MATHEMATICAL MODELS OF FLUIDS WITH STRUCTURE

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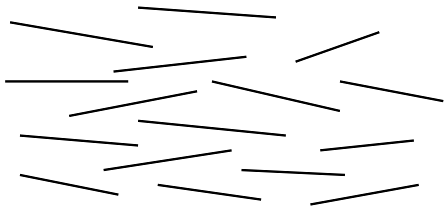
BALANCE LAWS

Mass: $\rho_t + div(\rho\,\mathbf{v}) = 0$

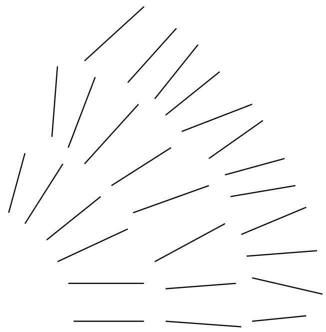
Linear Momentum: $\rho\,\dot{\mathbf{v}} - div(T) = \rho\,\mathbf{f}$

Angular Momentum: $\rho\dot{(J\boldsymbol{\omega})} - div(C) - rot(T) = \rho\,\mathbf{m}$

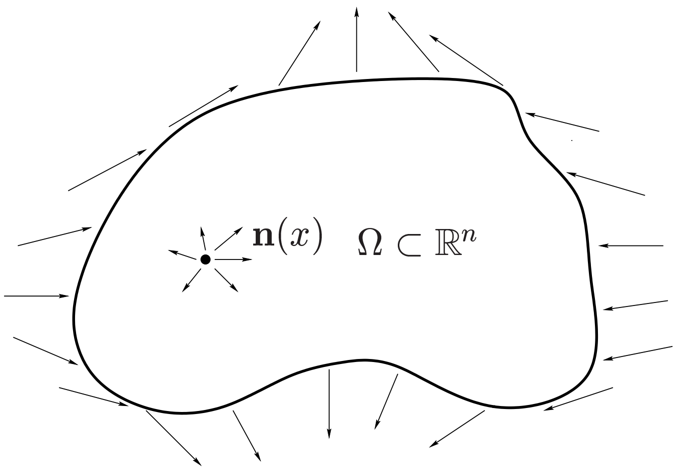
LIQUID CRYSTALS



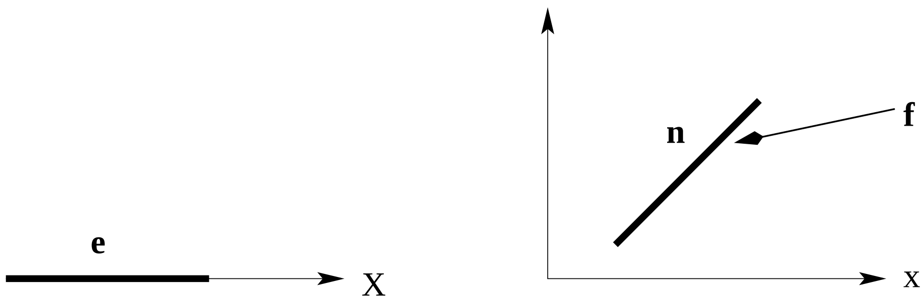
Nematic



Smectic



MOTION OF RIDGID RODS



$$x(X,t) = \mathbf{x}_c(t) + R(t)X = \mathbf{x}_c(t) + \lambda R(t)\mathbf{e}, \qquad \alpha \le \lambda \le \beta$$

Linear Momentum: Let $\mathbf{n} = R(t)\mathbf{e}$, then

$$M\ddot{\mathbf{x}}_c = \mathbf{F} \qquad \mathbf{F}(t) = \int_\alpha^\beta \mathbf{f}\left(\mathbf{x}_c(t) + \lambda\mathbf{n}(t)\right)\, d\lambda$$

Angular Momentum: Note: $|\mathbf{n}| = |\mathbf{e}| = 1$,

$$M\bar{r}^2(\mathbf{n} \times \ddot{\mathbf{n}}) = \mathbf{n} \times \mathfrak{m}$$

$$\boxed{M\bar{r}^2\ddot{\mathbf{n}} = \mathfrak{m} + \theta\mathbf{n}, \qquad \mathfrak{m}(t) = \int_\alpha^\beta \lambda\, \mathbf{f}\left(\mathbf{x}_c(t) + \lambda\mathbf{n}(t)\right)\, d\lambda}$$

$$M = \int_\alpha^\beta \rho(\lambda\mathbf{e})\, d\lambda \qquad M\bar{r}^2 = \int_\alpha^\beta \lambda^2\rho(\lambda\mathbf{e})\, d\lambda$$

DIRECTOR EQUATION

ANGULAR MOMENTUM: $\rho(\dot{J\boldsymbol{\omega}}) - div(C) - rot(T) = \rho\,\mathbf{m}$

CONSTITUTIVE ASSUMPTIONS

$\dot{\mathbf{n}} = \boldsymbol{\omega}\times\mathbf{n}, \qquad J = \bar{r}^2(I-\mathbf{n}\otimes\mathbf{n}), \qquad C_{ij} = \varepsilon_{i\alpha\beta}\mathbf{n}_\alpha\mathbb{C}_{\beta j}, \qquad \mathbf{m} = \mathbf{n}\times\mathfrak{m}$

$$\boxed{\rho\bar{r}^2\,\ddot{\mathbf{n}} - div(\mathbb{C}) - \mathbf{g} = \rho\,\mathfrak{m}}$$

CONSTITUTIVE RESTRICTION: $rot\left(\mathbb{C}(\nabla\mathbf{n})^T + T\right) = \mathbf{g}\times\mathbf{n}$

CONSTITUTIVE RELATIONS: $\mathcal{W}(\mathbf{n},\nabla\mathbf{n})$ frame indifferent,

$$T = -pI - \left[\frac{\partial\mathcal{W}}{\partial n_{k,j}}n_{k,i}\right] + \cdots + \alpha_2(\dot{\mathbf{n}} - W(\mathbf{v})\mathbf{n})\otimes\mathbf{n} + \nu D(\mathbf{v}) + \cdots$$

$$\mathbb{C} = \left[\frac{\partial\mathcal{W}}{\partial n_{i,j}}\right] \qquad \mathbf{g} = -\theta\mathbf{n} - \frac{\partial\mathcal{W}}{\partial\mathbf{n}} + \cdots$$

