

On Weingarten-Volterra defects

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October 14, 2017

Abstract

The kinematic theory of Weingarten-Volterra line defects is revisited, both at small and finite deformations. Existing results are clarified and corrected as needed, and new results are obtained. The primary focus is to understand the relationship between the disclination strength and Burgers vector of deformations containing a Weingarten-Volterra defect corresponding to different cut-surfaces.

1 Introduction

The question of characterizing the discontinuity of a deformation whose strain is locally compatible with a prescribed field on simple types of non-simply-connected domains is the main concern of this paper. Such questions originated in the works of Weingarten [Wei01, as translated by [Delb]] and Volterra [Vol07, as translated by [Dela]] in the setting of small deformations, and in those of Zubov [Zub97] and Casey [Cas04] in the context of finite deformations; related work is that of Yavari [Yav13], considering compatibility conditions, i.e. conditions for continuous deformations for a prescribed strain field, in general non-simply connected domains. Both finite and small deformations (i.e. linearized kinematics) are considered. A motivation for this paper is the recent emphasis on developing and understanding models of defects in materials.

We revisit classical results, provide alternative proofs, and correct some statements in the existing literature related to uniqueness of the disclination strength and Burgers vector of defects. We also provide new results related to the dependence of these objects on cut-surfaces.

A natural improvement of the work presented herein is to deduce the corresponding results for arbitrary non-simply connected domains, and make precise connections with the results of the metric differential geometric treatment of Kupferman, Moshe, and Solomon [KMS15]. Such a connection is desirable, as the differential geometric treatment does not involve notions of deformations of bodies and their discontinuities, while the continuum mechanics point of view, starting from Weingarten [Wei01], Volterra [Vol07], and Zubov [Zub97], is deeply rooted in the kinematics of deformation of 3-d bodies.

After this brief introduction, Section 2 provides the setting of the main questions asked in the paper. Section 3 considers the questions in the setting of small deformations. Section 4 considers the same questions for kinematics without approximation. The paper contains an Appendix collecting classical results on compatibility on simply-connected domains.

A somewhat mathematical style of presentation is adopted simply for the purpose of a better organization of definitions, assumptions, results, and remarks.

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2 The setting and the question of Weingarten's theorem

Definition 1 *By a region Ω we will mean a pathwise-connected open set in ambient 3-d Euclidean point space \mathcal{E}_3 , together with some or all of its boundary points [Kel54]. In contrast to standard continuum mechanics, we will need to consider non-compact bounded regions, generated by removing surfaces from compact regions.*

Definition 2 *By a deformation $\mathbf{y} : \Omega \rightarrow V_3$ we mean a C^1 mapping \mathbf{y} of Ω to the translation space V_3 of \mathcal{E}_3 , with pointwise positive determinant of its gradient, i.e. $\det(\text{grad } \mathbf{y}) > 0$. The displacement is defined as $\mathbf{u}(\mathbf{x}) := \mathbf{y}(\mathbf{x}) - \mathbf{x}$ for all $\mathbf{x} \in \Omega$.*

Given a *simply connected* region Ω and a prescribed twice continuously-differentiable, positive-definite, symmetric tensor field \mathbf{C} (a symmetric second order tensor field $\boldsymbol{\varepsilon}$) on it, it is a classical result of continuum mechanics, e.g. [Shi73], that a thrice-differentiable deformation (displacement) field can be constructed on it whose Right Cauchy-Green deformation (strain) tensor is the prescribed field \mathbf{C} ($\boldsymbol{\varepsilon}$), if the Riemann-Christoffel curvature tensor formed from \mathbf{C} (the St.-Venant tensor formed from $\boldsymbol{\varepsilon}$) vanishes, i.e.

$$\begin{aligned} \Gamma_{\alpha\beta}^\gamma &:= \frac{1}{2}(C^{-1})^{\gamma\mu} [C_{\alpha\mu,\beta} + C_{\beta\mu,\alpha} - C_{\alpha\beta,\mu}] \\ R_{\alpha\beta\rho}^\mu &:= \Gamma_{\alpha\beta,\rho}^\mu - \Gamma_{\alpha\rho,\beta}^\mu + \Gamma_{\gamma\rho}^\mu \Gamma_{\alpha\beta}^\gamma + \Gamma_{\gamma\beta}^\mu \Gamma_{\alpha\rho}^\gamma = 0 \\ \varepsilon_{il,km} - \varepsilon_{kl,im} - \varepsilon_{im,kl} + \varepsilon_{km,il} &= 0. \end{aligned} \tag{1}$$

The main purpose of Weingarten's theorem may be stated as understanding the obstruction to the above-mentioned construction of the deformation (displacement) field when the region is no longer simply-connected.

Definition 3 *When \mathbf{C} ($\boldsymbol{\varepsilon}$) satisfies conditions (1), we refer to it as locally compatible.*

Definition 4 *Given a \mathbf{C} ($\boldsymbol{\varepsilon}$) field on a region, any deformation \mathbf{y} (displacement \mathbf{u}) of the region that satisfies $(\text{grad } \mathbf{y})^T \text{grad } \mathbf{y} = \mathbf{C}$ ($(\text{grad } \mathbf{u})_{\text{sym}} = \boldsymbol{\varepsilon}$) is said to be (strain) compatible with \mathbf{C} ($\boldsymbol{\varepsilon}$) on the region.*

Definition 5 *Two deformations \mathbf{y}_1 and \mathbf{y}_2 of a region Ω are related by a rigid deformation if there exists a (proper) orthogonal tensor \mathbf{R} , constant on Ω , such that $\mathbf{y}_1(\mathbf{x}) - \mathbf{y}_1(\mathbf{z}) = \mathbf{R}[\mathbf{y}_2(\mathbf{x}) - \mathbf{y}_2(\mathbf{z})]$ for all $\mathbf{x}, \mathbf{z} \in \Omega$.*

We note that assuming \mathbf{R} to be a 'small' rotation in Definition 5 so that $\mathbf{R} \approx \mathbf{I} + \mathbf{W}$, with \mathbf{W} skew, results in the statement $\mathbf{u}_1(\mathbf{x}) - \mathbf{u}_1(\mathbf{z}) = \mathbf{u}_2(\mathbf{x}) - \mathbf{u}_2(\mathbf{z}) + \mathbf{W}[\mathbf{x} - \mathbf{z}] + \mathbf{W}[\mathbf{u}_2(\mathbf{x}) - \mathbf{u}_2(\mathbf{z})]$ for all $\mathbf{x}, \mathbf{z} \in \Omega$. In what follows, we will make the further assumption that $|\mathbf{W}[\mathbf{u}_2(\mathbf{x}) - \mathbf{u}_2(\mathbf{z})]|$ is small and define

Definition 6 *Two displacements \mathbf{u}_1 and \mathbf{u}_2 of a region Ω are related by an infinitesimally rigid deformation if there exists a skew symmetric tensor \mathbf{W} , constant on Ω , such that $\mathbf{u}_1(\mathbf{x}) - \mathbf{u}_1(\mathbf{z}) = \mathbf{u}_2(\mathbf{x}) - \mathbf{u}_2(\mathbf{z}) + \mathbf{W}[\mathbf{x} - \mathbf{z}]$ for all $\mathbf{x}, \mathbf{z} \in \Omega$.*

Remark 2.1 *Given two deformations related to each other by a (infinitesimal) rigid deformation, it is often analytically convenient to view the rigidity statement in the form $\mathbf{y}_1(\mathbf{x}) = \mathbf{R}\mathbf{y}_2(\mathbf{x}) + \mathbf{t}$ ($\mathbf{u}_1(\mathbf{x}) = \mathbf{u}_2(\mathbf{x}) + \mathbf{W}\mathbf{x} + \tilde{\mathbf{t}}$), where $\mathbf{t} := \mathbf{y}_1(\mathbf{z}) - \mathbf{R}\mathbf{y}_2(\mathbf{z})$ ($\tilde{\mathbf{t}} := \mathbf{u}_1(\mathbf{z}) - \mathbf{u}_2(\mathbf{z}) - \mathbf{W}\mathbf{z}$) for any $\mathbf{z} \in \Omega$ is a constant vector on Ω . However, it should be kept in mind that the constant \mathbf{t} ($\tilde{\mathbf{t}}$) so*

defined is not independent of the choice of the origin chosen to define position vectors when $\mathbf{R} \neq \mathbf{I}$ ($\mathbf{W} \neq \mathbf{0}$). Therefore it is not a constant in the physical sense, while the fundamental definition of a (infinitesimally) rigid deformation is a physical statement independent of the choice of an origin. This seemingly trivial point is of some importance in this paper (see Remark 3.3 and Sec. 4.3).

It is a classical result, see e.g. [Shi73], that if two continuous deformations (displacements) on the same region have identical, continuous right Cauchy-Green (strain) fields, then one is at most a rigid (infinitesimally rigid) deformation of the other. For the sake of completeness, we provide the main elements of the proofs of these classical results in Appendix A.

Definition 7 *By a (cut)-surface of Ω we mean a 2-d set of points in Ω lending itself to a smooth parametrization from an open square of \mathbb{R}^2 that provides a natural sense of orientation of the surface (through the choice of normal constructed from the parametrization). All surfaces will be assumed to be non self-intersecting.*

In what follows, we will consider two elementary types of non-simply connected regions. One will be a 3-dimensional body with a through-hole such that there are curves in the body that cannot be continuously shrunk to a point while staying within the body. Removing a *cut-surface* from the body connecting the inner hole to the outer boundary can render the body simply-connected, with the topology of a ball. The other type of non simply connected body is one with a toroidal hole in it. Removing a cut-surface from the exterior boundary of the body to the boundary of the hole again renders the body simply connected, the resulting body having the topology of a ball. Another alternative is to remove a cut-surface in the body that changes the toroidal hole into an opening with the topology of a connected cavity. See Fig. 1 for illustration of these concepts, also see [Nab87, p.16].

Assumption 1 *We consider any non-simply connected region Ω that is reduced to a simply connected region $\Omega_\tau := \Omega \setminus \tau$ by the removal of a single cut-surface, τ .*

Assumption 2 *For $\mathbf{x} \in \tau$ and a function ${}^\tau f$ defined on Ω_τ , we will assume that unique limits $\lim_{\mathbf{x}^\pm \rightarrow \mathbf{x}} {}^\tau f(\mathbf{x}^\pm)$, with $\mathbf{x} \in \tau$ and \mathbf{x}^\pm approaching \mathbf{x} from either side of the surface τ , exist; we will denote these limiting values as ${}^\tau f^\pm(\mathbf{x})$ for each $\mathbf{x} \in \tau$. We will also use the notation*

$$\llbracket {}^\tau f(\mathbf{x}) \rrbracket := {}^\tau f^+(\mathbf{x}) - {}^\tau f^-(\mathbf{x}).$$

We think of a sequence of points approaching $\mathbf{x} \in \tau$ from a ‘side’ in the intuitively natural way. If $\mathbf{n}(\mathbf{x})$ is the unit normal to τ at \mathbf{x} (arbitrarily choosing one alternative), we think of the sequence $\{\mathbf{x}_i^\pm\}$ as approaching \mathbf{x} from the \pm side if $(\mathbf{x}_i^\pm - \mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) \gtrless 0$ for all i .

The main question addressed by **Weingarten’s** theorem may now be stated as follows:

Given a non-simply connected region Ω as described above and a twice continuously differentiable, positive-definite, symmetric, locally compatible tensor field \mathbf{C} (a symmetric second order tensor field $\boldsymbol{\varepsilon}$) on it, characterize the ‘jump’ $\llbracket {}^\tau \mathbf{y}(\mathbf{x}) \rrbracket$ ($\llbracket {}^\tau \mathbf{u}(\mathbf{x}) \rrbracket$), of any deformation ${}^\tau \mathbf{y}$ (displacement ${}^\tau \mathbf{u}$) field compatible with \mathbf{C} ($\boldsymbol{\varepsilon}$) that can be constructed on Ω_τ . In particular, we will be interested in understanding to what extent the characterization of this jump is independent of points \mathbf{x} on a fixed cut-surface τ and to what extent the jump functions across different cut-surfaces may be related.

