Computing Interfacial Motions in Image Processing and Vision

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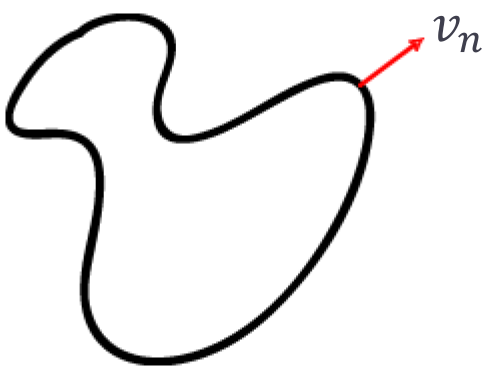
Objective

- Variety of variational models in image processing.
- Gradient descent: Curve / surface evolution:



Example:

- $v_n = \alpha(x)\kappa + \beta(x)$
- $v_n = -\alpha(x)(\kappa_{ss} + \frac{1}{2}\kappa^3) + \beta(x)\kappa + \gamma(x)$

- Algorithms for finding global minimizers.

Outline

- Monday: Problems, Models, Basic Facts.
- Tuesday: Diffuse Interface Methods:
  - Phase field
  - Threshold dynamics
  - Distance function dynamics
- Thursday:
  - Level sets
  - Redistancing: Fast marching
- Friday: Finding global minima via PDEs.
- Saturday: Network flows and graph cuts.

Monday

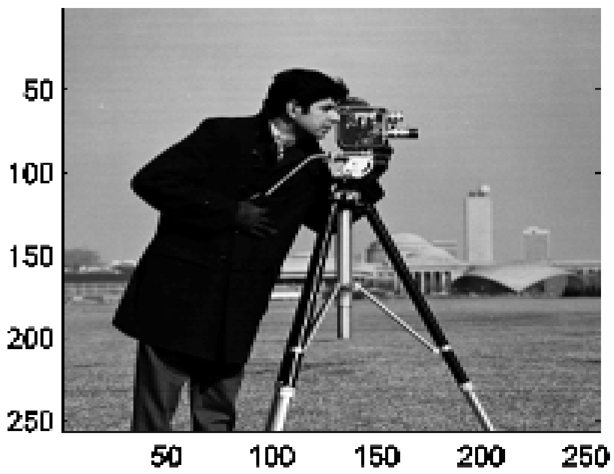
1. Image Segmentation
    1. Snakes: Active Contours
    2. Mumford-Shah Functional
    3. Piecewise Constant Mumford-Shah
    4. Segmentation with Depth: Nitzberg-Mumford-Shiota Model
2. Image Denoising
    1. Mean Curvature Flow of Level Lines
    2. Perona-Malik Scheme
    3. Perona-Malik and Mumford-Shah
    4. Rudin-Osher-Fatemi's Total Variation Model
3. Inpainting.

Representing Images

- Represent a gray-scale image by a function $f(x)$:

$$f(x): \Omega \rightarrow [0,1]$$

- Ω is the image domain = Computer screen (a rectangle).
- Value of $f(x)$ = Gray-scale intensity of pixel at location x.



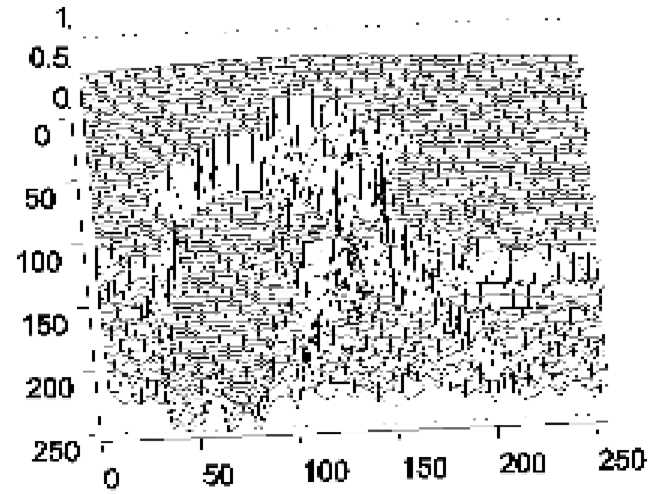


Image Segmentation

- Assumption: Image depicts a scene containing several objects.
- Segmentation: Divide image into distinct regions.
- Mathematically: Partition image domain Ω.

