Application of Optimal Transport to Evolutionary PDEs

5 -Applications

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2010 CNA Summer School
New Vistas in Image Processing and PDEs

Carnegie Mellon Center for Nonlinear Analysis, Pittsburgh, *June 7–12, 2010*

Outline

1. **Thin film equation as the gradient flow of the Dirichlet functional**
   - in collaboration with U.Gianazza, G.Toscani, D. Matthes, R. McCann

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Starting point: a family of 4th order equations in $\mathbb{R}^d$

We look for **non-negative** solutions to the nonlinear 4th order evolution PDEs

$$\partial_t \boldsymbol{u} + \mathsf{div}\Big(\boldsymbol{u} \ \mathrm{D}\big(\boldsymbol{u}^{\boldsymbol{\alpha}-1}\boldsymbol{\Delta}\boldsymbol{u}^{\boldsymbol{\alpha}}\big)\Big) = 0 \quad \text{in } (0,+\infty)\times\mathbb{R}^d, \qquad \boldsymbol{\alpha} \in [\mathbf{1/2}, \mathbf{1}],$$

with the initial condition

$$0 \le \boldsymbol{u}(0,\cdot) = u_0 \in L^1(\mathbb{R}^d), \quad \int_{\mathbb{R}^d} |x|^2 u_0 \, \mathrm{d}x < +\infty.$$

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$$\partial_t u + \mathsf{div}(\boldsymbol{m(u)}\,\mathrm{D}(\Delta u)) = 0, \qquad \text{where, e.g. } \boldsymbol{m(u) = u^m}$$

has been studied (mainly in dimension $d = 1,2,3$) by many authors: [Bernis-Friedman '90, Bertsch-Dal Passo-Garcke-Grün '98–'04; review: Becker-Grün '05.; asymptotic behaviour: Carrillo-Toscani '02, Carlen-Ulusoy '07]

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The generating functional is

$$\boldsymbol{\Phi(u) := \frac{1}{2}\int_{\mathbb{R}^d} |\mathrm{D}u|^2\,\mathrm{d}x}$$

The "Wasserstein gradient" of the Dirichlet functional

Standard technique: choose a vector field $\boldsymbol{\xi} \in \mathrm{C}_c^\infty(\mathbb{R}^d;\mathbb{R}^d)$ and the flow X

$$\frac{\mathrm{d}}{\mathrm{d}t}X_t(x) = \boldsymbol{\xi}(X_t(x)),\ X_0(x) = x; \qquad \boldsymbol{M_\varepsilon} := (X_\varepsilon)_\# \boldsymbol{M}; \quad \rightsquigarrow \quad \boxed{\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Phi(\boldsymbol{M_\varepsilon})_{|\varepsilon=0}.}$$

Wasserstein gradient $\boldsymbol{g} = -\boldsymbol{v}$: $\displaystyle\int_{\mathbb{R}^d} \langle \boldsymbol{g}, \boldsymbol{\xi}\rangle \,\mathrm{d}\boldsymbol{M} = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Phi(\boldsymbol{M_\varepsilon})_{|\varepsilon=0}.$

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It corresponds to the **weak formulation of the thin film equation**

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Discrete equation: $\boldsymbol{M^n_\tau} \leftrightarrow \boldsymbol{U^n_\tau}$

$$\int_{\mathbb{R}^d}\zeta(\boldsymbol{U^n_\tau} - \boldsymbol{U^{n-1}_\tau})\,\mathrm{d}x + \frac{\tau}{2}\int_{\mathbb{R}^d}\Delta^2\zeta\,(\boldsymbol{U^n_\tau})^2 - 2\mathrm{D}^2\zeta\,\mathrm{D}\boldsymbol{U^n_\tau}\cdot\mathrm{D}\boldsymbol{U^n_\tau} - \Delta\zeta\,|\mathrm{D}\boldsymbol{U^n_\tau}|^2\,\mathrm{d}x = \boldsymbol{o(\tau)}$$

Main problem

Discrete equation:

$$\int_{\mathbb{R}^d}\zeta\big(\boldsymbol{U}_{\boldsymbol{\tau}}^{\boldsymbol{n}}-\boldsymbol{U}_{\boldsymbol{\tau}}^{\boldsymbol{n-1}}\big)\,\mathrm{d}x+\frac{\tau}{2}\int_{\mathbb{R}^d}\Delta^2\zeta\,(\boldsymbol{U}_{\boldsymbol{\tau}}^{\boldsymbol{n}})^2-2\mathrm{D}^2\zeta\,\mathrm{D}\boldsymbol{U}_{\boldsymbol{\tau}}^{\boldsymbol{n}}\cdot\mathrm{D}\boldsymbol{U}_{\boldsymbol{\tau}}^{\boldsymbol{n}}-\Delta\zeta\,|\mathrm{D}\boldsymbol{U}_{\boldsymbol{\tau}}^{\boldsymbol{n}}|^2\,\mathrm{d}x=\boldsymbol{o(\tau)}$$

Strong compactness in $W^{1,2}$ in order to pass to the limit in the quadratic term

$$\int_{\mathbb{R}^d}2\mathrm{D}^2\zeta\,\mathrm{D}\boldsymbol{U}_{\boldsymbol{\tau}}^{\boldsymbol{n}}\cdot\mathrm{D}\boldsymbol{U}_{\boldsymbol{\tau}}^{\boldsymbol{n}}\,\mathrm{d}x$$

First variation along auxiliary flows

MAIN IDEA: take the first variation of the minimum problem

$$U_\tau^n \in \operatorname*{argmin}_V \frac{W^2(V, U_\tau^{n-1})}{2\tau} + \Phi(V)$$

along the (Wasserstein) gradient flow S^Ψ generated by other ***"good" auxiliary functionals*** $\boldsymbol{\Psi}$.