OSSEN FRANK ENERGY:

$$\mathcal{W}(\mathbf{n},\nabla\mathbf{n}) = (1/2)\left(k_1\,div(\mathbf{n})^2 + k_2\,(\mathbf{n}.curl(\mathbf{n}))^2 + k_3\,|\mathbf{n}\times curl(\mathbf{n})|^2\right)$$

NOTATION

$$D(\mathbf{v}) = \frac{1}{2}(\nabla\mathbf{v}+\nabla\mathbf{v}^T) \qquad \dot{\mathbf{n}} = \mathbf{n}_t + (\mathbf{v}.\nabla)\mathbf{n} \qquad (\nabla\mathbf{v}) = \left[\frac{\partial v_i}{\partial x_j}\right]$$

$$W(\mathbf{v}) = \frac{1}{2}(\nabla\mathbf{v}-\nabla\mathbf{v}^T) \qquad rot(T)_i = \epsilon_{i\alpha\beta}\,T_{\alpha\beta} \qquad \mathbf{a}\otimes\mathbf{b} = [a_i b_j]$$

ERICKSEN LESLIE EQUATIONS

$$\begin{aligned} \mathbf{v}_t + (\mathbf{v}.\nabla)\mathbf{v} \;&+\; \nabla p \\ &-\; div\left(\ldots + \nu D(\mathbf{v}) - \left[\frac{\partial \mathcal{W}}{\partial n_{k,j}} n_{k,i}\right]\right) = \mathbf{f} \end{aligned}$$

$$\nabla.\mathbf{v} = 0$$

$$\mathbf{n}_t + (\mathbf{v}.\nabla)\mathbf{n} + \ldots + \gamma\left\{\left(\frac{\partial \mathcal{W}}{\partial n_i}\right) - div\left[\frac{\partial \mathcal{W}}{\partial n_{i,j}}\right] + \theta\mathbf{n}\right\} = \mathfrak{m}$$

$$|\mathbf{n}| = 1$$

Lin–Liu Equations: Set $k_1 = k_2 = k_3, \ldots$

$$\mathcal{W}(\mathbf{n}, \nabla\mathbf{n}) = \frac{1}{2}|\nabla\mathbf{n}|^2$$

$$\boxed{\begin{gathered} \mathbf{v}_t + (\mathbf{v}.\nabla)\mathbf{v} + \nabla p - div\left(\nu D(\mathbf{v}) - \lambda \nabla\mathbf{d}^T \nabla\mathbf{d}\right) = 0 \\ \nabla.\mathbf{v} = 0 \\ \mathbf{d}_t + (\mathbf{v}.\nabla)\mathbf{d} - \gamma\left(\Delta\mathbf{d} - f(\mathbf{d})\right) = 0 \end{gathered}}$$

Penalty Function: $\mathbf{d} \sim \mathbf{n}$

$$F(\mathbf{d}) = \frac{1}{4\epsilon}\left(|\mathbf{d}|^2 - 1\right)^2 \qquad F : \mathbb{R}^n \to \mathbb{R}$$

$$\mathbf{f}(\mathbf{d}) = DF(\mathbf{d}) = \frac{1}{\epsilon}\left(|\mathbf{d}|^2 - 1\right)\mathbf{d} \qquad \mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$$

WEAK STATEMENTS OF THE LIQUID CRYSTAL PROBLEM

Lin & Liu's Weak Statement

Note:

- $div(\nabla \mathbf{d}^T \nabla \mathbf{d}) = (\nabla \mathbf{d})^T \Delta \mathbf{d} + \frac{1}{2}\nabla(|\mathbf{d}|^2)$
- $(\mathbf{v}.\nabla)\mathbf{d}.\mathbf{e} = \mathbf{e}^T(\nabla \mathbf{d})\mathbf{v}$

$$\int_\Omega \mathbf{v}_t.\mathbf{w} + (\mathbf{v}.\nabla)\mathbf{v}.\mathbf{w} - \tilde{p}\nabla.\mathbf{w} + \nu D(\mathbf{v})\cdot D(\mathbf{w}) + \lambda \Delta \mathbf{d}^T(\nabla \mathbf{d})\mathbf{w} = 0$$

$$\int_\Omega (\nabla.\mathbf{v})q = 0$$

$$\int_\Omega \mathbf{d}_t.\mathbf{e} + \mathbf{e}^T(\nabla \mathbf{d})\mathbf{v} - \gamma\left(\Delta \mathbf{d} - \mathbf{f}(\mathbf{d})\right).\mathbf{e} = 0$$

Bounds Independent of ϵ

$$\mathbf{w} = \mathbf{v} \qquad q = \tilde{p} \qquad \mathbf{e} = -\lambda(\Delta \mathbf{d} - \mathbf{f}(\mathbf{d}))$$

$$E(t) + \int_0^t \left(\nu \|D(\mathbf{v})\|^2_{L^2(\Omega)} + \lambda\gamma \|\Delta \mathbf{d} - \mathbf{f}(\mathbf{d})\|^2_{L^2(\Omega)}\right) = E(0)$$

$$E = \frac{1}{2}\|\mathbf{v}\|^2_{L^2(\Omega)} + \frac{\lambda}{2}\|\nabla \mathbf{d}\|^2_{L^2(\Omega)} + \lambda \int_\Omega \frac{1}{\epsilon}\left(|\mathbf{d}|^2 - 1\right)^2$$

