EXISTENCE FOR AN ALLEN-CAHN/CAHN-HILLIARD SYSTEM WITH DEGENERATE MOBILITY *

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Abstract. We prove existence in one space dimension of weak solutions for the Neumann problem for a degenerate parabolic system consisting of a fourth-order and a second-order equation with singular lower-order terms. This system arises in the description of phase separation and ordering in binary alloys.

1 Introduction

Our paper is concerned with the Allen-Cahn/Cahn-Hilliard system

$$
(\mathbf{P})\begin{cases}
u_t = [Q(u,v)(\mathcal{F}_u(u,v) - \varepsilon u_{xx})_x]_x & (x,t)\in\Omega_T\\
v_t = -Q(u,v)(\mathcal{F}_v(u,v) - \varepsilon v_{xx}) & (x,t)\in\Omega_T\\
u_x = v_x = Q(u,v)(\mathcal{F}_u(u,v) - \varepsilon u_{xx})_x = 0 & (x,t)\in S_T\\
u(x,0) = u_0(x), \quad v(x,0) = v_0(x) & x\in\Omega\\
(u,v)\in \overline{B} & (x,t)\in\Omega_T
\end{cases}
$$

where $\Omega \subset \mathbb{R}$ is a bounded open interval, $T>0$, $\Omega_T = \Omega\times(0,T)$ and $S_T = \partial\Omega\times(0,T)$. The homogeneous free energy $\mathcal{F}$ is assumed here to have the form

$$\mathcal{F}(u,v) = F(u+v) + F(u-v) + F(1-(u+v)) + F(1-(u-v)) + \alpha u(1-u) - \beta v^2,$$

where $F(s) = \frac{\Theta}{2} s\ln s$ and Θ denotes the absolute temperature; the function $\mathcal{F}$ is defined in the square

$$B = \{(u,v)\in\mathbb{R}^2 : 0 < u+v < 1,\ 0 < u-v < 1\},$$

and $Q(u,v)$, the mobility, is nonnegative.

***Key words:** nonlinear parabolic systems, degenerate systems, Allen-Cahn and Cahn-Hilliard equations, phase transitions.

The system (**P**) with constant mobility was introduced by Cahn and Novick-Cohen [6] in the description of simultaneous phase separation and order-disorder transition in a BCC Fe-Al binary alloy: u and v represent a conserved (typically an average concentration) and a non-conserved order parameter, respectively. See [6] for earlier relevant references. In [7], [13], Cahn and Novick-Cohen and Novick-Cohen developed formal asymptotics for the description of the large time behaviour for this system near the low temperature limit. The system (**P**) is also used by Cahn and Novick-Cohen in [8] as a framework in which to model the effects of solute drag on the motion of antiphase boundaries. In the present paper we shall make the further assumption that the mobility Q is given by

$$Q(u,v) = u(1-u)(\frac{1}{4} - v^2) =: Q_1(u)Q_2(v);$$

such a form satisfies the physical notion that the mobility should vanish at the pure phases and also turns out to be analytically convenient to consider.

In the case of nondegenerate mobilities, existence for the Cahn-Hilliard equation has been obtained by Blowey and Elliott [4] in the deep-quench limit; i.e. with an homogeneous free energy of the form

$$\mathcal{F}(u) = \begin{cases} u(1-u) & 0 \le u \le 1 \\ +\infty & \text{elsewhere,} \end{cases}$$

and by Barrett and Blowey [1] with a logarithmic free energy. The degenerate case has been studied by Elliott and Garcke [9] and Barrett, Blowey and Garcke [2] under general assumptions on Q and $\mathcal{F}$ which include the case

$$Q(u) = u(1-u),$$
$$\mathcal{F}(u) = F(u) + F(1-u) + \alpha u(1-u).$$

Recently, Elliott and Garcke [10] and Garcke and Novick-Cohen [11] have obtained existence results for respectively systems of Cahn-Hilliard equations with degenerate mobilities and the limiting geometric motions to which such systems give raise in the long time, low temperature limit. We mention that systems of Allen-Cahn/Cahn-Hilliard type with nonsingular lower-order terms and nondegenerate mobilities have been considered by Brochet, Hilhorst and Novick-Cohen in [5].

In considering Problem (**P**), some difficulty arises from the constraint $(u,v) \in \overline{B}$; indeed, note that the partial derivatives of $\mathcal{F}$ can be extended to be continuous functions with values in $\mathbb{R} \cup \{\pm\infty\}$ only in $\overline{B} \setminus \{(0, \pm\frac{1}{2}), (1, \pm\frac{1}{2})\}$, and not in $\overline{B}$. This difficulty can be handled in the case of positive mobilities via a-priori estimates in $L^1(\Omega_T)$ on the terms

$$F'(u+v), \ldots, F'(1-(u-v)) \tag{1.1}$$

which are logarithmic. This estimate implies that $(u,v) \in B$ almost everywhere in Ω_T in the case of positive mobilities. Some additional difficulty is given by the fact that the mobility degenerates only at the vertices of the box B, whereas the derivatives of $\mathcal{F}$ are singular over the whole boundary ∂B. Therefore there is no obvious way to bound a-priori the product of mobilities and derivatives of $\mathcal{F}$. For a single degenerate C-H equation such difficulties do not arise as the locus of the zeroes of the mobility and the singularities of the logarithmic terms appearing in the free energy is identical. Thus, this represents a main difference between the system which we consider here and the case of a single degenerate C-H equation [9]. On the other hand, the assumed form of Q allows the derivation of an estimate for v, namely

$$\int_\Omega \frac{1}{\left(\frac{1}{4}-v^2(t)\right)^{\beta-1}} \le c + \int_\Omega \frac{1}{\left(\frac{1}{4}-v_0^2\right)^{\beta-1}} \quad \text{if } \beta \in (1,4]. \tag{1.2}$$

This estimate permits control of the degeneracy of the system with respect to the variable v. It is worth noting that the estimate (1.2) only holds if one knows a-priori that the range of the solution is contained in the closure of the square. For this reason we first prove an existence result for the case of nondegenerate mobilities with a logarithmic free energy, whose solutions can be shown to satisfy $(u,v) \in \overline{B}$ by virtue of the estimates on terms of the type given in (1.1). Afterwards we derive an estimate in $L^1(\Omega_T)$ for the terms

$$u(1-u)vF'(u+v), \ldots, u(1-u)vF'(1-(u-v)) \tag{1.3}$$

which holds for arbitrarily positive mobilities and implies that the constraint $(u,v) \in B \cup \{(0,0),(1,0)\}$ is satisfied almost everywhere. Indeed, we are not able to exclude the possibility that the sets in Ω_T where $(u,v)=(0,0)$ or $(u,v)=(1,0)$ have positive measure – a possibility which has also not been excluded in the case of a single degenerate C-H equation with free energy and mobility of analogous form.

Throughout the paper we make the following assumption on the initial data:

$$\begin{gathered} u_0 \in H^1(\Omega), \quad v_0 \in H^1(\Omega), \\ (u_0, v_0) \in \overline{B}, \quad v_0 \in (-1/2, 1/2), \quad \overline{u_0} \in (0,1), \end{gathered} \tag{1.4}$$

where $\overline{f}$ denotes the mean value of f in Ω of a given $f \in L^1(\Omega)$. Let us introduce some notation which we shall use in the sequel. Given $u, v \in C(\overline{\Omega}_T)$, we define the sets

$$\begin{gathered} \mathcal{D}_\eta(u) = \{(x,t) \in \overline{\Omega}_T : \eta < u(x,t) < 1-\eta\}, \quad 0 \le \eta < \tfrac{1}{2}, \\ \mathcal{B}(u,v) = \{(x,t) \in \overline{\Omega}_T : (u,v) \in B\}, \end{gathered}$$

with the convention that $\mathcal{D}(u) = \mathcal{D}_0(u)$. We denote by $\langle \cdot, \cdot \rangle$ the duality pairing between $(H^1(\Omega))'$ and $H^1(\Omega)$.

Let us define our concept of a weak solution for Problem $(\mathbf{P})$ as follows.

Definition 1 *A triplet (u,v,w) is called a weak solution of $(\mathbf{P})$ if:*

(i) $u \in C^{0,\frac{1}{2},\frac{1}{8}}(\overline{\Omega}_T) \cap L^\infty(0,T;H^1(\Omega))$,
$v \in C^{0,\frac{1}{2},\frac{1}{4}}(\overline{\Omega}_T) \cap L^\infty(0,T;H^1(\Omega))$,
$w \in L^2(\Omega_T)$;

(ii) $u_{xx} \in L^2_{\mathrm{loc}}(\mathcal{D}(u))$, $v_{xx} \in L^2_{\mathrm{loc}}(\mathcal{D}(u))$;

(iii) $u_t \in L^2(0,T;(H^1(\Omega))')$, $v_t \in L^2(\Omega_T)$;

(iv) $(u,v) \in \overline{B}$, $v \in (-1/2,1/2)$;

(v) $\mathcal{F}_u(u,v) \in L^2_{\mathrm{loc}}(\mathcal{D}(u))$, $\mathcal{F}_v(u,v) \in L^2_{\mathrm{loc}}(\mathcal{D}(u))$;

(vi) $u(x,0) = u_0(x)$, $v(x,0) = v_0(x)$;

(vii) $u_x|_{S_T\cap\mathcal{D}(u)} = v_x|_{S_T\cap\mathcal{D}(u)} = 0$ *in* $L^2_{\mathrm{loc}}(S_T\cap\mathcal{D}(u))$;

(viii) (u,v,w) *solves the system* $(\mathbf{P})$ *in the following sense:*

$$\int_0^T \langle u_t,\phi\rangle\,dt = -\iint_{\Omega_T} [Q(u,v)]^{1/2} w\phi_x \quad \forall \phi \in L^2(0,T;H^1(\Omega)), \tag{1.5}$$

$$\iint_{\Omega_T} v_t\psi = -\iint_{\Omega_T} Q(u,v)(\mathcal{F}_v(u,v) - \varepsilon v_{xx})\psi \quad \forall \psi \in L^2(\Omega_T):\ \operatorname{supp}\psi \subset \mathcal{D}(u), \tag{1.6}$$

where $w = Q^{\frac{1}{2}}(u,v)(\mathcal{F}_u(u,v) - \varepsilon u_{xx})_x$ *in the sense that*

$$\iint_{\Omega_T} w\phi = -\iint_{\Omega_T} (\mathcal{F}_u(u,v) - \varepsilon u_{xx})\left([Q(u,v)]^{1/2}\phi\right)_x \tag{1.7}$$

for all $\phi \in L^2(0,T;H^1_0(\Omega))$ *with* $\operatorname{supp}\phi \subset \mathcal{D}(u)$.

Remark 1.1 The terms in the Definition are all well defined. In particular:

1. The traces of u_x, v_x in $S_T \cap \mathcal{D}(u)$ can be seen to be well defined by the following argument:
   Let $K \subset S_T \cap \mathcal{D}(u)$ compact; then $K \subseteq S_T \cap \mathcal{D}_\eta(u)$ for some $\eta > 0$. Since $u \in C(\overline{\Omega}_T)$, we can select $\zeta \in C^\infty(\overline{\Omega}_T)$ such that $0 \le \zeta \le 1$, $\zeta = 1$ in $\mathcal{D}_\eta(u)$ and $\zeta = 0$ in $\overline{\Omega}_T \setminus \mathcal{D}_{\eta/2}(u)$. Then $\zeta u \in L^2(0,T;H^2(\Omega))$ and therefore $(\zeta u)_x|_{S_T} \in L^2(0,T;L^2(\partial\Omega))$. It follows that
   $$u_x|_{S_T\cap K} \in L^2(S_T \cap K).$$
   The argument for v_x is the same.

