tartar@andrew.cmu.edu

Center for Nonlinear Analysis Summer School, May 31, 2001 - June 9, 2001
Multiscale Problems in Nonlinear Analysis

Does Nature minimize or conserve energy?

In 1848, STOKES[1] explained a discrepancy which CHALLIS[2] had noticed concerning some solutions of the equations of compressible gas dynamics, which POISSON[3] had obtained in an implicit form in 1808, by showing that solutions could approach a discontinuity in finite time; by using conservation of mass and conservation of momentum he had then correctly derived the jump conditions that discontinuous solutions must satisfy, and these are now interpreted as meaning that the discontinuous functions satisfy a partial differential equation in the sense of distributions, as developped by Laurent SCHWARTZ[4], following the pioneer work of Sergei SOBOLEV[5] and of Jean LERAY[6]. After STOKES, RIEMANN[7] derived independently the jump conditions in his thesis in 1860, but instead of being called the STOKES-RIEMANN conditions, the jump conditions are now named after RANKINE[8] and HUGONIOT[9]. However, when STOKES edited his complete works in 1880[10], he did not reproduce there his 1848 proof of the jump conditions, and instead he

---

[1] Sir George Gabriel STOKES, Irish-born mathematician, 1819-1903. He held the Lucasian chair (endowed by Henry LUCAS, English clergyman, 1610-1663) at Cambridge, England UK.

"On a difficulty in the theory of sound," Philos. Mag. **XXXIII** (1848), 349–356. *Classic papers in shock compression science*, 71–79, High-press. Shock Compression Condens. Matter, *Springer, New York*, 1998.

[2] James CHALLIS, English Astronomer, 1803-1882. He worked at Cambridge, England UK.

"On the velocity of sound," Philos. Mag. **XXXII** (1848), 494–499.

[3] Siméon Denis POISSON, French mathematician, 1781-1840. He worked in Paris, France.

"Mémoire sur la théorie du son," J. Éc. Polytech. **14** (1808), 319–392. *Classic papers in shock compression science*, 3–65, High-press. Shock Compression Condens. Matter, *Springer, New York*, 1998.

[4] Laurent SCHWARTZ, French mathematician, born in 1915. He received the FIELDS medal in 1950. He worked at Ecole Polytechnique, Palaiseau, France, and I had him as a teacher in 1965/66 (when École Polytechnique was in Paris).

John Charles FIELDS, Canadian mathematician, 1863-1932.

[5] Sergei L'vovich SOBOLEV, Russian mathematician, 1908-1989. He worked in Akademgorodok, Novosibirsk, Russia.

[6] Jean LERAY, French mathematician, 1906-1998. He received the WOLF prize in 1979. He held a chair (Théorie des équations différentielles et fonctionnelles, 1947-1978) at Collège de France, Paris, France.

Ricardo WOLF, German-born diplomat and philanthropist, 1887-1981. The WOLF foundation was established in 1976 with his wife, Francisca SUBIRANA-WOLF, 1900-1981, "to promote science and art for the benefit of mankind".

[7] Georg Friedrich Bernhard RIEMANN, German mathematician, 1826-1866. He worked in Göttingen, Germany.

"Über die Fortpssanzung ebener Luftwellen von endlicher Schwingungsweite," Abhandlungen der Gesellschaft der Wissenschafte zu Göttingen, Mathematisch-Physikalische Klasse 8, 43 (1860). Gesammelte Werke, 1876, p144.

[8] William John Macquorn RANKINE, Scottish engineer, 1820-1872. He worked in Glasgow, Scotland UK.

"On the thermodynamic theory of waves of finite longitudinal disturbance," Philos. Trans. **160** (1870), part II, 277–288. *Classic papers in shock compression science*, 133–147, High-press. Shock Compression Condens. Matter, *Springer, New York*, 1998.

[9] Pierre Henri HUGONIOT, French engineer, 1851-1887.

"Sur la propagation du mouvement dans les corps et spécialement dans les gaz parfaits, I, II" J. Éc. Polytech. **57** (1887), 3–97, **58** (1889), 1–125. *Classic papers in shock compression science*, 161–243, 245–358, High-press. Shock Compression Condens. Matter, *Springer, New York*, 1998.