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Look for good flows S^Ψ having $\boldsymbol{\Phi}$ as Lyapunov functional

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$$u_t := \mathsf{S}_t^\Phi(u_0) \text{ solves } \frac{d}{dt} u = -\nabla\Phi(u), \quad w_t := \mathsf{S}_t^\Psi(w_0) \text{ solves } \frac{d}{dt} w = -\nabla\Psi(w)$$

If $u_0 = w_0$ then we have the **"commutation"** identity

$$\frac{d}{d\varepsilon}\Phi(w_\varepsilon)\Big|_{\varepsilon=0^+} = \frac{d}{d\varepsilon}\Psi(u_\varepsilon)\Big|_{\varepsilon=0^+} \qquad \Big(= -\big\langle \nabla\Phi(w_0), \nabla\Psi(u_0)\big\rangle\Big)$$

RECIPE: if the derivative of the **(main) functional** Φ along the **(auxiliary) flow** S^Ψ is negative (up to lower order terms)

then Ψ is a **Lyapunov functional** for the **main flow** S^Φ (up to lower order terms).

Look for good flows S^Ψ having Φ as Lyapunov functional

A Lyapunov-type estimate at the discrete level in the Wasserstein space

Suppose that $\boldsymbol{\Psi}$ generates a good flow $\boldsymbol{w_t = \mathsf{S}_t^{\Psi}(w)}$ satisfying the EVI:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}W^2(\mathsf{S}_t^{\boldsymbol{\Psi}}(\boldsymbol{w}), z) \leq \boldsymbol{\Psi}(z) - \boldsymbol{\Psi}(\mathsf{S}_t^{\boldsymbol{\Psi}}(\boldsymbol{w})) \tag{EVI}$$

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We call $\mathcal{D}$ the dissipation of $\boldsymbol{\Phi}$ along $\mathsf{S}^{\boldsymbol{\Psi}}$

$$\mathcal{D}(\boldsymbol{w}) := \boxed{-}\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\boldsymbol{\Phi}(\mathsf{S}_\varepsilon^{\boldsymbol{\Psi}}(\boldsymbol{w}))\Big|_{\varepsilon=0^+} = \limsup_{\varepsilon\downarrow 0}\frac{\boldsymbol{\Phi}(\boldsymbol{w}) - \boldsymbol{\Phi}(\mathsf{S}_\varepsilon^{\boldsymbol{\Psi}}(\boldsymbol{w}))}{\varepsilon}$$

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Theorem (Discrete flow-interchange estimate)

If $\boldsymbol{U_\tau^n}$ is a minimizer of $V \mapsto \dfrac{W^2(V, \boldsymbol{U_\tau^{n-1}})}{2\tau} + \boldsymbol{\Phi}(V)$ then

$$\boldsymbol{\Psi}(\boldsymbol{U_\tau^n}) + \boldsymbol{\tau}\,\boldsymbol{\mathcal{D}(U_\tau^n)} \le \boldsymbol{\Psi}(\boldsymbol{U_\tau^{n-1}})$$

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PROOF:

$$0 \le \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\frac{W^2(\mathsf{S}_\varepsilon^{\boldsymbol{\Psi}}(\boldsymbol{U}_\tau^n), \boldsymbol{U}_\tau^{n-1})}{2\tau} + \boldsymbol{\Phi}(\mathsf{S}_\varepsilon^{\boldsymbol{\Psi}}(\boldsymbol{U}_\tau^n))\Big|_{\varepsilon=0+} \qquad \textit{(by the minimality of } \boldsymbol{U}_\tau^n\textit{)}$$

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PROOF:

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$$\leq \frac{\Psi(U_\tau^{n-1}) - \Psi(U_\tau^n)}{\tau} - \mathcal{D}(U_\tau^n) \qquad \textit{(by the EVI, with } z = U_\tau^{n-1},\ w = U_\tau^n\textit{)}$$

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Theorem (Discrete flow-interchange estimate)

If U^n_τ is a minimizer of $V \mapsto \dfrac{W^2(V, U^{n-1}_\tau)}{2\tau} + \Phi(V)$ then

$$\Psi(U^n_\tau) + \tau\,\mathcal{D}(U^n_\tau) \le \Psi(U^{n-1}_\tau) - \frac{\kappa}{2}W^2(U^n_\tau, U^{n-1}_\tau).$$

PROOF:

$$0 \le \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\frac{W^2(\mathsf{S}^\Psi_\varepsilon(U^n_\tau), U^{n-1}_\tau)}{2\tau} + \Phi(\mathsf{S}^\Psi_\varepsilon(U^n_\tau))\Big|_{\varepsilon=0+} \qquad \textit{(by the minimality of } U^n_\tau\textit{)}$$

$$\le \frac{\Psi(U^{n-1}_\tau) - \Psi(U^n_\tau)}{\tau} - \mathcal{D}(U^n_\tau) \qquad \textit{(by the EVI, with } z = U^{n-1}_\tau,\ w = U^n_\tau\textit{)}$$

Auxiliary flows for the thin film equation (II)

$\boldsymbol{\Phi(u) = \frac{1}{2}\int_{\mathbb{R}^d} |\mathrm{D}u|^2\,\mathrm{d}x}$ decays on the heat flow

$$\partial_t w - \Delta w = 0$$

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The heat equation is the Wasserstein gradient flow of the relative entropy functional $\mathcal{H}(w) := \int_{\mathbb{R}^d} w\log w \,\mathrm{d}x$.

