}$$

where $\mathcal{A} \coloneqq \{v \in H^1_{\rm loc}(\mathcal{R} \times (0, k_b h^\#)) : v(x,0) = m, v(x, k_b h^\#) = 0\}$. We claim that

$$\min_{\mathcal{A}} \int_\Omega |\nabla v|^2 \, d\boldsymbol{x} = \frac{\lambda c^2}{k_b h^\#}. \tag{4.14}$$

To see this, we notice that by Tonelli's theorem, Jensen's inequality and the fundamental theorem of calculus, for every $v \in \mathcal{A}$ we have that

$$\begin{aligned}\int_\Omega |\nabla v|^2 \, d\boldsymbol{x} &= \int_{-\lambda/2}^{\lambda/2} \int_0^{k_b h^\#} |\nabla v|^2 \, dy dx \geq \int_{-\lambda/2}^{\lambda/2} \int_0^{k_b h^\#} (\partial_y v)^2 \, dy dx \\ &\geq \int_{-\lambda/2}^{\lambda/2} k_b h^\# \left(\frac{1}{k_b h^\#} \int_0^{k_b h^\#} \partial_y v \, dy \right)^2 dx = \int_{-\lambda/2}^{\lambda/2} \frac{(v(x, k_b h^\#) - v(x,0)^2}{k_b h^\#} \, dx = \frac{\lambda m^2}{k_b h^\#},\end{aligned}$$

and moreover, the equality is achieved for

$$v(x,y) \coloneqq \frac{m}{k_b h^\#}(k_b h^\# - y)_+.$$

Let w be defined as in the proof of Theorem 1.1; we will show that

$$J_h(w) < \frac{\lambda m^2}{k_b h^\#}. \tag{4.15}$$

Notice that since $w \in \mathcal{K}_{\gamma(k_b h^\#)}$, by (4.13) and (4.14) this would give a contradiction. Observe that since $k_b < 1/2$, by (4.10) it follows that $k_b h^\# \leq q/2$. Consequently, by (3.16) and (3.17) we have

$$J_{k_b h^\#}(w) = m^2 \left(\frac{4\delta}{3\lambda} + \frac{\lambda}{\delta} \right) \log \left(1 + \frac{\delta}{\gamma_{k_b h^\#}} \right) + \frac{\lambda (k_b h^\#)^{2b+1}}{2b+1}.$$

Moreover, by (3.18)

$$J_{k_b h^\#}(w) < \frac{\lambda m^2}{\gamma_{k_b h^\#}} + \frac{\lambda (k_b h^\#)^{2b+1}}{2b+1} \leq \frac{2m^2\lambda}{q} + \frac{\lambda (k_b h^\#)^{2b+1}}{2b+1},$$

and so to prove (4.15) it is enough to show that

$$\frac{2\lambda m^2}{q} + \frac{\lambda (k_b h^\#)^{2b+1}}{2b+1} < \frac{\lambda m^2}{k_b h^\#}.$$

Multiplying both sides by the inverse of the right-hand side and substituting the value for $h^\#$ as in $(3.3)_1$, we can rewrite the previous inequality as

$$2k_b \frac{h^\#}{q} + \frac{k_b^{2b+2}(b+1)^{2b+2}}{(2b+1)b^{2b}} < 1. \tag{4.16}$$

To conclude it is enough to notice that (4.16), and hence (4.15), follows from (4.12) and (4.10). □

Remark 4.10. *We conjecture that it should be possible to remove assumption* (4.10) *from Lemma 4.9 without affecting its conclusion. Although the proof we presented relies heavily on the aforementioned assumption, we believe that this is more of a limitation of the method than a defining feature of the problem.*

4.5 Boundary regularity

By Corollary 2.5, for every $\gamma, h > 0$, every global minimizer of J_h in $\mathcal{K}_\gamma$ is locally Lipschitz continuous in Ω. Moreover, by Corollary 2.13, there exists a global minimizer (obtained as uniform limit of solutions to an opportunely regularized problem, see Theorem 2.12) that is Lipschitz continuous locally in $\overline{\Omega} \setminus (\partial\mathcal{R} \times \{\gamma\})$. This is the optimal regularity away from the singular set $\partial\mathcal{R} \times \{\gamma\}$. The regularity of the free boundaries in Ω is discussed in Theorem 4.1 and Corollary 4.5. In a recent paper, Chang-Lara and Savin (see [CLS17, Theorem 1.1]) showed that if Q is bounded away from zero the free boundary of a global minimizer detaches tangentially from a $C^{1,\alpha}$ regular portion of the fixed Dirichlet boundary, with $\alpha > 1/2$, as a $C^{1,1/2}$ regular hypersurface. Thanks to their result, in the next theorem we discuss the up to the boundary regularity for the free boundaries of global minimizers of J_h in dimension $N = 2$.