$$\Omega = \bigcup_j \Sigma_j$$

- Each Σ_j contains an object, and

$$\Sigma_i \cap \Sigma_j = \partial\Sigma_i \cap \partial\Sigma_j$$

- Edges in the image:

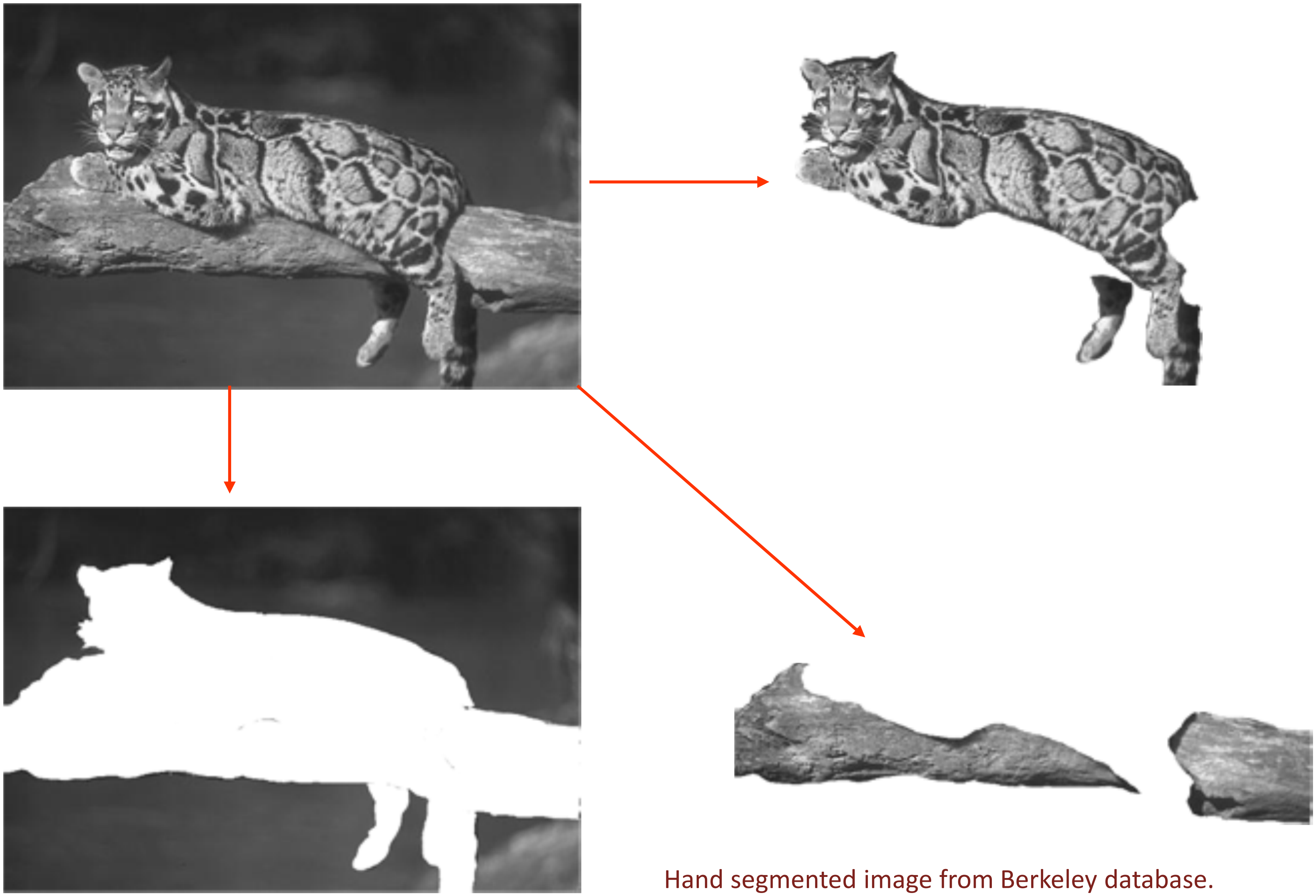
$$\bigcup_j \partial\Sigma_j$$

Image Segmentation

- Edges in the image: Boundaries of distinct objects.
- Assume: Different objects have different:
  - Color
  - Grayscale intensity
  - Texture
  - etc.
- Expect:
  - f is discontinuous, or
  - $|\nabla f|$ is very large

  at edges.