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In term of U_τ it corresponds to

$$\int_0^T\int_{\mathbb{R}^d} \big|\mathrm{D}^2 U_\tau\big|^2\,\mathrm{d}x\,\mathrm{d}t \le C.$$

Main result

Assume that the non-negative initial condition $u_0 \in L^1(\mathbb{R}^d)$ satisfies

$$\int_{\mathbb{R}^d} |x|^2 u_0(x)\,\mathrm{d}x < +\infty, \quad \mathcal{H}(u_0) = \int_{\mathbb{R}^d} u_0 \log u_0 \,\mathrm{d}x < +\infty.$$

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Theorem

There exists an infinitesimal subsequence of time steps $\tau_k \downarrow 0$ such that

$$\boldsymbol{U}_{\tau_k} \to \boldsymbol{u} \quad \textit{pointwise in } L^1(\mathbb{R}^d) \textit{ and in } L^2(0,T;W^{1,2}(\mathbb{R}^d)) \quad \textit{as } k \uparrow \infty$$

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$\boldsymbol{u} \in C^0([0,+\infty);L^1(\mathbb{R}^d)) \cap L^2_{\mathrm{loc}}([0,+\infty);W^{2,2}(\mathbb{R}^d))$ *is a non-negative global solution of the weak formulation of thin film equation*

$$\partial_t \boldsymbol{u} + \frac{1}{2}\Delta^2(\boldsymbol{u}^2) - \partial^2_{x_i\,x_j}(\partial_{x_i}\boldsymbol{u}\,\partial_{x_j}\boldsymbol{u}) - \frac{1}{2}\Delta|\mathrm{D}\boldsymbol{u}|^2 = 0$$

Outline

1. **Thin film equation as the gradient flow of the Dirichlet functional**
   - in collaboration with U.Gianazza, G.Toscani, D. Matthes, R. McCann

2. **The L^2-gradient flow of the simplest polyconvex functional**
   - in collaboration with L. Ambrosio, S. Lisini

3. **The sticky particle system**
   - in collaboration with L. Natile

Polyconvex functionals

$$\mathscr{F}(\boldsymbol{u}) = \int_\Omega F(D\boldsymbol{u})\,dx$$

where

$$F(A) = \Phi(A, M_2(A), \cdots, M_{d-1}(A), \det A), \quad \text{and } \Phi \text{ is convex;}$$
$$M_2(A), \cdots M_{d-1}(A), M_d(A) = \det A \quad \text{are the minors of } A.$$

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If Φ is superlinear then the functional $\mathscr{F}$ is lower semicontinuous in $L^2(\Omega;\mathbb{R}^d)$ [J. Ball].

Well posedness of the variational problems

$$\min_{\boldsymbol{U}} \frac{1}{2\tau}\int_\Omega |\boldsymbol{U} - \boldsymbol{U}_\tau^{n-1}|^2\,dx + \mathscr{F}(\boldsymbol{U})$$

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Nevertheless, no general results are known for gradient flows of polyconvex functionals and for their variational approximation

The "simplest" polyconvex functional

$$F(A) := \Phi(\det A), \qquad \mathscr{F}(\boldsymbol{u}) := \int_\Omega \Phi(\det D\boldsymbol{u}(x))\,dx$$

under the additional constraint that

$\boldsymbol{u}$ is a diffeomorphism between Ω and $\boldsymbol{u}(\Omega)$, $\det D\boldsymbol{u}(x) > 0$,
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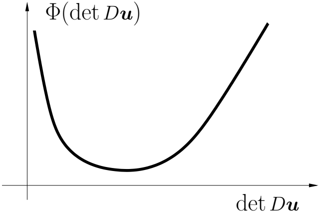

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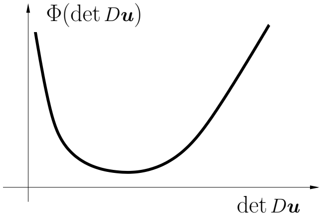

Difficulties (besides polyconvexity):

lack of coercivity ($\mathscr{F}$ controls only $\det D\boldsymbol{u}$)
lack of lower semicontinuity in $L^2(\Omega; \mathscr{U})$.

The form of the PDE

$$F(A) = \Phi(\det A), \quad DF(A) = \big(\mathsf{cof}\ A\big)^T \Phi'(\det A),$$

since

$$\frac{\partial \det A}{\partial A^i_\alpha} = (\mathsf{cof}\ A)^i_\alpha \quad \text{where} \quad \sum_\alpha A^i_\alpha (\mathsf{cof}\ A)^j_\alpha = \det A \delta_{ij} \quad \forall i, j.$$

$$\delta \mathscr{F}(\boldsymbol{u}, \boldsymbol{\xi}) = \int_\Omega \Phi'(\det D\boldsymbol{u})\, \mathsf{cof}\ D\boldsymbol{u} \cdot D\boldsymbol{\xi}\, dx$$

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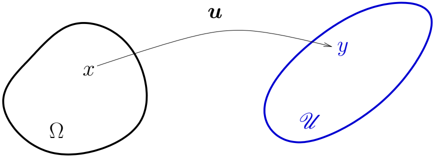
Gradient flow

$$\partial_t \boldsymbol{u} - \mathsf{div}\left(\Phi'(\det D\boldsymbol{u})\, \mathsf{cof}\ D\boldsymbol{u}\right) = 0$$

A differential approach [Evans, Gangbo, Savin]

Make the transformation

$$y = \boldsymbol{u}\,(x), \quad \rho\,(y) := \frac{1}{\det D\boldsymbol{u}\,(x)} = \frac{1}{\det D\boldsymbol{u}} \circ \boldsymbol{u}^{-1}(y) = \boldsymbol{u}_{\#}(\mathscr{L}^d|_{\Omega})$$



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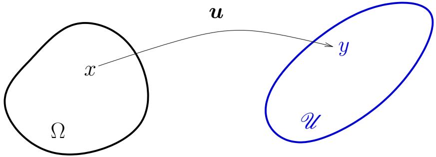

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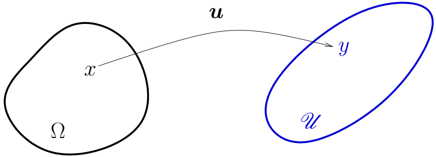