Theorem 4.11. *Given $b, m, \lambda > 0$, let $N = 2$ and consider $\Omega, J_h, \mathcal{K}_{\gamma_h}, u_0$ as in* (1.9), (1.12), (1.13) *and* (1.14)*, respectively, where the map $h \mapsto \gamma_h$ is given as in Theorem 1.1. Let u be a global minimizer of J_h in $\mathcal{K}_{\gamma_h}$.*

- *(i) If $h > h_{\rm cr}$ then $\partial\{u > 0\}$ is a curve of class $C^{1,1/2}$ locally in $\overline{\Omega} \setminus (\pm\lambda/2, \gamma_h)$.*
- *(ii) If $h < h_{\rm cr}$ then $\partial\{u > 0\}$ is a curve of class $C^{1,1/2}$ locally in $\overline{\Omega} \setminus \big((\pm\lambda/2, \gamma_h) \cup (\pm\lambda/2, h)\big)$.*
- *(iii) If $h = h_{\rm cr}$ then $\partial\{u > 0\}$ is a curve of class $C^{1,1/2}$ locally in $[-\lambda/2, \lambda/2] \times [0, h_{\rm cr}) \setminus (\pm\lambda/2, \gamma_h)$.*

4.6 Uniqueness and symmetry of global minimizers

Theorem 4.12. *Given $b, m, \lambda > 0$, let $\Omega, J_h, \mathcal{K}_{\gamma_h}, u_0$ be defined as in* (1.9), (1.12), (1.13) *and* (1.14)*, respectively, where the map $h \mapsto \gamma_h$ is given as in Theorem 1.1. Let $\{h_n\}_n \subset (0, \infty)$ be a strictly increasing (respectively, decreasing) sequence converging to h, and for every $n \in \mathbb{N}$ let u_n be a global minimizer of J_{h_n} in $\mathcal{K}_{\gamma_{h_n}}$. Then there exists a global minimizer of J_h $u \in \mathcal{K}_{\gamma_h}$ such that $u_n \to u \in H^1_{\rm loc}(\Omega)$ and uniformly on compact subsets of Ω. Moreover, if $\{\ell_n\}_n \subset (0, \infty)$ is another strictly increasing (respectively, decreasing) sequence converging to h and $v_n \in \mathcal{K}_{\gamma_{\ell_n}}$ are global minimizers of J_{ℓ_n}, then $v_n \to u$ in $H^1_{\rm loc}$ and uniformly on compact subsets of Ω.*

We begin by proving a preliminary lemma.

Lemma 4.13. *Given $b, m, \lambda > 0$, let $\Omega, J_h, \mathcal{K}_{\gamma_h}, u_0$ be defined as in* (1.9), (1.12), (1.13) *and* (1.14)*, respectively, where the map $h \mapsto \gamma_h$ is given as in Theorem 1.1. Let $\{h_n\}_n \subset (0, \infty)$ be a strictly monotone sequence converging to h and let $w \in \mathcal{K}_{\gamma_h}$ be such that $J_h(w) < \infty$. Then there exists a sequence $\{w_n\}_n$ such that $w_n \in \mathcal{K}_{\gamma_{h_n}}$ for every $n \in \mathbb{N}$ and $J_{h_n}(w_n) \to J_h(w)$ as $n \to \infty$.*

Proof. Notice that if $h_n \nearrow h$ then $w \in \mathcal{K}_{\gamma_{h_n}}$ for every $n \in \mathbb{N}$ and the result follows by considering the constant sequence $w_n = w$. Hence we assume that $h_n \searrow h$, set

$$\sigma_n := \frac{\gamma_h}{\gamma_{h_n}},$$

and define the rescaled function $w_n(\boldsymbol{x}', x_N) := w(\boldsymbol{x}', \sigma_n x_N)$. We then notice that $w_n \in \mathcal{K}_{\gamma_{h_n}}$ and by a change of variables

$$\begin{aligned} \int_\Omega |\nabla w_n|^2 \, d\boldsymbol{x} &= \int_\Omega |\nabla_{\boldsymbol{x}'} w(\boldsymbol{x}', \sigma_n x_N)|^2 + (\sigma_n \partial_{x_N} w(\boldsymbol{x}', \sigma_n x_N))^2 \, d\boldsymbol{x} \\ &= \int_\Omega \big(|\nabla_{\boldsymbol{x}'} w(\boldsymbol{x}', z)|^2 + (\sigma_n \partial_{x_N} w(\boldsymbol{x}', z))^2\big) \sigma_n^{-1} \, d\boldsymbol{x}' dz \\ &\to \int_\Omega |\nabla w(\boldsymbol{x}', z)|^2 \, d\boldsymbol{x}' dz, \end{aligned}$$