[10] STOKES George Gabriel Sir, *Mathematical and physical papers*, Reprinted from the original journals and transactions, with additional notes by the author. Cambridge, University Press, 1880-1905.

apologized for having made a mistake, because he had been (wrongly) convinced by Lord RAYLEIGH[11] and THOMSON[12] (later to become Lord KELVIN), that his discontinuous solutions were not physical: they did not conserve energy.

So in the third part of the 19^{th} Century, good physicists were adamant: energy is conserved!

KELVIN, RAYLEIGH and STOKES must have understood later that heat is a form of energy, and that the missing energy in STOKES's discontinuous solutions of (isentropic) gas dynamics is transformed into heat, which does make the temperature of the gas increase, but since that possibility is not allowed in the mathematical model this energy is "apparently lost"; in other terms, these three great scientists had not grasped yet why one needs a notion of "internal energy", although it seems that WATT[13] and CARNOT[14] had understood much earlier (and independently) that mechanical energy may be transformed into heat and heat may be transformed into mechanical energy, but with inherent limitations about the proportion of mechanical energy which can be recovered after it has been transformed into heat.

So in the fourth part of the 19^{th} Century, good physicists knew that energy is conserved, if one is careful to count all its forms[15], including heat.

One way for solving an equation of the form $-\Delta\, u = f$ with boundary conditions, an equation named after LAPLACE[16] or POISSON, is to invoke the DIRICHLET[17] principle, which consists in noticing that if a function u of class C^2 minimizes the functional J defined by $J(v) = \frac{1}{2}\int_\Omega |grad(v)|^2\,dx - \int_\Omega f\,v\,dx$ among all smooth functions having a given boundary value, then u satisfies $-\Delta\, u = f$ in Ω. The principle was named after DIRICHLET by RIEMANN, who had heard it from him, but it had been used before by GAUSS[18] and by GREEN[19], and the same principle is also named after THOMSON.

Could it be that any of these scientists believed that Nature is minimizing energy?

I do not think so. In those days the equation may have arisen from a question of Gravitation (using NEWTON[20]'s law of attraction), and in that case the potential energy for a finite number of masses must have been used and the limit for a density of masses must have been derived, probably first by LAPLACE and by POISSON; the same ideas were certainly applied to Magnetism, and to Electricity, but all these people must have known the basic truth of Classical Mechanics that total energy (kinetic plus potential) is conserved; they must have known that a local strict minimum of the potential energy gives rise to a stable equilibrium,

[11] John William STRUTT, 3^{rd} baron RAYLEIGH, English physicist, 1842-1919. He received the NOBEL prize in Physics in 1904. He held the CAVENDISH Professorship (named after Henry CAVENDISH, English physicist, 1731-1810) at Cambridge, England UK.

Alfred NOBEL, Swedish industrialist and philanthropist, 1833-1896. He created a fund to be used as awards "for people whose work most benefited humanity".

[12] William THOMSON, Irish-born physicist, 1824-1907. In 1892 he was made baron KELVIN of Largs, and thereafter known as Lord KELVIN. He worked in Glasgow, Scotland UK.

[13] James WATT, Scottish engineer, 1736-1819.

[14] Sadi Nicolas Léonard CARNOT, French engineer, 1796-1832.

[15] Soon after, EINSTEIN was going to observe that mass is also a form of energy, with the famous relation $e = m\,c^2$; energy over mass has the dimension of the square of a velocity, and the only intrinsic velocity is the velocity of Light c (299,792,458 meter/second), so once mass is understood to be energy, it is natural to compare this energy to $m\,c^2$, but physicists seem to have accepted without criticism that the proportionality factor must be 1 for all "particles", or maybe they consider that relation as a definition of "mass".

Albert EINSTEIN, German-born physicist, 1879-1955. He received the NOBEL prize in Physics in 1921. He worked at the Institute for Advanced Study, Princeton NJ.

[16] Pierre-Simon LAPLACE, French mathematician, 1749-1827. He was made count in 1806 and marquis in 1817. He worked in Paris, France.