Definition 8 *Given a cut-surface τ of Ω and a $\mathbf{C}(\boldsymbol{\varepsilon})$ field on Ω , we refer to the latter’s restriction to Ω_τ as $\mathbf{C}_\tau(\boldsymbol{\varepsilon}_\tau)$.*

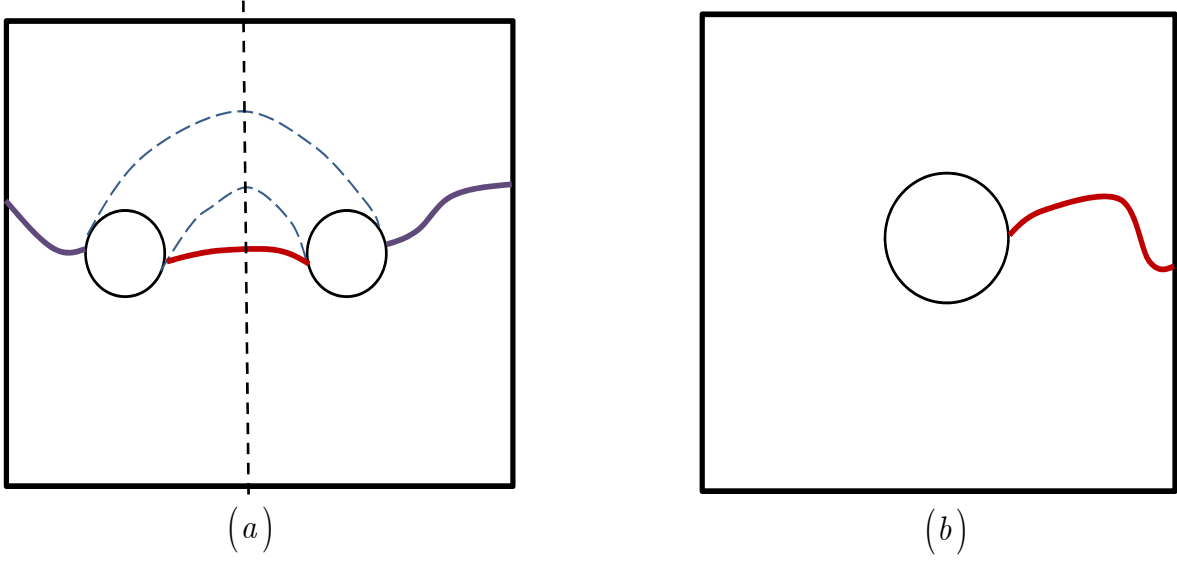


Figure 1: Cross-section of a body with (a) a toroidal hole and (b) a through hole. In (a), the whole body may be thought of as generated from a rotation of the cross-section by π radians about the dashed line. The red curve is the trace of one cut-surface and the purple one, that of another; each cut renders the body simply connected. The red cut surface produces a topological ball with a cavity. The purple cut produces a topological ball. Extracting the red cut surface in (b) from the cylinder with a through hole produces a simply connected topological ball.

Remark 2.2 We note that the construction of a family of thrice continuously differentiable deformations (displacements) with $(\text{grad}^\tau \mathbf{y})^T \text{grad}^\tau \mathbf{y} = \mathbf{C}_\tau$ ($(\text{grad}^\tau \mathbf{u})_{\text{sym}} = \boldsymbol{\varepsilon}_\tau$) on any Ω_τ is guaranteed; however, because of the non-simply connectedness of Ω , the limits of such a deformation (displacement) at points of the cut-surface τ from either side of τ may not match. The goal of the Weingarten theorem is to characterize the discontinuity of the deformation (displacement), when viewed as a function on Ω .

Definition 9 We say that a deformation ${}^\tau \mathbf{y}$ (displacement ${}^\tau \mathbf{u}$) on Ω_τ contains a **Weingarten-Volterra defect** if it displays a non-vanishing jump $[[{}^\tau \mathbf{y}]]$ ($[[{}^\tau \mathbf{u}]]$).

Remark 2.3 The role of any cut-surface τ in our considerations is to produce a simply-connected region from Ω . If Ω was simply connected to begin with, one could consider removing a cut-surface τ from it, but only of the type that would keep Ω_τ simply-connected (this has physical import in keeping the dislocation line within the body). It is clear from classical arguments (Appendix A.4) that any two differentiable deformations of Ω_τ with identical Right Cauchy Green fields are related to each other by at most a rigid deformation. Given a locally compatible \mathbf{C} field on Ω , any deformation, say ${}^\tau \mathbf{y}$, compatible with \mathbf{C}_τ , would necessarily differ from the restriction of any deformation \mathbf{y} compatible with \mathbf{C} on Ω to Ω_τ only by a rigid deformation of $\mathbf{y}(\Omega_\tau)$, i.e. ${}^\tau \mathbf{y}(\mathbf{x}^\pm) = \mathbf{R}\mathbf{y}(\mathbf{x}^\pm) + \mathbf{t}$ for $\mathbf{x}^\pm \in \Omega_\tau$, for some orthogonal tensor \mathbf{R} and vector \mathbf{t} , both constant on Ω_τ . Passing to the limit $\mathbf{x}^\pm \rightarrow \mathbf{x} \in \tau$ with sequences \mathbf{x}^+ and \mathbf{x}^- approaching \mathbf{x} from either side of the cut, we have that $[[{}^\tau \mathbf{y}]] = \mathbf{0}$. Thus, it is impossible for a deformation of a simply connected region Ω_τ induced from a simply connected Ω to display a Weingarten-Volterra defect if it is compatible, on Ω_τ , with a \mathbf{C}_τ induced from a locally compatible \mathbf{C} field on Ω . The ‘counterexample’ that Casey provides [Cas04, Example 1, p. 485] for this result appears to be related to the fact that Ω_τ in his example (induced

from a simply connected Ω) is not a path-connected region (allowed by his hypothesis adapted from [Gur73, p.42]), and therefore it would not be possible to conclude that \mathbf{y} and ${}^\tau\mathbf{y}$ in our construction are necessarily related by a single rigid deformation for such situations (see Appendix A.4).

Remark 2.4 *The essential content of the argument in Remark 2.3 was important to Volterra ([Vol07, as translated in [Dela]]; Volterra worked with the small deformation theory) in making the case that one could have a discontinuous elastic deformation only if the body was not simply connected or if its strain field contained a singularity in a simply connected domain (or both). We note that the motivation for the cut-surface in Volterra's arguments was to improve the topological situation by making a non-simply connected region into a simply-connected one (and not worse by taking a simply connected domain and making it disconnected by a through-cut-surface).*

3 Small deformation

3.1 Weingarten's theorem for small deformation

We give a proof of Weingarten's theorem that involves different arguments from those presented in Love and Nabarro [Lov44, Nab87].

Given a simply connected Ω_τ (induced from the non-simply connected Ω with a cylindrical/toroidal hole) and a locally compatible $\boldsymbol{\varepsilon}$ field on it, consider any displacement field ${}^\tau\mathbf{u}$ compatible with $\boldsymbol{\varepsilon}$ on Ω_τ . Appendix A.1 shows that there exists a family of such displacement fields, each member of which satisfies

$$\begin{aligned} \frac{1}{2}({}^\tau u_{i,j} + {}^\tau u_{j,i}) &= \varepsilon_{ij} \\ \frac{1}{2}({}^\tau u_{i,j} - {}^\tau u_{j,i}) &=: {}^\tau\omega_{ij} \\ {}^\tau\omega_{ik,l} &= \varepsilon_{il,k} - \varepsilon_{kl,i} =: E_{ikl} \end{aligned} \tag{2}$$

on Ω_τ . Strictly speaking, the $\boldsymbol{\varepsilon}$ appearing in (2) is $\boldsymbol{\varepsilon}_\tau$.

Remark 3.1 *Any two such displacement fields on Ω_τ compatible with the same strain field necessarily differ by an infinitesimally rigid deformation and, therefore, it follows from Remark 2.1 that their jump fields across τ are necessarily equal. It also follows that the jump in their infinitesimal rotation field across τ is equal.*

Definition 10 *For the purpose of this paper, we think of a curve as a 1-d set of points in Ω lending itself to a smooth parametrization from some interval in \mathbb{R} which provides a natural sense of direction on the curve. All curves will be assumed to be non self-intersecting.*

Consider a curve $\mathbf{c}_{(\mathbf{x},\mathbf{z})}$ on the surface τ joining \mathbf{x} and \mathbf{z} . Corresponding to \mathbf{c} , consider two other curves $\mathbf{c}_{(\mathbf{x}^+, \mathbf{z}^+)}$ and $\mathbf{c}_{(\mathbf{x}^-, \mathbf{z}^-)}$ in Ω_τ on either side of, and close to, τ . The curves run from \mathbf{x}^\pm to \mathbf{z}^\pm (in obvious notation), and in the following we will be thinking of limits of line integrals along \mathbf{c}^\pm as \mathbf{c}^\pm tend to \mathbf{c} .

We may write

$${}^\tau u_i(\mathbf{z}^\pm) - {}^\tau u_i(\mathbf{x}^\pm) = \int_{\mathbf{c}_{(\mathbf{x}^\pm, \mathbf{z}^\pm)}^\pm} (\varepsilon_{ij}(\mathbf{x}'^\pm) + {}^\tau\omega_{ij}(\mathbf{x}'^\pm)) dx_j'^\pm,$$

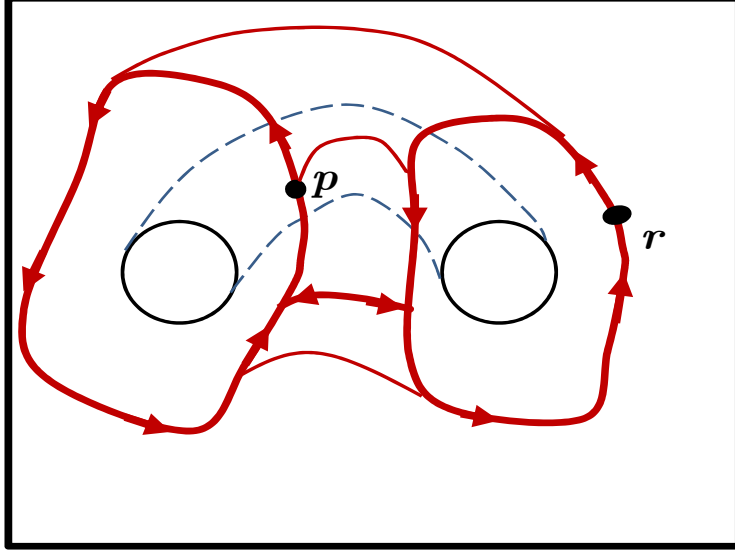


Figure 2: A surface with two non-contractible loops (through \mathbf{p} and \mathbf{r}) as edges.

and taking the limit as $\mathbf{c}_{(\mathbf{x}^\pm, \mathbf{z}^\pm)} \rightarrow \mathbf{c}_{(\mathbf{x}, \mathbf{z})}$ and then subtracting the \pm equations we obtain

$$\llbracket^\tau u_i(\mathbf{z}) \rrbracket = \llbracket^\tau u_i(\mathbf{x}) \rrbracket + \int_{\mathbf{c}_{(\mathbf{x}, \mathbf{z})}} \llbracket^\tau \omega_{ij}(\mathbf{x}') \rrbracket dx'_j. \quad (3)$$

Now, for each $\mathbf{x}' \in \tau$, $\llbracket^\tau \omega_{ij}(\mathbf{x}') \rrbracket$ can be expressed, using (2), as

$$\begin{aligned} \llbracket^\tau \omega_{ij}(\mathbf{x}') \rrbracket &= \lim_{\mathbf{x}'^\pm \rightarrow \mathbf{x}'} \int_{\mathbf{c}_{(\mathbf{x}'^-, \mathbf{x}'^+)}} (\tau \omega_{ij,m}(\mathbf{s})) ds_m \\ &= \lim_{\mathbf{x}'^\pm \rightarrow \mathbf{x}'} \int_{\mathbf{c}_{(\mathbf{x}'^-, \mathbf{x}'^+)}} [\varepsilon_{im,j}(\mathbf{s}) - \varepsilon_{jm,i}(\mathbf{s})] ds_m = \int_{\mathbf{l}_{\mathbf{x}'}} [\varepsilon_{im,j}(\mathbf{s}) - \varepsilon_{jm,i}(\mathbf{s})] ds_m \end{aligned} \quad (4)$$

where $\mathbf{c}_{(\mathbf{x}'^-, \mathbf{x}'^+)}$ is a curve from \mathbf{x}'^- to \mathbf{x}'^+ contained in Ω_τ , and $\mathbf{l}_{\mathbf{x}'}$ is a closed loop in Ω passing through \mathbf{x}' that pierces τ exactly once (i.e. the loop $\mathbf{l}_{\mathbf{x}'}$ goes around the hole in Ω).

Definition 11 For our purposes, a non-contractible loop through \mathbf{x} in a non-simply connected region Ω (as in Definition 1) is a closed curve in Ω that cannot be contracted to a point without exiting Ω ; and, it intersects only once some cut-surface containing \mathbf{x} .

Thus, the loop $\mathbf{l}_{\mathbf{x}'}$ is a non-contractible loop.

Remark 3.2 The last integral in (4) depends only on the loop $\mathbf{l}_{\mathbf{x}'}$ and the locally compatible strain field ε prescribed on Ω .