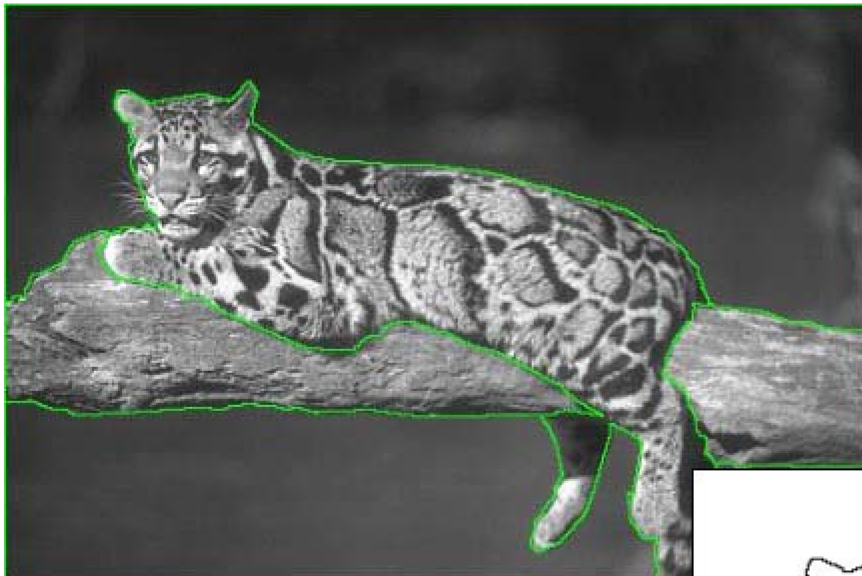
Image Segmentation

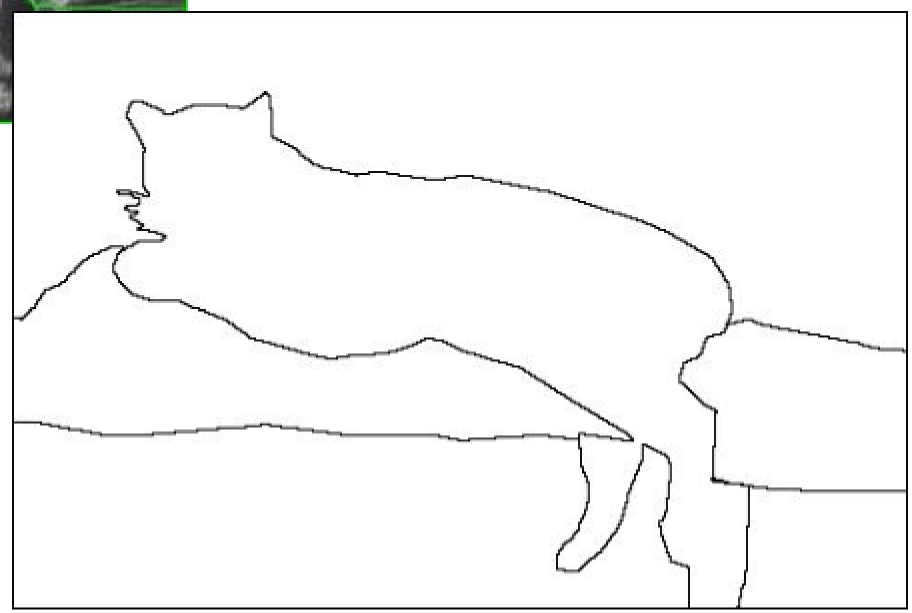


Hand segmented image from Berkeley database.