2. Since *(iv)* implies that $Q_2(v)$ is strictly positive, from *(i)* it then follows that the function $\left([Q(u,v)]^{1/2}\phi\right)_x$ has compact support in $\mathcal{D}(u)$ and belongs to $L^2(\Omega_T)$ for any $\phi \in L^2(0,T;H^1(\Omega))$ with $\operatorname{supp}\phi \subset \mathcal{D}(u)$. Therefore the integral at the right hand side of (1.7) makes sense.

The main result of the paper is the following

Theorem 1 *For all u_0, v_0 satisfying (1.4) there exists a weak solution of Problem* (**P**) *in the sense of Definition 1; in addition,*

$$|\mathcal{D}(u) \setminus \mathcal{B}(u,v)| = 0.$$

In Section 2 we obtain (by means of a Galerkin approximation) an existence result for suitably regularized systems; i.e., for systems with the form of Problem (**P**) but which have been made to be uniformly parabolic by assuming sufficient additional regularity properties for Q and $\mathcal{F}$. The proof of Theorem 1 is contained in Section 3, where by rescaling, without loss of generality we have assumed $\Theta = 2$: in subsection 3.1 we define a two-parameter family of approximating regular systems and derive an energy inequality yielding uniform estimates for the corresponding solutions; in subsection 3.2 we obtain intermediate results for nondegenerate systems based on a free energy which is now of the original logarithmic form, $\mathcal{F}(u,v)$; in subsection 3.3 we use these approximating solutions to construct a weak solution of Problem (**P**).

We observe that most of the results contained in subsection 3.1-3.2 remain valid in higher space dimensions. Actually it is the continuity of the solution that enables us to control the boundary of the box B uniformly with respect to the approximating procedure; however, the need for continuity forces our results to be restricted, up to now, to space dimension one.

2 Regularized systems

In this section we prove an existence theorem for the Allen-Cahn/Cahn-Hilliard system under additional regularity and positivity assumptions on the mobility and the homogeneous free energy. This result will be used later to construct approximating solutions to Problem (**P**).

Consider the following system:

$$(\mathbf{P'}) \begin{cases} u_t = [q_1(u,v)(f_1(u,v) - \varepsilon u_{xx})_x]_x & (x,t) \in \Omega_T \\ v_t = -q_2(u,v)(f_2(u,v) - \varepsilon v_{xx}) & (x,t) \in \Omega_T \\ v_x = u_x = u_{xxx} = 0 & (x,t) \in S_T \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x) & x \in \Omega \end{cases}$$

where q_i, f_i satisfy:
(H1) $q_i \in C(\mathrm{I\!R}^2; \mathrm{I\!R}^+)$, with $q_{\min} \leq q_i \leq q_{\max}$ for some $0 < q_{\min} \leq q_{\max}$;
(H2) $f_1 \in C^1(\mathrm{I\!R}^2; \mathrm{I\!R})$ and $f_2 \in C(\mathrm{I\!R}^2, \mathrm{I\!R})$, with $||f_1||_{C^1} + ||f_2||_{C^0} \leq F_0$ for some $F_0 > 0$.

Theorem 2.1 *Assuming (H1), (H2) and $u_0, v_0 \in H^1(\Omega)$, there exists a pair of functions (u,v) such that:*

(i) $u \in L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^3(\Omega)) \cap C(0,T;H^\lambda(\Omega))$, $\lambda < 1$;
(ii) $v \in L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega)) \cap C(0,T;H^\lambda(\Omega))$, $\lambda < 1$;
(iii) $u_t \in L^2(0,T;(H^1(\Omega))')$, $v_t \in L^2(\Omega_T)$;
(iv) $u(0) = u_0$ *and* $v(0) = v_0$ *in* $L^2(\Omega)$;
(v) $u_x|_{S_T} = v_x|_{S_T} = 0$ *in* $L^2(S_T)$;
(vi) (u,v) *solves* $(\mathbf{P}')$ *in the following sense:*

$$\int_0^T \langle u_t, \phi\rangle \, dt = -\iint_{\Omega_T} q_1(u,v)(f_1(u,v) - \varepsilon u_{xx})_x \phi_x \quad \forall \phi \in L^2(0,T;H^1(\Omega))$$
$$\iint_{\Omega_T} v_t \psi = -\iint_{\Omega_T} q_2(u,v)(f_2(u,v) - \varepsilon v_{xx})\psi \quad \forall \psi \in L^2(\Omega_T).$$

Proof. We apply a Galerkin approximation. Let us denote by ψ_i, $i \in \mathbf{N}$, the eigenfunctions of minus the Laplacian with Neumann boundary conditions:

$$\begin{cases} -\psi_i'' = \lambda_i \psi_i & x \in \Omega \\ \psi_i' = 0 & x \in \partial\Omega. \end{cases}$$

Without loss of generality, we assume that the eigenfunctions ψ_i are orthogonal in $H^1(\Omega)$, orthonormal in $L^2(\Omega)$, and that the first eigenvalue is zero ($0 = \lambda_1 < \lambda_2 \ldots$). The Galerkin ansatz for $(\mathbf{P}')$

$$u^N(x,t) = \sum_{i=1}^N a_i^N(t)\psi_i(x), \quad v^N(x,t) = \sum_{i=1}^N b_i^N(t)\psi_i(x)$$

yields the following initial value problem:

$$\frac{da_j^N}{dt} = \int_\Omega u_t^N \psi_j = -\int_\Omega q_1(u^N,v^N)(f_1(u^N,v^N) - \varepsilon u_{xx}^N)_x \psi_{jx} \tag{2.1}$$

$$\frac{db_j^N}{dt} = \int_\Omega v_t^N \psi_j = -\int_\Omega q_2(u^N,v^N)(f_2(u^N,v^N) - \varepsilon v_{xx}^N)\psi_j \qquad j = 1,\ldots,N \tag{2.2}$$

$$a_j^N(0) = (u_0,\psi_j)_{L^2(\Omega)}, \quad b_j^N(0) = (v_0,\psi_j)_{L^2(\Omega)}. \tag{2.3}$$

The problem has a local solution because the right hand side is continuous with respect to $(a_1^N,\ldots,a_N^N,b_1^N,\ldots,b_N^N)$. We now derive an a-priori estimate which allows us to infer global existence. Multiplying each equation of (2.1) by $-\lambda_j a_j(t)$ and summing (that is, using u_{xx}^N as a test function), we get

$$\frac{d}{dt}\frac{1}{2}\int_\Omega (u_x^N)^2 + \varepsilon \int_\Omega q_1(u_{xxx}^N)^2 = \int_\Omega q_1 f_{1u} u_x^N u_{xxx}^N + \int_\Omega q_1 f_{1v} v_x^N u_{xxx}^N;$$

using v^N_{xx} as a test function in (2.2) we obtain

$$\frac{d}{dt}\frac{1}{2}\int_\Omega (v^N_x)^2 + \varepsilon \int_\Omega q_2 (v^N_{xx})^2 = \int_\Omega q_2 f_2 v^N_{xx}.$$

Summing the two identities, using (H1)-(H2), and applying the Cauchy-Schwarz and Young's inequalities

$$\frac{d}{dt}\left[\int_\Omega (u^N_x)^2 + \int_\Omega (v^N_x)^2\right] + \int_\Omega (u^N_{xxx})^2 + \int_\Omega (v^N_{xx})^2 \le c_1 + c_2 \left[\int_\Omega (u^N_x)^2 + \int_\Omega (v^N_x)^2\right],$$

from which we obtain by a Gronwall argument

$$\int_\Omega (u^N_x)^2 + \int_\Omega (v^N_x)^2 + \iint_{\Omega_T} (u^N_{xxx})^2 + \iint_{\Omega_T} (v^N_{xx})^2 \le c_3(T). \tag{2.4}$$

Since $\psi_1' \equiv 0$, it follows from (2.1) that $\int_\Omega u^N_t = 0$; hence, by Poincaré inequality

$$\|u^N\|_{L^\infty(0,T;\,H^1(\Omega))} \le c_4(T). \tag{2.5}$$

The relation

$$\iint_{\Omega_T} (v^N_t)^2 = \iint_{\Omega_T} v^N_t q_2 (f_2 - \varepsilon v^N_{xx}) \le \left(\iint_{\Omega_T} (v^N_t)^2\right)^{1/2} \left(\iint_{\Omega_T} q_2^2 (f_2 - \varepsilon v^N_{xx})^2\right)^{1/2}$$

coupled with (2.4) gives

$$\|v^N_t\|_{L^2(\Omega_T)} \le c_5(T), \tag{2.6}$$

which in turn implies a bound on $b^N_1(t)$. Therefore, using Poincaré inequality and (2.4) we have

$$\|v^N\|_{L^\infty(0,T;\,H^1(\Omega))} \le c_6(T), \tag{2.7}$$

$$\|v^N\|_{L^2(0,T;\,H^2(\Omega))} \le c_6(T). \tag{2.8}$$

From (2.5) and (2.7), it follows that $(a^N_1, \ldots, a^N_N, b^N_1, \ldots, b^N_N)$ are uniformly bounded, and therefore there exists a solution of (2.1)-(2.3) in $(0,T)$.

In order to derive an estimate for u^N_t, we introduce the projection Π_N of $L^2(\Omega)$ onto $span\{\psi_1, \ldots, \psi_N\}$; for any $\psi \in L^2(0,T;H^1(\Omega))$ we obtain

$$\left|\iint_{\Omega_T} u^N_t \psi\right| = \left|\iint_{\Omega_T} q_1 (f_1 - \varepsilon u^N_{xx})_x (\Pi_N \psi)_x\right| \le c_7(T) \|\psi\|_{L^2(0,T;H^1(\Omega))}$$

(where (2.4) has been used); it follows that

$$\|u^N_t\|_{L^2(0,T;(H^1(\Omega))')} \le c_7(T). \tag{2.9}$$

Finally, since

$$\int_\Omega u^N_{xx} = 0, \tag{2.10}$$

the estimate (2.4), Poincaré's inequality and (2.5) yield

$$\|u^N\|_{L^2(0,T;H^3(\Omega))} \le c_8(T). \tag{2.11}$$

From the estimates (2.5)-(2.9) and (2.11), by well-known compactness results (see also [14]) it is possible to select a subsequence (still denoted by (u^N, v^N)) such that

$$\begin{array}{cccll}
u^N, v^N & \stackrel{*}{\rightharpoonup} & u, v & \text{in} & L^\infty(0,T;\ H^1(\Omega)), \\
u^N & \rightharpoonup & u & \text{in} & L^2(0,T; H^3(\Omega)), \\
v^N & \rightharpoonup & v & \text{in} & L^2(0,T; H^2(\Omega)), \\
u^N_t & \rightharpoonup & u_t & \text{in} & L^2(0,T; (H^1(\Omega))'), \\
v^N_t & \rightharpoonup & v_t & \text{in} & L^2(\Omega_T), \\
u^N, v^N & \longrightarrow & u, v & \text{in} & C(0,T; H^\lambda(\Omega)),\quad \lambda < 1, \\
u^N & \longrightarrow & u & \text{in} & L^2(0,T; H^\lambda(\Omega)),\quad \lambda < 3, \\
v^N & \longrightarrow & v & \text{in} & L^2(0,T; H^\lambda(\Omega)),\quad \lambda < 2.
\end{array}$$

In particular, the strong convergence in $C(0,T;L^2(\Omega))$ implies *(iv)*, and the last two convergence statements imply *(v)* in view of the continuous embedding

$$L^2(0,T;H^{1-\lambda}(\Omega)) \subset L^2(0,T;H^{\frac{1}{2}-\lambda}(\partial\Omega)),\quad \lambda > 0$$

applied to u^N_x, v^N_x with $\lambda < \frac{1}{2}$. □

Remark 2.2 Note that, using the same arguments as above with some minor modifications, Theorem 2.1 can be proved in any space dimension.