Consider two mutually non-intersecting, non-contractible loops passing through points $\mathbf{p}, \mathbf{r} \in \Omega$ and connect them by a surface in Ω that has them as edges (see Figure 2). A closed curve can always be constructed on this surface that includes these loops as segments and two overlapping parts, as shown in Fig. 2. Applying Stokes theorem on this curve with integrand that of last integral in (4), and noting the local compatibility of ε , we deduce that the loop integrals on $\mathbf{l}_{\mathbf{p}}$ and

l_r are equal, for the two points \mathbf{p}, \mathbf{r} chosen arbitrarily with the constraint of non-intersection of the loops as mentioned. Let us denote this important fact as

$$\int_{l_x} E_{ijm}(\mathbf{s}) ds_m = \int_{l_x} [\varepsilon_{im,j}(\mathbf{s}) - \varepsilon_{jm,i}(\mathbf{s})] ds_m =: \omega_{ij} \quad \forall \mathbf{x} \in \Omega, \quad (5)$$

where ω is a constant skew symmetric tensor on Ω and l_x is any non-contractible loop through \mathbf{x} . Thus,

Lemma 3.1 $[[^\tau \omega_{ij}]]$ in (3) is constant on τ , and takes the same value on all cut-surfaces τ of Ω .

Furthermore, the displacement jump across τ may be expressed as

$$[[^\tau \mathbf{u}(\mathbf{z})]] = [[^\tau \mathbf{u}(\mathbf{x})]] + \omega[\mathbf{z} - \mathbf{x}] \quad \forall \mathbf{z}, \mathbf{x} \in \tau. \quad (6)$$

Thus, thinking of \mathbf{x} as fixed and \mathbf{z} sweeping out τ and ${}^\tau \mathbf{u}^+$ and ${}^\tau \mathbf{u}^-$ as two displacements of the surface τ , (6) suggests that these two displacement fields of τ are related by an infinitesimally rigid deformation (cf. Definition 6). The statement (6) is **Weingarten's** result [Wei01, as translated in [Delb]] (see Nabarro [Nab87, pp 17-18] for another proof based on that in Love [Lov44], but in more convenient notation).

3.2 Volterra's "characteristic of the distortion"

Given two cut-surfaces τ' and τ in Ω and two points $\mathbf{x} \in \tau'$ and $\mathbf{y} \in \tau$ that can be linked by a curve in Ω which intersects τ' and τ only at \mathbf{x} and \mathbf{y} , respectively, we would now like to understand the relationship between the displacement jumps ${}^{\tau'} \mathbf{u}(\mathbf{x})$ and ${}^\tau \mathbf{u}(\mathbf{y})$. This question is motivated by statements in the classical literature starting from Volterra followed by Nabarro that the 'infinitesimally rigid deformation' characterizing a Volterra dislocation is independent of the cut surfaces τ and τ' . Before getting into the details, we first consider the statements from Volterra and Nabarro:

Volterra [Vol07, Chapter II, as translated in [Dela]] - "...if the multiply-connected elastic body is taken in its natural state then in order to bring it into a state of tension, one can perform the inverse operation - i.e. the sectioning that will render it simply connected - and then displace the two parts of each cut with respect to each other in such a manner that the relative displacements of the various pairs of pieces (which adhere to each other and which the cut has separated) are the resultants of translations and equal rotations; finally, re-establish the connectivity and the continuity along each cut, by subtracting or adding the necessary matter and welding the parts together. The set of these operations that relate to each cut may be called a *distortion* of the body and the six constants may be called the *characteristic of the distortion*.

...One may say, in addition, that the six characteristics of each distortion are not elements that depend upon the location where the cut has been executed.

Indeed, that same process that served to establish formulas (III) for us proved that if one takes two cuts in the body then one may transform the one into the other by a continuous deformation, so the constants that relate to one cut are equal to the constants that relate to the other.

It then follows that the characteristics of a distortion are not elements that are specific to each cut, but they depend exclusively on the geometrical nature of the space that is occupied by the body and the regular deformation to which it has been subjected."

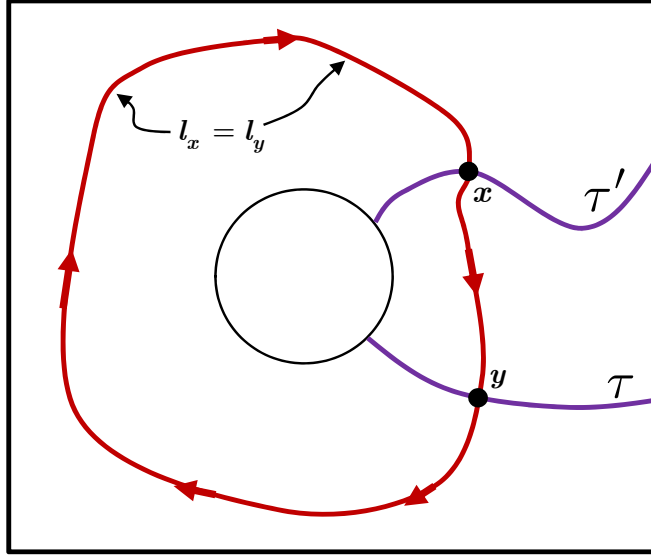


Figure 3: A non-contractible loop intersecting cut-surfaces τ' and τ at points x and y , respectively.

The “same process that served to establish formulas (III)” above refers to Volterra’s proof of Weingarten’s theorem, with the formulas stated as “Upon denoting the six constants across each section by l, m, n, p, q, r , we have:

$$(III) \quad U = l + ry - qz, \quad V = m + pz - rx, \quad W = n + qx - py,$$

where U, V, W represent the three Cartesian components of the displacement jump.

Nabarro [Nab87, p. 19] “Volterra showed that any dislocation of Weingarten’s type is equivalent to the dislocation produced by applying the same translation and rotation to the surfaces of any cut which can be continuously deformed into the original cut. This is proved by considering the body in its dislocated state, and showing that the same six constants b_i and d_{ij} are obtained by applying the preceding analysis to one cut or to the other.”

The “preceding analysis” Nabarro refers to is the proof of Weingarten’s theorem in his treatise [Nab87, pp 17-18] where b_i refers to the the displacement jump at an arbitrarily fixed point of a cut and d_{ij} refer to the components of the jump in the infinitesimal rotation tensor across the cut at that point.

I was unable to find, or deduce (primarily due to my inability in forming a precise statement of the problem), a proof of these statements of Volterra and Nabarro. What I was able to deduce is a relationship between displacement jumps of ‘corresponding’ points across two different cuts, with the sense of the correspondence defined in the opening paragraph of this Sec. 3.2. This is what is described in the following.

With reference to Figure 3,

$$\tau' u_i(\mathbf{x}^+) - \tau' u_i(\mathbf{x}^-) = \int_{\mathbf{c}_{(\mathbf{x}^-, \mathbf{x}^+)}} \left(\varepsilon_{ik}(\mathbf{s}) + \tau' \omega_{ik}(\mathbf{s}) \right) ds_k, \quad (7)$$

where $\mathbf{c}_{(\mathbf{x}^-, \mathbf{x}^+)}$ is any curve in Ω starting at \mathbf{x}^- and ending at \mathbf{x}^+ . Thinking of the curve parametrized by $t \in [0, 1]$ with $\mathbf{s}(0) = \mathbf{x}^-$ and $\mathbf{s}(1) = \mathbf{x}^+$ we have

$$\begin{aligned} \int_{\mathbf{c}_{(\mathbf{x}^-, \mathbf{x}^+)}} \tau' \omega_{ik}(\mathbf{s}) ds_k &= \int_0^1 \tau' \omega_{ik}(\mathbf{s}(t)) \frac{ds_k}{dt}(t) dt \\ &= \tau' \omega_{ik}(\mathbf{x}^+) x_k^+ - \tau' \omega_{ik}(\mathbf{x}^-) x_k^- - \int_{\mathbf{c}_{(\mathbf{x}^-, \mathbf{x}^+)}} s_j \tau' \omega_{ij,k} ds_k. \end{aligned}$$

Taking the limit $\mathbf{x}^\pm \rightarrow \mathbf{x}$, and using (4), (5), and (2) in (7) we obtain

$$\llbracket \tau' u_i(\mathbf{x}) \rrbracket = \omega_{ik} x_k + \int_{l_x} [\varepsilon_{ik}(\mathbf{s}) - s_j (\varepsilon_{ik,j}(\mathbf{s}) + \varepsilon_{jk,i}(\mathbf{s}))] ds_k, \quad (8)$$

where l_x represents a non-contractible loop passing through \mathbf{x} . Thus, the displacement jump at $\mathbf{x} \in \tau'$ depends on τ' only through \mathbf{x} , and otherwise on the constant ω on Ω , and the line integral along a non-contractible loop passing through \mathbf{x} of quantities depending only on the given strain field and the loop.

The displacement jump in $\tau \mathbf{u}$ at \mathbf{y} across τ may be expressed, following the same arguments to arrive at (8), as

$$\llbracket \tau u_i(\mathbf{y}) \rrbracket = \omega_{ik} y_k + \int_{l_y} [\varepsilon_{ik}(\mathbf{s}) - s_j (\varepsilon_{ik,j}(\mathbf{s}) + \varepsilon_{jk,i}(\mathbf{s}))] ds_k. \quad (9)$$

But, the hypothesis that $\mathbf{x} \in \tau'$ and $\mathbf{y} \in \tau$ can be linked by a curve means that we can always choose the non-contractible loop through \mathbf{x} to pass through $\mathbf{y} \in \tau$ as well so that the choice of $l_x = l_y$ is admissible. Therefore, (8) and (9) together imply that

$$\llbracket \tau' \mathbf{u}(\mathbf{x}) \rrbracket = \llbracket \tau \mathbf{u}(\mathbf{y}) \rrbracket + \omega[\mathbf{x} - \mathbf{y}] \quad \text{for } \mathbf{x} \in \tau' \text{ and } \mathbf{y} \in \tau. \quad (10)$$

When the surfaces τ and τ' can be mapped into each other by a continuous, 1-parameter family of surfaces (i.e. a homotopy), then $\mathbf{x} \in \tau'$ and $\mathbf{y} \in \tau$ can surely be linked by a curve. Thus for this situation, the displacement jumps across τ and τ' at corresponding points that map into each other by the homotopy are related by (10).

We also note that, keeping $\mathbf{x} \in \tau'$ fixed and choosing two points $\mathbf{y}, \mathbf{z} \in \tau$ which can be linked to $\mathbf{x} \in \tau'$ by (different) curves, (10) provides another proof of Weingarten's theorem (6).

3.3 Burgers vector of a dislocation and its cut-surface independence

While it is obvious that $\llbracket \tau \mathbf{u}(\mathbf{x}) \rrbracket$ in (6) is a constant translation vector for fixed $\mathbf{x} \in \tau$, it is clear that the choice of \mathbf{x} is arbitrary and choosing some other base point changes this constant when $\omega \neq \mathbf{0}$. Thus the rigid *translation* in the Weingarten theorem (6) is not well-defined when $\omega \neq \mathbf{0}$. We note that given the function $\tau \mathbf{u}$ and the family of all displacement fields on Ω_τ related to it by infinitesimally rigid deformations, the jump $\llbracket \tau \mathbf{u}(\mathbf{x}) \rrbracket$ for each fixed $\mathbf{x} \in \tau$ is unique within the family, see Remark 3.1.

Definition 12 *The Burgers vector of a Weingarten-Volterra defect of $\tau \mathbf{u}$ exists when $\llbracket \tau \omega \rrbracket = \mathbf{0}$ (in which case the defect is also called a dislocation), and is given by $\llbracket \tau \mathbf{u}(\mathbf{x}) \rrbracket$ for any $\mathbf{x} \in \tau$.*

Remark 3.3 *The Weingarten result (6) shows that the quantity*

$$\mathbf{b}_o^\tau := \llbracket \tau \mathbf{u}(\mathbf{x}) \rrbracket - \omega \mathbf{x} \quad (11)$$

is a mathematical constant for all $\mathbf{x} \in \tau$. This constancy however should not be interpreted as the physical Burgers vector of the Weingarten-Volterra defect since it depends on the choice of the origin invoked to define position vectors (see Remark 2.1 and cf. [Cas04, Zub97] - Zubov recognizes this problem [Zub97, p.19], but nevertheless adopts the definition [Zub97, Chapter 1.3]). To see one of the problematic implications of such a definition, it is physically reasonable to expect that within the family of displacement fields of Ω_τ that are related to each other by infinitesimally rigid deformations, the Burgers vector of a dislocation should, at most, be rotated and therefore maintain constant magnitude, and this is a physical statement where the choice of an origin to represent position vectors plays no role. It is easy to check that the expression \mathbf{b}_0^τ can be made to have arbitrary magnitude depending on the choice of origin when $\boldsymbol{\omega} \neq \mathbf{0}$.