Image Segmentation

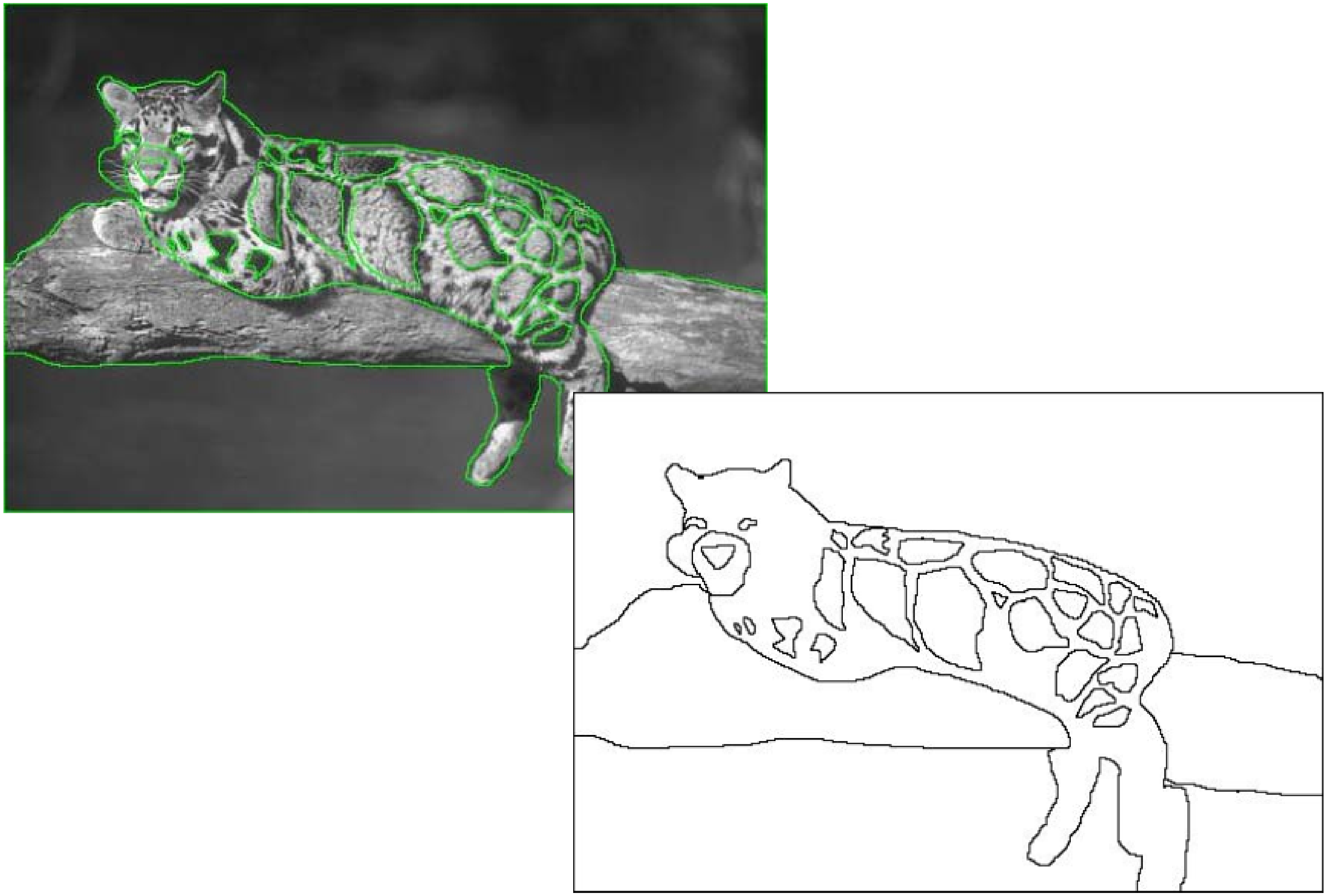
Segmentation and edge detection:



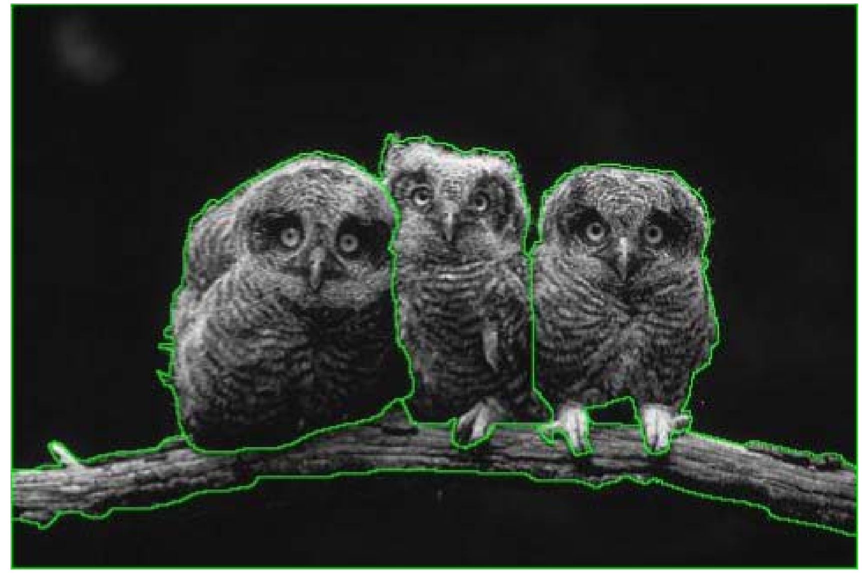


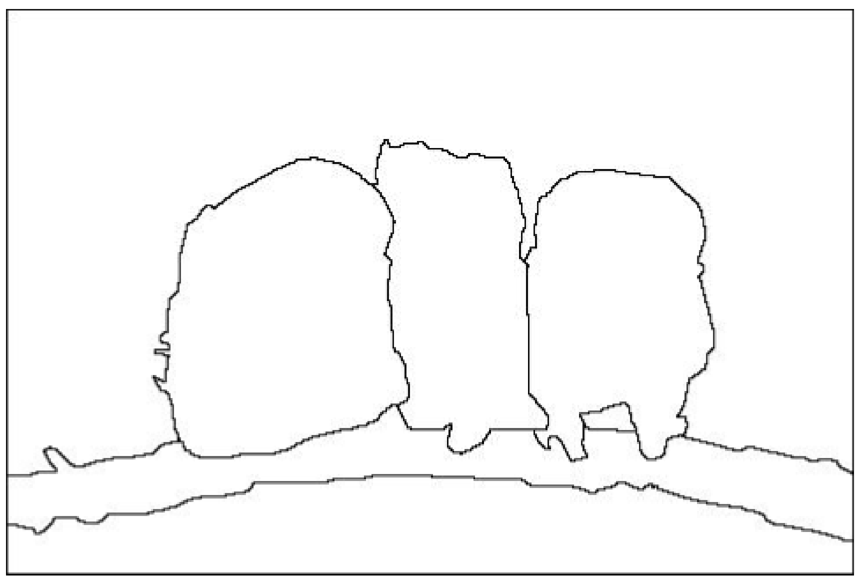
Hand segmented image from Berkeley database.

Segmentation is an ill-posed task.



It is necessary to specify the level of detail desired.





Hand segmented image from Berkeley image database.

Hand segmented image from Berkeley image database.

Snakes: Active Contours

- Kaas, Witkin, Terzopoulos (1992); Kichenassamy, Sapiro, Tannenbaum (1996)
- IDEA: Initialize a curve (snake) on the image domain Ω.

$$\gamma(s) = (\gamma_1(s), \gamma_2(s)) : [0,1] \rightarrow \Omega.$$

- Prescribe a normal speed to drive it towards edges.
- Edge detection:

$$|\nabla f| \text{ large } \Rightarrow \frac{1}{1 + |\nabla G_\sigma * f|^2} \approx 0$$

$$\text{where } G_\sigma(x) = \frac{1}{4\pi\sigma} e^{-\frac{|x|^2}{4\pi\sigma}}$$

- Gradient descent for:

$$\boxed{\int_0^1 \frac{1}{1 + |(\nabla G_\sigma * f)(\gamma)|^2} |\gamma'(s)| \, ds}$$

Snakes: Active Contours

- Global minimizer = A point. $\Rightarrow$ Look for a local minimizer.
- Start near, containing object of interest. Let curve shrink-wrap it.
- Alternatively, start small, in the interior of object of interest.
  - Provide an “expansion force”:

$$\text{Area in } \Gamma = \int_\Gamma \vec{\psi} \cdot d\vec{\sigma} = \int_0^1 \vec{\psi}(\gamma) \cdot \gamma'(s)\, ds$$