[17] Johann Peter Gustav LEJEUNE DIRICHLET, German mathematician, 1805-1859. He worked in Berlin, and then in Göttingen, Germany.

[18] Johann Carl Friedrich GAUSS, German mathematician, 1777-1855. He worked in Göttingen, Germany.

[19] George GREEN, English mathematician, 1793-1841. He was a miller and never held any academic position.

[20] Sir Isaac NEWTON, English mathematician, 1643-1727. He held the Lucasian chair at Cambridge, England UK.

but they could not have mistaken the minimization of potential energy as a physical process; in other words, they must have known that the DIRICHLET Principle is a useful mathematical trick[21].

So it seems better to consider that Nature does not minimize energy, but that the mathematical problem of minimizing a suitable functional may give information on the solutions of some physical processes which are not about minimizing anything.

Although Nature does not minimize energy, one may well arrive at something that looks like what Nature has created in a particular area by looking at the mathematical point of view of minimizing a particular functional, but one should not be fooled by this analogy and jump to conclusion too fast. If A implies B and one observes something which looks like B, then one must be an optimist to deduce that it must be because A is true, and physicists seem to be of an optimistic nature, but mathematicians would probably consider as intellectually challenged a student who thinks that if A implies B and B is true then A is true and they may think of showing the deluded student a large number of counter-examples (thinking that such a student would not even be convinced by seeing only one counter-example), so mathematicians seem to be of a pessimistic nature. If one observes a microstructure and one discovers that such a microstructure appears when one minimizes a particular functional, and if one claims that it is the reason why this particular microstructure occurs, one might be making precisely the mistake that mathematicians are warned against.

Although not related to a physical process, the DIRICHLET Principle has helped discover interesting mathematical structures. WEIERSTRASS[22] pointed out that the functional J might not attain its minimum, and this question may have been at the onset of his study of compactness. I think that the complete solution of the DIRICHLET Principle was one in the famous list of problems which HILBERT[23] proposed in 1900 at the International Congress of Mathematicians in Paris, France; it might have been the motivation of Sergei SOBOLEV for introducing his famous spaces, and for that purpose he may have considered the question of Electrostatics, a simplification of MAXWELL[24] equation; it might have been the motivation of Laurence C. YOUNG[25] for analyzing minimization problems without solutions and introducing a class of generalized solutions, for which one uses now the term *YOUNG measures.*

In the Summer 1978, I taught a course[26] at HERIOT[27]-WATT University in Edinburgh, Scotland, on Compensated Compactness, a theory which I had partly developed with François MURAT[28]; in that course, I

---

[21] Specialists of Numerical Analysis know that any unphysical principle is useful if it permits to compute quickly and accurately something that one is interested in, but there is no reason to mistake such a principle for reality.

[22] Karl Theodor Wilhelm WEIERSTRASS, German mathematician, 1815-1897. He worked in Berlin, Germany.

[23] David HILBERT, German mathematician, 1862-1943. He worked in Göttingen, Germany.

[24] James CLERK MAXWELL, Scottish physicist, 1831-1879. He held the first CAVENDISH Professorship of Physics at Cambridge, England UK.

[25] Laurence Chisholm YOUNG, English-born mathematician, 1905-2000. He worked at University of Wisconsin, Madison WI. His father, William Henry YOUNG (English-born mathematician, 1863-1942, who worked in Lausanne, Switzerland), and his mother, Grace CHISHOLM-YOUNG (English-born mathematician, 1868-1944), worked on many problems together, and some classical theorems attributed to YOUNG were their joint works.

YOUNG L. C., "Generalized curves and the existence of an attained absolute minimum in the calculus of variations," C. R. Soc. Sci. Lett. Varsovie, Classe III **30** (1937), 212–234. "Generalized surfaces in the calculus of variations, I, II. Ann. of Math. (2) **43**, (1942) 84–103, 530–544. *Lectures on the Calculus of Variation and Optimal Control Theory,* W. B. Saunders Co., Philadelphia-London-Toronto, Ont. 1969 xi+331 pp.

[26] TARTAR L. C., "Compensated compactness and applications to partial differential equations, *Nonlinear analysis and mechanics: Heriot-Watt Symposium, Vol. IV*, pp. 136–212, Res. Notes in Math., 39, *Pitman, Boston, Mass.-London*, 1979.

