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Mehrdad Massoudi & Phuoc X. Tran

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Mehrdad Massoudi · Phuoc X. Tran

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Abstract In this paper, we modify the thermodynamically compatible third-grade fluid model by introducing a shear-rate- and volume-fraction-dependent viscosity into the equation. With this new model, it is possible to predict not only the normal stress differences, but also the variable viscosity observed in many suspensions. We study the Couette–Poiseuille flow of such a fluid between two horizontal flat plates. The steady fully developed flow equations are made dimensionless and are solved numerically; the effects of different dimensionless numbers are discussed.

Keywords Third-grade fluids · Suspensions · Shear-rate- and volume-fraction-dependent viscosity · Continuum mechanics · Non-Newtonian fluids · Granular materials

List of symbols

$\mathbf{b}$	Body force vector
$\mathbf{D}$	Symmetric part of the velocity gradient
g	Acceleration due to gravity
H	Characteristic length
$\mathbf{I}$	Identity tensor
$\mathbf{L}$	Gradient of the velocity vector
t	Time
$\mathbf{T}$	Cauchy's stress tensor
U	Reference velocity
$\mathbf{x}$	Spatial position occupied at time t
y	Direction normal to the inclined plane
(Y) or $\overline{y}$	Dimensionless y
ϕ	Volume fraction
div	Divergence operator
∇	Gradient symbol

1 Introduction

Developing advanced coal-based fuel production with low pollutants is an important element for a cleaner environment and a more sustainable future. A great deal of effort has been directed toward relating rheological

M. Massoudi (✉) · P. X. Tran
National Energy Technology Laboratory (NETL), U.S. Department of Energy, 626 Cochrans Mill Road, P.O. Box 10940, Pittsburgh, PA 15236, USA
E-mail: massoudi@netl.doe.gov; mehrdad.massoudi@netl.doe.gov

properties of coal–water mixtures (CWM) to atomization quality. Rheological properties such as high shear viscosity, yield stress, viscoelasticity and extensional viscosity have been hypothesized as the key parameters in predicting slurry atomization quality. Atomization quality, defined in terms of a mean spray droplet, varies with nozzle type and geometry, atomizing air-to-fuel ratio, coal type, particle size distribution and concentration, and additive type and concentration. By selectively formulating slurries for rheological evaluation and atomization screening, the key rheological properties can be identified. For many fluids such as polymers, slurries and suspensions, some generalizations have been made to model shear-dependent viscosities. The most widely used models are known as the power-law models or the generalized Newtonian fluid models. These models are deficient in many ways: They cannot predict the normal stress differences or yield stresses; they cannot capture the memory or history effects. At the same time, the power-law models have been used for a variety of applications where the shear viscosity is not constant [1,2,12,14,45]. Many suspensions are formed by adding solid particles to a fluid. As the concentration of the particles increases, the rheological properties of the fluid change drastically. One of the main areas of interest in energy-related processes, such as power plants, atomization and alternative fuels, is the use of slurries, specifically coal–water or coal–oil slurries, as the primary fuel. Some studies indicate that the viscosity of coal–water mixtures depends not only on the volume fraction of solids and the mean size, and the size distribution of the coal, but also on the shear rate, since the slurry in many situations behaves as a shear-thinning fluid (see [32,41,50]).

With additional need for fossil fuels, the amount of waste materials and the environmental issues dealing with their disposal also increase. One of the promising approaches is the development of coal/waste co-firing technology. For co-firing, biomass has been considered as one of the fuels. Modeling these complex fluids remains a challenge for both engineers and mathematicians. From an engineering point of view, one could, for example, consider an effective viscosity for the total medium which is not just the constant viscosity of the carrier fluid but also depends upon the concentration of the suspended particles (see [10]). In reality, such systems are composed of a mixture of the carrier fluid and particles which can be spherical, rod-like or irregular in shape. In general, a homogeneous approximation can be used (see [28]). Many mathematical models have been proposed to treat the flow of slurries. The idea of modeling suspensions through their material properties in various contexts is not new and has been used in the past (see [21,33]).

Although we have motivated our studies based on coal–water slurries, our approach is general and the model that we propose for the suspension of particles infused in a fluid can be used for the cases when the amount of solid particles in the fluid is not so high that the particle–particle interactions can be ignored and the suspension can be modeled as a single-component nonlinear fluid. Otherwise, we need to use a two-phase approach (mixture theory).

In this paper, we modify the thermodynamically compatible third-grade fluid model by introducing a shear-rate- and volume-fraction-dependent viscosity into the constitutive equation. We study the Couette–Poiseuille flow of such a fluid between two horizontal flat plates The steady fully developed flow equations are made dimensionless and are solved numerically; the effects of different dimensionless numbers are discussed. The plan of the paper is: In Sect. 2, we present the governing equations, and in Sect. 3, a brief discussion of the constitutive relation for the stress tensor $\mathbf{T}$ is provided. In Sect. 4, the equations of motion for flow between two flat plates along with the numerical scheme are presented.