ρ solves the nonlinear diffusion PDE

$$\begin{cases} \partial_t \rho - \mathsf{div}(\rho D\phi'(\rho)) = 0 & \text{in } \mathscr{U} \times (0,+\infty), \\ \rho(x,0) = \rho_0(x) \text{ in } \mathscr{U}; \quad \partial_{\boldsymbol{n}} \rho = 0 & \text{on } \partial\mathscr{U} \times (0,+\infty) \end{cases}$$

where $\boxed{\phi(\rho) := \rho\Phi(1/\rho)}$

Recovering u

Step 1: put $\phi(\rho) := \rho\Phi(1/\rho)$

Recovering $\boldsymbol{u}$

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build the vector field

$$\boldsymbol{V}(t, y) = -\nabla \phi'(\rho_t(y))$$

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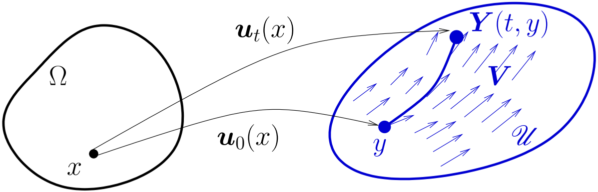
Step 5

$$\boxed{\boldsymbol{u}(t,x) = \boldsymbol{Y}(t, \boldsymbol{u}_0(x))}$$



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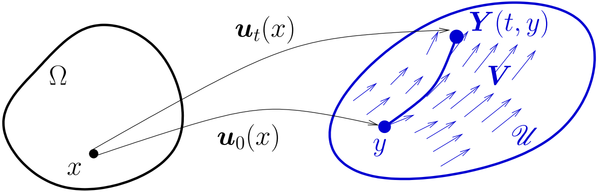

Main problem:

Prove that the L^2-Minimizing Movement scheme converges to this solution

Transporting the functional $\mathscr{F}$

$$\boxed{\mathscr{F}(\boldsymbol{u})} = \int_\Omega \Phi(\det D\boldsymbol{u}(x))\,dx = \int_{\mathscr{U}} \Phi(\det D\boldsymbol{u}(\boldsymbol{u}^{-1}(y)))\rho(y)\,dy$$

$$= \int_{\mathscr{U}} \Phi\Big(\frac{1}{\rho(y)}\Big)\rho(y)\,dy = \int_{\mathscr{U}} \phi(\rho(y))\,dy = \boxed{\mathscr{G}(\rho)}$$

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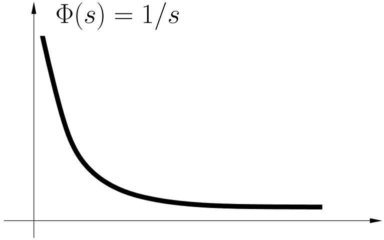

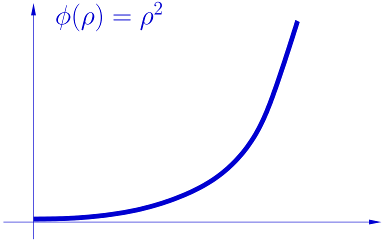

$\partial_t\rho - \Delta\rho^2 = 0$ Porous media equation

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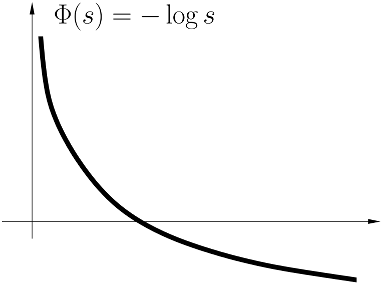

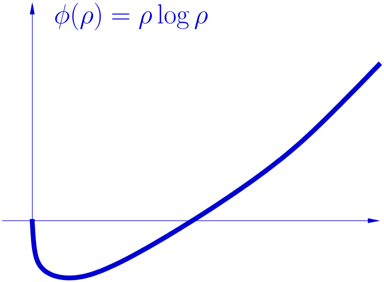

$\partial\rho - \Delta\rho = 0$ Heat equation

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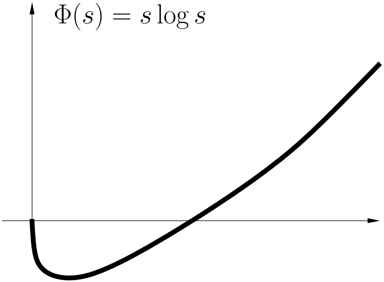

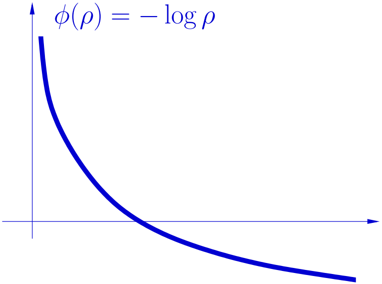

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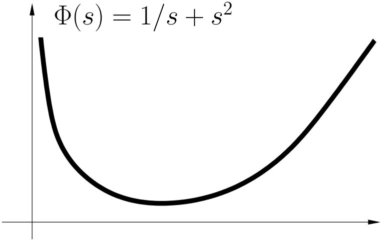

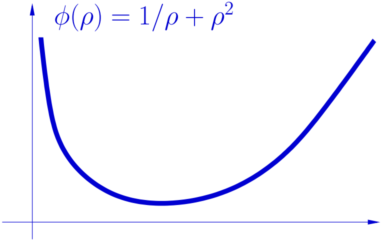

Transporting the variational problem

$$\boldsymbol{U} \rightsquigarrow R = \frac{1}{\det D\boldsymbol{U}} \circ \boldsymbol{U}^{-1}, \quad \begin{cases} \mathscr{F}(\boldsymbol{U}) = \int_\Omega \Phi(\det D\boldsymbol{U})\, dx = \\ \mathscr{G}(R) = \int_\Omega \phi(R)\, dy \end{cases}$$

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Given $\boldsymbol{U}_\tau^{n-1} \rightsquigarrow R_\tau^{n-1}$ find $\boldsymbol{U}^n \in \mathsf{Diff}(\Omega; \mathscr{U})$ solution of

$$\min_{\boldsymbol{U}} \mathscr{F}(\boldsymbol{U}) + \frac{1}{2\tau} \left\| \boldsymbol{U} - \boldsymbol{U}_\tau^{n-1} \right\|^2_{L^2(\Omega;\mathbb{R}^d)}$$