FINITE ELEMENT APPROXIMATIONS

$$\mathbf{v}_h \in \mathcal{U}_h \subset H_0^1(\Omega) \qquad p_h \in \mathcal{P}_h \subset L^2(\Omega)/\mathbb{R}$$

$$\mathbf{d}_h - \mathbf{d}_0 \in \mathcal{W}_h \subset H^2(\Omega) \cap H_0^1(\Omega)$$

$$\int_\Omega \left\{ \left(\frac{\mathbf{v} - \mathbf{v}^n}{\tau} \right) .\mathbf{w} + \frac{1}{2} \left[(\mathbf{v}.\nabla)\mathbf{v}.\mathbf{w} - (\mathbf{v}.\nabla)\mathbf{w}.\mathbf{v} \right] \right.$$

$$\left. + \ \nu D(\mathbf{v}) \cdot D(\mathbf{w}) - p(\nabla.\mathbf{w}) \ + \Delta \mathbf{d}^T (\nabla \mathbf{d}) \mathbf{w} \right\} = 0$$

$$\int_\Omega (\nabla.\mathbf{v}) q = 0$$

$$\int_\Omega \left\{ \nabla \left(\frac{\mathbf{d} - \mathbf{d}^n}{\tau} \right) .\nabla \mathbf{e} - \Delta \mathbf{e}^T (\nabla \mathbf{d}) \mathbf{v} + \gamma (\Delta \mathbf{d} - \mathbf{f}(\mathbf{d})).\Delta \mathbf{e} \right\} = 0$$

Theorem: (Liu & W) If the solution of the liquid crystal problem is sufficiently smooth, then

$$\|\mathbf{v}(t^n) - \mathbf{v}_h^n\|^2_{L^2(\Omega)} + \|\mathbf{d}(t^n) - \mathbf{d}_h^n\|^2_{H_0^1(\Omega)}$$

$$+ \ \sum_{m=0}^{n} \tau \left(\|\mathbf{v}(t^m) - \mathbf{v}_h^m\|^2_{H_0^1(\Omega)} + \|\mathbf{d}(t^m) - \mathbf{d}_h^m\|^2_{H^2(\Omega)} \right)$$

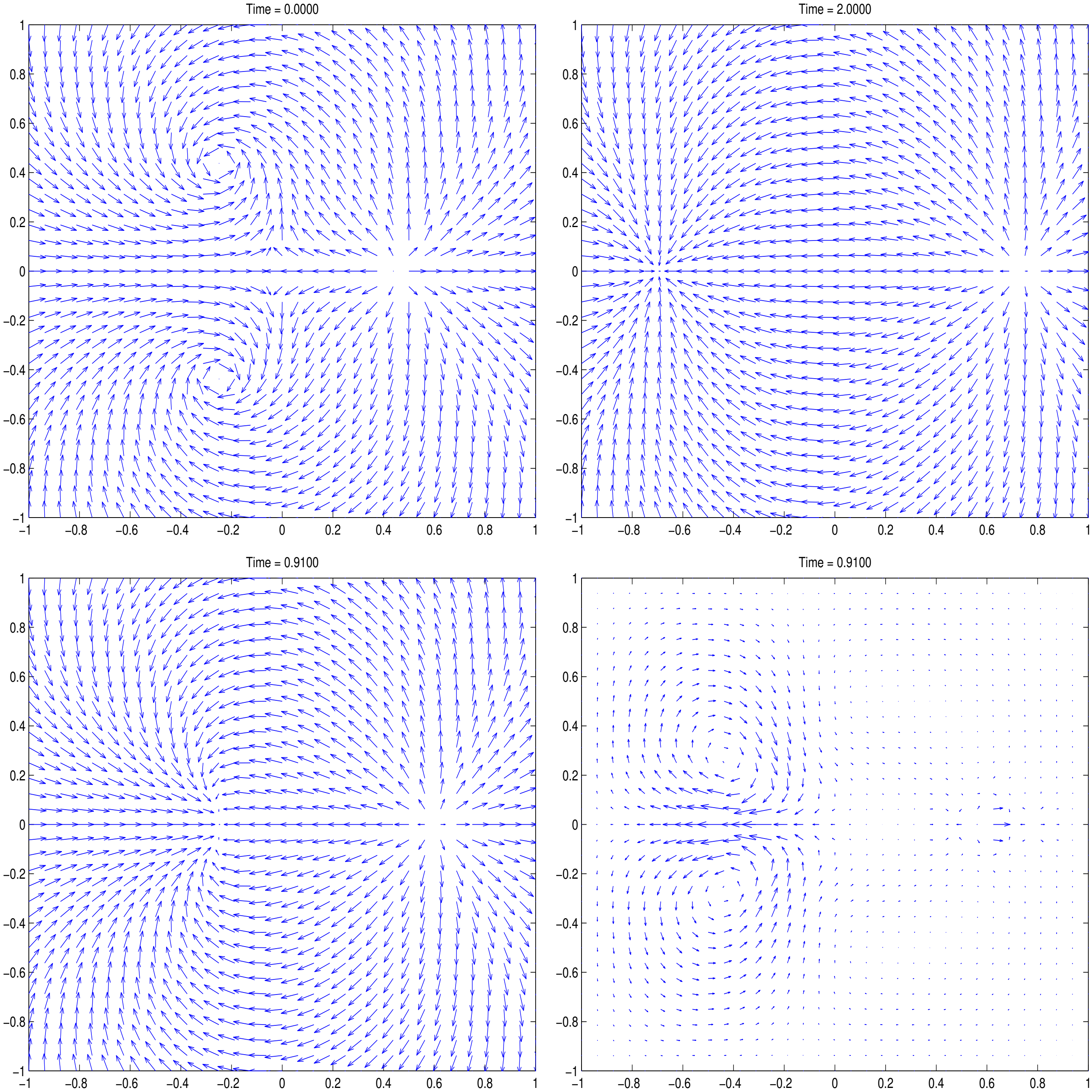
$$\leq \ C(\epsilon) \left(\tau + h^\alpha \right)^2$$

where α depends upon interpolation properties of $\mathcal{U}_h \times \mathcal{P}_h \times \mathcal{W}_h$:

$$\inf_{\mathbf{w}_h \in \mathcal{U}_h} \|\mathbf{w} - \mathbf{w}_h\|_{H_0^1(\Omega)} \ + \inf_{q_h \in \mathcal{P}_h} \|q - q_h\|_{L^2(\Omega)} + \inf_{\mathbf{e}_h \in \mathcal{W}_h} \|\mathbf{e} - \mathbf{e}_h\|_{H^2(\Omega)}$$

$$\leq \ C \left(\mathbf{w}, q, \mathbf{e} \right) h^\alpha$$

SIMULATION OF LIQUID CRYSTAL FLOWS



Space: 16×16, grid (Quad9 x Quad4 x Hermite4) elements

Time: $\tau = 1/100$, implicit Euler scheme

Parameters $\epsilon = 1/20$, $\rho = \mu = \lambda = \gamma = 1$

A MIXED METHOD

$$\mathbf{d}_t + (\mathbf{u}.\nabla)\mathbf{d} - \gamma(\Delta\mathbf{d} - \mathbf{f}(\mathbf{d})) = 0$$