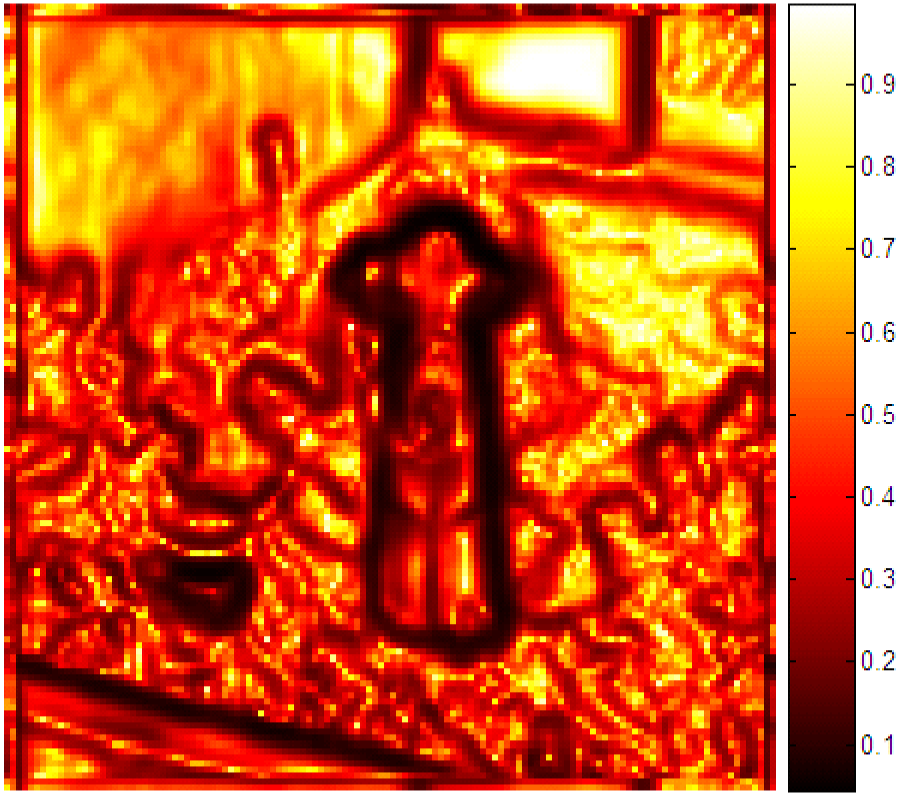
where

$$\vec{\psi}(x, y) = \frac{1}{2}(y, -x)$$

Snakes: Active Contours

Image f

Edge map $\frac{1}{1+|\nabla G_\sigma * f|^2}$



Snakes: Active Contours

Snakes: Active Contours

Gradient Descent: Perimeter

- $\gamma(s)$ arclength parametrization of $\partial\Sigma$, oriented counterclockwise.
- Let:

$$T(s) = \frac{\partial}{\partial s}\gamma(s) \;\text{ and }\; N(s) = \text{Outward unit normal.}$$

- We have:

$$\frac{\partial}{\partial s}T(s) = \kappa(s)N(s) \quad and \quad \frac{\partial}{\partial s}N(s) = -\kappa(s)T(s)$$

so that $\kappa(s) \leq 0$ for convex shapes by our convention.

- Consider the perturbation of $\gamma(s)$:

$$\gamma(s) + t\phi(s)N(s)$$

where $\phi(s): \partial\Sigma \to \boldsymbol{R}$.

Gradient Descent: Perimeter

- Compute the length:

$$L(\gamma + t\phi N) = \int_0^L \langle \gamma' + t\phi' N + t\phi N', \gamma' + t\phi' N + t\phi N' \rangle^{\frac{1}{2}}\, ds$$

$$= \int_0^L \langle T + t\phi' N - t\phi\kappa T, T + t\phi' N - t\phi\kappa T \rangle^{\frac{1}{2}}\, ds$$

$$= \int_0^L (1 - 2t\phi\kappa + t^2(\phi')^2 + t^2\phi^2\kappa^2)^{\frac{1}{2}}\, ds$$

- Differentiate w.r.t. t:

$$\frac{d}{dt} L(\gamma + t\phi N)\Big|_{t=0} = \int_0^L \phi(-\kappa)\, ds$$

- Thus, we see that

$$\boxed{\nabla L = -\kappa}$$

Gradient Descent: Weighted Perimeter

- Let $g(x)$ be a given positive *weight* function on Ω.
- Define the weighted length

$$L_g(\gamma) = \int_0^L g\big(\gamma(s)\big)\, ds$$

for a curve $\gamma(s)$ parametrized by arclength.

- As before:

$$L_g(\gamma + t\phi N) = \int_0^L g(\gamma + t\phi N)(1 - 2t\phi\kappa + t^2(\phi')^2 + t^2\phi^2\kappa^2)^{\frac{1}{2}}\, ds$$

- We find

$$\frac{d}{dt} L_g(\gamma + t\phi N)\Big|_{t=0} = \int_0^L \phi g(\gamma)(-\kappa) + \phi \nabla g \cdot N\, ds$$

Gradient Descent: Weighted Perimeter

- Thus, we get:

$$\boxed{\nabla L_g = -g\kappa + \nabla g \cdot N}$$

- Example: Geodesic active contours:

$$v_n = g\kappa - \nabla g \cdot N.$$

Mumford-Shah Segmentation Model

- Find the best piecewise smooth approximation, in least squares sense, to the given image:

$$\min_{\substack{u(x)\\K\subset\Omega}} \int_{\Omega\setminus K} |\nabla u|^2 \, dx + \mu \, \mathrm{Length}(K) + \lambda \int_{\Omega} (f-u)^2 \, dx$$

- Unknowns of the problem:
  - u(x): The piecewise smooth approximation. Smooth except across “edges” K.
  - K: The set of edges across which u(x) is allowed to be discontinuous.
  - λ: Acts as a scale parameter.