2 Governing equations

The balance laws, in the absence of thermochemical and electromagnetic effects, are the conservation of mass, linear momentum and angular momentum (see [49]). The conservation of mass in the Eulerian form is given by:

$$\frac{\partial \rho}{\partial t} + \mathrm{div}\,(\rho \mathbf{u}) = 0 \tag{1}$$

where $\partial/\partial t$ is the partial derivative with respect to time. The balance of linear momentum is

$$\rho \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = \mathrm{div}\mathbf{T} + \rho \mathbf{b} \tag{2}$$

where d/dt is the material time derivative given by $\mathrm{d}(.)/\mathrm{d}t = \partial(.)/\partial t + \left[\mathrm{grad}(.)\right]\mathbf{u}$, $\mathbf{b}$ is the body force, and $\mathbf{T}$ is the Cauchy stress tensor [13]. The balance of angular momentum (in the absence of couple stresses) yields the result that the Cauchy stress is symmetric. In order to 'close' the governing equations, we need constitutive relations for $\mathbf{T}$.

3 Constitutive equations

A constitutive relation is an equation which relates the unknowns, such as the stress tensor $\mathbf{T}$ to other quantities such as velocity, displacement and temperature, which are the actual quantities that are to be solved in the governing equations, or can be measured experimentally. If the unknown quantities, for example, $\mathbf{T}$, are explicitly related to the kinematical (velocity, or displacement) or thermal (temperature) quantities, then that constitutive relation is known as an *explicit* constitutive relation. In general, in the explicit scheme, we start with equations such as:

$$\mathbf{T} = \mathbf{T}\left(\mathbf{u}, \text{grad } \mathbf{u}, \ldots\right) \tag{3}$$

where $\mathbf{u}$ is the velocity vector. The selection of the dependent variables, i.e., the arguments in the function, is based on previous experience, if any, intuition or insight, and perhaps experimental observation. The most well-known examples for the stress tensor are the Newtonian (Navier–Stokes) fluid $\mathbf{T} = -p\mathbf{I} + (\lambda \text{ tr } \mathbf{D})\,\mathbf{I} + 2\mu\mathbf{D}$. In the Newtonian fluid case, p is the unknown pressure, λ and μ are material properties, and $\mathbf{D} = \frac{1}{2}\left[\mathbf{L} + \mathbf{L}^{\mathrm{T}}\right]$ where $\mathbf{L} = \text{grad } \mathbf{u}$.

If, however, it is not possible to directly relate the unknowns to the knowable (or measurable) quantities, then we obtain an *implicit* constitutive relation. Amongst the early examples of implicit constitutive relations in modern continuum mechanics, one can name Maxwell's fluid models or Oldroyd's fluid [30,31] models[1] and Truesdell's hypo-elastic [47,48] model. Rajagopal [34], in a much more generalized context, proposed an implicit relation for the stress tensor of a fluid where

$$\mathbf{F}(\mathbf{T}, \mathbf{D}) = \mathbf{0} \tag{4}$$

Clearly, from an analytical or computational perspective, explicit constitutive equations are preferred as they are much easier to deal with; however, from a practical point of view, there are many materials, such as viscoelastic type with a relaxation time, whose response characteristics require an implicit model.

In this paper, we will use the explicit scheme to model the stress tensor for a suspension composed of particles infused in a fluid. It should be emphasized that a suspension, in general, can be modeled in at least two distinct, yet related manners. One is to look at the suspension as a single continuum and model it using either continuum theories or develop models based on statistical theories or experimental observations. Doing so, one would obtain macroscopic or global values for terms such as velocity, pressure drop, temperature and flow rate without knowing anything about the interactions between the particles and the host fluid (see [20]). The second method is to use the techniques of multi-component materials (also known as multi-phase theories) and study the governing equations for each component (phase) and propose or derive constitutive relations for each phase. This methodology, although more complicated, would in theory provide more information about particle concentration, individual velocities, temperatures, etc. (see [22,23]). In this paper, we shall model the suspension as a single continuum (a nonlinear fluid).

Briefly, non-Newtonian fluids differ from Newtonian fluids in at least two ways: (1) They exhibit "normal stress differences," phenomena such as rod climbing and die swell which are manifestations of the stresses that develop orthogonal to the planes of shear and (2) they exhibit "shear thinning" or "shear thickening," which is the decrease or increase in viscosity with increasing shear rate, respectively. Furthermore, many non-Newtonian (nonlinear) fluids can exhibit yield stress, memory effects, viscoelastic effects, etc. One of the most widely used non-Newtonian models in the field of engineering is the "power-law" model (see [2]), which allows for the viscosity to depend on the velocity gradient. This model has been extensively used in coal–water slurries (see [44]). However, this model cannot predict the normal stress effects (see [43]). Although the power-law model adequately fits the shear stress and shear rate measurements for many non-Newtonian fluids, it cannot always be used to accurately predict the pressure loss data measured during transport of a coal–liquid mixture in a fuel delivery system (see [6]).