where in the last step we have used the fact that $\sigma_n \searrow 1$. Similarly one can show that

$$\int_\Omega \chi_{\{w_n>0\}}(\boldsymbol{x})(h_n - x_N)_+^{2b}\, d\boldsymbol{x} \to \int_\Omega \chi_{\{w>0\}}(\boldsymbol{x})(h - x_N)_+^{2b}\, d\boldsymbol{x},$$

and the result follows. □

Proof of Theorem 4.12. Assume that $h_n \searrow h$. We divide the proof into several steps.
Step 1: We begin by showing that there exists a subsequence of $\{u_n\}_n$ that converges weakly in $H^1_{\mathrm{loc}}(\Omega)$ to a function u that is a global minimizer of J_h in the class $\mathcal{K}_{\gamma_h}$. To this end, let $v\colon \mathbb{R}^N_+ \to \mathbb{R}$ be defined by

$$v(\cdot, x_N) := \frac{c}{\gamma_{h_1}}(\gamma_{h_1} - x_N)_+$$

(see (3.5)). Then $v \in \mathcal{K}_{\gamma_{h_n}}$ for every $n \in \mathbb{N}$ and in particular

$$\int_\Omega |\nabla u_n|^2\, d\boldsymbol{x} \le J_{h_n}(u_n) \le J_{h_n}(v) \le J_{h_1}(v) < \infty.$$

Hence $\{\nabla u_n\}_n$ is bounded in $L^2(\Omega; \mathbb{R}^N)$. Moreover since $u_n - v = 0$ on $\mathcal{R}$, by Poincaré's inequality we obtain

$$\int_{\Omega_r} |u_n - v|^2\, d\boldsymbol{x} \le C(\Omega_r) \int_{\Omega_r} |\nabla u_n - \nabla v|^2\, d\boldsymbol{x},$$

where $\Omega_r := \Omega \cap \{x_N < r\}$, with $r > 0$. This shows that $\{u_n\}_n$ is bounded in $H^1(\Omega_r)$ and thus, up to the extraction of a subsequence, $u_n \rightharpoonup u^r$ in $H^1(\Omega_r)$. If we now let $s > r$, up to extraction of a further subsequence, we have that $u_n \rightharpoonup u^r$ in $H^1(\Omega_r)$ and $u_n \rightharpoonup u^s$ in $H^1(\Omega_s)$. By the uniqueness of the weak limit we conclude that

$$u^r(\boldsymbol{x}) = u^s(\boldsymbol{x}) \text{ for } \mathcal{L}^N\text{-a.e. } \boldsymbol{x} \in \Omega_r.$$

By letting $r \nearrow \infty$ and by a diagonal argument, up to the extraction of a consecutive subsequences, this defines a function u such that for some $\{n_k\}_k \subset \mathbb{N}$

$$\begin{aligned} \nabla u_{n_k} &\rightharpoonup \nabla u \text{ in } L^2(\Omega, \mathbb{R}^N), \\ u_{n_k} &\to u \text{ in } L^2_{\mathrm{loc}}(\Omega), \\ u_{n_k} &\to u \text{ pointwise almost everywhere in } \Omega, \\ u_{n_k} &\to u \text{ in } L^2_{\mathrm{loc}}(\Gamma_\gamma), \end{aligned} \tag{4.17}$$

where Γ_γ is defined as in (1.15). In particular, this shows that $u \in \mathcal{K}_{\gamma_h}$. Moreover, we claim that up to the extraction of a subsequence which we don't relabel, $\{\chi_{\{u_{n_k}>0\}}\}_k$ converges weakly star in $L^\infty(\Omega)$ to a function ξ such that

$$\xi(\boldsymbol{x}) \ge \chi_{\{u>0\}}(\boldsymbol{x}) \text{ for } \mathcal{L}^N\text{-a.e. } \boldsymbol{x} \in \Omega. \tag{4.18}$$

Indeed, arguing as in the proof of Theorem 2.4, we observe that for every D compactly contained in $\{u > 0\}$

$$0 = \int_D (\chi_{\{u_{n_k}>0\}} - 1)u_{n_k}\, d\boldsymbol{x} \to \int_D (\xi - 1)u\, d\boldsymbol{x}.$$

Since $u > 0$ in D, then necessarily $\xi = 1$ $\mathcal{L}^N$-a.e. in D and hence in $\{u > 0\}$.