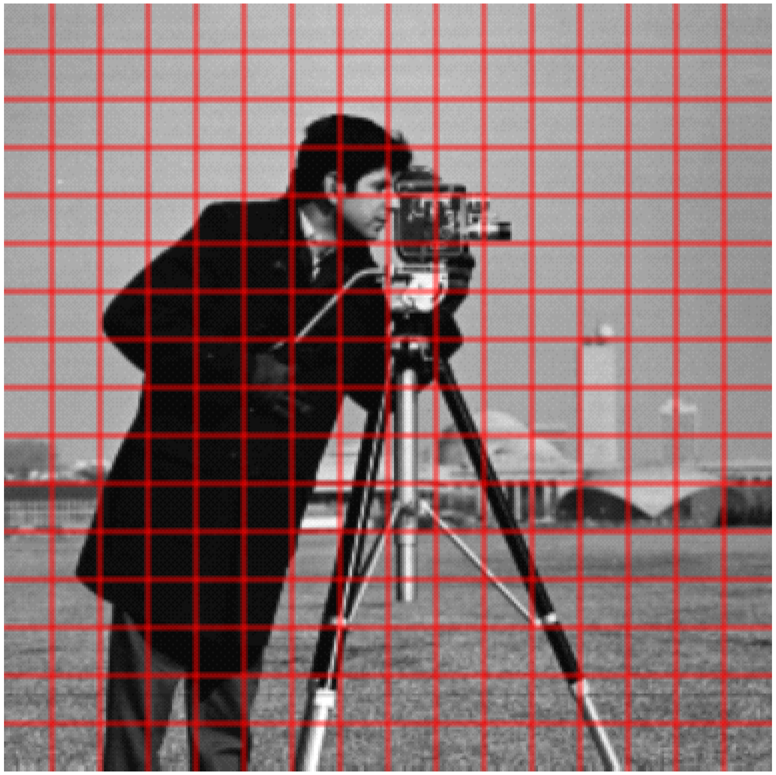
Mumford-Shah Segmentation Model

- $\int_{\Omega \setminus K} |\nabla u|^2 \, dx$: Ensures smoothness away from edges.
- $\mathrm{Length}(K)$: Prevents *oversegmentation.* Allows selection of scale.
- $\int_{\Omega} (f - u)^2 \, dx$: Fidelity terms. Ensures approximation of given image.

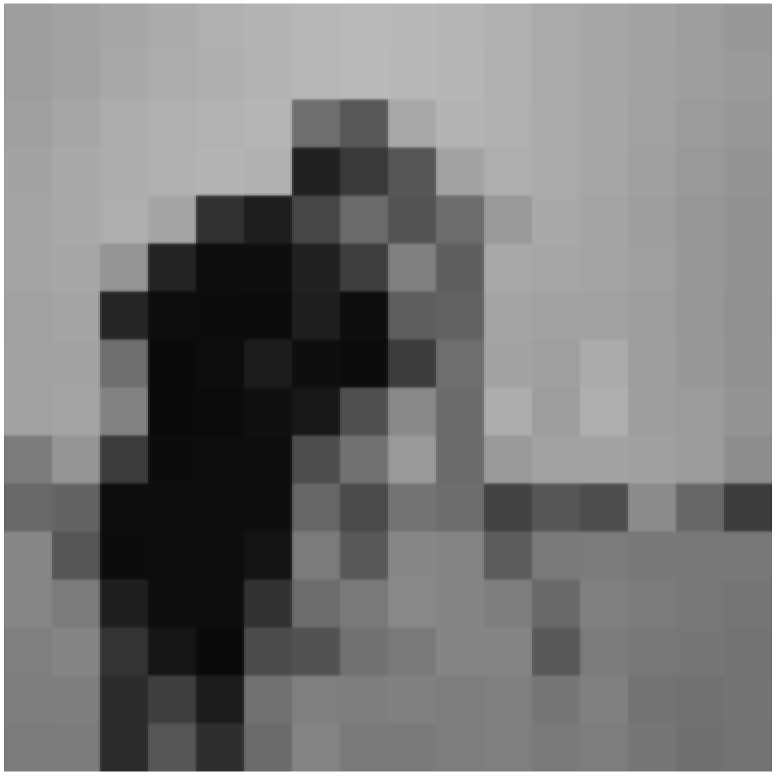
Advantages:

- No explicit edge detection needed.
- Does not require presence of prominent transitions f.
- Robust w.r.t. noise.