3 Proof of Theorem 1

3.1 Approximating systems

We shall approximate Problem (**P**) by regularized systems; i.e., systems with positive mobility and smooth bulk energy. For $\delta > 0$ we introduce positive mobilities $Q_\delta(u,v)$ by defining (see Fig. 1)

$$Q_{2\delta}(v) = \begin{cases} \left(\frac{1}{2}+\delta\right)^2 - v^2 & \text{if} \quad v \in \left[0, \frac{1}{2}+\frac{\delta}{2}\right], \\ \frac{\delta^2}{4}\left[1 + (1+\delta)\left(v-\frac{1}{2}\right)^{-1}\right] & \text{if} \quad v > \frac{1}{2}+\frac{\delta}{2}, \\ Q_{2\delta}(-v) & \text{if} \quad v < 0, \end{cases}$$

$$Q_{1\delta}(u) = Q_{2\delta}(u - \tfrac{1}{2}),$$

and setting

$$Q_\delta(u,v) = Q_{1\delta}(u)Q_{2\delta}(v).$$

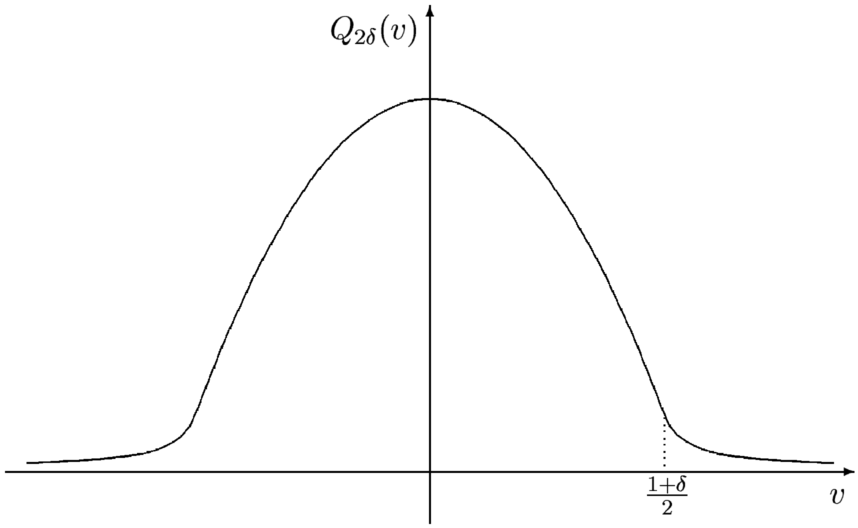

Fig. 1. The function $Q_{2\delta}(v)$.

With this choice we have $Q''_{i\delta} \in \mathrm{Lip}(\mathbb{R})$ and

$$\begin{gathered} \tfrac{\delta^2}{4} < Q_{i\delta} \le \left(\tfrac{1}{2}+\delta\right)^2, \\ |Q'_{i\delta}| \le 1+\delta, \\ Q_{i\delta} \longrightarrow Q_i \quad \text{uniformly in} \quad C^1(\operatorname{supp}[Q_i]_+). \end{gathered}$$

For $\sigma \in (0,1/2)$ we choose $F'_\sigma(s)$ such that (see Fig. 2)

$$F'_\sigma(s) = \begin{cases} \left[1-\frac{s-\sigma}{\sigma}\right]^{-1} + \ln\sigma, & \text{if} \quad s<\sigma, \\ \ln s + 1, & \text{if} \quad \sigma \le s \le 1-\sigma, \\ f_\sigma(s), & \text{if} \quad 1-\sigma < s < 2, \\ 1, & \text{if} \quad s \ge 2, \end{cases}$$

where $f_\sigma \in C^1([1-\sigma,2])$ has the following properties:

$$\begin{gathered} f_\sigma \le F', \quad f'_\sigma \ge 0, \\ f_\sigma(1-\sigma) = F'(1-\sigma), \quad f_\sigma(2) = 1, \\ f'_\sigma(1-\sigma) = F''(1-\sigma), \quad f'_\sigma(2) = 0. \end{gathered}$$

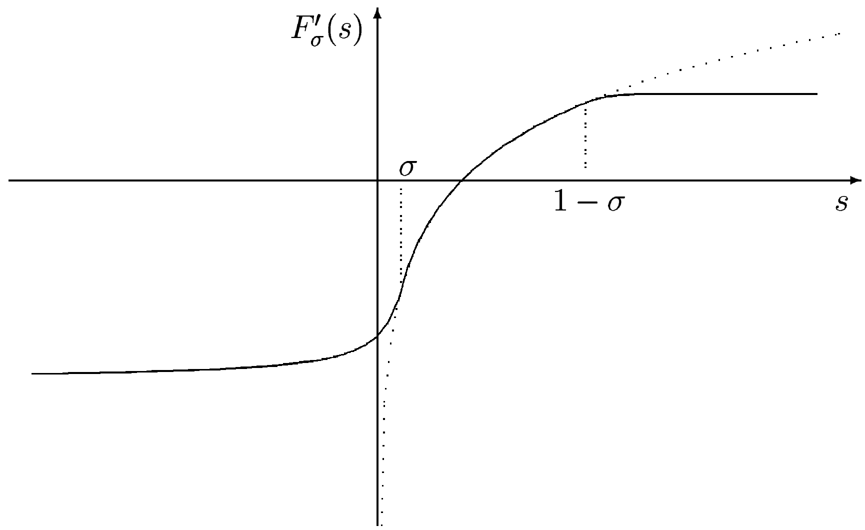

Fig. 2. The function $F'_\sigma(s)$ (the dotted curve is $F'(s)$).

Defining

$$F_\sigma(s) = -\frac{1}{e} + \int_{\frac{1}{e}}^{s} F'_\sigma(\xi)\, d\xi,$$

we have

$$\begin{aligned} F_\sigma \in C^2(\mathbb{R}), &\qquad F_\sigma(s) \le F(s) \ \text{ if } \ 0 < s, \\ F''_\sigma \ge 0, &\qquad F_\sigma(s) = F(s) \ \text{ if } \ \sigma \le s \le 1-\sigma. \end{aligned}$$

Note that F_σ is defined for all real values of s, whereas F is defined for $s > 0$ only. We also introduce U, V such that

$$\begin{gathered} \|U\|_{C^2(\mathbb{R})} \le U_0, \qquad \|V\|_{C^2(\mathbb{R})} \le V_0, \\ U(u) = \alpha u(1-u) \ \text{ if } 0 \le u \le 1, \qquad V(v) = -\beta v^2 \ \text{ if } \ -\tfrac{1}{2} \le v \le \tfrac{1}{2}, \end{gathered}$$

and we define the approximating homogeneous free energies as follows:

$$\mathcal{F}_\sigma(u,v) = F_\sigma(u+v) + F_\sigma(u-v) + F_\sigma(1-(u+v)) + F_\sigma(1-(u-v)) + U(u) + V(v).$$

We observe that

$$-U_0 - V_0 - \frac{4}{e} \le \mathcal{F}_\sigma(u,v) \ \text{ in } \mathbb{R}^2, \quad \text{and} \quad \mathcal{F}_\sigma(u,v) < U(u) + V(v) \ \text{ in } \overline{B}. \tag{3.1}$$

Applying Theorem 2.1, for each $\delta, \sigma > 0$ there exists a solution $(u_{\delta\sigma}, v_{\delta\sigma})$ of

$$(\mathbf{P}_{\delta\sigma}) \begin{cases} u_t = [Q_\delta(u,v)(\mathcal{F}_{\sigma u}(u,v) - \varepsilon u_{xx})_x]_x & (x,t) \in \Omega_T \\ v_t = -Q_\delta(u,v)(\mathcal{F}_{\sigma v}(u,v) - \varepsilon v_{xx}) & (x,t) \in \Omega_T \\ u_x = v_x = Q_\delta(u,v)(\mathcal{F}_{\sigma u}(u,v) - u_{xx})_x = 0 & (x,t) \in S_T \\ u(x,0) = u_0(x), \ \ v(x,0) = v_0(x) & x \in \Omega. \end{cases}$$

We shall hereafter denote by $(\mathbf{P}_\delta)$, Problem $(\mathbf{P}_{\delta\sigma})$ where $\mathcal{F}_\sigma$ is replaced by $\mathcal{F}$, and we shall denote by δ_0, σ_0 generic positive constants .
First of all we observe that

$$\overline{u_{\delta\sigma}(t)} = \overline{u_0} \in (0,1). \tag{3.2}$$

Lemma 3.1 *There exists a constant C_1 independent of δ, σ such that the following estimates hold for all $\delta < \delta_0$, $\sigma < \sigma_0$:*

(i) $\|u_{\delta\sigma}\|_{L^\infty(0,T;\,H^1(\Omega))} \leq C_1,$
(ii) $\|v_{\delta\sigma}\|_{L^\infty(0,T;\,H^1(\Omega))} \leq C_1,$
(iii) $\|Q_\delta^{1/2}(\mathcal{F}_{\sigma u} - \varepsilon u_{\delta\sigma xx})_x\|_{L^2(\Omega_T)} \leq C_1,$
(iv) $\|Q_\delta^{1/2}(\mathcal{F}_{\sigma v} - \varepsilon v_{\delta\sigma xx})\|_{L^2(\Omega_T)} \leq C_1,$
(v) $\|u_{\delta\sigma t}\|_{L^2(0,T;(H^1(\Omega))')} \leq C_1,$
(vi) $\|Q_\delta^{-1/2} v_{\delta\sigma t}\|_{L^2(\Omega_T)} \leq C_1,$
(vii) $\|\mathcal{F}_\sigma(u_{\delta\sigma}, v_{\delta\sigma})\|_{L^\infty(0,T;\,L^1(\Omega))} \leq C_1.$