To prove that u is a global minimizer of J_h in $\mathcal{K}_{\gamma_h}$, fix $r > 0$, let $w \in \mathcal{K}_{\gamma_h}$. If $J_h(w) = \infty$ there is nothing to show, hence we assume that $J_h(w) < \infty$ and consider $\{w_n\}_n$ as in Lemma 4.13. Then

$$\begin{aligned}\int_{\Omega_r} \left(|\nabla u|^2 + \chi_{\{u>0\}}(h - x_N)_+^{2b}\right) d\boldsymbol{x} &\leq \int_{\Omega} \left(|\nabla u|^2 + \xi(h - x_N)_+^{2b}\right) d\boldsymbol{x} \\ &\leq \liminf_{k\to\infty} J_{h_{n_k}}(u_{n_k}) \leq \lim_{k\to\infty} J_{h_{n_k}}(w_{n_k}) \\ &= J_h(w).\end{aligned} \tag{4.19}$$

Letting $r \nearrow \infty$, we conclude that $J_h(u) \leq J_h(w)$ for every $w \in \mathcal{K}_{\gamma_h}$.
Step 2: Taking $w = u$ in (4.19) yields

$$\int_{\Omega_r} \left(|\nabla u|^2 + \chi_{\{u>0\}}(h - x_N)_+^{2b}\right) d\boldsymbol{x} \leq \liminf_{k\to\infty} J_{h_{n_k}}(u_{n_k}) \leq \limsup_{k\to\infty} J_{h_{n_k}}(u_{n_k}) \leq J_h(u).$$

Letting $r \nearrow \infty$ we conclude that

$$J_h(u) = \lim_{k\to\infty} J_{h_{n_k}}(u_{n_k}).$$

On the other hand, by the lower semicontinuity of the L^2-norm and (4.18)

$$\int_{\Omega} |\nabla u|^2 \, d\boldsymbol{x} \leq \liminf_{k\to\infty} \int_{\Omega} |\nabla u_{n_k}|^2 \, d\boldsymbol{x},$$

and

$$\int_{\Omega} \chi_{\{u>0\}}(h - x_N)_+^{2b} \, d\boldsymbol{x} \leq \liminf_{k\to\infty} \int_{\Omega} \chi_{\{u_{n_k}>0\}}(h - x_N)_+^{2b} \, d\boldsymbol{x}.$$

Thus the previous two inequalities are necessarily equalities and therefore $u_{n_k} \to u$ in $H^1_{\mathrm{loc}}(\Omega)$. Moreover, by Theorem 4.4, $\{u_{n_k}\}_k$ is an increasing sequence of continuous functions with a continuous pointwise limit (see (4.17)). Hence, by Dini's convergence theorem, the convergence is uniform on compact subsets of Ω.
Step 3: Suppose by contradiction that the entire sequence does not converge to u in $H^1_{\mathrm{loc}}(\Omega)$. Then there are another subsequence $\{u_{n_j}\}_j$ and a minimizer w of J_h in $\mathcal{K}_{\gamma_h}$ such that $u_{n_j} \to w$ in $H^1_{\mathrm{loc}}(\Omega)$ and uniformly on compact subsets of Ω. By Theorem 4.4 we have that $u_{n_k} \leq w$ and $u_{n_j} \leq u$. Let x and r be such that $B_r(\boldsymbol{x})$ is compactly contained in the support of u. Then, passing to the limit as $k \to \infty$ and $j \to \infty$ in the previous inequalities we obtain $u = w$ in $B_r(\boldsymbol{x})$ and in particular $0 < u(\boldsymbol{x}) = w(\boldsymbol{x})$. Reasoning as in the proof of Theorem 4.4 we obtain that $u = w$ in Ω.

The same technique can be used to show the independence of the limiting minimizer on the sequences $\{h_n\}_n$ and $\{u_n\}_n$. This concludes the proof. □

Corollary 4.14. *Under the assumptions of Theorem 4.12, for every $h > 0$ there are two (possibly equal) global minimizers u_h^+, u_h^- of J_h in $\mathcal{K}_{\gamma_h}$ such that $u_h^- \leq u_h^+$ and if w is another global minimizer then $u_h^- \leq w \leq u_h^+$.*

Theorem 4.15. *Given $b, m, \lambda > 0$, let $\Omega, J_h, \mathcal{K}_{\gamma_h}, u_0$ be defined as in* (1.9), (1.12), (1.13) *and* (1.14)*, respectively, where the map $h \mapsto \gamma_h$ is given as in Theorem 1.1. Then there is a unique global minimizer of J_h in $\mathcal{K}_{\gamma_h}$ for all but countably many values of h.*

Proof. We define

$$\Lambda := \{h \in \mathbb{R}^+ : \text{ the minimization problem for } J_h \text{ in } \mathcal{K}_{\gamma_h} \text{ has at least two distinct solutions}\}.$$