If $W = \nabla\mathbf{d}$ then $(\mathbf{u}.\nabla)\mathbf{d} = W\mathbf{u}$ and $\Delta\mathbf{d} = divW$

$$W_t + \nabla\left(W\mathbf{u} - \gamma(divW - \mathbf{f}(\mathbf{d}))\right) = 0$$

Mixed Formulation

$$\begin{aligned}\int_\Omega \Big\{\mathbf{u}_t.\mathbf{v} \ &+\ (1/2)\left[(\mathbf{u}.\nabla)\mathbf{u}.\mathbf{v} - (\mathbf{u}.\nabla)\mathbf{v}.\mathbf{u})\right] \\ &+\ \nu D(\mathbf{u})\cdot D(\mathbf{v}) - p(\nabla.\mathbf{v}) + \lambda\, divW.(\nabla d)\mathbf{v}\Big\} = 0\end{aligned}$$

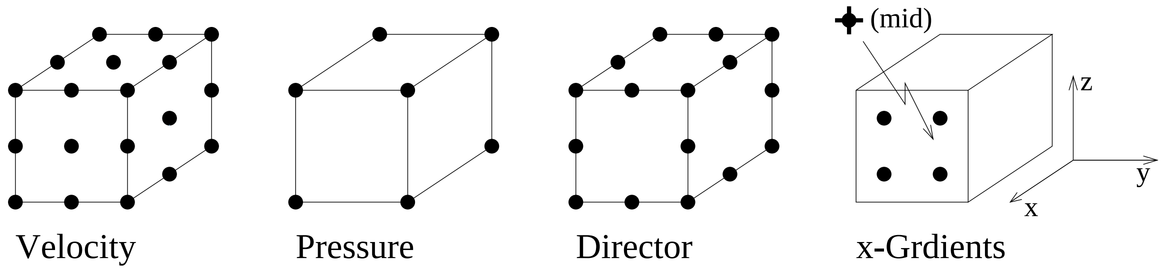
$$\int_\Omega (\nabla.\mathbf{u})q = 0$$

$$\int_\Omega \left\{W_t\cdot Z - (\nabla d)\mathbf{u}.divZ + \gamma\left(divW - \mathbf{f}(\mathbf{d})\right).divZ\right\} = 0$$

$$\int_\Omega \nabla\mathbf{d}\cdot\nabla\mathbf{e} + divW.\mathbf{e} = 0$$

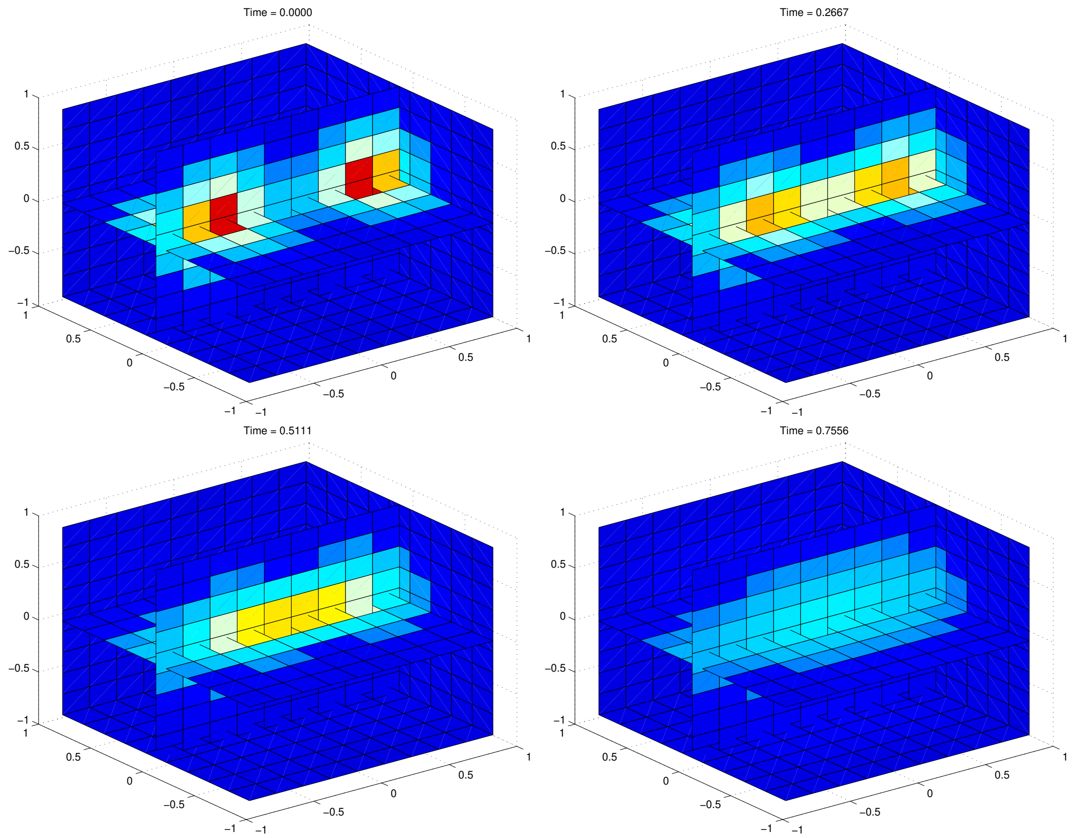
A 3D EXAMPLE (MIXED METHOD)

Elements:



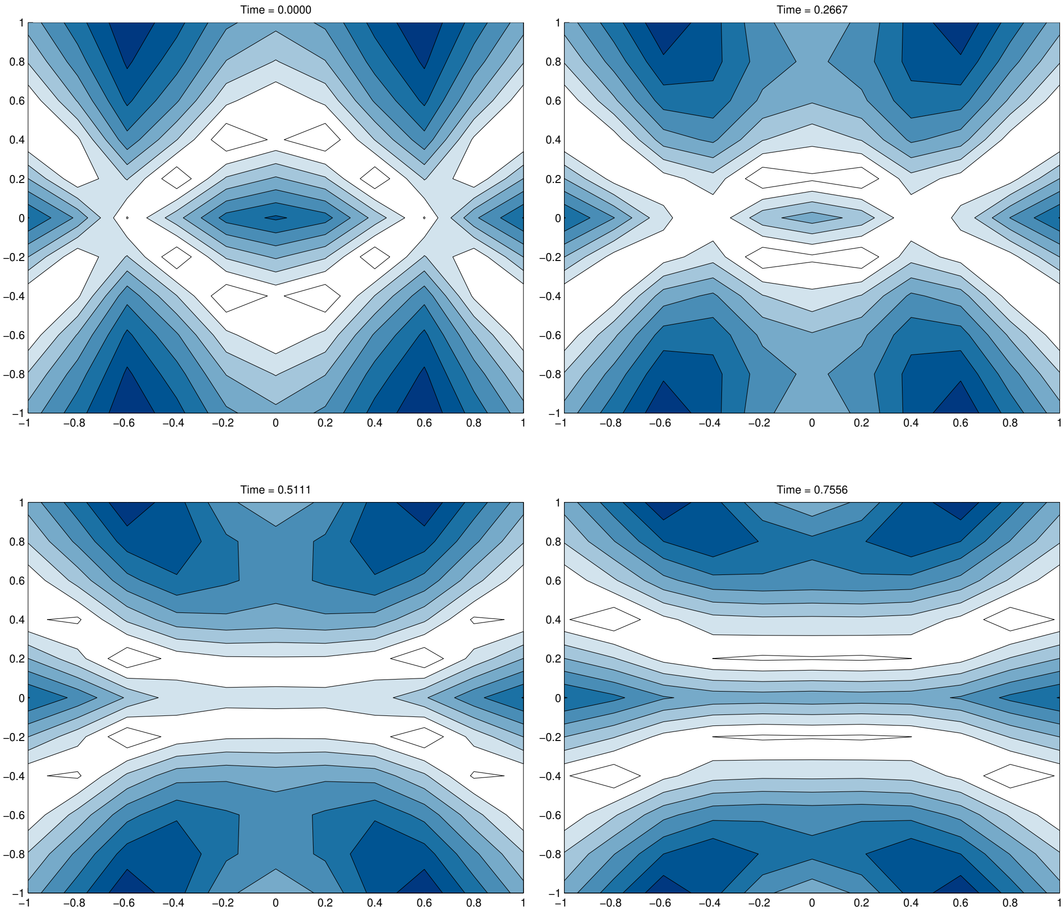
Energy: $e = (1/2)|u|^2 + (\lambda/2)|\nabla d|^2 + \lambda F(d)$

Annihilation of Singularities:



3D EXAMPLE (CONTINUED)

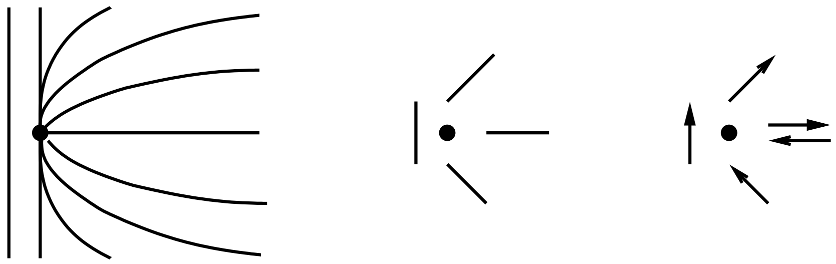
Simulation of Cross Polarized Light



Transmission in z-Direction
Polarizers at $\pm 45^o$

TENSOR MODELS (de Gennes)

Degree Half Singularities



Matrix Representation of $\mathbb{R}P^1 \sim S^1/\{\pm 1\}$

$$Q = \begin{bmatrix} a & b \\ b & -a \end{bmatrix} \qquad \text{eigenvalues} \qquad \pm\,\lambda = \sqrt{a^2+b^2},$$

then

$$Q + \lambda I = \mathbf{n} \otimes \mathbf{n}, \qquad \mathbf{n} = \pm\left(\frac{b}{\sqrt{\lambda - a}}, \sqrt{\lambda - a}\right), \qquad |\mathbf{n}|^2 = 2\lambda$$

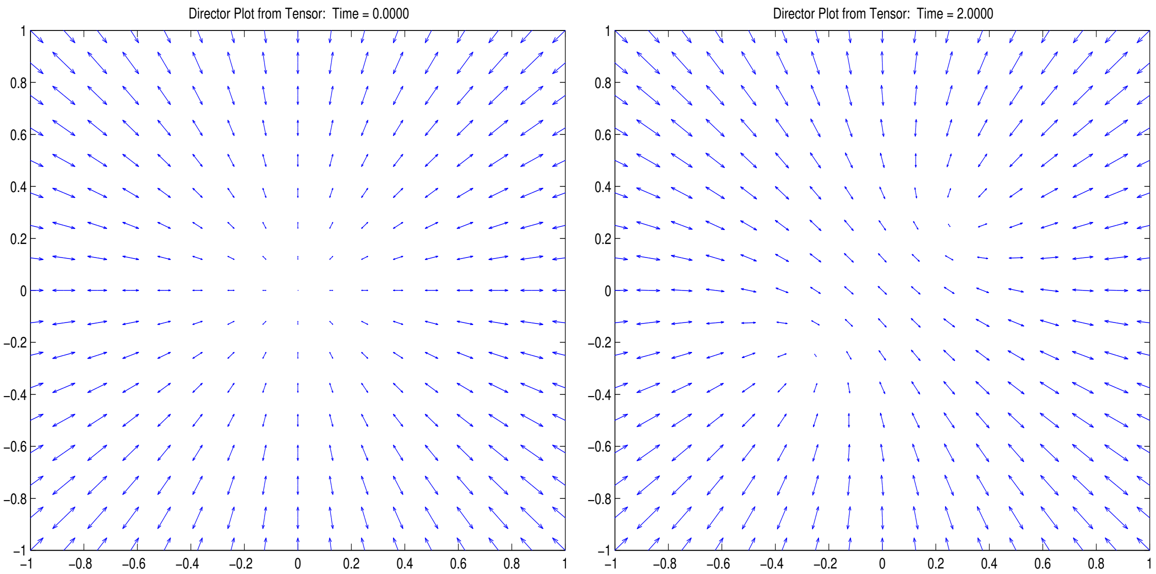
Lin-Liu Equations

$$\boxed{\begin{gathered} \mathbf{v}_t + (\mathbf{v}.\nabla)\mathbf{v} + \nabla p - div\left(\nu D(\mathbf{v}) - \lambda \nabla Q \otimes \nabla Q\right) = 0 \\ \nabla.\mathbf{v} = 0 \\ Q_t + (\mathbf{v}.\nabla)Q - \gamma\left(\Delta Q - f(Q)\right) = 0 \end{gathered}}$$