Perhaps the simplest model which can capture the normal stress effects is the second-grade fluid, or the Rivlin–Ericksen fluid of grade two [35,40,49]. This model has been used and studied extensively [4] and is a special case of fluids of differential type. One of the recent advances in the theoretical studies in rheology is the development of generalized differential fluid models. The simplicity of the form and the fact that these modified constitutive relations can be used to study shear thinning/thickening, the decrease/increase in viscosity with

[1] Morgan [29] suggested that in general, an implicit constitutive equation is a relation between a kinetic tensor (for example the stress tensor $\mathbf{T}$) and one kinematic tensor (for example $\mathbf{B}$, or $\mathbf{D}$) of the types [see Eqs. (3.5) and (3.6) of that paper]: $\mathbf{G}(\mathbf{T}, \mathbf{B}) = \mathbf{0}$; $\mathbf{H}(\mathbf{T}, \mathbf{D}) = \mathbf{0}$.

increasing/decreasing shear rate, as well as predicting normal stress differences, have opened the way for the solution to a series of engineering problems (see [9,16,25,26]). Before we propose the new modified third-grade fluid, we give a brief review of the third-grade fluid studied in detail by Fosdick and Rajagopal [7]. The Cauchy stress tensor for such a fluid is given by (see also [5,49]):

$$\mathbf{T} = -p\mathbf{I} + \mu\mathbf{A}_1 + \mathbf{S}_2 + \mathbf{S}_3 \tag{5}$$

where

$$\mathbf{S_2} = \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A_1^2} \tag{6}$$

$$\mathbf{S_3} = \beta_1\mathbf{A_3} + \beta_2\left(\mathbf{A}_2\mathbf{A}_1 + \mathbf{A}_1\mathbf{A}_2\right) + \boldsymbol{\beta_3}\left(\mathrm{tr}\mathbf{A_2}\right)\mathbf{A}_1 \tag{7}$$

and $\mathbf{A}_\mathrm{n}$ is given by

$$\mathbf{A}_\mathrm{n} = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{A}_{\mathrm{n}-1} + \mathbf{A_{n-1}}\mathbf{L} + \mathbf{L^T}\mathbf{A_{n-1}} \tag{8}$$

Fosdick and Rajagopal [7] showed that for an incompressible thermodynamically compatible fluid of grade three, this model reduces to

$$\mathbf{T} = -p\mathbf{I} + \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A_1^2} + \beta_3\left[\mathrm{tr}\mathbf{A_1^2}\right]\mathbf{A}_1 \tag{9}$$

where

$$\begin{aligned} &\mu > 0, \\ &\alpha_1 \geq 0, \\ &|\alpha_1 + \alpha_2| \leq \sqrt{24\mu\beta_3}, \\ &\beta_3 \geq 0 \end{aligned} \tag{10}$$

If we rewrite this equation as

$$\mathbf{T} = -p\mathbf{I} + \left(\mu + \beta_3\mathrm{tr}\mathbf{A}_1^2\right)\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2 \tag{11}$$

it can be seen that this equation can also be considered as a generalization of the standard second-grade fluid model[2] (see [4]) with an effective viscosity, μ_{eff}, given by (see [27])

$$\mu_{\mathrm{eff}} = \mu + \beta_3\mathrm{tr}\mathbf{A}_1^2 \tag{12}$$

Mansutti and Rajagopal [18] and Mansutti et al. [19] have used the power-law model, the generalized second-grade model and the thermodynamically compatible third-grade model to study the steady flows past a porous plate and also between intersecting plates. It is readily seen from this equation that the third-grade fluid model can not only predict the normal stress differences but also the shear-thickening effects, which emerges from the $\mathbf{S}_3$ portion of the tensor. It can also be related to a special power-law model, where in a simple shear flow, the shear stress τ is given by

$$\tau = \left[\mu + \beta_3\left(\frac{\mathrm{d}u}{\mathrm{d}y}\right)^2\right]\frac{\mathrm{d}u}{\mathrm{d}y} \tag{13}$$

[2] Criminale et al. [3] obtained an expression for $\mathbf{T}$, valid for any laminar shear flow:

$$\mathbf{T} = -p\mathbf{I} + \beta_1\mathbf{A_1} + \beta_2\mathbf{A_2} + \beta_3\left(\mathbf{A_1^2} + \frac{1}{2}\mathbf{A_2}\right); \tag{$*$}$$

where β_1, β_2 and β_3 are functions of Π, where $\Pi = \frac{1}{2}\mathrm{tr}\mathbf{A}_1^2$, and they are given by

$$\begin{aligned} \beta_1 &= \gamma_1 + 2\gamma_5\Pi + 4\gamma_7\Pi^2, \\ \beta_2 &= \gamma_2 + 0.5\gamma_3 + 2\left(\gamma_4 + \gamma_6\right) + 4\gamma_8\Pi^2, \\ \beta_3 &= \gamma_3, \\ \gamma_m &= \alpha_m\left(2\Pi, 0, 4\Pi^2, 8\Pi^3,\ 0,\ 2\Pi^2, 0, 4\Pi^4\right), \end{aligned}$$

The model given by Eq. (*) is known as CEF model. It can be seen that when $\beta_2 = 2\beta_3$, this equation reduces to the Reiner–Rivlin fluid model [38,39]. Now, since β_1, β_2 and β_3 can be assigned arbitrarily as function of Π, in theory, the CEF model can predict shear-thinning (or thickening) as well as normal stress effects.