[27] George HERIOT, Scottish goldsmith, 1563-1624.

[28] François MURAT, French mathematician, born in 1947. He works at CNRS (Centre National de la Recherche Scientifique) and Université Paris VI (Pierre et Marie CURIE), in Paris, France.

Pierre CURIE, French physicist, 1859-1906, and his wife Marie SLODOWSKA CURIE, Polish-born physiused YOUNG measures as a way to describe the constraints that were obtained by applying my Compensated Compactness Method, but I did not know at the time that Laurence C. YOUNG had introduced them, and I used the term *parametrized measures* which I had heard in the seminar of Robert PALLU DE LA BARRIÈRE[29] at IRIA (Institut de Recherche en Informatique et Automatique, Rocquencourt, France), where the idea in control theory was attributed to GHOUILA-HOURI[30], but they had been used in control theory before by Jack WARGA[31] and called relaxed controls. Although I was the first to use parametrized/YOUNG measures in a context of partial differential equations, I knew of some earlier work[32] on variational problems involving no derivatives by Henri BERLIOCCHI[33] and Jean-Michel LASRY[34]. In my course, it was not the YOUNG measures which were the important concept, and from my work on Homogenization with François MURAT I already knew about their limitations; in physical terms, YOUNG measures are just a precise mathematical way for talking about one point statistics, without falling into the usual trap of using probabilities for describing the laws of Nature.

Although MAXWELL had found his equation in his quest for unifying Electricity and Magnetism, his system of equations describes waves which propagate with the speed of Light[35]. Many systems of partial differential equations (of hyperbolic type) are used now as mathematical models of physical processes where energy is conserved, and waves explain an important phenomenon, that energy can be globally conserved but may nevertheless flow away, transported by waves; for the wave equation I had read some works in this direction in the early 70s by various combinations of authors, Peter LAX[36], Cathleen MORAWETZ[37], Ralph PHILLIPS[38], and Walter STRAUSS[39].

---

cist, 1867-1934, jointly received the NOBEL prize in Physics in 1903, and she also received the NOBEL prize in Chemistry in 1911.

[29] Robert PALLU DE LA BARRIÈRE, French mathematician. He worked at Université Paris VI (Pierre et Marie CURIE), Paris, France.

[30] Alain GHOUILA-HOURI, French mathematicien, -196?.

GHOUILA-HOURI Alain, "Sur la généralisation de la notion de commande d'un système guidable, Rev. Française Informat. Recherche Opérationnelle **1** (1967) no. 4, 7–32.

[31] Jack WARGA, American mathematician, born in 1922. He works at Northeastern University, Boston MA.

WARGA J., "Relaxed variational problems," "Necessary conditions for minimum in relaxed variational problems," J. Math. Anal. Appl. **4** (1962) 111–128, 129–145.

[32] BERLIOCCHI Henri & LASRY Jean-Michel, "Intégrandes normales et mesures paramétrées en calcul des variations, Bull. Soc. Math. France **101** (1973), 129–184.

[33] Henri BERLIOCCHI, French mathematician. He works at Université Paris XIII, Villetaneuse, France.

[34] Jean-Michel LASRY, French mathematician. He works at Université Paris IX Dauphine, Paris, France.

[35] Human eye is designed to "see" electromagnetic waves with wavelength 0.4 to 0.7 micron, corresponding to visible light with colour ranging from violet to red.

[36] Peter D. LAX, Hungarian-born mathematician, born in 1925. He received the WOLF prize in 1987. He works at the COURANT Institute of Mathematical Sciences at New York University, New York NY.

Richard COURANT, German-born mathematician, 1888-1972. He worked at New York University, New York NY.

[37] Cathleen SYNGE MORAWETZ, Canadian-born mathematician, born in 1923. She works at the COURANT Institute of Mathematical Sciences at New York University, New York NY. Her father, John Lighton SYNGE (Irish mathematician, 1897-1995, who worked in Dublin, Ireland), had been from 1946 to 1948 the Head of the Mathematics Department at the CARNEGIE Institute ot Technology in Pittsburgh PA, later to become CARNEGIE MELLON University.