We now prove the following assertion:

Theorem 3.1 *Given two cut-surfaces τ and τ' of Ω , a locally compatible strain field $\boldsymbol{\varepsilon}$ on Ω with $\boldsymbol{\omega} = \mathbf{0}$ (defined by (5)), and two displacement fields ${}^\tau \mathbf{u}$ and ${}^{\tau'} \mathbf{u}$ compatible with $\boldsymbol{\varepsilon}_\tau$ on Ω_τ and $\boldsymbol{\varepsilon}_{\tau'}$ on $\Omega_{\tau'}$, respectively, the Burgers vector, \mathbf{b} , of the dislocations of ${}^\tau \mathbf{u}$ and ${}^{\tau'} \mathbf{u}$ are equal and given by*

$$\mathbf{b} := \llbracket {}^\tau \mathbf{u}(\mathbf{x}) \rrbracket = \llbracket {}^{\tau'} \mathbf{u}(\mathbf{y}) \rrbracket, \quad \mathbf{x} \in \tau, \mathbf{y} \in \tau'.$$

Remark 3.4 *In the existing literature, a distinction between the fields ${}^\tau \mathbf{u}$ and ${}^{\tau'} \mathbf{u}$ is generally not made and it is assumed without proof that $\llbracket {}^\tau \mathbf{u}(\mathbf{x}) \rrbracket = \llbracket {}^{\tau'} \mathbf{u}(\mathbf{x}) \rrbracket$ when $\mathbf{x} \in \tau \cap \tau'$. We note that the domains of the two functions ${}^\tau \mathbf{u} : \Omega_\tau \rightarrow V_3$ and ${}^{\tau'} \mathbf{u} : \Omega_{\tau'} \rightarrow V_3$ are different, and it is not a priori obvious that the limiting value of the two displacement fields in question, or even their jumps, at any point of their corresponding cuts do not depend on the geometry of the respective cuts beyond the point of evaluation. In Proof 3 below, we fill this gap in the argument.*

We provide three different proofs of Theorem 3.1, with different levels of assumptions.

Proof 1: For $\mathbf{x}_0 \in \Omega_\tau \cap \Omega_{\tau'}$, from (2) we have that any ${}^\tau \boldsymbol{\omega}$ and ${}^{\tau'} \boldsymbol{\omega}$ may be expressed as

$$\begin{aligned} {}^\tau \omega_{ij}(\mathbf{z}) &= {}^\tau \omega_{ij}(\mathbf{x}_0) + \int_{\mathbf{c}(\mathbf{x}_0, \mathbf{z})} E_{ijk}(\mathbf{s}) ds_k \quad \text{for } \mathbf{z} \in \Omega_\tau \\ {}^{\tau'} \omega_{ij}(\mathbf{y}) &= {}^{\tau'} \omega_{ij}(\mathbf{x}_0) + \int_{\mathbf{c}'(\mathbf{x}_0, \mathbf{y})} E_{ijk}(\mathbf{s}) ds_k \quad \text{for } \mathbf{y} \in \Omega_{\tau'}, \end{aligned} \tag{12}$$

where \mathbf{c} is a curve in Ω_τ and \mathbf{c}' is a curve in $\Omega_{\tau'}$. Then for $\mathbf{x} \in \tau$ we have that $\llbracket {}^\tau \omega_{ij}(\mathbf{x}) \rrbracket = \int_{l_{\mathbf{x}}} E_{ijk}(\mathbf{s}) ds_k = \omega_{ij} = 0$ from (4) and (5), and similarly, for $\mathbf{x} \in \tau'$, $\llbracket {}^{\tau'} \omega_{ij}(\mathbf{x}) \rrbracket = 0$. This implies that both ${}^\tau \boldsymbol{\omega}$ and ${}^{\tau'} \boldsymbol{\omega}$ are actually continuous fields on Ω . We now have that both representations in (12) are actually valid on Ω and that

$${}^{\tau'} \boldsymbol{\omega}(\mathbf{x}) = {}^\tau \boldsymbol{\omega}(\mathbf{x}) + \mathbf{A} \quad \forall \mathbf{x} \in \Omega,$$

where \mathbf{A} is a constant skew-symmetric second order tensor on Ω . From (2) and the above relation we also have that

$$\begin{aligned} {}^\tau u_{i,j}(\mathbf{x}) &= {}^\tau \omega_{ij}(\mathbf{x}) + \varepsilon_{ij}(\mathbf{x}) \quad \forall \mathbf{x} \in \Omega_\tau \\ {}^{\tau'} u_{i,j}(\mathbf{x}) &= {}^\tau \omega_{ij}(\mathbf{x}) + A_{ij} + \varepsilon_{ij}(\mathbf{x}) \quad \forall \mathbf{x} \in \Omega_{\tau'}, \end{aligned}$$

so that for $\mathbf{x} \in \tau$ and $\mathbf{y} \in \tau'$ we have

$$\llbracket {}^\tau \mathbf{u}(\mathbf{x}) \rrbracket = \int_{l_{\mathbf{x}}} [{}^\tau \boldsymbol{\omega}(\mathbf{s}) + \boldsymbol{\varepsilon}(\mathbf{s})] ds; \quad \llbracket {}^{\tau'} \mathbf{u}(\mathbf{y}) \rrbracket = \int_{l_{\mathbf{y}}} [{}^\tau \boldsymbol{\omega}(\mathbf{s}) + \boldsymbol{\varepsilon}(\mathbf{s})] ds, \tag{13}$$

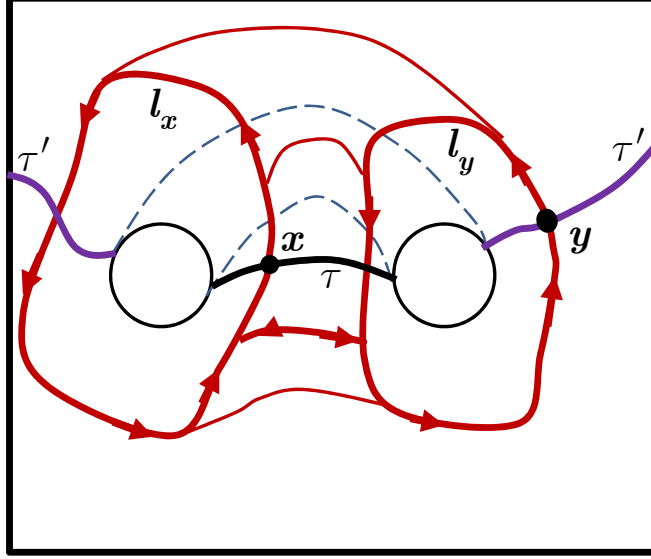


Figure 4: Non-contractible loop l_x intersects cut surface τ only once at x . Non-contractible loop l_y intersects cut surface τ' only once at y .

where l_x and l_y are non-contractible loops through x and y that intersect τ and τ' exactly once, respectively. But ${}^\tau\omega + \varepsilon =: \mathbf{H}$ is a continuous, and twice differentiable field on Ω that satisfies $H_{ik,l} = H_{il,k}$ on it by using ${}^\tau\omega_{ij,k} = E_{ijk}$. Then, by Stokes' theorem, we have that the two loop integrals in (13) are equal (by connecting them by a surface in Ω with the loops as boundaries and considering a closed curve on this surface with these loops as segments along with two overlapping parts, as in Fig. 4), and the proof is complete.

Proof 2: We assume that there is at least one point $x \in \tau'$ and $y \in \tau$ which can be connected by a curve in Ω . Then, consider (10) for $\omega = \mathbf{0}$. Finally, we apply Weingarten's theorem (6) to each surface. The proof is complete.

Proof 3: Assume that it is always possible to join τ and τ' by another cut-surface τ'' such that τ'' has common parts with τ and τ' . Let $x \in \tau \cap \tau''$ and $y \in \tau' \cap \tau''$. We also assume that it is possible to choose τ'' and a non-contractible loop l_x such that the loop intersects τ and τ'' exactly once at x and, similarly, a choice of l_y can be made w.r.t τ' and τ'' , intersecting both exactly once at y . Then, from the arguments leading up to (8), we conclude that $[[{}^\tau\mathbf{u}(x)]] = [[{}^{\tau''}\mathbf{u}(x)]]$ and $[[{}^{\tau'}\mathbf{u}(y)]] = [[{}^{\tau''}\mathbf{u}(y)]]$. But then, Weingarten's theorem (6) for $\omega = \mathbf{0}$ applied to all three surfaces completes the proof.

4 Finite deformation

The primary difference between the concepts and methods employed in proving Weingarten's theorem and associated results between the settings of infinitesimal and finite deformation kinematics is that the right hand side of the equation of integrability (14)₂ is not specified in terms of given data unlike (2)₃. This takes the great power afforded by Stokes' theorem essentially out of play in the case of discussing integrability for finite kinematics, using similar techniques as for the results of small deformation theory.

4.1 Weingarten's theorem for finite deformation

Given a simply connected Ω_τ (induced from the non-simply connected Ω with a cylindrical/toroidal hole) and a locally compatible \mathbf{C} field on it, consider any deformation ${}^\tau\mathbf{y}$ compatible with \mathbf{C}_τ on Ω_τ . Appendix A.3 shows that there exists a family of such deformation fields, each member of which satisfies

$$\begin{aligned} {}^\tau y_{,\alpha}^i &= {}^\tau F_\alpha^i \\ {}^\tau F_{\alpha,\beta}^i &= \Gamma_{\alpha\beta}^\rho {}^\tau F_\rho^i, \end{aligned} \tag{14}$$

where Γ is defined in (1). To be precise, the pair $({}^\tau\mathbf{y}, {}^\tau\mathbf{F})$ for fixed τ is not unique, but we do not make this distinction explicit in notation to keep it manageable (unless absolutely essential).

As in Section 3.1, consider a curve $\mathbf{c}_{(\mathbf{x},\mathbf{z})}$ on the surface τ joining \mathbf{x} and \mathbf{z} . Corresponding to \mathbf{c} , consider two other curves $\mathbf{c}_{(\mathbf{x}^+, \mathbf{z}^+)}$ and $\mathbf{c}_{(\mathbf{x}^-, \mathbf{z}^-)}$ in Ω_τ on either side of, and close to, τ . The curves run from \mathbf{x}^\pm to \mathbf{z}^\pm (in obvious notation), and in the following we will be thinking of limits of product integrals (see [DF84, Section 1.1] for definition) along \mathbf{c}^\pm as \mathbf{c}^\pm tend to \mathbf{c} , with both \mathbf{c}^\pm parametrized by $s \in [0, 1]$.

Definition 13 *Given a parametrized curve of position vectors $\mathbf{a} : [x, y] \rightarrow V_3$, $[x, y] \subset \mathbb{R}$, the 3×3 matrix-valued function of the parameter of the curve, $G^\mathbf{a}(s)$ is defined as*

$$(G^\mathbf{a})_\alpha^\rho(s) := \Gamma_{\alpha\beta}^\rho(\mathbf{a}(s)) \frac{d\mathbf{a}^\beta}{ds}(s);$$

the 3×3 invertible-matrix $\mathbf{P}^\mathbf{a}$ is defined by the product integral [DF84]

$$\mathbf{P}^\mathbf{a} = \prod_x^y e^{G^\mathbf{a}(s)ds}.$$

In standard matrix notation G_α^ρ and $(\mathbf{P}^\mathbf{a})_\alpha^\rho$ are to be interpreted as $G_{\rho\alpha}$ and $(\mathbf{P}^\mathbf{a})_{\rho\alpha}$ and our definition of the product integral corresponding to the matrix G is identical to the definition of the product integral corresponding to the matrix G^T given in [DF84, Theorem 1.1].