In many suspensions, the effects of particle concentration ϕ cannot be ignored. We propose the following modification to the effective coefficient of viscosity:

$$\mu_{\mathrm{eff}}(\phi, \mathbf{A_1}) = \beta_{30}\left(1 + a\phi + b\phi^2\right)\Pi^{\frac{m}{2}} \tag{14}$$

where

$$\Pi = \frac{1}{2}\mathrm{tr}\mathbf{A}_1^2 \tag{15}$$

where a and b are constants. The function $\phi(\mathbf{x}, t)$ is called the volume fraction distribution (concentration) and has the property $0 \leq \phi(\mathbf{x}, t) < \phi_{\max} < 1$. The function $\phi(\mathbf{x}, t)$ is represented as a continuous function of position and time; in reality, $\phi(\mathbf{x}, t)$ in such a system is either one or zero at any position and time, depending upon whether one is pointing to a granule or to the void space at that position. The dependence of the viscosity term on the volume fraction is based on the numerical simulation studies of [51,52]. Thus, the modified form of the thermodynamically compatible third-grade fluid where the coefficient of viscosity is a function not only of volume fraction, but also of shear rate, is:

$$\mathbf{T} = -p\mathbf{I} + \beta_{30}(1 + a\phi + b\phi^2)\Pi^{\frac{m}{2}}\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2 \tag{16}$$

It is this equation that we will use in our analysis.[3] It also needs to be mentioned that when using the grade fluids the order of differential equations is raised due to the introduction of higher-order derivates into the equations. As a result, in general, one needs additional boundary conditions.

3.1 Special cases of this model

(i) If, $\alpha_1 = \alpha_2 = 0$, then Eq. (16) reduces to

$$\mathbf{T} = -p\mathbf{I} + \beta_{30}\left(1 + a\phi + b\phi^2\right)\Pi^{\frac{m}{2}}\mathbf{A}_1 \tag{17}$$

Now, if there are no particles, that is, $\phi = 0$, then Eq. (17) reduces to

$$\mathbf{T} = -p\mathbf{I} + \beta_{30}\Pi^{\frac{m}{2}}\mathbf{A}_1 \tag{18}$$

which is the standard power-law model, where if m $>$ 0, the suspension is shear thickening, if m $<$ 0, the suspension is shear thinning, and if m = 0, the fluid behaves as a Navier–Stokes fluid.

(ii) If $m = 0$, and $\phi \neq 0$, then Eq. (17) reduces to

$$\mathbf{T} = -p\mathbf{I} + \beta_{30}\left(1 + a\phi + b\phi^2\right)\mathbf{A}_1 \tag{19}$$

Now, if $b = 0$ and $a = 2.5$, this equation would reduce to the classical expression derived by of Einstein, namely

$$\mu_{\mathrm{eff}} = \beta_{30}(1 + 2.5\phi) \tag{20}$$

(iii) Now, if there are no particles, $\phi = 0$, then Eq. (16) reduces to

$$\mathbf{T} = -p\mathbf{I} + \beta_{30}\Pi^{\frac{m}{2}}\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2 \tag{21}$$

This is the same as a generalization of the second-grade fluid due to [16,17] (for a detailed discussion of this, see [15,24]). Obviously, when $m = 0$, this equation reduces to the standard second-grade fluid model. It also needs to be mentioned that second-grade fluids (or higher-order models) raise the order of differential

[3] Note that the simplest expression for the nonlinear behavior of fluids is that of the generalized power-law model where $\mathbf{T} = -p\mathbf{I} + \mu_0\left(\mathrm{tr}\mathbf{A_1^2}\right)^m\mathbf{A_1}$ when $m < 0$, the fluid is shear thinning, and if $m > 0$, the fluid is shear thickening. This equation is a subclass of the model presented here.

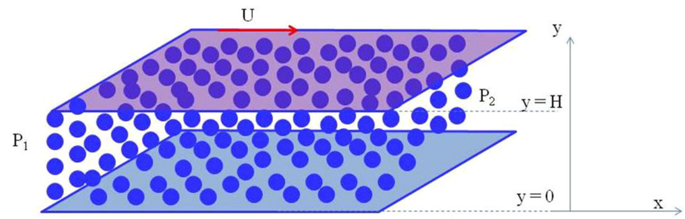

Fig. 1 Configuration of Poiseuille–Couette flow of suspensions between two flat plates. The *lower plate* is stationary and the *upper plate* is moving with a constant velocity U

equations by introducing higher-order derivates into the equations. As a result, in general, one needs additional boundary conditions; for a discussion of this issue, see Rajagopal [36] and Rajagopal and Kaloni [37]. In the next section, we shall use Eq. (16) to study the flow of a suspension between two horizontal flat plates.