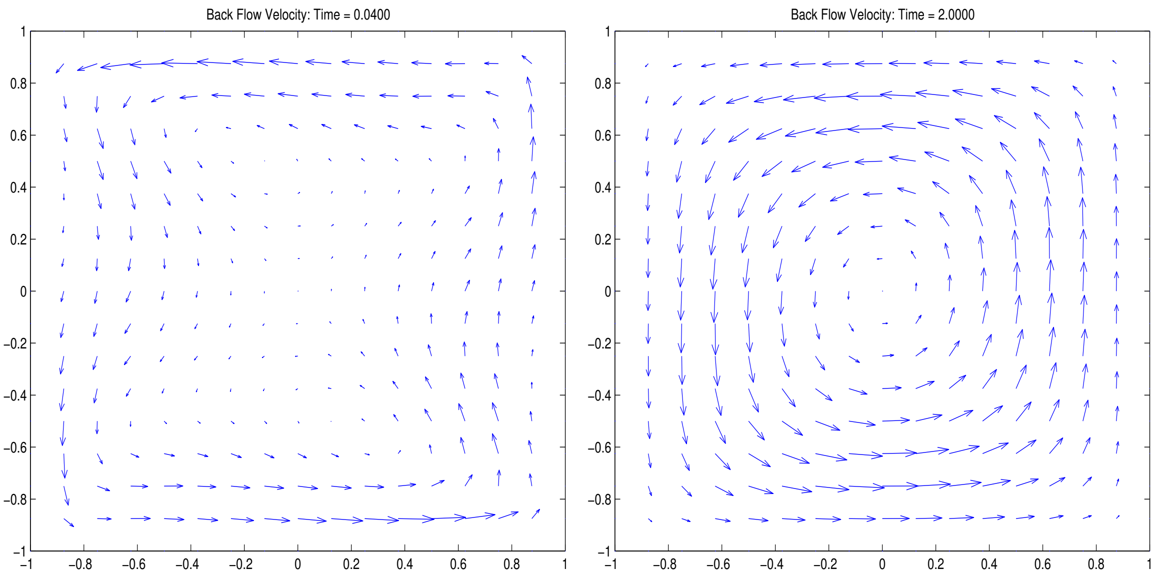
$$f(Q) = \frac{1}{\epsilon}\left(|Q|^2 - 1\right) Q, \qquad |Q| = \sqrt{2}\lambda, \qquad (\nabla Q \otimes \nabla Q)_{ij} = Q_{\alpha\beta,i} Q_{\alpha\beta,j}$$

TENSOR MODEL SIMULATION

DEGREE ONE SIGULARITY IN A SHEAR FLOW



Director Field



Backflow Velocity Field

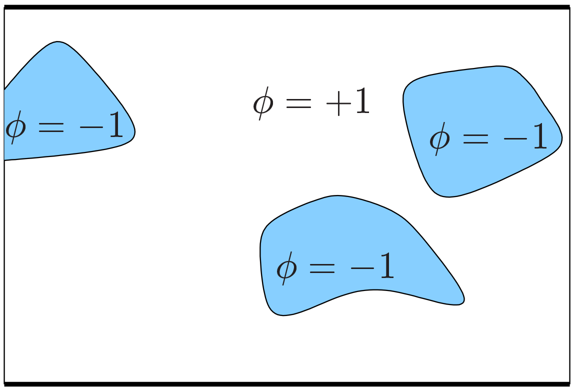
Space: 8×8, grid (Quad9 x Quad4 x Hermite4) elements

Time: $\tau = 1/100$, implicit Euler scheme

Parameters $\epsilon = 1/2$, $\rho = \mu = \lambda = \gamma = 1$

EQUATIONS FOR FLUID-FLUID MIXTURES
(R. DiPerna & P. L. Lions)

Let $\phi(x,t) = \pm 1$ in the thin/thick fluid



Momentum Equation: (weak form)

$$\int_\Omega \rho(\phi)\left(\mathbf{v}_t + (\mathbf{v}.\nabla)\mathbf{v}\right).\mathbf{w} - p\nabla.\mathbf{w} + \mu(\phi) D(\mathbf{v})\cdot D(\mathbf{w}) = \int_\Omega \rho(\phi) f.\mathbf{w}$$

Balance of Mass: $\nabla.\mathbf{v} = 0$

Convection of Particles: $\phi_t + \mathbf{v}.\nabla\phi = 0$

Viscosity and density:

$$\rho(\phi) = \frac{1+\phi}{2}\rho_f + \frac{1-\phi}{2}\rho_s \quad \text{and} \quad \mu(\phi) = \frac{1+\phi}{2}\mu_f + \frac{1-\phi}{2}\mu_s$$

Energy Estimate:

$$\frac{1}{2}\int_\Omega \rho(\phi)|\mathbf{v}|^2\Big|_{t=T} + \int_0^T\int_\Omega \mu(\phi)|D(\mathbf{v})|^2 = \frac{1}{2}\int_\Omega \rho(\phi)|\mathbf{v}|^2\Big|_{t=0}$$

Rigid Particles: Let $\mu_s \to \infty$.