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Problem: given a density R in $\mathscr{U}$ and $\boldsymbol{U}_\tau^{n-1} \rightsquigarrow R_\tau^{n-1}$ solve

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Optimal transportation

$$\text{Minimize} \quad \int_\Omega \left|\boldsymbol{U} - \boldsymbol{U}_\tau^{n-1}\right|^2 dx \quad \text{under the constraint} \quad \boldsymbol{U} \rightsquigarrow R.$$

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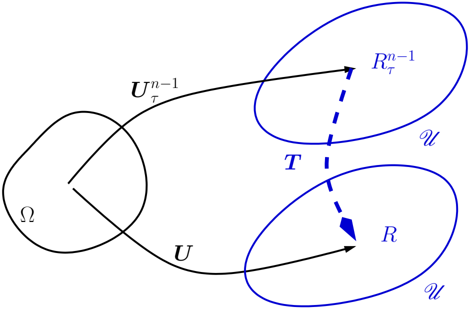

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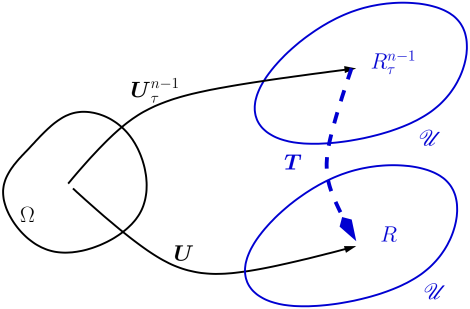

$$\int_\Omega \left|\boldsymbol{U} - \boldsymbol{U}_\tau^{n-1}\right|^2 dx = \int_\Omega \left|\boldsymbol{T}(\boldsymbol{U}_\tau^{n-1}) - \boldsymbol{U}_\tau^{n-1}\right|^2 dx$$
$$= \int_{\mathscr{U}} \left|\boldsymbol{T}(y) - y\right|^2 R_\tau^{n-1}(y)\, dy$$

A Wasserstein gradient flow

The piecewise constant intepolant R_τ of the discrete solution of the variational algorithm

$$\min_{\boldsymbol{U}} \mathscr{F}(\boldsymbol{U}) + \frac{1}{2\tau}\|\boldsymbol{U} - \boldsymbol{U}_\tau^{n-1}\|^2_{L^2(\Omega;\mathbb{R}^d)} = \min_{R} \mathscr{G}(R) + \frac{1}{2\tau} W^2(R, R_\tau^{n-1})$$

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converge to the solution of the nonlinear PDE

$$\begin{cases} \partial_t \rho + \mathsf{div}(\rho \boldsymbol{v}) = 0 \quad \text{in } \mathscr{U} \times (0,+\infty) & \text{(continuity equation)} \\ \boldsymbol{v} = -\nabla \phi'(\rho) & \text{(Nonlinear condition)} \\ \rho(y,0) = \rho_0(y), \quad \partial_{\boldsymbol{n}} \rho = 0 & \text{on } \partial\mathscr{U} \times (0,+\infty). \end{cases}$$

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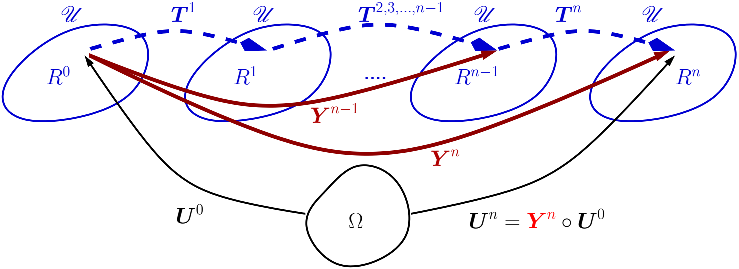
Optimal error estimate:

$$\sup_t W^2(R_\tau(t), \rho(t)) \leq \tau \mathscr{G}(\rho_0)$$

Iterated optimal transport maps



$$\min_R \int_{\mathscr{U}} \phi(R)\, dy + \frac{1}{2\tau} W^2(R, R_\tau^{n-1}) \quad \rightsquigarrow \quad R_\tau^n$$

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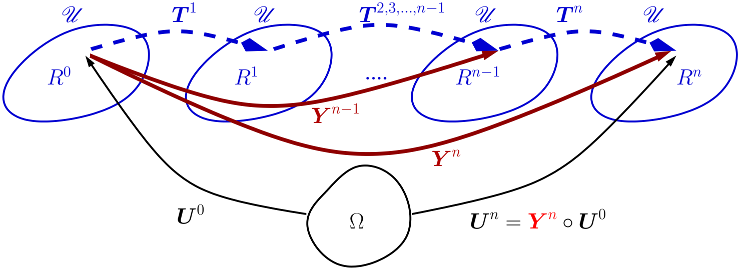

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R_τ^n, $\boldsymbol{Y}_\tau^n$ solve the PDE.

$$\frac{\boldsymbol{Y}_\tau^n - \boldsymbol{Y}_\tau^{n-1}}{\tau} = \boldsymbol{V}_\tau^n(\boldsymbol{Y}_\tau^n), \qquad \boldsymbol{V}_\tau^n = -\nabla\phi'(R_\tau^n)$$

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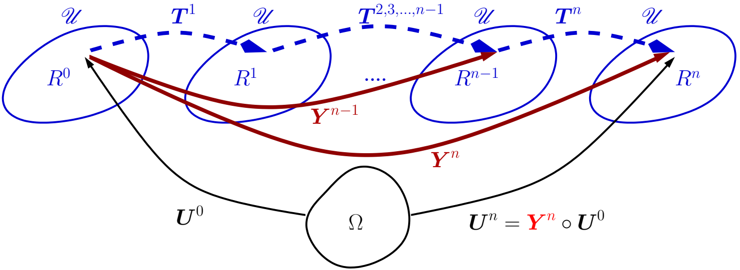