4 Poiseuille–Couette flow between two flat plates

We are interested in the Poiseuille–Couette flow of a suspension between two long parallel plates (see Fig. 1). For the problem under consideration, we make the following assumptions:

(i) The motion is steady,
(ii) The constitutive equation for the stress tensor is given by Eq. (16),
(iii) The volume fraction and the velocity fields are of the form.

$$\phi = \phi(y) \tag{22a}$$
$$\mathbf{u} = u(y)\mathbf{i} \tag{22b}$$

In the absence of the body forces, and with the above assumptions, the conservation of mass is automatically satisfied and the momentum equations are given by

$$\frac{\partial}{\partial x}\left[-p + \alpha_2\left(\frac{\mathrm{d}u}{\mathrm{d}y}\right)^2\right] + \frac{\partial}{\partial y}\left[\beta_{30}\left(1 + a\phi + b\phi^2\right)\left(\left|\frac{\mathrm{d}u}{\mathrm{d}y}\right|^2\right)^{m/2}\frac{\mathrm{d}u}{\mathrm{d}y}\right] = 0 \tag{23}$$

$$\frac{\partial}{\partial x}\left[\beta_{30}\left(1 + a\phi + b\phi^2\right)\left(\left|\frac{\mathrm{d}u}{\mathrm{d}y}\right|^2\right)^{m/2}\frac{\mathrm{d}u}{\mathrm{d}y}\right] + \frac{\partial}{\partial y}\left[-p + (2\alpha_1 + \alpha_2)\left(\frac{\mathrm{d}u}{\mathrm{d}y}\right)^2\right] = 0 \tag{24}$$

$$\frac{\partial p}{\partial z} = 0 \tag{25}$$

Let us define

$$-\hat{p} = -p + (2\alpha_1 + \alpha_2)\left(\frac{\mathrm{d}u}{\mathrm{d}y}\right)^2 \tag{26}$$

Since ϕ and u depend on y only, the first term of Eq. (24) is equal to zero and Eq. (24) reduces to

$$\frac{\partial}{\partial y}\left[-p + (2\alpha_1 + \alpha_2)\left(\frac{\mathrm{d}u}{\mathrm{d}y}\right)^2\right] = \frac{\partial \hat{p}}{\partial y} = 0 \tag{27}$$

And Eq. (23) reduces to

$$\frac{\partial \hat{p}}{\partial x} = \frac{\partial}{\partial y}\left[\beta_{30}\left(1 + a\phi + b\phi^2\right)\left(\left|\frac{\mathrm{d}u}{\mathrm{d}y}\right|^2\right)^{m/2}\frac{\mathrm{d}u}{\mathrm{d}y}\right] \tag{28}$$

where $\partial \hat{p}/\partial x$ is the prescribed pressure gradient. These equations are subject to the boundary conditions,

$$\text{at } y = 0: \quad u = 0 \tag{29a}$$
$$\text{at } y = H: \quad u = U \tag{29b}$$

Let us define the dimensionless distance $\bar{y}$, the velocity $\bar{u}$ and the viscosity $\bar{\mu}$ by the following equations (see [9]):

$$\bar{y} = \frac{y}{H}; \tag{30a}$$

$$\bar{u} = \frac{u}{U}; \tag{30b}$$

$$\bar{\mu} = \frac{\beta_{30}}{\mu_0} \tag{30c}$$

where U is the velocity of the upper plate, μ_0 is a reference viscosity, and H is the distance between the two plates. With these definitions, the dimensionless form of Eq. (28) becomes

$$\frac{\mathrm{d}}{\mathrm{d}\bar{y}}\left[\left(1 + a\phi + b\phi^2\right)\left(\left|\frac{\mathrm{d}\bar{u}}{\mathrm{d}\bar{y}}\right|^2\right)^{m/2}\frac{\mathrm{d}\bar{u}}{\mathrm{d}\bar{y}}\right] = C \tag{31}$$

where C is a parameter representing the effects of the upper plate velocity, the plate gap, the pressure gradient and the suspension viscosity and is given by

$$C = \frac{1}{\bar{\mu}}\frac{\partial \hat{p}}{\partial x}\left(\frac{H}{\mu_o}\right)\left[\frac{H}{U}\right]^{m+1} \tag{32}$$

The dimensionless boundary conditions are

$$\text{at } \bar{y} = 0: \quad \bar{u} = 0 \tag{33a}$$

$$\text{at } \bar{y} = 1: \quad \bar{u} = 1 \tag{33b}$$

Equation (31) can be rewritten as

$$\frac{\mathrm{d}^2\bar{u}}{\mathrm{d}\bar{y}^2} = \frac{C + (a + 2b\phi)\frac{\mathrm{d}\phi}{\mathrm{d}\bar{y}}\left(-\frac{\mathrm{d}\bar{u}}{\mathrm{d}\bar{y}}\right)^{m+1}}{\left(1 + a\phi + b\phi^2\right)(m+1)\left(-\frac{\mathrm{d}\bar{u}}{\mathrm{d}\bar{y}}\right)^{m}}; \quad \text{when } \frac{\mathrm{d}\bar{u}}{\mathrm{d}\bar{y}} \prec 0 \tag{34}$$

$$\frac{\mathrm{d}^2\bar{u}}{\mathrm{d}\bar{y}^2} = \frac{C - (a + 2b\phi)\frac{\mathrm{d}\phi}{\mathrm{d}\bar{y}}\left(\frac{\mathrm{d}\bar{u}}{\mathrm{d}\bar{y}}\right)^{m+1}}{\left(1 + a\phi + b\phi^2\right)(m+1)\left(\frac{\mathrm{d}\bar{u}}{\mathrm{d}\bar{y}}\right)^{m}}; \quad \text{when } \frac{\mathrm{d}\bar{u}}{\mathrm{d}\bar{y}} \succ 0 \tag{35}$$