Along such curves we have

$$\begin{aligned} {}^\tau F_{\alpha,\beta}^i(\mathbf{c}^\pm(s)) \frac{d(\mathbf{c}^\pm)^\beta}{ds}(s) &= \Gamma_{\alpha\beta}^\rho(\mathbf{c}^\pm(s)) \frac{d(\mathbf{c}^\pm)^\beta}{ds}(s) {}^\tau F_\rho^i(\mathbf{c}^\pm(s)) \\ \implies \frac{d({}^\tau F_\alpha^i \circ \mathbf{c}^\pm)}{ds}(s) &= (G^{\mathbf{c}^\pm})_\alpha^\rho(s) {}^\tau F_\rho^i(\mathbf{c}^\pm(s)). \end{aligned} \tag{15}$$

Equation (15)₂ implies that ${}^\tau F_\alpha^i(\mathbf{z}^\pm)$ has the representation shown below in (16) by the representation and uniqueness of solutions of linear systems of ordinary differential equations with prescribed initial data proved in [DF84, Theorem 2.1, pp 12-13] (of course, uniqueness can also be proved by the Gronwall inequality). Thus,

$${}^\tau F_\alpha^i(\mathbf{z}^\pm) = \left[\prod_{s=0}^{s=1} e^{G^{\mathbf{c}^\pm}(s)ds} \right]_\alpha^\rho {}^\tau F_\rho^i(\mathbf{x}^\pm) \iff {}^\tau \mathbf{F}(\mathbf{z}^\pm) = {}^\tau \mathbf{F}(\mathbf{x}^\pm) \mathbf{P}^{\mathbf{c}^\pm}, \tag{16}$$

so that

$${}^\tau \mathbf{F}(\mathbf{z}^+) [{}^\tau \mathbf{F}(\mathbf{z}^-)]^{-1} = {}^\tau \mathbf{F}(\mathbf{x}^+) \mathbf{P}^{\mathbf{c}^+} \left[\mathbf{P}^{\mathbf{c}^-} \right]^{-1} [{}^\tau \mathbf{F}(\mathbf{x}^-)]^{-1}.$$

Then, in the limit $\mathbf{c}^\pm \rightarrow \mathbf{c}$, we have $\mathbf{P}^{\mathbf{c}^\pm} = \mathbf{P}^{\mathbf{c}^\mp} = \mathbf{P}^{\mathbf{c}}$ and therefore

$${}^\tau \mathbf{Q} := {}^\tau \mathbf{F}^+(\mathbf{z}) [{}^\tau \mathbf{F}^-(\mathbf{z})]^{-1} = {}^\tau \mathbf{F}^+(\mathbf{x}) [{}^\tau \mathbf{F}^-(\mathbf{x})]^{-1} \quad \forall \mathbf{x}, \mathbf{z} \in \tau. \quad (17)$$

The compatibility of ${}^\tau \mathbf{y}$ with \mathbf{C}_τ implies that ${}^\tau \mathbf{F}$ is invertible. Since $\mathbf{C} =: \mathbf{U}^2$ is continuous on Ω , ${}^\tau \mathbf{F}^\pm = {}^\tau \mathbf{R}^\pm \mathbf{U}$ on τ with ${}^\tau \mathbf{R}^\pm$ proper orthogonal, and hence ${}^\tau \mathbf{Q} = {}^\tau \mathbf{R}^+ [{}^\tau \mathbf{R}^-]^T$ is a proper orthogonal tensor.

Considering now the first equation in (14) we have

$$\begin{aligned} {}^\tau y_{,\alpha}^i(\mathbf{c}^\pm(s)) \frac{d(\mathbf{c}^\pm)^\alpha}{ds}(s) &= {}^\tau F_\alpha^i(\mathbf{c}^\pm(s)) \frac{d(\mathbf{c}^\pm)^\alpha}{ds}(s) \\ \implies {}^\tau \mathbf{y}^\pm(\mathbf{z}) - {}^\tau \mathbf{y}^\pm(\mathbf{x}) &= \lim_{\mathbf{c}^\pm \rightarrow \mathbf{c}} {}^\tau \mathbf{y}(\mathbf{z}^\pm) - {}^\tau \mathbf{y}(\mathbf{x}^\pm) = \lim_{\mathbf{c}^\pm \rightarrow \mathbf{c}} \int_0^1 {}^\tau \mathbf{F}(\mathbf{c}^\pm(s)) \frac{d\mathbf{c}^\pm}{ds}(s) ds. \end{aligned}$$

Using (17) we obtain

$$\begin{aligned} {}^\tau \mathbf{y}^+(\mathbf{z}) - {}^\tau \mathbf{y}^+(\mathbf{x}) &= \int_0^1 {}^\tau \mathbf{F}^+(\mathbf{c}(s)) \frac{d\mathbf{c}}{ds}(s) ds \\ &= {}^\tau \mathbf{Q} \int_0^1 {}^\tau \mathbf{F}^-(\mathbf{c}(s)) \frac{d\mathbf{c}}{ds}(s) ds = {}^\tau \mathbf{Q} \left(\lim_{\mathbf{c}^- \rightarrow \mathbf{c}} \int_0^1 {}^\tau \mathbf{F}(\mathbf{c}^-(s)) \frac{d\mathbf{c}^-}{ds}(s) ds \right) \\ &= {}^\tau \mathbf{Q} \left(\lim_{\mathbf{c}^- \rightarrow \mathbf{c}} [{}^\tau \mathbf{y}(\mathbf{z}^-) - {}^\tau \mathbf{y}(\mathbf{x}^-)] \right) \\ &= {}^\tau \mathbf{Q} [{}^\tau \mathbf{y}^-(\mathbf{z}) - {}^\tau \mathbf{y}^-(\mathbf{x})] \quad \forall \mathbf{z}, \mathbf{x} \in \tau. \end{aligned} \quad (18)$$

Thus, thinking of ${}^\tau \mathbf{y}^+$ and ${}^\tau \mathbf{y}^-$ as two deformations of the surface τ , (18) suggests that these two deformation fields of τ are related by a rigid deformation (cf. Definition 5). *This is a proof of Weingarten's theorem at finite deformation, essentially due to Zubov [Zub97, Sec. 1.3] (also see Casey [Cas04] for a different proof).*

Definition 14 ${}^\tau \mathbf{y} \mathbf{Q}$, or ${}^\tau \mathbf{Q}$ when there is no ambiguity in the rotation of which deformation is being referred to, (ω) is defined to be the disclination strength of the Weingarten-Volterra defect of ${}^\tau \mathbf{y}$ (${}^\tau \mathbf{u}$).

Remark 4.1 Let ${}^\tau \mathbf{y}_1$ and ${}^\tau \mathbf{y}_2$ be two deformations compatible with \mathbf{C}_τ on Ω_τ , with deformation gradient fields ${}^\tau \mathbf{F}_2$ and ${}^\tau \mathbf{F}_1$. Then, necessarily, there exists a constant orthogonal tensor \mathbf{R}^* on Ω_τ such that ${}^\tau \mathbf{F}_2 = \mathbf{R}^* {}^\tau \mathbf{F}_1$ and therefore ${}^\tau \mathbf{Q}_2 = \mathbf{R}^* {}^\tau \mathbf{Q}_1 \mathbf{R}^{*T}$. Thus, in contrast with the small deformation case (see Remark 3.1), the disclination strength of the Weingarten-Volterra defect of ${}^\tau \mathbf{y}$, ${}^\tau \mathbf{y} \mathbf{Q}$, is not constant for all deformations ${}^\tau \mathbf{y}$ of Ω_τ compatible with \mathbf{C}_τ that are rigidly related to each other.

Zubov [Zub97, p. 20] claims that his “vector of finite rotation” is uniquely determined by the field \mathbf{C} (in our notation) for a doubly-connected domain in nonlinear elasticity, which implies from [Zub97, Equation (1.3.5)] that ${}^\tau \mathbf{Q}$ must be too. The demonstration above shows that this is not the case.¹

¹As an aside, Zubov's notation is non-standard, e.g. the action of a tensor \mathbf{A} on a vector \mathbf{b} is written as $\mathbf{b} \cdot \mathbf{A}$; for Q^M (curvilinear) coordinates on the current configuration with position vectors represented as \mathbf{R} and q_s as coordinates on the reference configuration with position vectors represented as \mathbf{r} , the deformation gradient is written as $\left(\frac{\partial Q^M}{\partial q^s} \right) \mathbf{r}^s \otimes \mathbf{R}_M$, where \mathbf{r}^s represents (an element of) the dual basis in the reference configuration corresponding to coordinates q_s , and \mathbf{R}_M is the natural basis in the current configuration (instead of the more standard notation that would be $\left(\frac{\partial Q^M}{\partial q^s} \right) \mathbf{R}_M \otimes \mathbf{r}^s$; the correspondence of upper and lower case letters with objects on the current and reference configuration in this footnote also follows Zubov's notation).

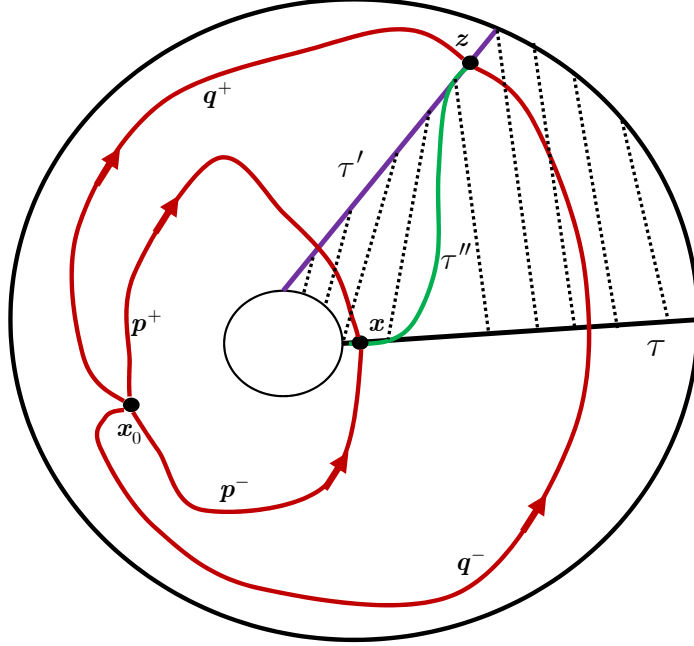


Figure 5: \mathbf{x}_0 placed in the shaded region would not satisfy the hypotheses of the argument in Sec. 4.2.

4.2 A condition for τ -independence of ${}^\tau Q$

With reference to Remark 3.4, Zubov [Zub97, p. 19] does not make a distinction between the fields ${}^\tau \mathbf{y} : \Omega_\tau \rightarrow V_3$ and ${}^{\tau'} \mathbf{y} : \Omega_{\tau'} \rightarrow V_3$ compatible with \mathbf{C}_τ and $\mathbf{C}_{\tau'}$, respectively, for two different cut-surfaces τ and τ' . He also assumes that ${}^\tau Q(\mathbf{x}) = {}^{\tau'} Q(\mathbf{x})$ if $\mathbf{x} \in \tau \cap \tau'$ without proof. Clearly, Remark 4.1 suggests that this need not be true without further conditions, even when $\tau = \tau'$. In this section we define a sufficient condition that ensures the cut-surface independence of ${}^\tau Q$.

Let τ , τ' , and τ'' be cut-surfaces that render Ω simply connected. Let $\tau \cap \tau''$ and $\tau' \cap \tau''$ both be non-empty. Assume there exist a point $\mathbf{x}_0 \in \Omega$ and an $\mathbf{x} \in \tau \cap \tau''$ such that they can be connected by two paths \mathbf{p}^+ and \mathbf{p}^- , both contained in $\Omega_\tau \cup \{\mathbf{x}\}$ and in $\Omega_{\tau''} \cup \{\mathbf{x}\}$, where \mathbf{p}^+ and \mathbf{p}^- approach \mathbf{x} from opposite sides of τ and τ'' . Similarly, assume that \mathbf{x}_0 can be connected to a $\mathbf{z} \in \tau' \cap \tau''$ by two paths \mathbf{q}^+ and \mathbf{q}^- , both contained in $\Omega_{\tau'} \cup \{\mathbf{z}\}$ and in $\Omega_{\tau''} \cup \{\mathbf{z}\}$, where \mathbf{q}^+ and \mathbf{q}^- approach \mathbf{z} from opposite sides of τ' and τ'' - see Figure. 5 for a realization of these conditions. Consider deformations ${}^\tau \mathbf{y} : \Omega_\tau \rightarrow V_3$, ${}^{\tau'} \mathbf{y} : \Omega_{\tau'} \rightarrow V_3$ and ${}^{\tau''} \mathbf{y} : \Omega_{\tau''} \rightarrow V_3$ with deformation gradient fields ${}^\tau \mathbf{F}$, ${}^{\tau'} \mathbf{F}$, and ${}^{\tau''} \mathbf{F}$, respectively, with

$${}^\tau \mathbf{F}(\mathbf{x}_0) = {}^{\tau'} \mathbf{F}(\mathbf{x}_0) = {}^{\tau''} \mathbf{F}(\mathbf{x}_0) = \mathbf{F}_0. \quad (19)$$

Under these hypotheses, arguments identical to arriving at (15), (16), and (17) imply

$$\begin{aligned} {}^\tau \mathbf{F}^\pm(\mathbf{x}) &= \mathbf{F}_0 \mathbf{P}^{\mathbf{p}^\pm} = {}^{\tau''} \mathbf{F}^\pm(\mathbf{x}) \\ {}^{\tau'} \mathbf{F}^\pm(\mathbf{z}) &= \mathbf{F}_0 \mathbf{P}^{\mathbf{q}^\pm} = {}^{\tau''} \mathbf{F}^\pm(\mathbf{z}), \end{aligned}$$

and using (17) we obtain

$${}^\tau Q(\mathbf{x}) = {}^{\tau'} Q(\mathbf{y}) = {}^{\tau''} Q(\mathbf{z}) \quad \forall \mathbf{x} \in \tau, \forall \mathbf{y} \in \tau', \forall \mathbf{z} \in \tau'', \quad (20)$$

for the $(\cdot)\mathbf{Q}$ generated from deformations $(\cdot)\mathbf{y}$ that satisfy (19) on their respective regions $\Omega_{(\cdot)}$, with \mathbf{x}_0 and the cut-surfaces, τ , τ' , and τ'' satisfying the hypotheses mentioned above in the second paragraph of this section 4.2.