Proof. Choosing $\phi = (\mathcal{F}_{\sigma u} - \varepsilon u_{\delta\sigma xx})$ and $\psi = (\mathcal{F}_{\sigma v} - \varepsilon v_{\delta\sigma xx})$ as test functions in the equations for $u_{\delta\sigma}$ and $v_{\delta\sigma}$ respectively, we obtain

$$\begin{aligned}
&\frac{\varepsilon}{2}\int_\Omega u_{\delta\sigma x}^2(t) + \frac{\varepsilon}{2}\int_\Omega v_{\delta\sigma x}^2(t) + \int_\Omega \mathcal{F}_\sigma(u_{\delta\sigma}(t), v_{\delta\sigma}(t)) + \\
&+\iint_{\Omega_t} Q_\delta(\mathcal{F}_{\sigma u} - \varepsilon u_{\delta\sigma xx})_x^2 + \iint_{\Omega_t} Q_\delta(\mathcal{F}_{\sigma v} - \varepsilon v_{\delta\sigma xx})^2 = \\
&= \frac{\varepsilon}{2}\int_\Omega u_{0x}^2 + \frac{\varepsilon}{2}\int_\Omega v_{0x}^2 + \int_\Omega \mathcal{F}_\sigma(u_0, v_0)
\end{aligned}$$

for almost every $t \in (0,T]$ (see [9] for a detailed proof). Using (1.4) and (3.1) we obtain

$$\begin{aligned}
&\frac{\varepsilon}{2}\int_\Omega u_{\delta\sigma x}^2(t) + \frac{\varepsilon}{2}\int_\Omega v_{\delta\sigma x}^2(t) + \int_\Omega \mathcal{F}_\sigma(u_{\delta\sigma}(t), v_{\delta\sigma}(t)) + \\
&+\iint_{\Omega_t} Q_\delta(\mathcal{F}_{\sigma u} - \varepsilon u_{\delta\sigma xx})_x^2 + \iint_{\Omega_t} Q_\delta(\mathcal{F}_{\sigma v} - \varepsilon v_{\delta\sigma xx})^2 \leq c_9
\end{aligned} \tag{3.3}$$

which implies *(iii)*, *(iv)* and *(vii)* since $-\frac{1}{e} \leq \mathcal{F}_\sigma$. Using Poincaré inequality and (3.2), *(i)* is also verified. To prove *(vi)*, we choose $\psi = v_{\delta\sigma t}/Q_\delta(u_{\delta\sigma}, v_{\delta\sigma})$ as a test function in the second equation of $(\mathbf{P}_{\delta\sigma})$, which yields

$$\iint_{\Omega_T} \frac{v_{\delta\sigma t}^2}{Q_\delta(u_{\delta\sigma}, v_{\delta\sigma})} = -\iint_{\Omega_T} (\mathcal{F}_{\sigma v} - \varepsilon v_{\delta\sigma xx}) v_{\delta\sigma t} \leq \left(\iint_{\Omega_T} Q_\delta(\mathcal{F}_{\sigma v} - \varepsilon v_{\delta\sigma xx})^2\right)^{\frac{1}{2}} \left(\iint_{\Omega_T} \frac{v_{\delta\sigma t}^2}{Q_\delta}\right)^{\frac{1}{2}}.$$

Since $Q_\delta \leq 1$ for δ small enough, in particular it holds that

$$\int_\Omega v_{\delta\sigma}^2(t) \leq 2\int_\Omega v_0^2 + 2t \iint_{\Omega_t} v_{\delta\sigma t}^2 \leq c_{10}$$

which together with (3.3) implies *(ii)*. Finally, *(v)* follows since

$$\left| \iint_{\Omega_T} u_{\delta\sigma t}\phi \right| \leq \left(\iint_{\Omega_T} Q_\delta(\mathcal{F}_{\sigma u} - \varepsilon u_{\delta\sigma xx})_x^2 \right)^{1/2} \left(\iint_{\Omega_T} \phi_x^2 \right)^{1/2}$$

for all $\phi \in L^2(0,T;H^1(\Omega))$. $\square$

Arguing in a standard fashion (see [3] for a proof) we have

Corollary 3.2 *There exists a constant C_2 independent of $0 < \sigma < \sigma_0$, $0 < \delta < \delta_0$ such that*

$$\begin{aligned} \|u_{\delta\sigma}\|_{C^{0,\frac{1}{2},\frac{1}{8}}(\overline{\Omega}_T)} &\leq C_2, \\ \|v_{\delta\sigma}\|_{C^{0,\frac{1}{2},\frac{1}{4}}(\overline{\Omega}_T)} &\leq C_2. \end{aligned} \tag{3.4}$$

3.2 Systems with positive mobility

Throughout this subsection we fix $\delta > 0$, and assume the mobility to be given by the form $Q_\delta(u,v)$ introduced in subsection 3.1. However, we remark that the results in this subsection are independent of the particular form of the mobility, and hold for any $\tilde{Q}(u,v) \in C(\mathbb{R}^2;\mathbb{R})$ with $0 < Q_{\min} \leq \tilde{Q} \leq Q_{\max}$.

By Corollary 3.2 we can extract a subsequence (still denoted by $(u_{\delta\sigma}, v_{\delta\sigma})$) such that

$$\begin{gathered} (u_{\delta\sigma}, v_{\delta\sigma}) \to (u_\delta, v_\delta) \text{ uniformly in } \overline{\Omega}_T \text{ as } \sigma \to 0 \\ u_\delta \in C^{0,\frac{1}{2},\frac{1}{8}}(\overline{\Omega}_T), \quad v_\delta \in C^{0,\frac{1}{2},\frac{1}{4}}(\overline{\Omega}_T). \end{gathered} \tag{3.5}$$

We now demonstrate that the limit (u_δ, v_δ) lies within the square B.

Lemma 3.3 *For all $0 < \delta < \delta_0$,*

$$|\Omega_T \setminus \mathcal{B}(u_\delta, v_\delta)| = 0.$$

Proof. Let N denote minus the inverse of the Laplacian with zero Neumann boundary conditions: given $f \in (H^1(\Omega))'_{\text{null}} := \{f \in (H^1(\Omega))' : \langle f, 1\rangle = 0\}$, we define $Nf \in H^1(\Omega)$ as the unique solution of

$$\begin{aligned} &\int_\Omega (Nf)'\psi' = \langle f, \psi\rangle \quad \forall \psi \in H^1(\Omega) \\ &\int_\Omega Nf = 0. \end{aligned}$$

By (3.2) and Lemma 3.1 *(i)*, $N(u_{\delta\sigma}(t) - \overline{u_{\delta\sigma}})$ is well defined. Choosing $\phi = Q_\delta^{-1} N(u_{\delta\sigma} - \overline{u_{\delta\sigma}})$ as a test function in the equation for $u_{\delta\sigma}$, we have

$$\int_0^T \langle u_{\delta\sigma t}, Q_\delta^{-1} N(u_{\delta\sigma} - \overline{u_{\delta\sigma}})\rangle \, dt =$$

$$= -\iint\limits_{\Omega_T} (\mathcal{F}_{\sigma u} - \varepsilon u_{\delta\sigma xx})_x (N(u_{\delta\sigma} - \overline{u_{\delta\sigma}}))_x + \iint\limits_{\Omega_T} Q_\delta^{-1} (\mathcal{F}_{\sigma u} - \varepsilon u_{\delta\sigma xx})_x Q_{\delta x} =$$

$$= -\iint\limits_{\Omega_T} (u_{\delta\sigma} - \overline{u_{\delta\sigma}}) \mathcal{F}_{\sigma u} - \varepsilon \iint\limits_{\Omega_T} u_{\delta\sigma x}^2 + \iint\limits_{\Omega_T} Q_\delta^{-1} (\mathcal{F}_{\sigma u} - \varepsilon u_{\delta\sigma xx})_x Q_{\delta x}.$$

Choosing $\psi = Q_\delta^{-1} v_{\delta\sigma}$ as a test function in the equation for $v_{\delta\sigma}$,

$$\iint\limits_{\Omega_T} Q_\delta^{-1} v_{\delta\sigma} v_{\delta\sigma t} = -\iint\limits_{\Omega_T} v_{\delta\sigma} \mathcal{F}_{\sigma v} - \varepsilon \iint\limits_{\Omega_T} v_{\delta\sigma x}^2.$$

Summing the two above identities, we obtain

$$\iint\limits_{\Omega_T} (u_{\delta\sigma} - \overline{u_{\delta\sigma}}) \mathcal{F}_{\sigma u} + \iint\limits_{\Omega_T} v_{\delta\sigma} \mathcal{F}_{\sigma v} =$$

$$= -\int_0^T \langle u_{\delta\sigma t}, Q_\delta^{-1} N(u_{\delta\sigma} - \overline{u_{\delta\sigma}}) \rangle \, dt - \varepsilon \iint\limits_{\Omega_T} u_{\delta\sigma x}^2 +$$

$$+ \iint\limits_{\Omega_T} Q_\delta^{-1} (\mathcal{F}_{\sigma u} - \varepsilon u_{\delta\sigma xx})_x Q_{\delta x} - \iint\limits_{\Omega_T} Q_\delta^{-1} v_{\delta\sigma} v_{\delta\sigma t} - \varepsilon \iint\limits_{\Omega_T} v_{\delta\sigma x}^2.$$

The estimates in Lemma 3.1, the lower bound for Q_δ, the fact that $\|Q_\delta\|_{C^1} \leq 2$ for δ sufficiently small, and the definition of N imply

$$\iint\limits_{\Omega_T} [(u_{\delta\sigma} - \overline{u_{\delta\sigma}}) \mathcal{F}_{\sigma u} + v_{\delta\sigma} \mathcal{F}_{\sigma v}] \leq c_{11} \tag{3.6}$$

(where $c_{11} = c_{11}(\delta)$). We now exploit the sign property of the integrand at the left hand side. To this purpose, we observe that the following identity holds for any $c \in \mathbb{R}$:

$$\begin{aligned} (u-c)\mathcal{F}_{\sigma u}(u,v) + v\mathcal{F}_{\sigma v}(u,v) = \{&(u+v)\,[F'_\sigma(u+v) - 1] + (u-v)\,[F'_\sigma(u-v) - 1] + \\ &+(1-(u+v))\,[F'_\sigma(1-(u+v)) - 1] + \\ &+(1-(u-v))\,[F'_\sigma(1-(u-v)) - 1]\} + \\ &-cF'_\sigma(u+v) - cF'_\sigma(u-v) - (1-c)F'_\sigma(1-(u+v)) - \\ &-(1-c)F'_\sigma(1-(u-v)) + (u-c)U'(u) + vV'(v) + 2. \end{aligned} \tag{3.7}$$

The terms inside the double brackets are bounded from below since

$$\begin{array}{rcll} 0 & \leq & s\,[F'_\sigma(s) - 1], & s \leq 0, \\ -\frac{1}{e} - \frac{1}{2} & \leq & \sigma\,[\ln\sigma - 1] \leq s\,[F'_\sigma(s) - 1] \leq 0, & 0 < s \leq \sigma, \\ -\frac{1}{e} & \leq & s\ln s = s\,[F'_\sigma(s) - 1] \leq 0, & \sigma \leq s \leq 1-\sigma, \\ -2 & \leq & s\,[F'_\sigma(s) - 1] \leq 0, & 1-\sigma \leq s \leq 2, \\ 0 & = & s\,[F'_\sigma(s) - 1], & s \geq 2 \end{array}$$

for $\sigma \in (0,1/2)$. Since U', V' are uniformly bounded, setting $c = \overline{u_{\delta\sigma}} \equiv \overline{u_0} \in (0,1)$ in (3.7) and noting that $F'_\sigma \leq 1$, it follows from (3.6) that

$$-\iint\limits_{\Omega_T} [F'_\sigma(u_{\delta\sigma}+v_{\delta\sigma}) + F'_\sigma(u_{\delta\sigma}-v_{\delta\sigma}) + \\ + F'_\sigma(1-(u_{\delta\sigma}+v_{\delta\sigma})) + F'_\sigma(1-(u_{\delta\sigma}-v_{\delta\sigma}))] \leq c_{12}. \tag{3.8}$$

To complete the proof, suppose by contradiction $|\Omega_T \setminus \mathcal{B}(u_\delta, v_\delta)| > 0$. Since the argument can be repeated for each side of the box B, we can assume without loss of generality that the set

$$A = \{(x,t) \in \Omega_T : u_\delta + v_\delta \leq 0\}$$

has positive measure. Since $F'_\sigma \leq 1$, the estimate (3.8) gives

$$-\iint_A F'_\sigma(u_{\delta\sigma}+v_{\delta\sigma}) \leq c_{13}.$$

Note, however, that the uniform convergence of $u_{\delta\sigma}$ and $v_{\delta\sigma}$ implies that

$$\forall\lambda > 0 \; \exists\, \sigma_\lambda : u_{\delta\sigma}+v_{\delta\sigma} \leq \lambda \quad \forall x \in A,\; \sigma < \sigma_\lambda;$$

therefore, due to the convexity of F_σ, $F'_\sigma(u_{\delta\sigma}+v_{\delta\sigma}) \leq F'_\sigma(\lambda)$. Hence

$$c_{13} \geq -\lim_{\sigma\to 0} \iint_A F'_\sigma(\lambda) = -|A|(\ln\lambda + 1),$$

which leads to a contradiction for λ sufficiently small. □

In the next Lemma we derive additional estimates which will allow us to pass to the limit as $\sigma \to 0$ in Problem $(\mathbf{P}_{\delta\sigma})$. To simplify the notation we define the function

$$\varphi_\sigma(s) = F''_\sigma(s) + F''_\sigma(1-s),$$

which is positive in view of the definition of F'_σ.