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R_τ^n, Y_τ^n solve the PDE. How to pass to the limit?

$$\frac{Y_\tau^n - Y_\tau^{n-1}}{\tau} = V_\tau^n(Y_\tau^n), \qquad V_\tau^n = -\nabla\phi'(R_\tau^n)$$

Convergence of the iterated maps

Main problem:

$$\begin{array}{lllll} & \frac{d}{dt}\boldsymbol{Y}_\tau(t,y) = \boldsymbol{V}_\tau(t,\boldsymbol{Y}_\tau(t,y)), & & \boldsymbol{V}_\tau(t,y) = -\nabla\phi'(R_\tau(t,y)) & \\ \text{as } \tau\to 0 & \downarrow \qquad\qquad \downarrow & & \downarrow \qquad\qquad \downarrow & ? \\ & \frac{d}{dt}\boldsymbol{Y}(t,y) = \boldsymbol{V}(t,\boldsymbol{Y}(t,y)), & & \boldsymbol{V}(t,y) = -\nabla\phi'(\rho(t,y)) & \end{array}$$

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- **convergence of the energy:**

$$\lim_{\tau\downarrow 0}\int_0^T\!\!\int_{\mathscr{U}} \big|\boldsymbol{V}_\tau(t,y)\big|^2 R_\tau(t,y)\,dy\,dt = \int_0^T\!\!\int_{\mathscr{U}} \big|\boldsymbol{V}(t,y)\big|^2 \rho(t,y)\,dy\,dt$$

A first result: convergence of flows

Suppose that $\boldsymbol{V}_\tau, \boldsymbol{Y}_\tau, \mu_\tau = \rho_\tau \mathscr{L}^d$ are given with

$$\frac{d}{dt}\boldsymbol{Y}_\tau(t,y) = \boldsymbol{V}_\tau(t, \boldsymbol{Y}_\tau(t,y)), \qquad \mu_{\tau,t} = (\boldsymbol{Y}_\tau(t,\cdot))_\# \mu_{\tau,0}$$

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- The limit ODE admits a unique solution for μ_0-a.e. $y \in \mathscr{U}$.

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- The limit ODE admits a unique solution for μ_0-a.e. $y \in \mathscr{U}$.

Then there exists a unique flow $\boldsymbol{Y}$ solving

$$\dot{\boldsymbol{Y}}(t,y) = \boldsymbol{V}(t, \boldsymbol{Y}(t,y)), \quad \boldsymbol{Y}(0,y) = y$$

$$\lim_{\tau\downarrow 0} \int_0^T \max_t \left|\boldsymbol{Y}_\tau(t,y) - \boldsymbol{Y}(t,y)\right|^2 d\mu_0(y) = 0.$$

Reconstruction of the gradient flow of $\mathscr{F}$

Suppose that $\rho_0 \in C^\alpha(\overline{\mathscr{U}})$, $\mathscr{G}(\rho_0) = \int_{\mathscr{U}} \phi(\rho_0)\,dy < +\infty$.

- The discrete transports $\boldsymbol{Y}_\tau$ converge to $\boldsymbol{Y}$ in the sense of $L^2(\mathscr{U}; L^\infty(0,T))$

$$\lim_{\tau\downarrow 0} \int_0^T \max_t \left| \boldsymbol{Y}_\tau(t,y) - \boldsymbol{Y}(t,y) \right|^2 \rho_0(y)\,dy = 0.$$

and the discrete solutions $\boldsymbol{U}_\tau(t,x) = \boldsymbol{Y}_\tau(t,\boldsymbol{u}_0(x))$ converge to $\boldsymbol{u}(t,x) = \boldsymbol{Y}(t,\boldsymbol{u}_0(x))$.

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Suppose that $\rho_0 \in C^\alpha(\overline{\mathscr{U}})$, $\mathscr{G}(\rho_0) = \int_{\mathscr{U}} \phi(\rho_0)\, dy < +\infty$.

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- ρ is the unique solution of the nonlinear diffusion equation

$$\begin{cases} \partial_t\rho - \mathsf{div}(\rho D\phi'(\rho)) = 0 & \text{in } \mathscr{U}, \\ \rho(y,0) = \rho_0(y), \qquad \partial_{\boldsymbol{n}}\rho = 0 & \text{on } \partial\mathscr{U} \end{cases}$$

Outline

1. **Thin film equation as the gradient flow of the Dirichlet functional**
   - in collaboration with U.Gianazza, G.Toscani, D. Matthes, R. McCann

2. **The L^2-gradient flow of the simplest polyconvex functional**
   - in collaboration with L. Ambrosio, S. Lisini

3. **The sticky particle system**
   - in collaboration with L. Natile

Starting point: motion of a finite number of particles.

Discrete particle model

N particles $\quad P_i := (m_i, x_i, v_i), \quad i = 1, \ldots, N,$
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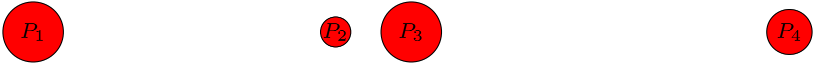

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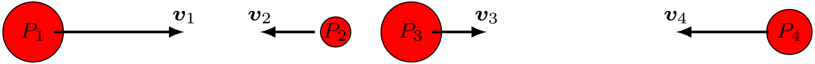

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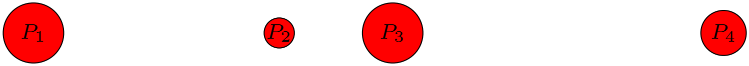

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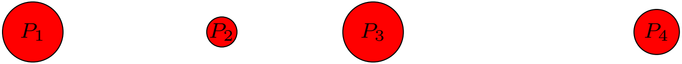

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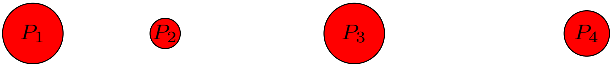

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Measure-theoretic description

We thus have:
a **(finite) sequence of collision times** $0 < t^1 < t^2 < \dots$
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$$v_t(x_2) - v_t(x_1) \le \frac{1}{t}(x_2 - x_1) \quad \text{for } \rho_t\text{-a.e. } x_1, x_2 \in \mathbb{R},\ x_1 \le x_2.$$

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Consider a sequence of discrete initial data $\mu_0^n := (\rho_0^n, \rho_0^n v_0^n)$ converging to $\mu_0 = (\rho_0, \rho_0 v_0)$ in a suitable measure-theoretic sense and let $\mu_t^n = (\rho_t^n, \rho_t^n v_t^n)$ be the (discrete) solution of SPS.