We claim that

$$\Lambda = \bigcup_{m=1}^{\infty} \bigcup_{n=1}^{\infty} \left\{ h \in (m-1, m] : \sup\{|u_h^+(\boldsymbol{x}) - u_h^-(\boldsymbol{x})| : \boldsymbol{x} \in \mathcal{R} \times (0, \gamma_h/2)\} \geq \frac{1}{n} \right\}.$$

We recall that by Corollary 4.14, $h \in \Lambda$ if and only if $u_h^- \neq u_h^+$. To prove the claim it is enough to notice that if $u_h^- = u_h^+$ in $\mathcal{R} \times (0, \gamma_h/2)$ then the equality holds everywhere in Ω. Let

$$\Lambda_{m,n} := \left\{ h \in (m-1, m] : \sup\{|u_h^+(\boldsymbol{x}) - u_h^-(\boldsymbol{x})| : \boldsymbol{x} \in \mathcal{R} \times (0, \gamma_h/2)\} \geq \frac{1}{n} \right\};$$

we observe that it is enough to show that $\Lambda_{1,n}$ is countable for every $n \in \mathbb{N}$ and that $\Lambda_{m,n}$ is finite for every $m, n \in \mathbb{N}$ with $m \geq 2$.

Fix $m, n \in \mathbb{N}$ with $m \geq 2$ and assume by contradiction that $\Lambda_{m,n}$ has infinite cardinality. Then we can find a sequence $\{h_i\}_i \subset h_{m,n}$ and $h \in [m-1, m]$ such that $\{h_i\}_i$ converges strictly monotonically to h. By Theorem 4.12, there exists a function u such that $u_{h_i}^-, u_{h_i}^+ \to u$ in $H^1_{\mathrm{loc}}(\Omega)$ and uniformly in the compact set of $\mathcal{R} \times [0, \gamma_h/2]$. In turn, for i large enough we have that

$$|u_i^+(\boldsymbol{x}) - u_i^-(\boldsymbol{x})| \leq |u_i^+(\boldsymbol{x}) - u(\boldsymbol{x})| + |u(\boldsymbol{x}) - u_i^-(\boldsymbol{x})| < \frac{1}{n}$$

for all $\boldsymbol{x} \in \mathcal{R} \times (0, \gamma_h/2)$. We notice that this is in contradiction with the definition of $h_{m,n}$. On the other hand, if $m = 1$ we can write

$$\Lambda_{1,n} = \bigcup_{i=2}^{\infty} \left\{ h \in \left(\frac{1}{i}, 1\right] : \sup\{|u_h^+(\boldsymbol{x}) - u_h^-(\boldsymbol{x})| : \boldsymbol{x} \in \mathcal{R} \times (0, \gamma_h/2)\} \geq \frac{1}{n} \right\}.$$

We can then set

$$\Lambda_{1,n,i} := \left\{ h \in \left(\frac{1}{i}, 1\right] : \sup\{|u_h^+(\boldsymbol{x}) - u_h^-(\boldsymbol{x})| : \boldsymbol{x} \in \mathcal{R} \times (0, \gamma_h/2)\} \geq \frac{1}{n} \right\}$$

and repeat the same argument as above to prove that $\Lambda_{1,n,i}$ is finite for every $i \geq 2$. This concludes the proof. □

Theorem 4.16. *Let u_h^+, u_h^- be as in Corollary 4.14. Then u_h^+, u_h^- are symmetric with respect to the coordinate hyperplanes $\{x_i = 0\}$ and the maps*

$$x_i \in [0, \lambda/2] \mapsto u_h^+(\boldsymbol{x}), \quad x_i \in [0, \lambda/2] \mapsto u_h^-(\boldsymbol{x})$$

are decreasing for $i = 1, \ldots, N-1$.

Proof. **Step 1:** Let $h \in \mathbb{R}^+ \setminus \Lambda$ where Λ is defined as in Theorem 4.15 and let u_h be the unique global minimizer of J_h in $\mathcal{K}_{\gamma_h}$. For $i = 1, \ldots, N-1$, let w_i be the function obtain by applying to u_h an even reflection about the hyperplane $\{x_i = 0\}$, i.e.

$$w_i(\boldsymbol{x}) := \begin{cases} u_h(-x_1, x_2, \ldots, x_N) & \text{if } i = 1, \\ u_h(x_1, \ldots, -x_i, \ldots, x_N) & \text{if } i \geq 2. \end{cases}$$