Equations (34) and (35) describe the dimensionless velocity for the Poiseuille–Couette flow of a suspension between two flat plates. To integrate these equations, one needs to either calculate (based on other equations or methods) or know (based on experiments) the volume fraction distribution. From an engineering point of view, generally, the volume fraction (or concentration) profile is determined via experiments (see, e.g., [8], p. 662). However, depending on the loading of the particles, that is, how high or low the particle concentration is, one can also model the complex fluid using various approaches. For example, Phillips et al. [33] suggested a Convection–Diffusion type equation for calculating the concentration. At higher concentration, methods in bulk transport (see [46]) can be used.

Kabir et al. [11] used the explicit finite element method to simulate a dense flow of steel particles and titanium oxide (TiO_2) powder in a rough gravity-free Couette shear cell with no externally applied load. They reported that the flow was dilute near the walls with the moving wall having the least solid fraction due to the high-energy particle–wall collisions. The particles continuously had their kinetic energy dissipated by the inelastic collisions in the internal region of the cell, which lead to stagnant particles and a denser (i.e., high solid fraction) granular field. For the present work, we express their calculated data using Eq. (36) as shown in Fig. 2.

$$\phi = 0.6351 + 1.555\ \bar{y} - 5.5829\ \bar{y}^2 + 8.108\ \bar{y}^3 - 4.197\ \bar{y}^4 \tag{36}$$

In our present calculations, we simply use Eq. (36) as an example for the solid fraction distribution. We also let $b = 0$ and $a = 2.5$ which is the classical Einstein model for the suspension viscosity. We, therefore, concentrate our attention only on the effects of the non-Newtonian nature of the suspension and the competition

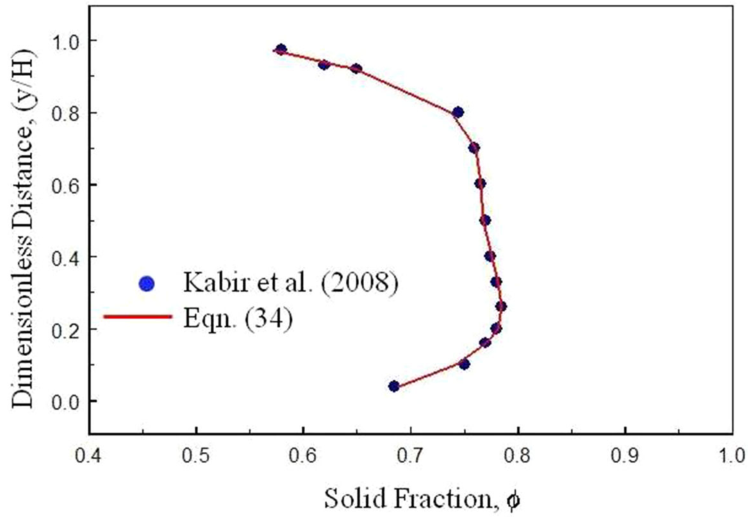

Fig. 2 Distribution of the solid fraction across the gap between two flat plates; the *lower plate* is stationary and the *upper plate* is moving

between the pressure gradient and the shear thinning or thickening on the velocity field. We use two values for m ($m = 1$ and $m = -2$), and we vary C from -6.5 to 6.5. In most engineering applications, the volume fraction profile is obtained via some experiments, and in the suspension approach, it is rarely calculated. What we have done in this study, is to use the numerical simulation of a similar problem (see [11]) and obtain a correlation for the volume fraction which is used in our study. Alternatively, if we had used a more complicated scheme such as the multi-phase approach (see [22,23]), we would have been able to actually solve for the volume fraction. The results are presented in the next section.

5 Results and discussion

Figure 3 shows typical velocity profiles for various values of C and for $m = 1$ (the shear-thickening case) and $m = -2.0$ (the shear-thinning case). The positive values of C represent the cases where the pressure gradient is against the direction of the shearing velocity, while the negative values are for the case in which the pressure gradient is in the direction as the shearing velocity. When $C = 0$, that is, when no pressure gradient is imposed and the flow is driven only by the upper plate, the velocity distribution is linear. This is the typical Couette flow between two flat plates where the velocity profile is the same for both Newtonian and non-Newtonian fluids. When C is not equal to zero, the flow is driven by the pressure gradient as well as the upper moving plate. As a result, since C represents the competition between the pressure gradient and the shearing motion, the velocity profiles become nonlinear and dependent on the magnitude of C. Thus, for $C < \pm 1$, the effect of the moving plate is more significant, and the velocity profiles still retain their linearity throughout the gap. When $C > 1$, the effect of the pressure gradient becomes more significant. For the shear-thickening fluid ($m > 0$), the flow becomes faster, and the fluid near the upper plate moves faster than the upper plate velocity as C increases negatively. On the other hand, as C increases positively, the flow near the lower plate is retarded and a reversed flow could prevail for high values of C. For the shear-thinning case, ($m < 0$), the results on the velocity profiles are reversed, that is, the retarded flow occurs if the values of C increase positively. If C increases negatively, the flow becomes faster. For the range of the values used for C, however, both the reversed flow and the faster flow (than the moving plate) did not occur.