Remark 4.2 *Related to the discussion in [Zub97, Sec. 1.3] of the vector of finite rotation, Zubov mentions ‘initial conditions’ [Zub97, Equation (1.2.5), Sec. 1.2] specifiable at any point of Ω . The point \mathbf{x}_0 where ‘initial conditions’ (19) may be specified for the proof above requires further specification of a topological nature (see Fig. 5).*

Remark 4.3 *Erroneous remarks about the results (18) and (20) are made in a footnote in [AF15, p.146]. While the footnote does not affect the developments of [AF15] in any way, the latter dealing with a different, but broadly related, geometric construct for line defects than that of Weingarten-Volterra, nevertheless, the error is entirely regretted.*

Remark 4.4 *The contrast in the generality of the result (20) for large deformations and the corresponding result, Remark 3.1, for small deformations is to be noted. In particular, the small deformation result requires no specification of ‘initial conditions.’*

4.3 Burgers vector of a dislocation and its dependence on the cut-surface

It is clear from (18) that when ${}^\tau\mathbf{Q} = \mathbf{I}$, the jump in ${}^\tau\mathbf{y}$ across τ is constant on τ as well as being a well-defined physical constant independent of the origin, unlike ${}^\tau\mathbf{y}^+ - {}^\tau\mathbf{Q}{}^\tau\mathbf{y}^-$ which is also a constant on τ but not independent of the choice of origin (cf. [Cas04, p. 481], [Zub97, p. 19]). Hence, we define

Definition 15 *The Burgers vector of a Weingarten-Volterra defect of ${}^\tau\mathbf{y}$ exists when ${}^\tau\mathbf{Q} = \mathbf{I}$ (in which case the defect is also called a dislocation), and is given by $[[{}^\tau\mathbf{y}(\mathbf{x})]]$ for any $\mathbf{x} \in \tau$.*

Remark 4.5 *Let ${}^\tau\mathbf{y}_1$ and ${}^\tau\mathbf{y}_2$ be two deformations compatible with \mathbf{C}_τ on Ω_τ . Then, necessarily, there exists a constant orthogonal tensor \mathbf{R}^* on Ω_τ such that ${}^\tau\mathbf{y}_2 = \mathbf{R}^*{}^\tau\mathbf{y}_1 + \mathbf{t}$, with \mathbf{t} a constant on Ω_τ and therefore $[[{}^\tau\mathbf{y}_2]] = \mathbf{R}^*[[{}^\tau\mathbf{y}_1]]$ on τ , both jumps not necessarily constant on the cut-surface τ .*

When ${}^\tau\mathbf{y}_1\mathbf{Q} = \mathbf{I} \iff {}^\tau\mathbf{y}_2\mathbf{Q} = \mathbf{I}$ by Remark 4.1, $[[{}^\tau\mathbf{y}_1]]$ and $[[{}^\tau\mathbf{y}_2]]$ are the constant Burgers vectors of the two deformations ${}^\tau\mathbf{y}_1$ and ${}^\tau\mathbf{y}_2$, respectively, and are related by the rotation \mathbf{R}^* linking the two deformations; thus they are not uniquely determined by the \mathbf{C} field on Ω (cf. [Zub97, p. 20]).

We prove the following assertion:

Theorem 4.1 *Consider two cut-surfaces τ and τ' of Ω , a locally compatible strain field \mathbf{C} on Ω , and two deformations ${}^\tau\mathbf{y}$ and ${}^{\tau'}\mathbf{y}$ compatible with \mathbf{C}_τ on Ω_τ and with $\mathbf{C}_{\tau'}$ on $\Omega_{\tau'}$, respectively, satisfying ${}^\tau\mathbf{Q} = \mathbf{I}$ and ${}^{\tau'}\mathbf{Q} = \mathbf{I}$. Suppose there exists a non-contractible loop $\mathbf{l}_\mathbf{x} \subset \Omega$ passing through $\mathbf{x} \in \Omega$ that intersects both τ and τ' exactly once. Then the Burgers vector of the dislocations of ${}^\tau\mathbf{y}$ and ${}^{\tau'}\mathbf{y}$ are related by*

$$[{}^\tau\mathbf{R}(\mathbf{x})]^T [[{}^\tau\mathbf{y}(\mathbf{s})]] = [{}^{\tau'}\mathbf{R}(\mathbf{x})]^T [[{}^{\tau'}\mathbf{y}(\mathbf{z})]], \quad \mathbf{s} \in \tau, \mathbf{z} \in \tau', \quad (21)$$

where ${}^\tau\mathbf{R}$ and ${}^{\tau'}\mathbf{R}$ are the rotation tensor fields from the polar decompositions of ${}^\tau\mathbf{F} = {}^\tau\mathbf{R}\mathbf{U}$ and ${}^{\tau'}\mathbf{F} = {}^{\tau'}\mathbf{R}\mathbf{U}$, respectively. In particular, the magnitudes of the two Burgers vectors are equal.

Furthermore, if the values of ${}^\tau\mathbf{F}(\mathbf{x}^*) = {}^{\tau'}\mathbf{F}(\mathbf{x}^*) = \mathbf{F}_0$ for some $\mathbf{x}^* \in \Omega$, then the Burgers vectors of the dislocations of ${}^\tau\mathbf{y}$ and ${}^{\tau'}\mathbf{y}$ are equal.

Proof: Since ${}^\tau\mathbf{Q} = {}^{\tau'}\mathbf{Q} = \mathbf{I}$, ${}^\tau\mathbf{F}$ and ${}^{\tau'}\mathbf{F}$, solutions of (14)₂ on Ω_τ and $\Omega_{\tau'}$, respectively, are actually continuous functions on Ω ; for the same reason, it is clear from (18) that it suffices to prove the theorem statement for one $\mathbf{s} \in \tau$ and one $\mathbf{z} \in \tau'$.

It is also clear that

$$\llbracket {}^\tau\mathbf{y}(\mathbf{s}^*) \rrbracket = \int_{\mathbf{l}_x} {}^\tau\mathbf{F} \circ \mathbf{l}(t) \frac{d\mathbf{l}}{dt}(t) dt; \quad \llbracket {}^{\tau'}\mathbf{y}(\mathbf{z}^*) \rrbracket = \int_{\mathbf{l}_x} {}^{\tau'}\mathbf{F} \circ \mathbf{l}(t) \frac{d\mathbf{l}}{dt}(t) dt,$$

where \mathbf{s}^* is the point at which \mathbf{l}_x intersects τ and \mathbf{z}^* is the point at which \mathbf{l}_x intersects τ' .

We think of the closed loop \mathbf{l}_x as parametrized by $t \in [0, 1]$ with $\mathbf{l}(0) = \mathbf{l}(1) = \mathbf{x}$ and denote the portion of the curve corresponding to the interval $[0, t], t \leq 1$ as \mathbf{l}_t . Then

$${}^\tau\mathbf{F} \circ \mathbf{l}(t) = {}^\tau\mathbf{F}(\mathbf{x})\mathbf{P}^{\mathbf{l}_t}; \quad {}^{\tau'}\mathbf{F} \circ \mathbf{l}(t) = {}^{\tau'}\mathbf{F}(\mathbf{x})\mathbf{P}^{\mathbf{l}_t}$$

so that

$${}^\tau\mathbf{F} \circ \mathbf{l}(t) = {}^\tau\mathbf{F}(\mathbf{x}) \left[{}^{\tau'}\mathbf{F}(\mathbf{x}) \right]^{-1} {}^{\tau'}\mathbf{F} \circ \mathbf{l}(t) = {}^\tau\mathbf{R}(\mathbf{x}) \left[{}^{\tau'}\mathbf{R}(\mathbf{x}) \right]^T {}^{\tau'}\mathbf{F} \circ \mathbf{l}(t),$$

and we have the desired result (21) by performing a line integral of both sides of the expression along the loop \mathbf{l}_x .

Since ${}^\tau\mathbf{F}$ and ${}^{\tau'}\mathbf{F}$ are both continuous functions on Ω and satisfy

$$\left[{}^\tau\mathbf{F}(\mathbf{r}) - {}^{\tau'}\mathbf{F}(\mathbf{r}) \right] = \left[{}^\tau\mathbf{F}(\mathbf{x}^*) - {}^{\tau'}\mathbf{F}(\mathbf{x}^*) \right] \mathbf{P}^{\mathbf{c}},$$

where \mathbf{c} is a parametrized curve from \mathbf{x}^* to each $\mathbf{r} \in \Omega$, then, whenever ${}^\tau\mathbf{F}(\mathbf{x}^*) = {}^{\tau'}\mathbf{F}(\mathbf{x}^*) = \mathbf{F}_0$, we have uniqueness, i.e. ${}^\tau\mathbf{F} = {}^{\tau'}\mathbf{F}$ on Ω , which further implies that ${}^\tau\mathbf{R}(\mathbf{x}) = {}^{\tau'}\mathbf{R}(\mathbf{x})$ and hence the Burgers vector of ${}^\tau\mathbf{y}$ and ${}^{\tau'}\mathbf{y}$ are equal.

Remark 4.6 Let $\mathbf{l}_z \subset \Omega$ also be a non-contractible loop passing through $\mathbf{z} \in \Omega$ that intersects both τ and τ' exactly once. Denoting the (constant on τ) Burgers vector of ${}^\tau\mathbf{y}$ as ${}^\tau\mathbf{b}$ and the Burgers vector for ${}^{\tau'}\mathbf{y}$ as ${}^{\tau'}\mathbf{b}$, (21) implies

$$\left[{}^\tau\mathbf{R}(\mathbf{x}) - {}^\tau\mathbf{R}(\mathbf{z}) \right]^T {}^\tau\mathbf{b} = \left[{}^{\tau'}\mathbf{R}(\mathbf{x}) - {}^{\tau'}\mathbf{R}(\mathbf{z}) \right]^T {}^{\tau'}\mathbf{b}.$$

Thus, the change in the action of the (transposed) rotation of ${}^\tau\mathbf{y}$ on the latter's Burgers vector in moving from $\mathbf{x} \in \Omega$ to $\mathbf{z} \in \Omega$ is equal to the change in the action of the (transposed) rotation of ${}^{\tau'}\mathbf{y}$ on its Burgers vector for the same movement in point of evaluation.

While it is natural in the context of dislocations to assume that ${}^\tau\mathbf{Q} = \mathbf{I}$ and ${}^{\tau'}\mathbf{Q} = \mathbf{I}$, we now consider an argument which shows that assuming only one of these conditions implies the other in many circumstances, without invoking any ‘initial conditions’ of the type (19), Sec. 4.2.

