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Overdetermined boundary value problems

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In the theory of elliptic boundary value problems one prescribes usually m boundary conditions for solutions of an equation of order $2m$. If more conditions are prescribed, this usually restricts the shape of the boundary.

i) The following problem is still open (and known as Schiffer's conjecture): Suppose that Ω is a bounded domain. If $u \not\equiv 0$ solves the overdetermined problem

$$\begin{aligned} \Delta u + \gamma u =& 0 \qquad \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} =& 0 \qquad \text{on } \partial\Omega, \\ u = const. =& c \qquad \text{on } \partial\Omega, \end{aligned}$$

does this imply that Ω is a ball? This conjecture is related to the so-called Pompeiu problem, see for instance [L1]. As indicated in [K2] and [L2] it is also related to overdetermined eigenvalue problems for plates, and Pohozaev identities that support the conjecture can be found in [K1] and [L2].

ii) For a bounded connected domain Ω let u now be a solution of

$$\begin{aligned} -\Delta u =& 1 \qquad \text{in } \Omega, \\ u =& 0 \qquad \text{on } \partial\Omega, \\ \frac{\partial u}{\partial \nu} = const. =& c \qquad \text{on } \partial\Omega. \end{aligned}$$

Then Ω must be a ball. This was first shown by Serrin in [S] in a seminal paper via a method that is now known as moving plane method. An entirely different proof was given by Weinberger [W]. Under suitable structural assumptions both methods of proof extend to quasilinear elliptic equations, see [FGK], [FK]. An extension of Weinberger's method to a fourth order equation was given in [B].

iii) In Bernoulli-type problems one studies (p-)harmonic functions with Dirichlet and Neumann conditions on a free part of the boundary and Dirichlet conditions on a fixed part of the boundary. Here the fixed boundary influences the shape of the free boundary. One example, in which the behaviour of solutions as $p \to \infty$ and $p \to 1$ is studied, can be found in [KS].

References

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