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- Pioneering ideas which lies (more or less explicitly) at the core of the papers by E-Rykov-Sinai and Brenier-Grenier have been introduced by
  - Shnirelman '86 and further clarified by
  - Andrievwsky-Gurbatov-Soboelvskiĭ '07 in a formal way.
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For every probability measure $\rho \in \mathcal{P}(\mathbb{R})$ we introduce the **cumulative distribution function**

$$M_\rho(x) := \rho\big((-\infty, x]\big), \quad x \in \mathbb{R}, \quad \text{so that } \rho = \partial_x M_\rho \quad \text{in } \mathscr{D}'(\mathbb{R}).$$

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Theorem (Brenier-Grenier '96)

M is the unique entropy solution of the scalar conservation law

$$\partial_t M + \partial_x A(M) = 0 \quad \textit{in } \mathbb{R} \times (0, +\infty)$$

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Monotone rearrangement

Point of view of 1-dimensional optimal transport: instead of using the cumulative distribution function $M_\rho(x) = \rho\big((-\infty, x]\big)$, we

represent each probability measure ρ by its monotone rearrangement $X_\rho : (0,1) \to \mathbb{R}$

$$X_\rho(w) := \inf\Big\{x \in \mathbb{R} : M_\rho(x) > w\Big\} \quad w \in (0,1)$$

which is the so-called pseudo-inverse of M_ρ.

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$$X_\rho(w) := \inf \Big\{ x \in \mathbb{R} : M_\rho(x) > w \Big\} \quad w \in (0,1)$$

which is the so-called pseudo-inverse of M_ρ.

The map X_ρ is **nondecreasing and right-continuous** and it pushes the Lebesgue measure $\lambda := \mathscr{L}^1|_{(0,1)}$ on $(0,1)$ onto ρ.

Wasserstein distance and the L^2 isometry

The map $\rho \mapsto X_\rho$ is a **one-to-one correspondence** between

the space $\mathcal{P}_2(\mathbb{R})$ of probability measures with finite quadratic moment $\mathfrak{m}_2(\rho) = \int_{\mathbb{R}} |x|^2 \, \mathrm{d}\rho(x) < +\infty$

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In this way $\rho \leftrightarrow X_\rho$ is an **isometry** between $(\mathcal{P}_2(\mathbb{R}), W_2)$ and $(\mathcal{K}, \|\cdot\|_{L^2(0,1)})$.

A metric space for the measure-momentum couples $(\rho, \rho v)$

We consider the space of couples $(\rho, \rho v)$, with $\rho \in \mathcal{P}_2(\mathbb{R})$ and $v \in L^2_\rho(\mathbb{R})$:

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$\mu_n = (\rho_n, \rho_n v_n)$ *converges to* $\mu = (\rho, \rho v)$ *in* $\mathcal{V}_2(\mathbb{R})$ *if and only if*

$$W_2(\rho_n, \rho) \to 0, \quad \rho_n v_n \rightharpoonup \rho v \quad \textit{weakly in } \mathcal{M}(\mathbb{R}), \quad \int_{\mathbb{R}} |v_n|^2 \, d\rho_n \to \int_{\mathbb{R}} |v|^2 \,\mathrm{d}\rho.$$

The fundamental estimate

Let $\mathscr{V}_{\rm discr}(\mathbb{R})$ the collection of all the discrete measures in $\mathcal{V}_2(\mathbb{R})$ and let us denote by $\mathscr{S}_t : \mathscr{V}_{\rm discr}(\mathbb{R}) \to \mathscr{V}_{\rm discr}(\mathbb{R})$ the map associating to any discrete initial datum $(\rho_0, \rho_0 v_0) \in \mathscr{V}_{\rm discr}$ the solution $(\rho_t, \rho_t v_t)$ of the (discrete) sticky-particle system.
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Theorem (Stability with respect to the initial data)

Let $\mu_t^\ell = (\rho_t^\ell, \rho_t^\ell v_t^\ell) = \mathscr{S}_t[\mu_0^\ell]$, $\ell = 1, 2$, be the solutions of the (discrete) sticky-particle system with initial data $\mu_0^\ell \in \mathscr{V}_{\mathrm{discr}}(\mathbb{R})$.

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$$\int_0^t U_2^2(\mu_r^1, \mu_r^2)\,{\rm d}r \le C(1+t)\Big([\mu^1]_2 + [\mu^2]_2\Big)\Big(W_2(\rho_0^1, \rho_0^2) + U_2(\mu_0^1, \mu_0^2)\Big),$$

for a suitable "universal" constant C independent of t and the data.