Let $\mathbf{x} \in \tau$ and $\mathbf{z} \in \tau'$. Assume that a cut-surface τ'' exists that contains both \mathbf{x} and \mathbf{z} . Consider points \mathbf{x}^+ and \mathbf{x}^- on opposite sides of τ'' and near it. Consider curves $\mathbf{l}_{(\mathbf{x}^-, \mathbf{x}^+)}$ and $\mathbf{q}_{(\mathbf{x}^+, \mathbf{x}^-)}$ in $\Omega_{\tau''}$ connecting \mathbf{x}^- to \mathbf{x}^+ and \mathbf{x}^+ to \mathbf{x}^- , respectively. Then, there exists ${}^{\tau''}\mathbf{F}$ on $\Omega_{\tau''}$ satisfying (14) and

$${}^{\tau''}\mathbf{F}(\mathbf{x}^-) = {}^{\tau''}\mathbf{F}(\mathbf{x}^-) \mathbf{P}^{\mathbf{l}_{(\mathbf{x}^-, \mathbf{x}^+)}} \mathbf{P}^{\mathbf{q}_{(\mathbf{x}^+, \mathbf{x}^-)}} \implies \left[\mathbf{P}^{\mathbf{q}_{(\mathbf{x}^+, \mathbf{x}^-)}} \right]^{-1} = \mathbf{P}^{\mathbf{l}_{(\mathbf{x}^-, \mathbf{x}^+)}} ,$$

and assuming the natural definition [DF84, Definition 1.4] that $\left[\mathbf{P}^{\mathbf{q}_{(\mathbf{x}^-, \mathbf{x}^+)}} \right]^{-1} = \mathbf{P}^{\mathbf{q}_{(\mathbf{x}^+, \mathbf{x}^-)}}$ we have

$$\mathbf{P}^{\mathbf{l}_{(\mathbf{x}^-, \mathbf{x}^+)}} = \mathbf{P}^{\mathbf{q}_{(\mathbf{x}^-, \mathbf{x}^+)}} .$$

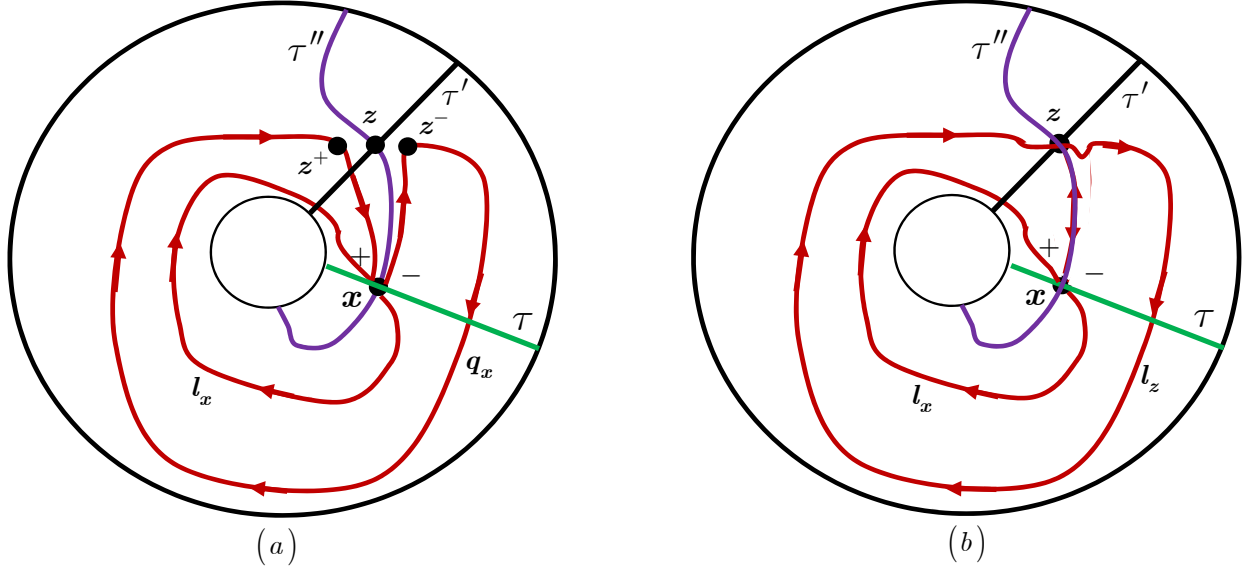


Figure 6: An example when $P^{l_x} = P^{q(x,z)} P^{l_z} P^{q(z,x)}$ for $x \in \tau$ and $z \in \tau'$. $P^{l_x} = I$ iff $P^{l_z} = I$.

Passing to the limit $x^\pm \rightarrow x$, we have that

$$P^{q_x} = P^{l_x}, \quad (22)$$

where q_x, l_x are parametrized, non-contractible loops in Ω starting and ending at x with ‘identical orientation’ (defined by invoking the cut-surface τ'' in Ω passing through x , and both l_x and q_x traversing from the ‘- side’ to the ‘+ side’ of τ''), and both contained in $\Omega_{\tau''} \cup \{x\}$.

Suppose now we introduce points z^+ and z^- on the curve $q(x^-, x^+)$ as shown in Figure 6a. The product integral has the multiplicative property

$$P^{q(x^-, x^+)} = P^{q(x^-, z^-)} P^{q(z^-, z^+)} P^{q(z^+, x^+)}$$

(see [DF84, Theorem 1.5, p.11]). Now choose a sequence of curves based on $q(x^-, x^+)$ in such a way that the segments $q(x^-, z^-)$ and $q(z^+, x^+)$ approach the curve joining x to z , the latter entirely contained in τ'' , but with opposite sense of traversal and in the limit we have

$$P^{q_x} = P^{q(x,z)} P^{l_z} P^{q(z,x)}, \quad (23)$$

where l_z is a non-contractible loop in Ω starting and ending at z , running from the ‘-’ side to the ‘+’ side of τ'' , and contained in $\Omega_{\tau''} \cup \{z\}$, and we note that $[P^{q(x,z)}]^{-1} = P^{q(z,x)}$.

It seems natural (and we assume this) that l_x can be chosen in such a way that it intersects both τ and τ'' exactly once at x and, similarly, l_z can be chosen in such a way that it intersects both τ' and τ'' exactly once at z (see Fig. 6b). Then ${}^\tau Q = I$ implies that $P^{l_x} = I$ (since ${}^\tau F(x^+) = {}^\tau F(x^-) P^{l(x^-, x^+)}$), which, along with (22), implies $P^{q_x} = I$, and then it can be inferred, using (23), that ${}^{\tau'} Q = I$ (since ${}^{\tau'} F(z^+) = {}^{\tau'} F(z^-) P^{q(z^-, z^+)}$).

Remark 4.7 For any solution $({}^\tau y, {}^\tau F)$ of (14) on Ω_τ , ${}^\tau Q(x) = I$ if and only if $P^{l_x} = I$, for any $x \in \tau$ a cut-surface of Ω and l_x a parametrized, non-contractible loop in Ω starting and ending at x .

For \mathbf{l}_x running from the ‘-’ side to the ‘+’ of τ ,

$$\begin{aligned} {}^\tau \mathbf{Q} &= {}^\tau \mathbf{F}^-(\mathbf{x}) \mathbf{P}^{\mathbf{l}_x} [{}^\tau \mathbf{F}^-(\mathbf{x})]^{-1} \\ [{}^\tau \mathbf{R}^-(\mathbf{x})]^T {}^\tau \mathbf{Q} [{}^\tau \mathbf{R}^-(\mathbf{x})] &= \mathbf{U}(\mathbf{x}) \mathbf{P}^{\mathbf{l}_x} [\mathbf{U}(\mathbf{x})]^{-1} \end{aligned} \quad (24)$$

for all $\mathbf{x} \in \tau$. The function on the right hand side of (24)₁ is constant on τ , even though $\mathbf{P}^{\mathbf{l}_x}$, as \mathbf{x} varies on τ , is not, and neither is $\mathbf{U}(\mathbf{x}) \mathbf{P}^{\mathbf{l}_x} [\mathbf{U}(\mathbf{x})]^{-1}$, where \mathbf{U} is the right stretch tensor field of $\mathbf{C} = \mathbf{U}^2$ on Ω . The work of Shield [Shi73], along with the use of the product integral, can provide an explicit characterization of the rotation field, ${}^\tau \mathbf{R}$, of ${}^\tau \mathbf{y}$ on Ω_τ (and hence the values ${}^\tau \mathbf{R}^-$) in terms of the prescribed strain field and an ‘initial condition’; (24)₂ also provides a characterization of the variation of the ${}^\tau \mathbf{R}^-$ field on τ in terms of given data, with an explicit indication of the nonlocality involved.

Acknowledgments

I am very grateful to Reza Pakzad and Raz Kupferman for their valuable comments and discussion. I thank Reza for reading the whole paper and Raz for taking a look at the finite deformation part. I also thank Reza for showing me alternative proofs of the main results of this paper for bodies in two space dimensions with a single hole without involving any line or product integrals, but capitalizing only on appropriate statements of rigidity. I acknowledge the support of the Center for Nonlinear Analysis at Carnegie Mellon and grants ARO W911NF-15-1-0239 and NSF-CMMI-1435624.

Appendix

A Strain Compatibility on a simply connected region

For the sake of completeness we collect some classical results related to questions of strain compatibility in the following sections. *All regions considered in these appendices are simply connected, unless mentioned otherwise.* We repeatedly use the argument that if $f_{,k} = 0$ on the domain then f is a constant, which uses the fact that the region in question is path connected.

A.1 Small deformation

Theorem A.1 *Given a C^2 second-order tensor field ε on a simply connected region, it is necessary and sufficient for the existence of a C^3 displacement field \mathbf{u} on it satisfying $(\text{grad } \mathbf{u})_{\text{sym}} = \varepsilon$ that the St.-Venant compatibility condition (1)₃ be satisfied.*

Proof: Necessity - u_i exists satisfying

$$\begin{aligned} u_{i,k} &= \varepsilon_{ik} + \omega_{ik}; \quad \varepsilon_{ik} = \frac{1}{2}(u_{i,k} + u_{k,i}); \quad \omega_{ik} := \frac{1}{2}(u_{i,k} - u_{k,i}) \\ \omega_{ik,l} &= \frac{1}{2}(u_{i,kl} + u_{l,ki} - u_{l,ki} - u_{k,li}) = e_{il,k} - e_{kl,i}. \end{aligned} \quad (25)$$

The infinitesimal rotation $\boldsymbol{\omega}$ is twice continuously differentiable and therefore, $\omega_{ik,lm} - \omega_{ik,ml} = 0$ implies (1)₃.

Sufficiency - We assume (1)₃ is satisfied and define

$$\begin{aligned} E_{ikl} &:= \varepsilon_{il,k} - \varepsilon_{kl,i} \\ \omega_{ik}(\mathbf{x}) &= W_{ik} + \int_{\mathbf{c}(\mathbf{x}_0, \mathbf{x})} E_{ikl}(\mathbf{c}) dc_l, \end{aligned} \tag{26}$$

where $\mathbf{c}_{(\mathbf{x}_0, \mathbf{x})}$ is some path in the region from \mathbf{x}_0 to \mathbf{x} and \mathbf{W} is a arbitrary skew-symmetric tensor. Since $E_{ikl,m} = E_{ikm,l}$ due to (1)₃, ω as defined in (26)₂ is independent of path, and defines a smooth function that satisfies

$$\omega_{ik,l} = E_{ikl} \tag{27}$$

on the region which is unique up to the choice of $\omega(\mathbf{x}_0) = \mathbf{W}$. Since (27) and (26)₁ imply

$$\varepsilon_{ik,l} + \omega_{ik,l} - (\varepsilon_{il,k} + \omega_{il,k}) = 0,$$

the definition

$$u_i(\mathbf{x}) = u_i^0 + \int_{\mathbf{c}(\mathbf{x}_0, \mathbf{x})} (\varepsilon_{ik}(\mathbf{c}) + \omega_{ik}(\mathbf{c})) dc_k,$$

for any \mathbf{u}^0 an arbitrary constant vector, is independent of the path $\mathbf{c}_{(\mathbf{x}_0, \mathbf{x})}$ chosen to connect \mathbf{x}_0 and \mathbf{x} , and hence defines a smooth displacement field whose symmetrized gradient equals the given ε field.

A.2 Rigidity for smooth infinitesimal deformations

Theorem A.2 *If two C^1 displacement fields on a region have identical strain fields, then they differ at most by an infinitesimally rigid deformation. The region here need not be simply-connected.*

Proof: Let \mathbf{u}^1 and \mathbf{u}^2 be the two displacement fields with strain fields ε^1 and ε^2 and rotation field ω^1 and ω^2 , respectively, defined from the corresponding displacement fields by relations (25) and let $\varepsilon^1 = \varepsilon^2$. Then

$$\omega_{ij,k}^1 - \omega_{ij,k}^2 = \varepsilon_{ik,j}^1 - \varepsilon_{jk,i}^2 = 0,$$

using a similar computation as in (25), and therefore $\mathbf{W} := \omega^1 - \omega^2$ is a constant skew-symmetric tensor on the region (which is path-connected). We then have

$$u_{i,j}^1 - u_{i,j}^2 = W_{ij},$$

and integrating along paths from an arbitrarily fixed \mathbf{z} to all points \mathbf{x} in the region and rearranging terms we obtain

$$\mathbf{u}^1(\mathbf{x}) - \mathbf{u}^1(\mathbf{z}) = \mathbf{u}^2(\mathbf{x}) - \mathbf{u}^2(\mathbf{z}) + \mathbf{W}[\mathbf{x} - \mathbf{z}].$$

Since \mathbf{z} was arbitrarily fixed, this implies that \mathbf{u}^1 and \mathbf{u}^2 are related by an infinitesimally rigid deformation by Definition 6.

A.3 Finite deformation

The treatment here is from Sokolnikoff [Sok58]. While the considerations below relate to fundamental relations in Riemannian geometry, we intentionally emphasize the purely algebraic fact that if the functions h, g, x, y (defined below) satisfy (28), then they necessarily satisfy (31) and the definition (29) implies (32).