Andrew CARNEGIE, Scottish-born businessman and philanthropist, 1835-1919.

Andrew William MELLON, American financier and philanthropist, 1855-1937.

[38] Ralph Saul PHILLIPS, American mathematician, 1913-1998. He worked at STANFORD University, Stanford CA.

Leland STANFORD, American businessman, 1824-1893.

[39] Walter Alexander STRAUSS, American mathematician, born in 1937. He works at BROWN University, Providence RI.

There is another way for the energy to be conserved while it may seem to disappear. It may happen from a nonlinear effect, like for the creation of shocks in quasilinear systems, and more precisely for (hyperbolic) systems of conservation laws; this question generalizes the particular situations which STOKES and RIEMANN had studied, and it was put into a contemporary mathematical framework by Peter LAX in the 50s[40]. A good way to learn about this area now is to look at the recent book[41] of my good friend Constantine DAFERMOS[42], but it should be noted that it was a book[43] by COURANT and FRIEDRICHS[44] who had stemmed interest[45] about these questions in the academic community.

More generally, there are situations where the information goes to infinity, but instead of infinity in the space variable x, it is infinity in the dual space of the FOURIER[46] variable ξ, i.e. energy flows into smaller wavelengths (higher frequencies).

In elastic structures, at least the two aspects occur, and one part of the energy is lost away through the support, and another part of the energy is trapped into high frequency vibrations, and this often happens because of a small scale in some direction, as one uses thin plates and beams in buildings; in real buildings there is an added damping due to the viscoelastic properties of concrete, which one may interpret as resulting from its microstructure by an Homogenization process, as I heard from Henri SANCHEZ-PALENCIA[47], and his explanation is based on a coupling between a (linearly) elastic material and a viscous fluid trapped in

---

Nicholas BROWN Jr., American merchant, 1769-1841.

LAX Peter D. & MORAWETZ Cathleen S. & PHILLIPS Ralph S., "The exponential decay of solutions of the wave equation in the exterior of a star-shaped obstacle," Bull. Amer. Math. Soc. **68** (1962) 593–595. "Exponential decay of solutions of the wave equation in the exterior of a star-shaped obstacle," Comm. Pure Appl. Math. **16** (1963) 477–486.

MORAWETZ Cathleen S., "The decay of solutions of the exterior initial-boundary value problem for the wave equation," Comm. Pure Appl. Math. **14** (1961) 561–568. "The limiting amplitude principle," Comm. Pure Appl. Math. **15** (1962) 349–361.

STRAUSS Walter A., "Local exponential decay of a group of conservative nonlinear operators," J. Functional Analysis **6** (1970) 152–156.

[40] LAX Peter D., "Weak solutions of nonlinear hyperbolic equations and their numerical computation," Comm. Pure Appl. Math. **7**, (1954). 159–193. "Hyperbolic systems of conservation laws. II. Comm. Pure Appl. Math. **10** (1957) 537–566.

[41] DAFERMOS Constantine M., *Hyperbolic conservation laws in continuum physics*, Grundlehren der Mathematischen Wissenschaften, 325. Springer-Verlag, Berlin, 2000. xvi+443 pp. ISBN 3-540-64914-X

[42] Constantine M. DAFERMOS, Greek-born mathematician, born in 1941. He works at BROWN University, Providence RI.

[43] COURANT R. & FRIEDRICHS K. O., *Supersonic Flow and Shock Waves* Interscience Publishers, Inc., New York, N. Y. 1948. xvi+464 pp. Applied Mathematical Sciences, Vol. 21. Springer-Verlag, New York-Heidelberg, 1976. xvi+464 pp.

[44] Kurt Otto FRIEDRICHS, German-born mathematician, 1901-1982. He worked at the COURANT Institute of Mathematical Sciences at New York University, New York NY.

[45] It is the historical section in this book which triggered my interest in reading more about the origin of ideas, some of them I had learned as a student without any precise attribution. Cathleen MORAWETZ had the charge of editing the book when she was a graduate student; a few years ago she told me that other historical informations had to be discarded due to a restriction in space.