Notice that $w_i \in \mathcal{K}_{\gamma_h}$ and $J_h(w) = J_h(u_h)$. Thus, since by assumption J_h has exactly one global minimizer in $\mathcal{K}_{\gamma_h}$, it must be the case that $u_h = w_i$ for every i. This proves that u_h is symmetric with respect to the hyperplanes $\{x_i = 0\}$ for $i = 1, \dots, N-1$, and in particular the support of u_h in Ω coincides with its Steiner symmetrizations with the respect to the same hyperplanes. Let u_h^* be the symmetric decreasing rearrangement of u_h with respect to the variables $x_1, \dots, x_{N-1}$ (see [Fri88, Definition 7.1]). Then $u_h^* \in \mathcal{K}_{\gamma_h}$ and by a repeated iteration of the Pólya-Szegö inequality (see [Fri88, Theorem 7.1]), together with Tonelli's theorem, we obtain

$$\int_\Omega |\nabla u_h^*|^2 \, d\boldsymbol{x} \le \int_\Omega |\nabla u_h|^2 \, d\boldsymbol{x}.$$

By definition of u_h^*, for every $x_N > 0$

$$\int_{\mathcal{R}} \chi_{\{u_h^* > 0\}}(\boldsymbol{x}', x_N) \, d\boldsymbol{x}' = \int_{\mathcal{R}} \chi_{\{u_h > 0\}}(\boldsymbol{x}', x_N) \, d\boldsymbol{x}'$$

and thus, again by Tonelli's theorem,

$$\begin{aligned} \int_\Omega \chi_{\{u_h^* > 0\}} (h - x_N)_+^{2b} \, d\boldsymbol{x} &= \int_0^h (h - x_N)_+^{2b} \int_{\mathcal{R}} \chi_{\{u_h^* > 0\}}(\boldsymbol{x}', x_n) \, d\boldsymbol{x}' \, dx_N \\ &= \int_0^h (h - x_N)_+^{2b} \int_{\mathcal{R}} \chi_{\{u_h > 0\}}(\boldsymbol{x}', x_n) \, d\boldsymbol{x}' \, dx_N \\ &= \int_\Omega \chi_{\{u_h > 0\}} (h - x_N)_+^{2b} \, d\boldsymbol{x}. \end{aligned}$$

Consequently, $J_h(u_h^*) \le J_h(u_h)$, which in turn implies that $u_h \equiv u_h^*$.
Step 2: If $h \in \Lambda$, consider a sequence $\{h_n\}_n \subset \mathbb{R} \setminus \Lambda$ such that $h_n \nearrow h$ and let u_n be the unique minimizer of J_{h_n} in $\mathcal{K}_{\gamma_{h_n}}$. Then, $u_{h_n} \equiv u_{h_n}^*$ and by Theorem 4.12 it follows that u_h^+ has all the desired properties. By considering a sequence $\{h_n\}_n \subset \mathbb{R} \setminus \Lambda$ such that $h_n \searrow h$, we obtain the analogous result for u_h^-. □

Remark 4.17. *Let $u \in \mathcal{K}_{\gamma_h}$ be a global minimizer of J_h and assume that u is symmetric with respect to the coordinate hyperplanes $\{x_i = 0\}$, $i = 1, \dots, N-1$.*

- (*i*) *Then $u \equiv u^*$, where u^* is the symmetric decreasing rearrangement of u with respect to the variables $x_1, \dots, x_{N-1}$. Indeed, by minimality we have that the equality holds in the Pólya-Szegö inequality and thus $x_i \mapsto u(\boldsymbol{x})$ is decreasing in $[0, \lambda/2]$ for every $i = 1, \dots, N-1$ by* [Kaw85, Theorem 2.13].

- (*ii*) *In view of (i), the free boundary of u can be described by the graph of a function*

$$x_1 = g(x_2, \dots, x_N), \qquad \text{in } (0, \lambda/2)^{N-2} \times \mathbb{R}_+ \times (0, \lambda/2).$$

Indeed, one has that $x_1 \mapsto u(\boldsymbol{x})$ is decreasing in $[0, \lambda/2]$ and hence we can write

$$\partial\{u > 0\} \cap (0, \lambda/2)^{N-1} \times \mathbb{R}_+ = \{\boldsymbol{x} : x_1 = g(x_2, \dots, x_N)\},$$

where

$$g(x_2, \dots, x_N) := \sup\{x_1 : u(\boldsymbol{x}) > 0\}. \tag{4.20}$$

Corollary 4.18. *Given $b, m, \lambda > 0$, let $\Omega, J_h, \mathcal{K}_{\gamma_h}, u_0$ be defined as in* (1.9), (1.12), (1.13) *and* (1.14), *respectively, where the map $h \mapsto \gamma_h$ is given as in Theorem 1.1. Then there exists a global minimizer of J_h in $\mathcal{K}_{\gamma_h}$ whose support in Ω is $N-1$-connected.*