Let Ω_x and Ω_y be two 3-d coordinate patches, i.e. open bounded regions of \mathbb{R}^3 , and let $y : \Omega_x \rightarrow \Omega_y$ be a C^2 diffeomorphism with a C^2 inverse that we denote by $x : \Omega_y \rightarrow \Omega_x$. Let $h : \Omega_y \rightarrow \mathbb{R}^{3 \times 3}$ and $g : \Omega_x \rightarrow \mathbb{R}^{3 \times 3}$ be two prescribed matrix fields with range in the set of symmetric, positive definite 3×3 matrices. In the following all indices range over the set $\{1, 2, 3\}$. Assume

$$h_{ij} = x_{,i}^\alpha (g_{\alpha\beta} \circ x) x_{,j}^\beta \iff g_{\alpha\beta} = y_{,\alpha}^i (h_{ij} \circ y) y_{,\beta}^j \quad (28)$$

holds. Then, using the symmetry of $g_{\alpha\beta}$ (and switching some dummy indices),

$$\begin{aligned} h_{ij,k} &= \left(x_{,ik}^\alpha x_{,j}^\beta + x_{,i}^\beta x_{,jk}^\alpha \right) (g_{\alpha\beta} \circ x) + x_{,i}^\alpha x_{,j}^\beta x_{,k}^\gamma (g_{\alpha\beta,\gamma} \circ x) \\ h_{ik,j} &= \left(x_{,ij}^\alpha x_{,k}^\beta + x_{,i}^\beta x_{,kj}^\alpha \right) (g_{\alpha\beta} \circ x) + x_{,i}^\alpha x_{,k}^\gamma x_{,j}^\beta (g_{\alpha\gamma,\beta} \circ x) \\ h_{jk,i} &= \left(x_{,ji}^\alpha x_{,k}^\beta + x_{,j}^\beta x_{,ki}^\alpha \right) (g_{\alpha\beta} \circ x) + x_{,j}^\beta x_{,k}^\gamma x_{,i}^\alpha (g_{\beta\gamma,\alpha} \circ x). \end{aligned}$$

Now define $\hat{\Gamma} : \Omega_x \rightarrow \mathbb{R}^{3 \times 3 \times 3}$ and $\hat{H} : \Omega_y \rightarrow \mathbb{R}^{3 \times 3 \times 3}$ as

$$\begin{aligned} \hat{H}_{ijk} &:= \frac{1}{2} (h_{ik,j} + h_{jk,i} - h_{ij,k}) \\ \hat{\Gamma}_{\alpha\beta\gamma} &:= \frac{1}{2} (g_{\alpha\gamma,\beta} + g_{\beta\gamma,\alpha} - g_{\alpha\beta,\gamma}). \end{aligned} \quad (29)$$

Then,

$$\hat{H}_{ijk} = x_{,i}^\alpha x_{,j}^\beta x_{,k}^\gamma \left(\hat{\Gamma}_{\alpha\beta\gamma} \circ x \right) + x_{,ji}^\alpha x_{,k}^\beta (g_{\alpha\beta} \circ x).$$

Next we define $H : \Omega_y \rightarrow \mathbb{R}^{3 \times 3 \times 3}$ and $\Gamma : \Omega_x \rightarrow \mathbb{R}^{3 \times 3 \times 3}$ by

$$H_{ij}^k = h^{km} \hat{H}_{ijm} \quad \text{and} \quad \Gamma_{\alpha\beta}^\rho = g^{\rho\nu} \hat{\Gamma}_{\alpha\beta\nu},$$

where h^{km} and $g^{\alpha\beta}$ are the components of the matrices h^{-1} and g^{-1} , respectively. We note that $\hat{H}, H, \hat{\Gamma}, \Gamma$ are all symmetric in their lower first two indices. Noting from (28) that

$$h^{km} = \left(g^{\alpha\beta} \circ x \right) \left(y_{,\alpha}^k \circ x \right) \left(y_{,\beta}^m \circ x \right),$$

we obtain

$$H_{ij}^k = \left(y_{,\alpha}^k \circ x \right) x_{,ji}^\alpha + \left(y_{,\rho}^k \circ x \right) x_{,i}^\alpha x_{,j}^\beta \left(\Gamma_{\alpha\beta}^\rho \circ x \right),$$

which, after rearrangement of terms, yields

$$x_{,ji}^\mu = x_{,k}^\mu H_{ij}^k - x_{,i}^\alpha x_{,j}^\beta \left(\Gamma_{\alpha\beta}^\mu \circ x \right). \quad (30)$$

Of course, the computations above indicate that interchanging the list $(x, y, g, g^{-1}, h, h^{-1}, \hat{\Gamma}, \hat{H}, \Gamma, H)$ by $(y, x, h, h^{-1}, g, g^{-1}, \hat{H}, \hat{\Gamma}, H, \Gamma)$ in the above formulae is admissible. We thus have

$$y_{,\alpha\beta}^i = y_{,\gamma}^i \Gamma_{\alpha\beta}^\gamma - y_{,\alpha}^j y_{,\beta}^k (H_{jk}^i \circ y). \quad (31)$$

Another result we will need for our compatibility argument to follow is as follows:

$$\begin{aligned} \hat{\Gamma}_{\alpha\beta\gamma} + \hat{\Gamma}_{\beta\gamma\alpha} &= g_{\alpha\beta,\gamma} \\ \left(g_{\alpha\beta} g^{\beta\mu} \right)_{,\gamma} &= 0 \end{aligned}$$

so that

$$g_{,\gamma}^{\rho\mu} = -g^{\rho\alpha} \left(\hat{\Gamma}_{\alpha\gamma\beta} + \hat{\Gamma}_{\beta\gamma\alpha} \right) g^{\beta\mu} = - \left(g^{\rho\alpha} \Gamma_{\alpha\gamma}^{\mu} + g^{\beta\mu} \Gamma_{\beta\gamma}^{\rho} \right),$$

which implies

$$g_{,\mu}^{\alpha\beta} = - \left(g^{\alpha\gamma} \Gamma_{\gamma\mu}^{\beta} + g^{\gamma\beta} \Gamma_{\gamma\mu}^{\alpha} \right). \quad (32)$$

Let Ω be a simply connected region and let $\mathbf{C} : \Omega \rightarrow \mathcal{P}_{sym}$ be a prescribed C^2 field on Ω , where \mathcal{P}_{sym} is the set of all positive-definite, symmetric tensors on V_3 . Let Ω_x be a Rectangular Cartesian coordinate patch parametrizing Ω and $C : \Omega \rightarrow \mathbb{R}^{3 \times 3}$ be the component map of \mathbf{C} with respect to the rectangular Cartesian basis of the parametrization of Ω by Ω_x .

Theorem A.3 *If there exists a deformation $\mathbf{y} : \Omega \rightarrow \mathcal{E}_3$ satisfying*

$$(\text{grad } \mathbf{y})^T \text{grad } \mathbf{y} = \mathbf{C}, \quad (33)$$

then (1)₂ is satisfied in Ω . Conversely, if the matrix field C satisfies (1)₂, then there exists a deformation $\mathbf{y} : \Omega \rightarrow \mathcal{E}_3$ that satisfies (33).

Proof: Necessity of (1)₂ for (33) - Let $y : \Omega_x \rightarrow \Omega_y \subset \mathbb{R}^3$ be the coordinate map representing the parametrization of $\mathbf{y}(\Omega)$ by the same Rectangular Cartesian system for \mathcal{E}_3 used to parametrize Ω . Then

$$y_{,\alpha}^i \delta_{ij} y_{,\beta}^j = C_{\alpha\beta}$$

holds. Making the identification of $h = I$ and $g = C$ in (28), we have $H = 0$ and (31) implies

$$\begin{aligned} y_{,\alpha}^i &= F_{\alpha}^i \\ F_{\alpha,\beta}^i &= \Gamma_{\alpha\beta}^{\gamma} F_{\gamma}^i. \end{aligned} \quad (34)$$

Due to the smoothness of \mathbf{C} and (34) we obtain

$$\begin{aligned} y_{,\alpha,\beta\rho}^i &= y_{\alpha,\rho\beta}^i \\ \implies F_{\gamma,\rho}^i \Gamma_{\alpha\beta}^{\gamma} + F_{\gamma}^i \Gamma_{\alpha\beta,\rho}^{\gamma} - F_{\gamma,\beta}^i \Gamma_{\alpha\rho}^{\gamma} - F_{\gamma}^i \Gamma_{\alpha\rho,\beta}^{\gamma} &= 0 \\ \implies F_{\mu}^i \left(\Gamma_{\alpha\beta,\rho}^{\mu} - \Gamma_{\alpha\rho,\beta}^{\mu} + \Gamma_{\gamma\rho}^{\mu} \Gamma_{\alpha\beta}^{\gamma} - \Gamma_{\gamma\beta}^{\mu} \Gamma_{\alpha\rho}^{\gamma} \right) &= 0 \end{aligned}$$

and since F is an invertible matrix (due to $\mathbf{C} \in \mathcal{P}_{sym}$), (1)₂ holds.

Sufficiency of (1)₂ for (33) - By a theorem of Thomas [Tho34] (also the Froebenius theorem in the differential geometry literature), we have that a solution to (34) exists, with freely specifiable value of F and y at arbitrarily chosen points of Ω_x , if $\Gamma_{\alpha\beta}^{\gamma} = \Gamma_{\beta\alpha}^{\gamma}$ (which holds by definition of Γ), and (1)₂ hold.

Specify the value of F at an arbitrarily chosen point $x_0 \in \Omega_x$ such that $F_{\alpha}^i(x_0) F_{\beta}^j(x_0) = C_{\alpha\beta}(x_0)$, written alternatively as $F_0^T F_0 = C_0 \implies F_0 C_0^{-1} F_0^T = I$. For an F field satisfying (34) with the identification $C = g$ in (29) and (32), consider now

$$\left(C^{\alpha\beta} F_{\alpha}^i F_{\beta}^j \right)_{,\mu} = \left(C_{,\mu}^{\alpha\beta} + C^{\gamma\beta} \Gamma_{\gamma\mu}^{\alpha} + C^{\alpha\gamma} \Gamma_{\gamma\mu}^{\beta} \right) F_{\alpha}^i F_{\beta}^j = 0,$$

by (32). Thus we have that

$$F C^{-1} F^T = I \implies C = F^T F \quad \text{on } \Omega_x,$$

and defining $\mathbf{y} = y^i \mathbf{e}_i$ where $\mathbf{e}_i, i = 1, 2, 3$ is the (spatially constant) natural basis of the Rectangular Cartesian coordinate system used to parametrize Ω , we find that \mathbf{y} satisfies (33).

A.4 Rigidity for smooth finite deformations

Let $\mathbf{y}^* : \Omega \rightarrow \mathcal{E}_3$ and $\mathbf{z} : \Omega \rightarrow \mathcal{E}_3$ be two C^2 deformations of Ω . Consider a rectangular Cartesian parametrization of \mathcal{E}_3 under which Ω maps to the set of coordinates $\Omega_x \subset \mathbb{R}^3$, $\mathbf{y}^*(\Omega)$ maps to Ω_y , and $\mathbf{z}(\Omega)$ maps to Ω_z . Let $y^* : \Omega_x \rightarrow \Omega_y$, $z : \Omega_x \rightarrow \Omega_z$, $y : \Omega_z \rightarrow \Omega_y$, and $x : \Omega_z \rightarrow \Omega_x$ be the corresponding deformations, represented in coordinates. Ω need not be simply connected.

Theorem A.4 *If \mathbf{y}^* and \mathbf{z} have the same right Cauchy-Green tensor fields then they are rigidly related to each other.*

Proof: We have, following [Shi73],

$$\begin{aligned} y_{,j}^{*i} y_{,k}^{*i} &= z_{,j}^m z_{,k}^m = C_{jk} \\ \Rightarrow (y_{,m}^i \circ z) z_{,j}^m (y_{,n}^i \circ z) z_{,k}^n &= z_{,j}^m z_{,k}^m \\ \Rightarrow y_{,p}^i y_{,q}^i &= y_{,p}^\alpha \delta_{\alpha\beta} y_{,q}^\beta = x_{,p}^j (z_{,j}^m \circ x) (z_{,k}^m \circ x) x_{,q}^k = \delta_p^m \delta_q^m = \delta_{pq}, \end{aligned} \tag{35}$$

and identifying x , h , and $g \circ x$ in (28) with y , I , and I , respectively, here, we have from (30) that

$$y_{,pq}^\alpha = 0 \text{ on } \Omega_z.$$

Due to the path connectedness of Ω_z (induced from Ω_x), this implies that there exists a constant matrix R satisfying $y_{,p}^\alpha = R_p^\alpha$ on Ω_z with $R^T R = I$ from (35)₃. Integrating along an arbitrarily chosen path from $z(w_0) \in \Omega_z$ to $z(w) \in \Omega_z$ for $w_0, w \in \Omega_x$, we obtain

$$y^\alpha \circ z(w) = y^\alpha \circ z(w_0) + R_p^\alpha [z^p(w) - z^p(w_0)] \implies y^*(w) = y^*(w_0) + R[z(w) - z(w_0)]$$

for any $w \in \Omega_x$ and $w_0 \in \Omega_x$ and therefore \mathbf{y}^* and \mathbf{z} are rigidly related to each other by Definition 5.

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