[46] Jean-Baptiste Joseph FOURIER, French mathematician, 1768-1830. He was prefect in Grenoble under NAPOLÉON.

Napoléon BONAPARTE, French general and statesman, 1769-1821; he proclaimed himself emperor, under the name NAPOLÉON I, 1804-1814 (and 100 days in 1815).

[47] Enrique Evariste SANCHEZ-PALENCIA, Spanish-born mathematician, born in 1941. He works at CNRS (Centre National de la Recherche Scientifique) and Université Paris VI (Pierre et Marie CURIE), Paris, France.

SANCHEZ-PALENCIA Enrique, *Nonhomogeneous media and vibration theory* Lecture Notes in Physics, 127. Springer-Verlag, Berlin-New York, 1980. ix+398 pp. ISBN 3-540-10000-8.

cavities inside[48]. However, I then heard of another scenario in a talk of Joseph KELLER[49], at a meeting at IMA (Institut for Mathematics and its Applications, Minneapolis MN) in 1985; he mentioned the effect of inclusions (of different elastic materials), together with a small coupling due to taking into account the nonlinear geometric effect in Elasticity, as a way to explain an origin of damping and the flow of energy to higher frequencies.

It is possible that the energy trapped in high frequency vibrations transforms more easily into heat, but it is possible that one has been calling heat too many different kinds of energy, and this is one difficulty for putting Thermodynamics into a sound basis. In the traditional description, heat is taken away by conduction, diffusion, convection of the fluid surrounding the structure and radiation from the boundary (according to the blackbody radiation formula of PLANCK[50], which implies a total radiated energy following the law[51] in T^4 which STEFAN[52] had found earlier); part of the energy trapped in high frequency is also transmitted as waves to the fluid surrounding the structure, some being audible[53].

It must be noted that in some situations physicists call noise these particular high frequency oscillations which others are trying to describe, and of course calling it noise will not help explaining what it is, and postulating that it follows some rules which probabilists like is certainly not the way to understand the phenomenon.

I like to call science-fiction elastic materials those inexistant materials which only have an elastic range and do not show any onset of plasticity or apparition of cracks, whatever forces are applied to it. In the late 70s, I made the mistake of assuming that such materials could exist in principle, although I had failed to find a good class of strain-stress relations for them; of course, one has to impose that the evolution problem is well posed (and the mathematical methods for studying this question still seem quite inadequate), and I also expected this class to be stable by Homogenization[54]. A few years later, Gianni DAL MASO[55] had mentioned to me that one can say something about the stored energy functional of the limiting material by using the method of Γ-convergence which Ennio DE GIORGI[56] had developed, and until a few years ago, I did find that it was an advantage of the Γ-convergence approach that it could talk about the stored energy functional of such a limiting material, whose strain-stress relation I had been unable to characterize; however, I believe now that no such material exist (which has to satisfy some invariance properties), and that the Γ-convergence result has answered another question, which may not be very physical.

I like to call ultra-science-fiction elastic materials those different inexistant materials which instanta-

---

[48] Indeed, the water content in concrete is known to be related to its viscoelastic properties; in particular the viscoelastic properties of concrete change with time, and this aging may be due to excessive drying.

[49] Joseph Bishop KELLER, American mathematician, born in 1923. He received the WOLF prize for 1996/97. He works at STANFORD University, Stanford CA.

[50] Max Karl Ernst Ludwig PLANCK, German physicist, 1858-1947. He received the NOBEL prize in Physics in 1918. He worked in Berlin, Germany.

[51] The absolute temperature T is 273.15 + temperature in degrees CELSIUS.

Anders CELSIUS, Swedish astronomer, 1701-1744. He worked in Uppsala, Sweden.

CELSIUS had put the boiling point of water at 0 and the freezing point at 100 degrees. This was reversed by LINNÉ who invented the scale that is used now, under CELSIUS's name.

Carolus LINNAEUS (Carl VON LINNÉ), Swedish naturalist, 1707-1778.

[52] Josef STEFAN, Austrian physicist, 1835-1893. He worked in Vienna, Austria.

[53] Human ear is designed to "hear" acoustic waves with frequency 20 to 20,000 Hertz.