Lemma 3.4 *Let $\delta > 0$ be fixed. Then there exists a constant C_3 which is independent of σ such that*

(i) $$\|\mathcal{F}_{\sigma v}(u_{\delta\sigma}, v_{\delta\sigma})\|_{L^2(\Omega_T)} \leq C_3,$$

(ii) $$\|u_{\delta\sigma xx}\|_{L^2(\Omega_T)} \leq C_3,$$

(iii) $$\|v_{\delta\sigma xx}\|_{L^2(\Omega_T)} \leq C_3,$$

(iv) $$\iint\limits_{\Omega_T} \varphi_\sigma(u_{\delta\sigma}+v_{\delta\sigma})(u_{\delta\sigma}+v_{\delta\sigma})_x^2 + \iint\limits_{\Omega_T} \varphi_\sigma(u_{\delta\sigma}-v_{\delta\sigma})(u_{\delta\sigma}-v_{\delta\sigma})_x^2 \leq C_3.$$

Proof. From Lemma 3.1 *(iv)* we have

$$\iint_{\Omega_T} \mathcal{F}_{\sigma v}^2 + 2\varepsilon \iint_{\Omega_T} (\mathcal{F}_{\sigma v})_x v_{\delta\sigma x} + \varepsilon^2 \iint_{\Omega_T} v_{\delta\sigma xx}^2 \le c_{14}. \tag{3.9}$$

Defining $H_{\delta\sigma} = \mathcal{F}_{\sigma u} - \varepsilon u_{\delta\sigma xx}$, since $u_{\delta\sigma x}|_{S_T} = 0$

$$\int_\Omega H_{\delta\sigma}(t) = \int_\Omega \mathcal{F}_{\sigma u}(u_{\delta\sigma}(t), v_{\delta\sigma}(t)),$$

and from Lemma 3.1 *(iii)*,

$$\iint_{\Omega_T} H_{\delta\sigma x}^2 \le c_{15}.$$

Hence we can write

$$\begin{aligned}\iint_{\Omega_T} \mathcal{F}_{\sigma u}^2 + 2\varepsilon \iint_{\Omega_T} (\mathcal{F}_{\sigma u})_x u_{\delta\sigma x} + \varepsilon^2 \iint_{\Omega_T} u_{\delta\sigma xx}^2 = \iint_{\Omega_T} H_{\delta\sigma}^2 = \\ = \iint_{\Omega_T} (H_{\delta\sigma} - \overline{H_{\delta\sigma}})^2 + \iint_{\Omega_T} \overline{H_{\delta\sigma}}^{\,2} \le c_P \iint_{\Omega_T} H_{\delta\sigma x}^2 + \iint_{\Omega_T} \mathcal{F}_{\sigma u}^2,\end{aligned}$$

where c_P denotes the Poincaré coefficient. Summing this inequality with (3.9) we obtain

$$\iint_{\Omega_T} \mathcal{F}_{\sigma v}^2 + 2\varepsilon \iint_{\Omega_T} \left[(\mathcal{F}_{\sigma u})_x u_{\delta\sigma x} + (\mathcal{F}_{\sigma v})_x v_{\delta\sigma x}\right] + \varepsilon^2 \iint_{\Omega_T} \left[u_{\delta\sigma xx}^2 + v_{\delta\sigma xx}^2\right] \le c_{16}.$$

From this estimate, the Lemma follows recalling that $\varphi_\sigma > 0$ and observing that

$$\begin{aligned}(\mathcal{F}_{\sigma u})_x u_{\delta\sigma x} + (\mathcal{F}_{\sigma v})_x v_{\delta\sigma x} = \varphi_\sigma(u_{\delta\sigma} + v_{\delta\sigma})(u_{\delta\sigma} + v_{\delta\sigma})_x^2 + \\ + \varphi_\sigma(u_{\delta\sigma} - v_{\delta\sigma})(u_{\delta\sigma} - v_{\delta\sigma})_x^2 + U''(u_{\delta\sigma})u_{\delta\sigma x}^2 + V''(v_{\delta\sigma})v_{\delta\sigma x}^2\end{aligned} \tag{3.10}$$

and using the definition of U, V and Lemma 3.1 *(i)*,*(ii)* to bound the last two terms. □

We can now state the following result.

Proposition 3.5 *For each $0 < \delta < \delta_0$ there exists a triplet $(u_\delta, v_\delta, w_\delta)$ such that:*

(i) $u_\delta, v_\delta \in L^\infty(0,T;\, H^1(\Omega))$ *uniformly in* δ*;*
(ii) $u_\delta \in C^{0,\frac{1}{2},\frac{1}{8}}(\overline{\Omega}_T)$ *and* $v_\delta \in C^{0,\frac{1}{2},\frac{1}{4}}(\overline{\Omega}_T)$ *uniformly in* δ*;*
(iii) $u_{\delta t} \in L^2(0,T;(H^1(\Omega))')$ *and* $Q_\delta^{-\frac{1}{2}} v_{\delta t} \in L^2(\Omega_T)$ *uniformly in* δ*;*
(iv) $u_{\delta xx}, v_{\delta xx} \in L^2(\Omega_T)$*;*
(v) $|\Omega_T \setminus \mathcal{B}(u_\delta, v_\delta)| = 0$*;*
(vi) $\mathcal{F}_v(u_\delta, v_\delta) \in L^2(\Omega_T)$ *and* $\mathcal{F}_u(u_\delta, v_\delta) \in L^2_{\mathrm{loc}}(\mathcal{D}(u_\delta))$*;*
(vii) $w_\delta \in L^2(\Omega_T)$ *uniformly in* δ*;*

(viii) $u_\delta(x,0) = u_0(x)$, $v_\delta(x,0) = v_0(x)$;
(ix) $u_{\delta x}|_{S_T} = v_{\delta x}|_{S_T} = 0$ *in* $L^2(S_T)$;
(x) $(u_\delta, v_\delta, w_\delta)$ *solves* $(\mathbf{P}_\delta)$ *in the following sense:*

$$\int_0^T \langle u_{\delta t}, \phi\rangle \, dt = -\iint_{\Omega_T} Q_\delta^{\frac{1}{2}}(u_\delta, v_\delta) w_\delta \phi_x \qquad \forall \phi \in L^2(0,T; H^1(\Omega)), \tag{3.11}$$

$$\iint_{\Omega_T} v_{\delta t}\psi = -\iint_{\Omega_T} Q_\delta(u_\delta, v_\delta)(\mathcal{F}_v(u_\delta, v_\delta) - \varepsilon v_{\delta xx})\psi \qquad \forall \psi \in L^2(\Omega_T), \tag{3.12}$$

where $w_\delta = Q_\delta^{\frac{1}{2}}(\mathcal{F}_u - \varepsilon u_{\delta xx})$ *in the sense that*

$$\iint_{\Omega_T} w_\delta \phi = -\iint_{\Omega_T} (\mathcal{F}_u(u_\delta, v_\delta) - \varepsilon u_{\delta xx})(Q_\delta^{\frac{1}{2}}(u_\delta, v_\delta)\phi)_x \tag{3.13}$$

for all $\phi \in L^2(0,T; H_0^1(\Omega))$ *with compact support in* $\mathcal{D}(u_\delta)$.

Proof. Let $\{(u_{\delta\sigma}, v_{\delta\sigma})\}$ be the sequence given in (3.5). By Lemmas 3.1 and 3.4, it can be seen that

$$\begin{array}{rcll}
u_{\delta\sigma}, v_{\delta\sigma} & \overset{*}{\rightharpoonup} & u_\delta, v_\delta & \text{in} \quad L^\infty(0,T;\, H^1(\Omega)), \\
u_{\delta\sigma}, v_{\delta\sigma} & \rightharpoonup & u_\delta, v_\delta & \text{in} \quad L^2(0,T; H^2(\Omega)), \\
u_{\delta\sigma t} & \rightharpoonup & u_{\delta t} & \text{in} \quad L^2(0,T; (H^1(\Omega))'), \\
Q_\delta^{-\frac{1}{2}}(u_{\delta\sigma}, v_{\delta\sigma}) v_{\delta\sigma t} & \rightharpoonup & Q_\delta^{-\frac{1}{2}}(u_\delta, v_\delta) v_{\delta t} & \text{in} \quad L^2(\Omega_T), \\
Q_\delta^{\frac{1}{2}}(u_{\delta\sigma}, v_{\delta\sigma})(\mathcal{F}_{\sigma u} - \varepsilon u_{\delta\sigma xx})_x & \rightharpoonup & w_\delta & \text{in} \quad L^2(\Omega_T).
\end{array}$$

Recalling also Corollary 3.2 and Lemma 3.3, *(i)*-*(v)* and *(vii)* now follow. Lemma 3.4 *(i)* implies that

$$\mathcal{F}_{\sigma v}(u_{\delta\sigma}, v_{\delta\sigma}) \rightharpoonup g \ \text{ in } \ L^2(\Omega_T);$$

since $\mathcal{F}_{\sigma v}(u_{\delta\sigma}, v_{\delta\sigma})$ converges pointwise in $\mathcal{B}(u_\delta, v_\delta)$ and $|\Omega_T \setminus \mathcal{B}(u_\delta, v_\delta)| = 0$, it follows that $g = \mathcal{F}_v(u_\delta, v_\delta)$, which proves the first part of *(vi)*. In order to prove the second part, we recall that

$$\begin{aligned}
\mathcal{F}_{\sigma u}(u,v) &= F'_\sigma(u+v) + F'_\sigma(u-v) - F'_\sigma(1-(u+v)) - F'_\sigma(1-(u-v)) + U'(u) \\
\mathcal{F}_{\sigma v}(u,v) &= F'_\sigma(u+v) - F'_\sigma(u-v) - F'_\sigma(1-(u+v)) + F'_\sigma(1-(u-v)) + V'(v).
\end{aligned}$$