FLUID-FLUID MIXTURES WITH SURFACE TENSION

$$\rho\left(\mathbf{v}_t + (\mathbf{v}.\nabla)\mathbf{v}\right) + \nabla p - div(\mu D(\mathbf{v})) + \epsilon\Delta\phi\nabla\phi = 0$$

$$\nabla.\mathbf{v} = 0$$

$$\phi_t + \mathbf{v}.\nabla\phi - \epsilon^\alpha(\epsilon\Delta\phi - (1/\epsilon)F'(\phi)) = 0$$

Formal Asymptotics: $\mathcal{S}(t) = \{x \in \Omega \mid \phi(x,t) = 0\}$

$$\phi_t + \mathbf{v}.\nabla\phi = 0$$

$$\int_\Omega (\epsilon\Delta d - (1/\epsilon)F'(\phi))\nabla\phi.\mathbf{w} \sim \int_{\mathcal{S}} -(4/3)\,\kappa\,\mathbf{w}.\mathbf{n}$$

Jump Condition: $[T]\mathbf{n} = \kappa\mathbf{n}$, where $T = -pI + \nu D(\mathbf{v})$.

Energy Estimate: If $\rho_t + \mathbf{v}.\nabla\rho = 0$, then

$$\frac{d}{dt}\int_\Omega \left((\rho/2)|\mathbf{v}|^2 + (\epsilon/2)|\nabla\phi|^2 + (1/\epsilon)F(\phi)\right)$$
$$+ \int_\Omega \left(\nu|D(\mathbf{v})|^2 + \epsilon^\alpha(\epsilon\Delta\phi - (1/\epsilon)F'(\phi))^2\right) = 0$$

MESOSCOPIC MODELS

Stochastic Equations: State space $\mathbb{R}^d$,

$$dX(t) = \mathbf{a}(X,t)\,dt + B(X,t)\,d\mathbf{w}(t), \qquad X(0) = x_0$$

Fokker-Planck Equation:

$$\boxed{f_t = div\left((1/2)\nabla(BB^T f) + \mathbf{a}\, f\right)}$$

$$\int_A f(x,t)\,dx = meas\{\omega \mid X(t;\omega) \in A\} \qquad A \subset \mathbb{R}^d$$

General Form: If $\mathcal{A} = \mathcal{A}(f)$ is the "free energy" then

$$\boxed{f_t = div\left(f\,\nabla\left(\frac{\delta\mathcal{A}}{\delta f}\right)\right)}$$

Energy Estimate/Dissipation Relation

$$\mathcal{A}_t = -\int f\left|\nabla\left(\frac{\delta\mathcal{A}}{\delta f}\right)\right|^2$$

Gradient flow in $(\mathcal{M}, d_w)$: $d_w(.,.)$ is the Wasserstein metric on the set of probability measures $\mathcal{M}$ (F. Otto).

$$\mathcal{E}(f) = \frac{1}{2\tau}d_w^2(f, f^n) + \mathcal{A}(f)$$

MESOSCOPIC MODELS

Example (Einstein) $f(t, x)$ is the distribution function for the location of of spherical particles in a dilute solution.

$$f_t = div\left(\frac{1}{\xi}\left(k_b T \nabla f + f \nabla \psi\right)\right),$$

- T is temperature, k_b is a Boltzmann constant
- ψ is a potential for body forces (eg. gravity)
- Gibbs Thompson entropy: $\mathcal{A} = (1/\xi)\int_{\mathbb{R}^d}(k_b T \ln(f) + \psi) f \, dx$
- $\xi = 6\pi\nu a$ is the constant relating force and velocity in steady state motion of particles of radius a in a viscous fluid.

APPLICATIONS

- Polymer solutions: flexible molecules; multi-chain molecules; etc.
- Tensor models for liquid crystals: Rod-like molecules with state space $\mathcal{S}^2$ give rise to "tensor models" which extend the Ericksen-Leslie theory

- Fluid/Solid Mixtures? Can computations at the microscopic level be used to estimate the statistics and/or potential for a macroscopic mixture model?

MESOSCOPIC MODELS
(Doi & Marrucci)

Balance of Momentum:

$$\int_{\Omega} \{\rho(\mathbf{v}_t + \mathbf{v}.\nabla\mathbf{v}).\mathbf{w} - p\, div(\mathbf{w}) + \mu D(\mathbf{v}) \cdot D(\mathbf{w})\} = \int_{\Omega} \rho\, \mathbf{f}.\mathbf{w} - \frac{\delta \mathcal{A}}{\delta f} \nabla f.\mathbf{w}$$

$$f_t + \mathbf{v}.\nabla f = div\left(f\, \nabla \left(\frac{\delta \mathcal{A}}{\delta f}\right)\right)$$

Energy Estimate:

$$\frac{d}{dt}\int_{\Omega} \big((1/2)\rho|\mathbf{v}|^2 + \mathcal{A}(f)\big) + \int_{\Omega} \mu|D(\mathbf{v})|^2 + |\nabla(\delta\mathcal{A}/\delta f)|^2 = 0$$

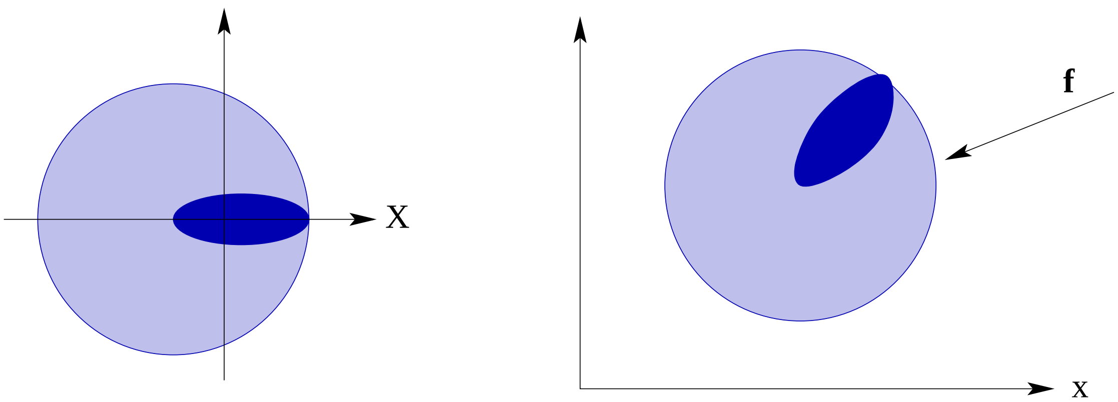
Hamilton's Principle: The momentum equation and the Fokker Planck equation render the Hamiltonian

$$\mathcal{H}(x, f) = \int_0^T \int_{\Omega} \big((1/2)\rho|\mathbf{v}|^2 + \mathcal{A}(f)\big)$$

stationary under variations of $x = \boldsymbol{\chi}(X, t)$ and f respectively (note that $\mathbf{v} = \dot{x}(X, t)$).