Proof. Let $u \in \mathcal{K}_{\gamma_h}$ be any symmetric global minimizer of J_h. Observe that $\{\boldsymbol{x} \in \Omega : u(\boldsymbol{x}) > 0\}$ is open and connected (by minimality) and therefore path-connected. Let $\boldsymbol{z} \in \Omega$ be such that $u(\boldsymbol{z}) = 0$ and assume without loss of generality that $\boldsymbol{z} \in [0, \lambda/2] \times \mathbb{R}_+$. By Remark 4.17, $u \equiv u^*$; thus

$$u(\boldsymbol{x}) = 0 \quad \text{for every } \boldsymbol{x} \in \prod_{i=1}^{N-1} [z_i, \lambda/2] \times \{z_N\}$$

and the result readily follows. □

Having established the convergence of monotone sequences of minimizers in Theorem 4.12, we now investigate the type of convergence of the associated free boundaries. Our proof is based on standard techniques that are more commonly used in the study of the convergence to a blow-up limit.

Theorem 4.19. *Given $b, m, \lambda > 0$, let $\Omega, J_h, \mathcal{K}_{\gamma_h}, u_0$ be defined as in* (1.9), (1.12), (1.13) *and* (1.14)*, respectively, where the map $h \mapsto \gamma_h$ is given as in Theorem 1.1. Let $\{h_n\}_n \subset (0, \infty)$ be a monotone sequence that converges to $h > 0$. For every $n \in \mathbb{N}$, let u_n be a global minimizer of J_{h_n} in $\mathcal{K}_{\gamma_{h_n}}$ and consider u_h^+, u_h^- as in Corollary 4.14. The following statements hold:*

- *(i) if $h_n \searrow h$ then $\partial\{u_n > 0\} \to \partial\{u_h^- > 0\}$ in Hausdorff distance locally in Ω;*
- *(ii) if $h_n \nearrow h$ then $\partial\{u_n > 0\} \to \partial\{u_h^+ > 0\}$ in Hausdorff distance locally in $\mathcal{R} \times (0, h)$;*
- *(iii) if $h_n \searrow h$ (respectively $h_n \nearrow h$) then $\chi_{\{u_n>0\}} \to \chi_{\{u_h^->0\}}$ (respectively to $\chi_{\{u_h^+>0\}}$) in $L^1_{\mathrm{loc}}(\mathcal{R}\times (0, h))$.*

Proof. (*i*) Let $h_n \searrow h > 0$ and consider a ball $B_r(\boldsymbol{x}) \subset \Omega$ such that $B_r(\boldsymbol{x}) \cap \partial\{u_h^- > 0\} = \emptyset$. Then either $u_h^- \equiv 0$ in $B_r(\boldsymbol{x})$ or $u_h^- > 0$ in $B_r(\boldsymbol{x})$. By Theorem 4.4 we have that for every $n \in \mathbb{N}$ $\{u_n > 0\} \subset \{u_h^- > 0\}$; thus if $u_h^- \equiv 0$ in $B_r(\boldsymbol{x})$ so does u_n for every $n \in \mathbb{N}$. In particular, this implies that

$$B_{r/2}(\boldsymbol{x}) \cap \partial\{u_n > 0\} = \emptyset. \tag{4.21}$$

On the other hand, if $u_h^- > 0$ in $B_r(\boldsymbol{x})$, since by Theorem 4.12 we have that $\{u_n\}_n$ converges uniformly to u_h^- in $B_{r/2}(\boldsymbol{x})$, then for n sufficiently large

$$u_n(\boldsymbol{x}) \geq \frac{1}{2} \min\left\{u_h^-(\boldsymbol{y}) : \boldsymbol{y} \in \overline{B_{r/2}(\boldsymbol{x})}\right\} > 0$$

for every $\boldsymbol{x} \in B_{r/2}(\boldsymbol{x})$ and hence (4.21) is satisfied.

Conversely, if $B_r(\boldsymbol{x}) \cap \partial\{u_n > 0\} = \emptyset$ then for all n sufficiently large we have that either $u_n > 0$ in $B_r(\boldsymbol{x})$ or $u_n = 0$ in $B_r(\boldsymbol{x})$. Assume first that $u_m > 0$ in $B_r(\boldsymbol{x})$ for some $m \in \mathbb{N}$. Then, by Theorem 4.4, $u_n > 0$ in $B_r(\boldsymbol{x})$ for every $n \geq m$ and therefore u_h^- is harmonic in $B_{r/2}(\boldsymbol{x})$ being the uniform limit of harmonic functions. Consequently, either $u_h^- > 0$ in $B_{r/2}(\boldsymbol{x})$ or $u_h^- = 0$ in $B_{r/2}(\boldsymbol{x})$. In both cases

$$B_{r/2}(\boldsymbol{x}) \cap \partial\{u_h^- > 0\} = \emptyset. \tag{4.22}$$

On the other hand, if $u_n \equiv 0$ in $B_{r/2}(\boldsymbol{x})$ for every $n \in \mathbb{N}$ then also $u_h^- \equiv 0$ in $B_{r/2}(\boldsymbol{x})$. This shows that (4.22) is also satisfied in case. By a standard compactness argument one can show that $\partial\{u_n > 0\} \to \partial\{u_h^- > 0\}$ in Hausdorff distance locally in Ω.