Heinrich Rudolf HERTZ, German physicist, 1857-1894. He worked in Bonn, Germany.

[54] "Weak convergence in nonlinear partial differential equations," *Existence Theory in Nonlinear Elasticity,* 209–218. *The University of Texas at Austin,* 1977. "Nonlinear constitutive relations and homogenization," *Contemporary developments in continuum mechanics and partial differential equations* (*Proc. Internat. Sympos., Inst. Mat., Univ. Fed. Rio de Janeiro, Rio de Janeiro,* 1977), *pp.* 472–484. *North-Holland Math. Studies,* 30, *North-Holland, Amsterdam-New York,* 1978.

[55] Gianni DAL MASO, Italian mathematician. He works at SISSA (International School for Advanced Studies) in Trieste, Italy.

[56] Ennio DE GIORGI, Italian mathematician, 1928-1996. He received the WOLF prize in 1990. He worked at Scuola Normale Superiore, Pisa, Italy.

neously discover a minimum of their potential energy; this is usually the class which is studied by those who pretend to study Elasticity but never mention the term stress.

Although ideal (science-fiction) elastic materials are conservative, it is clear that real materials are slightly dissipative, but the way a real material tends to dissipate its kinetic energy and apparently stop in an equilibrium configuration is certainly not by looking for a global minimum of the potential energy; of course, if there is only one equilibrium solution satisfying the desired constraints it will be the global minimum, and an example[57] of such a situation has been shown in Elasticity by Robin KNOPS[58] and Charles STUART[59].

I have left aside an argument which is sometimes used for explaining the interest in configurations of lowest energy, which is to invoke Statistical Physics; only systems in thermal equilibrium are supposed to exist in this framework, and the state of a system is indexed by the absolute temperature T, and the rule says that there is a "probability" to find the system in a state of energy w, which is proportional to $exp\big(-\frac{w}{k\,T}\big)$, where k is the BOLTZMANN[60] constant[61]. Of course, the basic rule of this game makes no sense but for large systems whose parts are connected enough to interact, and it is worth observing that this rule has been invented by analogy with the equilibrium solutions of BOLTZMANN equation, where the parts are identical "particles" which collide at a sufficient rate in order to arrive at a thermal equilibrium quickly[62]; those who like this framework sometime argue that when the temperature T tends to 0, a system tends to occupy only the positions of lowest energy (with equal probabilities if there are a few of them), and therefore an expansion near zero temperature is mentioned, where one first considers the global minima of the energy w.

I do not think that there is much Physics in a model where there is no time. In Classical Mechanics, one often invokes NOETHER[63]'s theorem which links symmetries and conservation laws, and it is precisely the invariance by translation in time which is related to the conservation of total energy, and therefore one should not be surprised that the partisans of Nature minimizing energy live in a world where time does not exist. Some may pretend that time is not so important for their models because they are only interested in stationary solutions, but they should then certainly avoid mentioning the question of stability of the equilibrium model that they study, because the only possible meaning of stability is related to the precise evolution equation that is satisfied, at least since the work of LYAPUNOV[64], and I had read in the

---

[57] KNOPS R. J. & STUART C. A., "Quasiconvexity and uniqueness of equilibrium solutions in nonlinear elasticity," Arch. Rational Mech. Anal. **86** (1984), no. 3, 233–249.

[58] Robin J. KNOPS, English mathematician, born in 1932. He works at HERIOT-WATT University in Edinburgh, Scotland UK.

[59] Charles A. STUART, Scottish-born mathematician. He works at EPFL (École Polytechnique Fédérale de Lausanne) in Lausanne, Switzerland.

[60] Ludwig BOLTZMANN, Austrian physicist, 1844-1906. He worked in Graz and Vienna, Austria, and also in Leipzig, Germany.

[61] $k = 1.3807\ 10^{-23}$ JOULE/KELVIN, and of course the JOULE is the unit for energy, NEWTON×meter, or kilogram×meter^2×second^{-2}.

James Prescot JOULE, English physicist, 1818-1889.