Evolution semigroup

Theorem (The evolution semigroup in $\mathscr{V}_2(\mathbb{R})$)

- *The semigroup $\mathscr{S}_t$ can be uniquely extended by density to a right-continuous semigroup (still denoted $\mathscr{S}_t$) of strongly-weakly continuous transformations in $\mathscr{V}_2(\mathbb{R})$, thus satisfying*

$$\mathscr{S}_{s+t}[\mu] = \mathscr{S}_s[\mathscr{S}_t[\mu]] \quad \forall\, s, t \geq 0, \qquad \lim_{t \downarrow 0} D_2(\mathscr{S}_t[\mu], \mu) = 0. \tag{2}$$

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- *$(\rho_t, \rho_t v_t) = \mathscr{S}_t[\mu]$, $\mu \in \mathscr{V}_2(\mathbb{R})$, is a distributional solution of Euler system satisfying Oleinik entropy condition.*

A gradient flow formulation in $\mathcal{P}_2(\mathbb{R})$

The semigroup $\mathscr{S}_t$ can also be characterized by the **(metric) gradient flow $\mathscr{G}_\tau$ of the (-1)-geodesically convex functional**

$$\Phi(\rho) := -\frac{1}{2} W_2^2(\rho, \rho_0)$$

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If $\mu_t = (\rho_t, \rho_t v_t) = \mathscr{S}_t(\rho_0, \rho_0 v_0)$ is a solution of SPS then the rescaling $\tau = \log t$, $\hat{\mu}_\tau = \mu_t$, $\hat{\rho}_\tau = \rho_t$ satisfy

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The (rescaled) semigroup $\mathscr{G}$ provides a **displacement extrapolation**, i.e. a canonical way to extend Wasserstein geodesics after collisions.

A simple example

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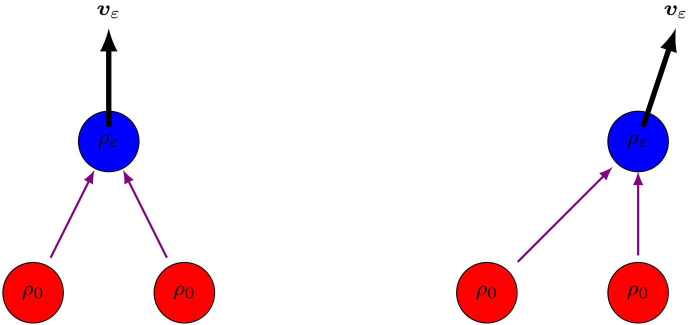

Non-local effects in the multi-dimensional case

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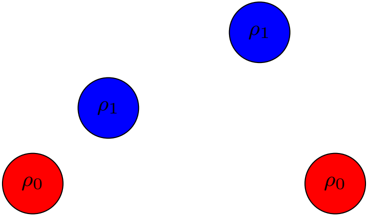

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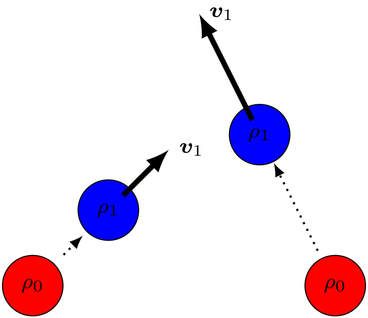

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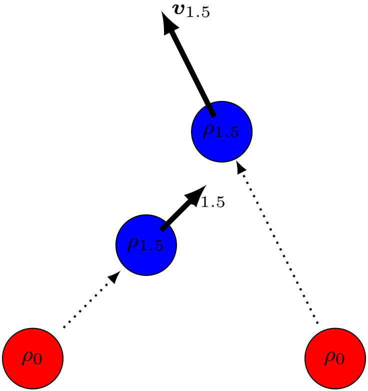

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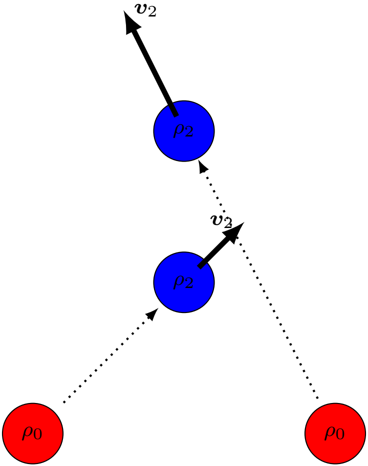

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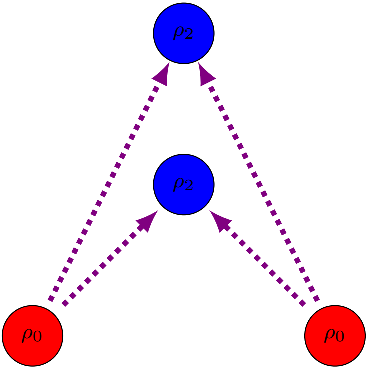

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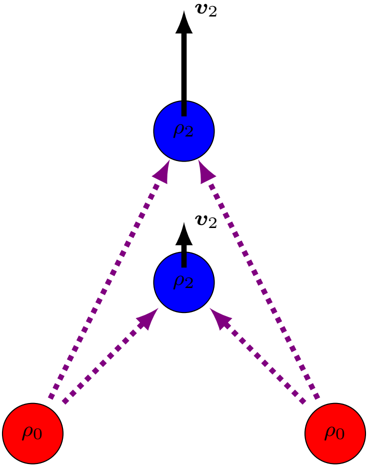

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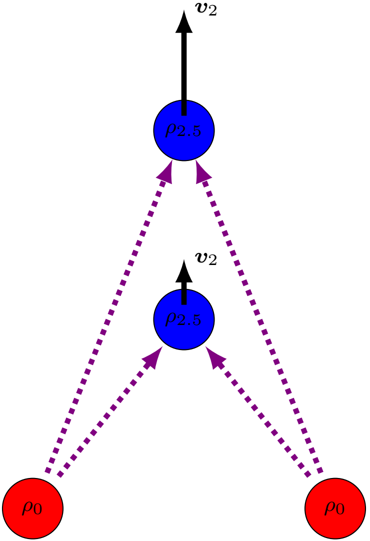

Non-local interaction can be avoided only in the 1-dimensional case.

Extensions

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- (in collaboration with W. Gangbo and M. Westdickenberg) Adding a force induced by a potential V

$$\begin{cases} \partial_t \rho + \partial_x(\rho\, v) = 0, \\ \partial_t(\rho\, v) + \partial_x(\rho\, v^2) = \boxed{-\rho\, \partial_x V}. \end{cases}$$

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Open problems:

- The SPS in the multidimensional case.
- The displacement-extrapolation problem.