This structure suggests that each term of $\mathcal{F}_{\sigma u}(u_{\delta\sigma}, v_{\delta\sigma}))$ is uniformly bounded in $L^2(\Omega_T)$ away from $\{u_\delta = 0\}$ and $\{u_\delta = 1\}$. We prove this in detail for the first term. Let $K \subset \mathcal{D}(u_\delta)$ be compact; then

$$K \subset \{\eta < u_\delta < 1 - \eta\}, \quad \frac{\eta}{2} < u_{\delta\sigma} < 1 - \frac{\eta}{2} \ \text{ in } \ K$$

for a suitable $\eta > 0$ and for σ sufficiently small. Let us consider the sets

$$K'_\sigma = K \cap \{u_{\delta\sigma} + v_{\delta\sigma} < \eta/2\}, \quad K''_\sigma = K \setminus K'_\sigma;$$

in K''_σ we have

$$0 \geq F'_\sigma(u_{\delta\sigma} + v_{\delta\sigma}) - 1 \geq F'_\sigma(\eta/2) - 1 \geq c_{17}. \tag{3.14}$$

In order to estimate $F'_\sigma(u_{\delta\sigma} + v_{\delta\sigma})$ in K'_σ, observe that the following relation holds:

$$\begin{aligned} 0 \geq F'_\sigma(u+v) - 1 &\geq F'_\sigma(u+v) + F'_\sigma(1-(u-v)) - 2 = \\ &= \mathcal{F}_{\sigma v} + F'_\sigma(u-v) + F'_\sigma(1-(u+v)) - V'(v) - 2. \end{aligned}$$

From its definition, in K'_σ we have $u_{\delta\sigma} - v_{\delta\sigma} > \eta/2$ and $1 - (u_{\delta\sigma} + v_{\delta\sigma}) > 1 - \eta/2$; hence

$$\begin{aligned} 0 \geq F'_\sigma(u+v) - 1 \geq &\ \mathcal{F}_{\sigma v} + F'_\sigma(\eta/2) + F'_\sigma(1-\eta/2) - V'(v) - 2 \geq \\ \geq &\ \mathcal{F}_{\sigma v} + c_{18} \end{aligned}$$

which, together with (3.14), implies

$$\iint_K [F'_\sigma(u_{\delta\sigma} + v_{\delta\sigma}) - 1]^2 \leq c_{19} + 2 \iint_{\Omega_T} \mathcal{F}^2_{\sigma v}.$$

Repeating the same argument for the other terms, we conclude that

$$\iint_K \mathcal{F}^2_{\sigma u}(u_{\delta\sigma}, v_{\delta\sigma}) \leq c_{20} + 8 \iint_{\Omega_T} \mathcal{F}^2_{\sigma v}$$

and therefore

$$\mathcal{F}_{\sigma u}(u_{\delta\sigma}, v_{\delta\sigma}) \rightharpoonup \mathcal{F}_u(u_\delta, v_\delta) \quad \text{in} \ \ L^2_{\text{loc}}(\mathcal{D}(u_\delta)), \tag{3.15}$$

which completes the proof of *(vi)*.
The property *(viii)* is straightforward; by compactness we have that

$$u_{\delta\sigma} \longrightarrow u_\delta \ \text{ in } \ L^2(0,T; H^{2-\lambda}(\Omega)), \quad \lambda > 0, \tag{3.16}$$

$$v_{\delta\sigma} \longrightarrow v_\delta \ \text{ in } \ L^2(0,T; H^{2-\lambda}(\Omega)), \quad \lambda > 0, \tag{3.17}$$

which imply *(ix)*.
To prove *(x)*, we pass to the limit as $\sigma \to 0$ in the first equation of $(\mathbf{P}_{\delta\sigma})$, obtaining

$$\int_0^T \langle u_{\delta t}, \phi \rangle \, dt = -\iint_{\Omega_T} Q_\delta^{\frac{1}{2}}(u_\delta, v_\delta) w_\delta \phi_x \quad \forall \phi \in L^2(0,T; H^1(\Omega));$$

passing to the limit in the second equation of $(\mathbf{P}_{\delta\sigma})$ we have

$$\iint_{\Omega_T} v_{\delta t}\psi = -\iint_{\Omega_T} Q_\delta(u_\delta, v_\delta)(\mathcal{F}_v(u_\delta, v_\delta) - \varepsilon v_{\delta xx})\psi \quad \forall \psi \in L^2(\Omega_T).$$

To identify w_δ, we observe that for all $\sigma > 0$ and all $\phi \in L^2(0,T; H^1_0(\Omega))$ the following identity holds:

$$\iint_{\Omega_T} Q_\delta^{\frac{1}{2}}(u_{\delta\sigma}, v_{\delta\sigma})(\mathcal{F}_{\sigma u} - \varepsilon u_{\delta\sigma xx})_x \phi = -\iint_{\Omega_T} (\mathcal{F}_{\sigma u}(u_{\delta\sigma}, v_{\delta\sigma}) - \varepsilon u_{\delta\sigma xx})(Q_\delta^{\frac{1}{2}}(u_{\delta\sigma}, v_{\delta\sigma})\phi)_x. \tag{3.18}$$

By (3.15), (3.16) and (3.17) we can pass to the limit on the right hand side of (3.18), provided ϕ is compactly supported in $\mathcal{D}(u_\delta)$. □

3.3 The degenerate system

Proposition 3.5 allows us to select a subsequence, which we still denote by $(u_\delta, v_\delta, w_\delta)$, such that

$$\begin{gathered} u_\delta, v_\delta \to u, v \ \text{ uniformly in } \overline{\Omega}_T, \\ u \in C^{0,\frac{1}{2},\frac{1}{8}}(\overline{\Omega}_T), \ \ v \in C^{0,\frac{1}{2},\frac{1}{4}}(\overline{\Omega}_T), \\ w_\delta \rightharpoonup w \ \text{ in } L^2(\Omega_T). \end{gathered} \tag{3.19}$$

We already know by Lemma 3.3 that $(u,v) \in \overline{B}$; in the following Lemma we prove an estimate which shall enable us to improve these bounds. Let χ_A denote the characteristic function of the set A, and let

$$G(u,v) = v\left[\ln(u-v) + \ln(1-(u+v))\right]\chi_{v\geq 0} - v\left[\ln(u+v) + \ln(1-(u-v))\right]\chi_{v\leq 0}.$$

Lemma 3.6 *For all $0 < \delta < \delta_0$, $G(u_\delta, v_\delta) \in L^2(\Omega_T)$, and for $\beta \in (1,4]$ there exist two positive constants β_0, C_4 independent of δ such that for any $t \in (0,T]$*

$$\begin{aligned} &\tfrac{1}{\beta-1}\int_\Omega Q_{2\delta}^{1-\beta}(v_\delta(t)) - 2\iint_{\Omega_t} Q_{1\delta}(u_\delta)Q_{2\delta}^{1-\beta}(v_\delta)G(u_\delta,v_\delta) + \\ &\quad + \varepsilon\beta_0\iint_{\Omega_t} Q_{1\delta}(u_\delta)Q_{2\delta}^{-\beta}(v_\delta)v_{\delta x}^2 \leq \tfrac{1}{\beta-1}\int_\Omega Q_{2\delta}^{1-\beta}(v_0) + C_4 + C_4\iint_{\Omega_t} Q_{2\delta}^{1-\beta}(v_\delta). \end{aligned} \tag{3.20}$$

Proof. We choose $\psi = -Q_{2\delta}(v_\delta)^{-\beta}Q'_{2\delta}(v_\delta)\chi_{[0,t]}$ as a test function in (3.12). Observe that, since $(u_\delta, v_\delta) \in \overline{B}$, we have

$$Q'_{1\delta}(u_\delta) = 1 - 2u_\delta, \quad Q'_{2\delta}(v_\delta) = -2v_\delta.$$

Since $v_{\delta t} \in L^2(\Omega_T)$, for all $t \in (0,T]$ we obtain

$$\begin{aligned} \frac{1}{\beta-1}\int_\Omega Q_{2\delta}^{1-\beta}(v_\delta(t)) - \frac{1}{\beta-1}\int_\Omega Q_{2\delta}^{1-\beta}(v_0) &= -\iint_{\Omega_t} Q_{2\delta}^{-\beta}Q'_{2\delta}v_{\delta t} = \\ &= -2\iint_{\Omega_t} Q_{1\delta}Q_{2\delta}^{1-\beta}v_\delta\mathcal{F}_v + 2\varepsilon\iint_{\Omega_t} Q_{1\delta}Q_{2\delta}^{1-\beta}v_\delta v_{\delta xx}. \end{aligned} \tag{3.21}$$

From the definition of G and $\mathcal{F}_v$ we can write the identity

$$\begin{aligned} G(u_\delta,v_\delta) = &-v_\delta\mathcal{F}_v(u_\delta,v_\delta) + v_\delta\ln(u_\delta+v_\delta)\chi_{v_\delta>0} - v_\delta\ln(u_\delta-v_\delta)\chi_{v_\delta<0} + \\ &-v_\delta\ln(1-(u_\delta+v_\delta))\chi_{v_\delta<0} + v_\delta\ln(1-(u_\delta-v_\delta))\chi_{v_\delta>0} + v_\delta V'(v_\delta). \end{aligned}$$

Since $u_\delta \geq 0$, we get

$$0 \geq v_\delta\ln(u_\delta+v_\delta)\chi_{v_\delta>0} \geq v_\delta\ln(v_\delta)\chi_{v_\delta>0} \geq -1/e,$$

and similar inequalities hold for the other logarithmic terms on the right hand side. Hence

$$0 \geq G(u_\delta,v_\delta) \geq -\frac{4}{e} - v_\delta\mathcal{F}_v(u_\delta,v_\delta) + v_\delta V'(v_\delta), \tag{3.22}$$

which implies that $G(u_\delta, v_\delta) \in L^2(\Omega_T)$. Substituting (3.22) into (3.21), we obtain

$$\begin{aligned} &\frac{1}{\beta-1}\int_\Omega Q_{2\delta}^{1-\beta}(v_\delta(t)) - \frac{1}{\beta-1}\int_\Omega Q_{2\delta}^{1-\beta}(v_0) \le \\ &\qquad \le 2\iint_{\Omega_t} Q_{1\delta}Q_{2\delta}^{1-\beta}G + c_{21}\iint_{\Omega_t} Q_{2\delta}^{1-\beta} + 2\varepsilon\iint_{\Omega_t} Q_{1\delta}Q_{2\delta}^{1-\beta} v_\delta v_{\delta xx}. \end{aligned} \tag{3.23}$$

It remains to estimate the last integral at the right hand side of (3.23). Integrating by parts, we write

$$\begin{aligned} \iint_{\Omega_t} Q_{1\delta}Q_{2\delta}^{1-\beta} v_\delta v_{\delta xx} &= -\iint_{\Omega_t} Q'_{1\delta}Q_{2\delta}^{1-\beta} v_\delta u_{\delta x} v_{\delta x} + \\ &\quad -\iint_{\Omega_t} Q_{1\delta}Q_{2\delta}^{-\beta}\left[Q_{2\delta} + 2(\beta-1)v_\delta^2\right] v_{\delta x}^2 =: I_1 + I_2. \end{aligned}$$

A straightforward calculation shows that

$$Q_2(v) + 2(\beta-1)v^2 \ge \beta_0 := \min\left\{\frac{\beta-1}{2}, \frac{1}{4}\right\} > 0, \quad -\frac{1}{2} \le v \le \frac{1}{2}.$$