(*ii*) Let $h_n \nearrow h$ and consider a ball $B_r(\boldsymbol{x}) \subset \mathcal{R} \times (0, h)$ such that $B_r(\boldsymbol{x}) \cap \partial\{u_h^+ > 0\} = \emptyset$. As before, either $u_h^+ \equiv 0$ in $B_r(\boldsymbol{x})$ or $u_h^+ > 0$ in $B_r(\boldsymbol{x})$. If $u_h^+ > 0$ in $B_r(\boldsymbol{x})$, by Theorem 4.4, $u_n > 0$ in $B_r(\boldsymbol{x})$ for every $n \in \mathbb{N}$. Therefore (4.22) holds. On the other hand, if $u^+ = 0$, for every $\delta > 0$

we can find m such that $u_n \le \delta$ in $B_{3r/4}(\boldsymbol{x})$ for every $n \ge m$. Hence, for $\delta = \delta(r)$ sufficiently small and $n \ge m$,

$$\frac{1}{\frac{3}{4}r} \fint_{B_{3r/4}(\boldsymbol{x})} u_n \, d\mathcal{H}^{N-1} \le \frac{4\delta}{3r} \le C(2/3) \left(h - x_N - \frac{2}{3}\frac{3}{4}r \right)^b .$$

Then we can conclude from Proposition 4.2 that $u_n \equiv 0$ in $B_{r/2}(\boldsymbol{x})$, proving that (4.21) holds. The rest of the proof follows as in the previous case, therefore we omit the details.

(*iii*) Let $h_n \searrow h > 0$ and let K be a compact subset of $\mathcal{R} \times (0, h)$. If $\operatorname{dist}(K, \partial\{u_h^- > 0\}) > 0$ then either $u_h^- \equiv 0$ in K or $u_h^- > 0$ in K. Reasoning as the proof of (*i*), we can conclude that either $u_n \equiv 0$ in K for every n or $u_n > 0$ in K for n sufficiently large; hence in this case there is nothing to prove. Therefore, we can assume that $K \cap \partial\{u_h^- > 0\} \ne \emptyset$. By (*i*), for every $0 < \eta < d_K := \operatorname{dist}(K, \partial(\mathcal{R} \times (0, h)))$ we can find $m = m(\eta, K)$ such that if $n \ge m$ then

$$\partial\{u_n > 0\} \cap K \subset \mathcal{N}_\eta(\partial\{u_h^- > 0\}),$$

where for any set $A \subset \Omega$, $\mathcal{N}_\eta(A)$ represents the tubular neighborhood of A of width η, i.e.

$$\mathcal{N}_\eta(A) := \{\boldsymbol{x} \in \Omega : \operatorname{dist}(\boldsymbol{x}, A) < \eta\}.$$

Observe that by Proposition 4.2, for every ball $B_r(\boldsymbol{x}) \subset K$ with center on $\partial\{u_h^- > 0\}$

$$\frac{1}{r} \fint_{\partial B_r(\boldsymbol{x})} u_h^- \, d\mathcal{H}^{N-1} \ge C(1/2)(h - x_N - r/2)_+^b > C(1/2)(d_K)^b.$$

Similarly, by [AC81, Lemma 3.2] (see also [AL12, Theorem 3.1]), there is a constant $C_{\max}$ such that

$$\frac{1}{r} \fint_{\partial B_r(\boldsymbol{x})} u_h^- \, d\mathcal{H}^{N-1} \le C_{\max}(h - x_N + r)^b < C_{\max}(2h)^b.$$

Hence we are in a position to apply [AC81, Representation theorem 4.5] to conclude that

$$\mathcal{H}^{N-1}(\partial\{u_h^- > 0\} \cap K) < \infty.$$

Since $\chi_{\{u_n > 0\}} \to \chi_{\{u_h^- > 0\}}$ in $L^1(K \setminus \mathcal{N}_\eta(\partial\{u_h^- > 0\}))$ and since

$$\mathcal{L}^N(\mathcal{N}_\eta(\partial\{u_h^- > 0\}) \cap K) \le 2\eta \mathcal{H}^{N-1}(\partial\{u_h^- > 0\} \cap K),$$

letting $\eta \to 0^+$ in the previous estimate concludes the proof.

The proof of (*iii*) for a monotonically increasing sequence $h_n \nearrow h$ is almost identical, thus we omit the details. □

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