The effect of C on the flow can be clearly seen in Fig. 4 where the derivatives of the velocity at the lower and upper plates as a function of C are given. It can be seen that as C increases in a positive sense, the derivatives of the velocity with shear-thickening behavior decrease, indicating that the flow is retarded near the lower plate while for a shear-thinning fluid the flow is faster as indicated by the decrease in the derivative of the velocity at the upper plate. As C increases negatively, the effect is reversed, that is, the flow of fluid with shear-thickening behavior becomes faster, while that of a shear-thinning fluid becomes retarded.

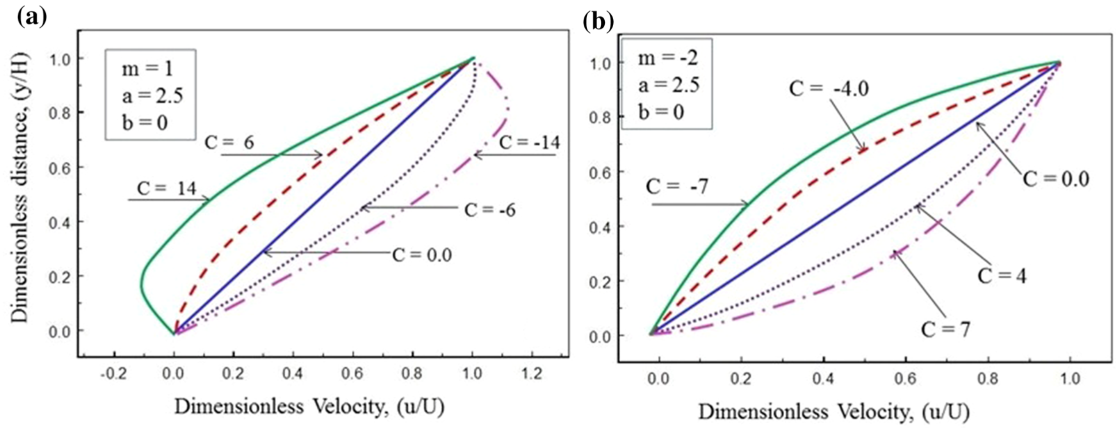

Fig. 3 **a** Typical dimensionless velocity profiles for various values of C $(m = 1.5)$. **b** Typical dimensionless velocity profiles for various values of C $(m = -2)$

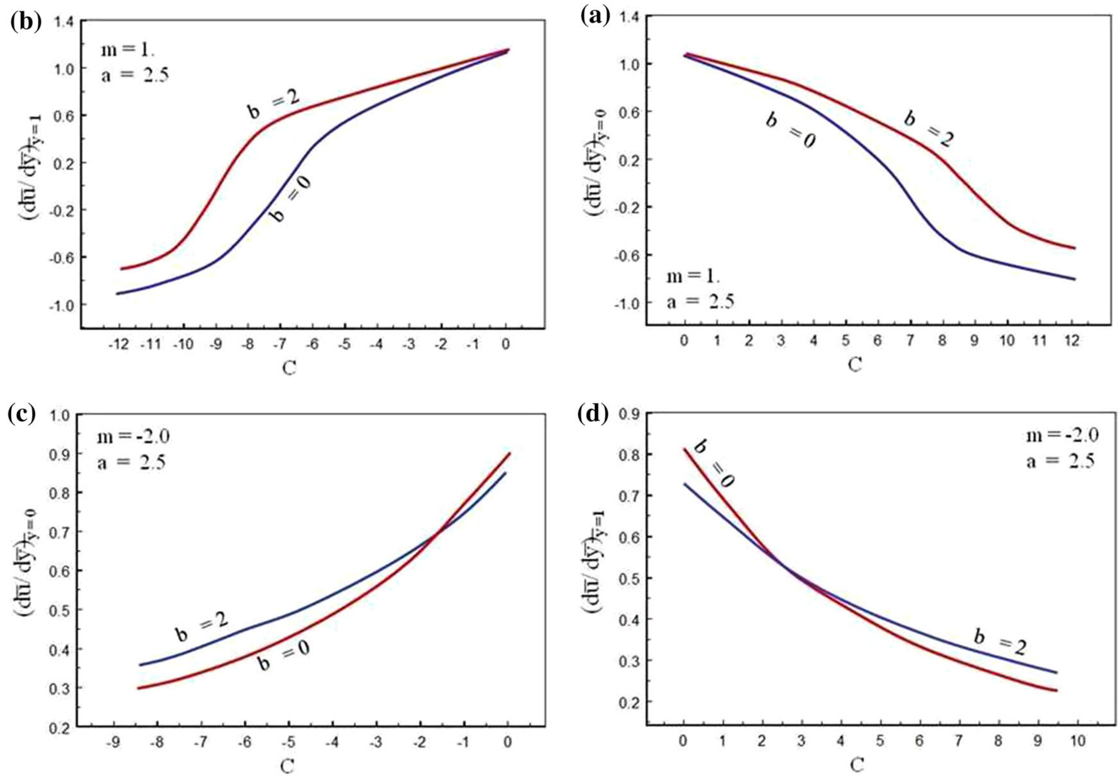

Fig. 4 Derivatives of the dimensionless velocity at the plate surfaces as a function of **C** (**a**, **c** derivative at the surface of the lower plate; **b**, **d** derivative at the surface of the upper plate)