Since, by its definition, $Q_{2\delta}(v) \ge Q_2(v)$ for $v \in [-1/2, 1/2]$, it follows therefore that

$$I_2 \le -\beta_0 \iint_{\Omega_t} Q_{1\delta}Q_{2\delta}^{-\beta} v_{\delta x}^2.$$

Applying Cauchy-Schwarz and Young's inequalities to I_1, we have

$$|I_1| \le \frac{1}{2}\beta_0 \iint_{\Omega_t} Q_{1\delta}Q_{2\delta}^{-\beta} v_{\delta x}^2 + c_{22}\iint_{\Omega_t} Q_{1\delta}^{-1}(Q'_{1\delta})^2 Q_{2\delta}^{2-\beta} v_\delta^2 u_{\delta x}^2.$$

Since $(u_\delta, v_\delta) \in \overline{B}$, we can make use of the following simple properties of B:

$$(u,v) \in \overline{B} \Longrightarrow \begin{cases} |v| \le 2u(1-u) \\ |1-2u| \le 4(\frac{1}{4} - v^2), \end{cases}$$

which imply that

$$\begin{aligned} |v_\delta| \le 2u_\delta(1-u_\delta) \le 2Q_{1\delta}, \\ (Q'_{1\delta})^2 = (1-2u_\delta)^2 \le 16(\tfrac{1}{4} - v_\delta^2)^2 \le 16 Q_{2\delta}^2. \end{aligned}$$

Therefore

$$|I_1| \le \frac{1}{2}\beta_0 \iint_{\Omega_t} Q_{1\delta}Q_{2\delta}^{-\beta} v_{\delta x}^2 + c_{23}\iint_{\Omega_t} |v_\delta| Q_{2\delta}^{4-\beta} u_{\delta x}^2 \le \frac{1}{2}\beta_0 \iint_{\Omega_t} Q_{1\delta}Q_{2\delta}^{-\beta} v_{\delta x}^2 + c_{24}$$

since $\beta \le 4$. Collecting the estimates of I_1 and I_2, we finally obtain from (3.23)

$$\begin{aligned} \frac{1}{\beta-1}\int_\Omega Q_{2\delta}^{1-\beta}(v_\delta(t)) - \frac{1}{\beta-1}\int_\Omega Q_{2\delta}^{1-\beta}(v_0) \le \\ \le c_{25} + 2\iint_{\Omega_t} Q_{1\delta}Q_{2\delta}^{1-\beta}G + c_{21}\iint_{\Omega_t} Q_{2\delta}^{1-\beta} - \varepsilon\beta_0\iint_{\Omega_t} Q_{1\delta}Q_{2\delta}^{-\beta}v_{\delta x}^2, \end{aligned} \tag{3.24}$$

which completes the proof of the Lemma. □

Corollary 3.7 *There exists a constant $d > 0$ such that*

$$v(x,t) \in \left[-\frac{1}{2}+d, \frac{1}{2}-d\right] \quad \forall (x,t) \in \overline{\Omega}_T; \tag{3.25}$$

in addition,

$$|\mathcal{D}(u) \setminus \mathcal{B}(u,v)| = 0.$$

Proof. Applying a Gronwall argument to (3.20) with $\beta = 3$ and noting that $G(u_\delta, v_\delta) \le 0$, we obtain

$$\int_\Omega Q_{2\delta}^{-2}(v_\delta(t)) - \iint_{\Omega_t} Q_{1\delta}(u_\delta)Q_{2\delta}^{-2}(u_\delta)G(u_\delta,v_\delta) + \iint_{\Omega_t} Q_{1\delta}(u_\delta)Q_{2\delta}^{-3}(v_\delta)v_{\delta x}^2 \le c_{26}. \tag{3.26}$$

Suppose by contradiction that $v(x_0,t_0) = -\frac{1}{2}$. The uniform convergence of v_δ implies that for any $\gamma > 0$ there exists $\delta_\gamma > 0$ such that

$$|v_\delta(x,t_0) + \frac{1}{2}| \le |v_\delta(x,t_0) - v(x,t_0)| + |v(x,t_0) + \frac{1}{2}| \le \gamma + C_1|x-x_0|^{1/2}$$

for all $\delta < \delta_\gamma$, $x \in \overline{\Omega}$. Therefore, by (3.26) and (3.22)

$$c_{26} \ge \int_\Omega \left(\left(\frac{1}{2}+\delta\right)^2 - v_\delta^2\right)^{-2} \ge \int_\Omega (1+\delta)^{-2}\left(\gamma + C_1|x-x_0|^{1/2} + \delta\right)^{-2} \quad \forall \delta < \delta_\gamma,$$

which implies

$$c_{26} \ge \int_\Omega (\gamma + C_1|x-x_0|^{1/2})^{-2}$$

a contradiction for γ sufficiently small. Thus $v > -\frac{1}{2}$ in $\overline{\Omega}_T$. Similarly we may show that $v < \frac{1}{2}$ in $\overline{\Omega}_T$, and the first assertion follows.

To prove the second statement, suppose by contradiction that the set $\mathcal{D}(u) \setminus \mathcal{B}(u,v)$ has positive measure. Without loss of generality we can assume that the set

$$A = \{(x,t) \in \Omega_T : u + v = 0,\ u > 0\}$$

has positive measure. Then, defining $A' = A \cap \mathcal{D}_\eta(u)$, we have $|A'| \ge |A|/2$ for η sufficiently small. By (3.19) and Lemma 3.3 it follows that

$$\forall\, \gamma > 0\ \exists\, \delta_\gamma > 0 \text{ s.t. } 0 < u_\delta + v_\delta < \gamma \text{ and } |u_\delta - u| < \gamma \text{ a.e. in } A',\ \forall\, \delta < \delta_\gamma; \tag{3.27}$$

it is now easy to check that almost everywhere in A'

$$\frac{2\eta}{3} < u_\delta < \frac{1}{2} + \frac{\eta}{3} \quad \text{and} \quad v_\delta < -\frac{\eta}{3} \qquad \forall\, \gamma \in \left(0, \frac{\eta}{3}\right),\ \delta < \delta_\gamma. \tag{3.28}$$

From (3.26) and the definition of G, we obtain

$$\iint_{\Omega_T} Q_{1\delta} Q_{2\delta}^{-2} |v_\delta \ln(u_\delta + v_\delta)| \chi_{v_\delta \le 0} \le -\iint_{\Omega_T} Q_{1\delta} Q_{2\delta}^{-2} G(u_\delta, v_\delta) \le c_{26}. \tag{3.29}$$

Since

$$|v_\delta \ln(u_\delta + v_\delta) \chi_{v_\delta > 0}| \le |v_\delta \ln(v_\delta) \chi_{v_\delta > 0}| \le \frac{1}{e},$$

using (3.29) and (3.25) we conclude that

$$\iint_{\Omega_T} Q_{1\delta} |v_\delta \ln(u_\delta + v_\delta)| \le c_{27},$$

and a contradiction now follows for γ sufficiently small, taking (3.28) into account and arguing as in the proof of Lemma 3.3. □

In order to pass to the limit as $\delta \to 0$ we need an estimate independent of δ on the second derivatives of u_δ, v_δ and for the singular terms $\mathcal{F}_u(u_\delta, v_\delta)$, $\mathcal{F}_v(u_\delta, v_\delta)$. The next Lemma provides such an estimate on any compact subset of $\mathcal{D}(u)$.

Lemma 3.8 *For any $K \subset \mathcal{D}(u)$ compact there exist positive constants δ_K and C_K such that the following estimates hold for all $\delta < \delta_K$:*

$$\begin{aligned} &\|u_{\delta xx}\|_{L^2(K)} \le C_K, \qquad \|\mathcal{F}_u(u_\delta, v_\delta)\|_{L^2(K)} \le C_K,\\ &\|v_{\delta xx}\|_{L^2(K)} \le C_K, \qquad \|\mathcal{F}_v(u_\delta, v_\delta)\|_{L^2(K)} \le C_K. \end{aligned}$$

Proof. Let $K \subset \mathcal{D}(u)$ compact; then $K \subseteq \mathcal{D}_\eta(u)$ for some $\eta > 0$. Since there exists a positive distance between $\mathcal{D}_\eta(u)$ and $\overline{\Omega}_T \setminus \mathcal{D}_{\eta/2}(u)$, it is possible to select $\zeta \in C^\infty(\overline{\Omega}_T)$ such that $0 \le \zeta \le 1$, $\zeta = 1$ in $\mathcal{D}_\eta(u)$, $\zeta = 0$ in $\overline{\Omega}_T \setminus \mathcal{D}_{\eta/2}(u)$ and $\zeta_x = 0$ over S_T.
Choosing $\psi = [Q_\delta(u_{\delta\sigma}, v_{\delta\sigma})]^{-1} \zeta^2 v_{\delta\sigma xx}$ as a test function in the second equation in $(\mathbf{P}_{\delta\sigma})$, we have

$$\iint_{\Omega_T} \zeta^2 Q_\delta^{-1} v_{\delta\sigma xx} v_{\delta\sigma t} = -\iint_{\Omega_T} \zeta^2 (\mathcal{F}_{\sigma v} - \varepsilon v_{\delta\sigma xx}) v_{\delta\sigma xx}. \tag{3.30}$$

By the choice of ζ, Corollary 3.7 and the equicontinuity of $\{u_{\delta\sigma}\}$ and $\{v_{\delta\sigma}\}$, the following properties hold for $0 < \delta < \delta_K$ and $0 < \sigma < \sigma(\delta)$:

$$\begin{aligned} u_{\delta\sigma} &\in \left[\tfrac{\eta}{4}, 1 - \tfrac{\eta}{4}\right] \ \text{ in } \operatorname{supp}\zeta,\\ v_{\delta\sigma} &\in \left[-\tfrac{1}{2} + \tfrac{d}{4}, \tfrac{1}{2} - \tfrac{d}{4}\right]. \end{aligned}$$

Hence

$$\left|\iint\limits_{\Omega_T} \zeta^2 Q_\delta^{-1} v_{\delta\sigma xx} v_{\delta\sigma t}\right| \le \frac{1}{2\varepsilon} Q_{1\delta}^{-1}\left(\frac{\eta}{4}\right) Q_{2\delta}^{-1}\left(\frac{1}{2}-\frac{d}{4}\right) \iint\limits_{\Omega_T} \zeta^2 Q_\delta^{-1} v_{\delta\sigma t}^2 + \frac{\varepsilon}{2}\iint\limits_{\Omega_T} \zeta^2 v_{\delta\sigma xx}^2,$$

where we have used the Cauchy-Schwarz and Young's inequalities. Therefore, it follows from (3.30) and Lemma 3.1 *(vi)* that

$$\frac{\varepsilon}{2}\iint\limits_{\Omega_T} \zeta^2 v_{\delta\sigma xx}^2 - \iint\limits_{\Omega_T} \zeta^2 \mathcal{F}_{\sigma v} v_{\delta\sigma xx} \le c_{28} \quad \forall\ \delta < \delta_K,\ \sigma < \sigma(\delta). \tag{3.31}$$