Stress Tensor: $\mathcal{A} = \mathcal{A}(f, \nabla f)$ gives stress $(\delta\mathcal{A}/\delta\nabla f) \otimes \nabla f$

RIGID BODY MECHANICS



$$x(X,t) = \mathbf{x}_c(t) + R(t)X \qquad X \in \Omega_r, \qquad R^T R = I$$

Linear Momentum:

$$M\ddot{\mathbf{x}}_c = \mathbf{F} \qquad \mathbf{F}(t) = \int_{\Omega(t)} \mathbf{f}(x)\, dx$$

Angular Momentum: Let $\int_{\Omega_r} \rho(X)X = 0$,

$$(J\boldsymbol{\omega})^{\cdot} = \mathbf{L}_c \qquad \mathbf{L}_c(t) = \int_{\Omega(t)} (x - \mathbf{x}_c(t)) \times \mathbf{f}(x,t)\, dx$$

$$J = R(t)J_r R(t)^T \qquad J_r = \int_{\Omega_r} \rho(X) \left(|X|^2 I - X \otimes X\right) dX$$

Rotation:

$$\boxed{\dot{R} = W(\boldsymbol{\omega})\, R}$$

$$W(\boldsymbol{\omega})\mathbf{v} = \boldsymbol{\omega} \times \mathbf{v} \qquad W(\boldsymbol{\omega}) = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

EQUATIONS FOR SOLID-FLUID MIXTURES

Let $\phi(x,t) = \pm 1$ in the fluid/solid

Momentum Equation:

$$\int_\Omega \rho(\phi)\left(\mathbf{v}_t + (\mathbf{v}.\nabla)\mathbf{v}\right).\mathbf{w} - p\nabla.\mathbf{w} + T(\phi)\cdot\nabla\mathbf{w} = \int_\Omega \rho(\phi) f.\mathbf{w}$$

Stress: $T(\phi) = \dfrac{1+\phi}{2}T_f + \dfrac{1-\phi}{2}T_s$

Newtonian Fluid: $T_f = \mu_f D(\mathbf{v})$

Viscoelastic Solid:

$$T_s = \mu_s D(\mathbf{v}) + (1/J)S(F)F^T$$

$S = [\partial\mathcal{W}/\partial F]$ is the Piola Kirchoff stress, $F = [\partial x/\partial X]$ is the deformation gradient, and $J = det(F) = 1$ for incompressible materials.

Deformation Gradient:

$$\boxed{F_t + (\mathbf{v}.\nabla)F = \nabla\mathbf{v}\,F}$$

Energy Estimate:

$$\frac{1}{2}\int_\Omega \rho(\phi)|\mathbf{v}|^2 + (1-\phi)(1/J)\mathcal{W}(F)\Big|_{t=0}^{T} + \int_0^T\int_\Omega \mu(\phi)|D(\mathbf{v})|^2 = 0$$

LINEAR VISCO-ELASTIC PARTICLES

Ansatz: Polar decomposition $F = RU$ takes the form

$$F = R(I + E), \qquad \text{strain } E = E^T \text{ is small}$$

Linearized Strain Equation: $W(\mathbf{v}) = (1/2)\left((\nabla\mathbf{v}) - (\nabla\mathbf{v})^T\right)$

$$\boxed{E_t + (\mathbf{v}.\nabla)E = R^T D(\mathbf{v})R, \qquad R_t + (\mathbf{v}.\nabla)R = W(\mathbf{v})R}$$

Momentum Equation:

$$\int_\Omega \rho(\phi)\left(\mathbf{v}_t + (\mathbf{v}.\nabla)\mathbf{v}\right).\mathbf{w} \;-\; p\nabla.\mathbf{w}$$
$$+\mu(\phi)D(\mathbf{v})\cdot D(\mathbf{w}) \;+\; R\mathbb{C}(E)R^T \cdot D(\mathbf{w}) = \int_\Omega \rho(\phi)\mathbf{f}.\mathbf{w}$$

Elasticity Tensor: $\mathbb{C} : \mathbb{R}^{d\times d} \to \mathbb{R}^{d\times d}$ is the elasticity tensor, $\mathbb{C} = D^2\mathcal{W}(I)$, $\mathbb{C}(E)_{ij} = \mathbb{C}_{ijkl}E_{kl}$

Energy Estimate:

$$\frac{1}{2}\int_\Omega \rho(\phi)|\mathbf{v}|^2 + |E|^2_{\mathbb{C}}\Big|_{t=0}^{T} + \int_0^T\int_\Omega \mu(\phi)|D(\mathbf{v})|^2 = \int_0^T\int_\Omega \rho(\phi)\mathbf{f}.\mathbf{v}$$

LINEAR VISCO-ELASTIC PARTICLES

Regularized Rotations: Mollify $W(\mathbf{v})$

$$R_{\epsilon_t} + (\mathbf{v}.\nabla)R_\epsilon = W(\mathbf{v}_\epsilon)R_\epsilon \qquad \text{where} \qquad \epsilon = O(|E|)$$

$$\|R - R_\epsilon\|_{L^\infty[0,T;L^1(\Omega)]} \leq C|\chi_s W(\mathbf{v})|_{L^1[0,T;BV(\Omega)]}\,\epsilon$$

Theorem: (Liu & W) Weak solutions of the equations for the motion of linear elastic particles in an incompressible Newtonian fluid (with regularized rotations) exist and satisfy

$$\mathbf{v} \in L^\infty[0,T;L^2(\Omega)] \cap L^2[0,T;H_0^1(\Omega)],$$

$$\phi, R \in L^\infty[0,T;L^\infty(\Omega)], \qquad E \in L^\infty[0,T;L^2(\Omega)],$$

and the energy estimate (with inequality).