Flow behaviors similar to those described above are also obtained when $b \neq 0$. Since b is the coefficient that describes the dependence of the viscosity on the volume fraction of the particles and C is the parameter that represents the competition between the pressure gradient and the shearing velocity, varying values of b would result in changing the magnitude of C at which the flow is reversed or becomes faster than the velocity of the moving upper plate. For example, from Fig. 4a, b it is clear that with $b = 0$, $(\mathrm{d}\bar{u}/\mathrm{d}\bar{y})_{\bar{y}=0}$ becomes negative when C is about 6.5 while for $b = 2$ this occurs for C up to 8.5. On the other hand, as C increases negatively, the flow becomes faster, $(\mathrm{d}\bar{u}/\mathrm{d}\bar{y})_{\bar{y}=1} < 0$, at $C = -6.75$ when $b = 0$, and this occurred when the value of C was about -9.5 for the case when $b = 2$.

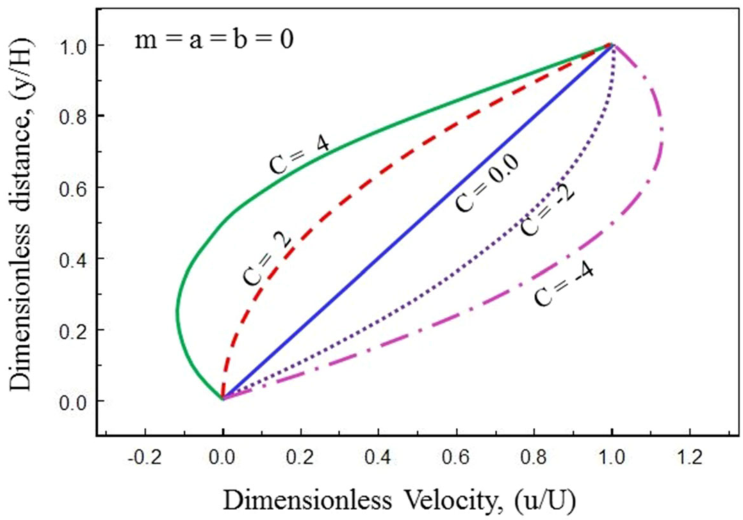

Fig. 5 Effects of parameter C on the dimensionless velocity profiles of Poiseuille–Couette flow of a Newtonian fluid

If in Eq. 31 we set $a = b = m = 0$, we obtain the following analytical solution for the dimensionless velocity

$$\bar{u} = C\frac{\bar{y}^2}{2} + \left(1 - \frac{C}{2}\right)\bar{y} \tag{37}$$

This is the velocity profile for the generalized Couette flow problem for a Navier–Stokes fluid (see Schlichting ([42], p. 85). We have plotted the results in Fig. 5 to show that in the limiting case, our equations reduce to the standard version of the Navier–Stokes equation.

6 Conclusions

In this paper, we modify the constitutive relation for a thermodynamically compatible third-grade fluid by suggesting that the shear viscosity can depend on the shear rate and the volume fraction of particles suspended in the fluid. For the concentration (volume fraction) field, we use the numerical simulation of Kabir et al. [11] to obtain a polynomial expression. The Couette–Poiseuille flow of this suspension is studied, and the dimensionless forms of the governing equations are solved numerically. Qualitatively, we can see that since we have assumed these parameters to be functions of the volume fraction, there is a strong nonlinearity in the equations, and therefore, numerically it will be more difficult to obtain solutions. This simple boundary value problem with all the basic assumptions specified should serve as a limiting case for more complicated flow geometries and flow conditions.

The positive values of C represent the cases where the pressure gradient is against the direction of the shearing velocity, while the negative values are for the case in which the pressure gradient is in the direction as the shearing velocity. When $C = 0$, that is, when no pressure gradient is imposed and the flow is driven only by the upper plate, the velocity distribution is linear. This is the typical Couette flow between two flat plates where the velocity profile is the same for both Newtonian and non-Newtonian fluids. When C is not equal to zero, the flow is driven by the pressure gradient as well as the upper moving plate. As a result, since C represents the competition between the pressure gradient and the shearing motion, the velocity profiles become nonlinear and dependent on the magnitude of C. Obviously, the effects of slip at the wall, particle shape, etc. are important issues which need to be studied.

Finally, since we have non-dimensionalized our equations, we cannot really compare the simulation with any given suspension of coal slurries as the material parameters have been absorbed into the dimensionless numbers. The current study was not intended to be an engineering-type modeling; instead, it is a parametric study into this nonlinear model. The main reason for doing a parametric study via non-dimensionalizing the equations of motion is that we can gain some insight into a class of problems.

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