We now choose $\phi = \zeta^2 \tilde{Q}_{1\delta}(u_{\delta\sigma})[Q_{2\delta}(v_{\delta\sigma})]^{-1}$ as a test function in the first equation of $(\mathbf{P}_{\delta\sigma})$, where $\tilde{Q}_{1\delta}(u)$ is defined by

$$\tilde{Q}_{1\delta}(u) = \int\limits_{\frac{1}{2}}^{u} \frac{1}{Q_{1\delta}(s)}\, ds,$$

obtaining

$$\begin{aligned} \int\limits_0^T \langle u_{\delta\sigma t}, \zeta^2 \tilde{Q}_{1\delta} Q_{2\delta}^{-1}\rangle\, dt = &-\iint\limits_{\Omega_T} \zeta^2 u_{\delta\sigma x}(\mathcal{F}_{\sigma u} - \varepsilon u_{\delta\sigma xx})_x + \\ &+\iint\limits_{\Omega_T} \zeta^2 Q_{1\delta}\tilde{Q}_{1\delta} Q_{2\delta}^{-1} Q_{2\delta}' v_{\delta\sigma x}(\mathcal{F}_{\sigma u} - \varepsilon u_{\delta\sigma xx})_x - \\ &-\iint\limits_{\Omega_T} (\zeta^2)_x Q_{1\delta}\tilde{Q}_{1\delta}(\mathcal{F}_{\sigma u} - \varepsilon u_{\delta\sigma xx})_x =: I_1 + I_2 + I_3. \end{aligned} \tag{3.32}$$

It easily follows from the energy estimates in Lemma 3.1 and the definition of ζ that

$$\left|\int\limits_0^T \langle u_{\delta\sigma t}, \zeta^2 \tilde{Q}_{1\delta} Q_{2\delta}^{-1}\rangle\, dt\right| + |I_2| + |I_3| \le c_{29} \quad \forall\ \delta < \delta_K,\ \sigma < \sigma(\delta).$$

Let $\gamma > 0$ be a constant to be chosen later: integrating I_1 by parts and using the Cauchy-Schwarz and Young's inequalities we find that

$$I_1 \le \iint\limits_{\Omega_T} \zeta^2 \mathcal{F}_{\sigma u} u_{\delta\sigma xx} - \frac{\varepsilon}{2}\iint\limits_{\Omega_T} \zeta^2 u_{\delta\sigma xx}^2 + \frac{\gamma}{2}\iint\limits_{\Omega_T} \zeta^2 \mathcal{F}_{\sigma u}^2 + 2\left(\varepsilon + \frac{1}{\gamma}\right)\iint\limits_{\Omega_T} (\zeta_x)^2 u_{\delta\sigma x}^2.$$

Note that the last integral on the right hand side is uniformly bounded. Because of the definition of ζ and arguing as in the proof of Proposition 3.5, we find that

$$\frac{\gamma}{2}\iint\limits_{\Omega_T} \zeta^2 \mathcal{F}_{\sigma u}^2 \le c_{30} + 4\gamma \iint\limits_{\Omega_T} \zeta^2 \mathcal{F}_{\sigma v}^2 \tag{3.33}$$

We now proceed to bound the right hand side of (3.33):

$$4\gamma \iint_{\Omega_T} \zeta^2 \mathcal{F}_{\sigma v}^2 \leq 8\gamma \iint_{\Omega_T} \zeta^2 (\mathcal{F}_{\sigma v} - \varepsilon v_{\delta\sigma xx})^2 + 8\gamma\varepsilon^2 \iint_{\Omega_T} \zeta^2 v_{\delta\sigma xx}^2 \leq$$
$$\leq 8\gamma c_{31} \iint_{\Omega_T} \zeta^2 Q_\delta (\mathcal{F}_{\sigma v} - \varepsilon v_{\delta\sigma xx})^2 + 8\gamma\varepsilon^2 \iint_{\Omega_T} \zeta^2 v_{\delta\sigma xx}^2,$$

and therefore, choosing $\gamma < 1/(32\varepsilon)$ and recalling Lemma 3.1 *(iv)* it turns out that

$$4\gamma \iint_{\Omega_T} \zeta^2 \mathcal{F}_{\sigma v}^2 \leq c_{32} + \frac{\varepsilon}{4} \iint_{\Omega_T} \zeta^2 v_{\delta\sigma xx}^2. \tag{3.34}$$

Summing (3.31) and (3.32) and using the estimates above we have

$$\frac{\varepsilon}{2} \iint_{\Omega_T} \zeta^2 u_{\delta\sigma xx}^2 + \frac{\varepsilon}{4} \iint_{\Omega_T} \zeta^2 v_{\delta\sigma xx}^2 - \iint_{\Omega_T} \zeta^2 \left[\mathcal{F}_{\sigma u} u_{\delta\sigma xx} + \mathcal{F}_{\sigma v} v_{\delta\sigma xx}\right] \leq c_{33}. \tag{3.35}$$

Integrating by parts the last integral at the left hand side and recalling the identity (3.10), we obtain

$$-\iint_{\Omega_T} \zeta^2 \left[\mathcal{F}_{\sigma u} u_{\delta\sigma xx} + \mathcal{F}_{\sigma v} v_{\delta\sigma xx}\right] = -\iint_{\Omega_T} (\zeta^2)_{xx} \mathcal{F}_\sigma + \iint_{\Omega_T} \zeta^2 \left[U''(u_{\delta\sigma}) u_{\delta\sigma x}^2 + V''(v_{\delta\sigma}) v_{\delta\sigma x}^2\right] +$$
$$+ \iint_{\Omega_T} \zeta^2 \left[\varphi_\sigma(u_{\delta\sigma} + v_{\delta\sigma})(u_{\delta\sigma} + v_{\delta\sigma})_x^2 + \varphi_\sigma(u_{\delta\sigma} - v_{\delta\sigma})(u_{\delta\sigma} - v_{\delta\sigma})_x^2\right].$$

Since $\mathcal{F}_\sigma$ is uniformly bounded in $L^1(\Omega_T)$ by Lemma 3.1 *(vii)*, the second integral is uniformly bounded and the third is positive, we obtain

$$\iint_{\Omega_T} \zeta^2 \left[\mathcal{F}_{\sigma u} u_{\delta\sigma xx} + \mathcal{F}_{\sigma v} v_{\delta\sigma xx}\right] \leq c_{34}.$$

Therefore, from (3.35), we can conclude that

$$\frac{\varepsilon}{2} \iint_{\Omega_T} \zeta^2 u_{\delta\sigma xx}^2 + \frac{\varepsilon}{4} \iint_{\Omega_T} \zeta^2 v_{\delta\sigma xx}^2 \leq c_{35} \quad \forall\ \delta < \delta_K,\ \sigma < \sigma(\delta).$$

Combining this estimate with (3.34) and (3.33) we obtain, moreover, that

$$\iint_{\Omega_T} \zeta^2 \mathcal{F}_{\sigma u}^2 \leq c_{36}, \quad \text{and} \quad \iint_{\Omega_T} \zeta^2 \mathcal{F}_{\sigma v}^2 \leq c_{37}.$$

The assertion now follows by means of the definition of ζ, the lower semicontinuity of L^2-norms and the arbitrariness of K. □

Proof of Theorem 1. Let (u_δ, v_δ) be the sequence given in (3.19). Using Proposition 3.5 and Lemma 3.8 we have

$$\begin{array}{cccc} u_\delta, v_\delta & \stackrel{*}{\rightharpoonup} & u, v & \text{in} \quad L^\infty(0,T;\, H^1(\Omega)), \\ u_{\delta xx}, v_{\delta xx} & \rightharpoonup & u_{xx}, v_{xx} & \text{in} \quad L^2_{\text{loc}}(\mathcal{D}(u)), \\ u_{\delta t} & \rightharpoonup & u_t & \text{in} \quad L^2(0,T;(H^1(\Omega))'), \\ v_{\delta t} & \rightharpoonup & v_t & \text{in} \quad L^2(\Omega_T), \end{array}$$

and *(i)*-*(iii)* follow. Property *(iv)* is a consequence of Corollary 3.7. From Lemma 3.8 we obtain

$$\mathcal{F}_u(u_\delta, v_\delta) \rightharpoonup g_1 \quad \text{in} \quad L^2_{\text{loc}}(\mathcal{D}(u)),$$
$$\mathcal{F}_v(u_\delta, v_\delta) \rightharpoonup g_2 \quad \text{in} \quad L^2_{\text{loc}}(\mathcal{D}(u)).$$

Since $|\mathcal{D}(u) \setminus \mathcal{B}(u,v)| = 0$ and $u_\delta, v_\delta \to u, v$ uniformly in $\overline{\Omega}_T$, we have

$$\mathcal{F}_u(u_\delta, v_\delta) \longrightarrow \mathcal{F}_u(u,v) \quad \text{a.e. in } \mathcal{D}(u),$$

and therefore $g_1 = \mathcal{F}_u(u,v)$. By the same argument $g_2 = \mathcal{F}_v(u,v)$ and *(v)* is proved. Property *(vi)* is straightforward. To prove *(vii)*, let $K \subset \mathcal{D}(u)$ compact; then $K \subseteq \mathcal{D}_\eta(u)$ for some $\eta > 0$. Since $u \in C(\overline{\Omega}_T)$, we can select $\zeta \in C^\infty(\overline{\Omega}_T)$ such that $0 \le \zeta \le 1$, $\zeta = 1$ in $\mathcal{D}_\eta(u)$ and $\zeta = 0$ in $\overline{\Omega}_T \setminus \mathcal{D}_{\eta/2}(u)$. Hence we have

$$||\zeta u_\delta||_{L^2(0,T;H^2(\Omega))} \le c_{38},$$
$$||(\zeta u_\delta)_t||_{L^2(0,T;(H^1(\Omega))')} \le c_{39},$$

which implies that

$$(\zeta u_\delta)_x \longrightarrow (\zeta u)_x \quad \text{in} \quad L^2(0,T;H^{1-\lambda}(\Omega)), \quad \lambda > 0. \tag{3.36}$$

The continuous embeddings

$$L^2(0,T;H^{1-\lambda}(\Omega)) \subset L^2(0,T;H^{\frac{1}{2}-\lambda}(\partial\Omega)) \subset L^2(0,T;L^2(\partial\Omega)), \quad \lambda < \frac{1}{2},$$

yield

$$(\zeta u_\delta)_x|_{S_T} \to (\zeta u)_x|_{S_T} \quad \text{in} \quad L^2(S_T)$$

and therefore $u_x|_{S_T \cap K} = 0$. Now *(vii)* follows applying the same argument to v_x. Finally we pass to the limit as $\delta \to 0$ in equations (3.11)-(3.13) to prove *(viii)*. The limit is straightforward in equations (3.11) and (3.12). With respect to (3.13) we observe that

$$(Q_\delta^{\frac{1}{2}}\phi)_x = \frac{1}{2} Q_{1\delta}^{-\frac{1}{2}} Q'_{1\delta} Q_{2\delta}^{\frac{1}{2}} u_{\delta x}\phi + \frac{1}{2} Q_{2\delta}^{-\frac{1}{2}} Q'_{2\delta} Q_{1\delta}^{\frac{1}{2}} v_{\delta x}\phi + Q_\delta^{\frac{1}{2}}\phi_x$$

and, for all $\phi \in L^2(0,T;H^1_0(\Omega))$ with compact support in $\mathcal{D}(u)$, each term converges strongly in $L^2(\Omega_T)$ because of (3.36) (with the analogous convergence for v) and the fact that $v \in (-\frac{1}{2}, \frac{1}{2})$. This completes the proof of the Theorem. $\square$

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