[62] However, specialists of Plasma Physics have observed that in their experiments lighter electrons tend to settle quickly to some temperature, while heavier ions tend to settle quickly to another temperature, and their experiments do not last long enough for these two temperatures to come together. It would be very silly either for a specialist of meteorology to use a one temperature model for the atmosphere. These obvious considerations are but one of the many known defects of using equilibrium models for deducing some properties of the real world; one should be aware that this is the point of view which has been taught for generations, that there is an equation of state valid all the time, although it has been deduced from the sole knowledge of equilibria, and part of the scientific problem is not only to ascertain if some equations have a solution and how to select the physical solution when there are many, but to question the validity of these equations that one has studied and construct better physical models of reality.

[63] Emmy Amalie NOETHER, German-born mathematician, 1882-1935. Until 1933 she worked in Göttingen, Germany, and then in Bryn Mawr PA.

[64] Aleksandr Mikhaĭlov LYAPUNOV, Russian mathematician, 1857-1918.

correspondence[65] between MITTAG-LEFFLER[66] and POINCARÉ[67] that a different notion of stability had been used at the beginning; it would be very silly for a partisan of Nature minimizing energy, to use the term stability and mistake the gradient flow corresponding to that minimization for the evolution equation governing the system.

I do not think that there is much Physics either in a model where there is no space. Once one knows that energy is conserved and may change its form, the problem is to understand where it is located, in space and time or in the dual frequency space. It is precisely when one wants to localize elastic energy in space that one discovers the CAUCHY[68] stress tensor; one should notice that this stress tensor is natural in the physical Eulerian[69] point of view, and that in the less realistic mathematical Lagrangian[70] point of view (also studied by EULER) it corresponds to the PIOLA[71]-KIRCHHOFF[72] stress tensor.

The mathematical difficulty for keeping track of all the energy is that one needs to develop efficient mathematical tools for analyzing where the energy has gone, in space-time as well as in the dual variables. For doing that one certainly needs to use the relation between different scales which I have developed in the early 70s, based on the use of various types of weak convergence, like H-convergence[73], developped with François MURAT, which extends the notion of G-convergence[74], which Sergio SPAGNOLO[75] had developed in the late 60s, helped with the insight of Ennio DE GIORGI.

---

[65] *La correspondance entre Henri Poincaré et Gösta Mittag-Leffler* Publications des Archives Henri-Poincaré. Birkhäuser Verlag, Basel, 1999. 421 pp. ISBN 3-7643-5992-7.

[66] Magnus Gösta MITTAG-LEFFLER, Swedish mathematician, 1846-1927. He worked in Stockholm, Sweden.

[67] Jules Henri POINCARÉ, French mathematician, 1854-1912. He worked in Paris, france.

[68] Augustin Louis CAUCHY, French mathematician, 1789-1857. He was made baron in 1838. He worked in Paris, France.

[69] Leonhard EULER, Swiss-born mathematician, 1707-1783. He worked in St. Petersburg, Russia.

[70] Giuseppe Lodovico LAGRANGIA (Joseph Louis LAGRANGE), Italian-born mathematician, 1736-1813. He was made count in 1808. He worked in Paris, France.

[71] Gabrio PIOLA, Italian physicist, 1794-1850. He worked in Milan, Italy.

[72] Gustav Robert KIRCHHOFF, German physicist, 1824-1887. He worked in Berlin, Germany.

[73] MURAT F., "H-convergence," *Séminaire d'analyse fonctionnelle et numérique*, Université d'Alger, 1977-78. Translated into English as MURAT F. & TARTAR L., "H-convergence," *Topics in the mathematical modelling of composite materials*, 21–43, Progr. Nonlinear Differential Equations Appl., 31, *Birkhäuser Boston, Boston, MA*, 1997.

[74] SPAGNOLO S., "Sul limite delle soluzioni di problemi di Cauchy relativi all'equazione del calore," Ann. Scuola Norm. Sup. Pisa (3) **21** (1967), 657–699. "Sulla convergenza di soluzioni di equazioni paraboliche ed ellitiche," Ann. Scuola Norm. Sup. Pisa (3) **22** (1968), 571–597.

[75] Sergio SPAGNOLO, Italian mathematician. He works at Universitá di Pisa, Pisa, Italy.