Mumford-Shah Segmentation Model



$\approx$

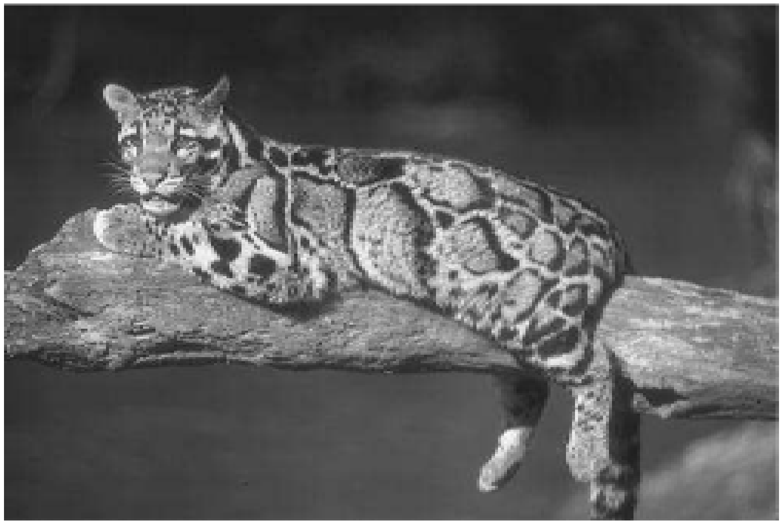


Without the length term, the Mumford-Shah function can be trivially minimized:

$$\inf_{u,K} \int_{\Omega \setminus K} |\nabla u|^2 \, dx + \lambda \int_{\Omega} (f-u)^2 \, dx = 0$$

Mumford-Shah Segmentation Model

Example:



An image and its piecewise smooth approximation found by minimizing the Mumford-Shah functional.

Mumford-Shah Segmentation Model

$$\min_{\substack{u(x)\\K\subset\Omega}} \int_{\Omega\setminus K} |\nabla u|^2\,dx + \mu\,\mathrm{Length}(K) + \lambda \int_{\Omega} (f-u)^2\,dx$$

- Consider the limit:

$$\mu = \mu_0\varepsilon, \qquad \lambda = \lambda_0\varepsilon, \qquad \varepsilon \to 0^+.$$

- $|\nabla u| \approx 0$ in $\Omega \setminus K$.

$$\Rightarrow \quad \text{Piecewise constant } u.$$

- Model becomes:

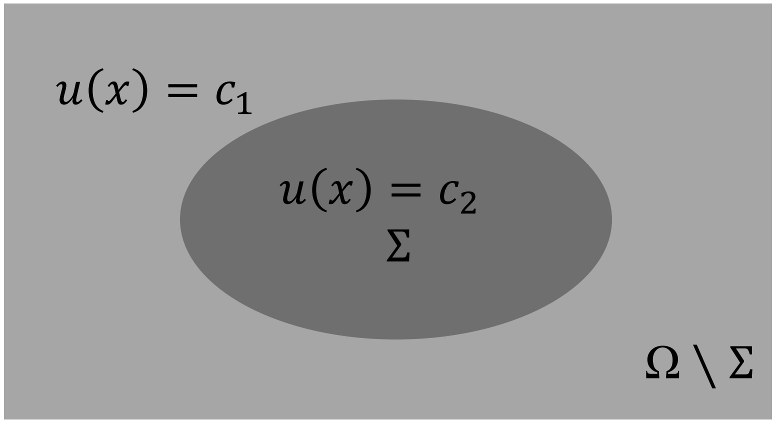
$$\min_{\substack{\coprod_j \Sigma_j = \Omega\\ c_j}} \sum_j \left\{ \mathrm{Length}(\partial\Sigma_j) + \alpha \int_{\Sigma_j} (c_j - f)^2\,dx \right\}$$

- No a priori restriction on number of regions Σ_j.
- However, # of regions bdd. in terms of α.

Piecewise Constant Mumford-Shah Model

Chan-Vese (2000):

- Approximate the image by a piecewise constant function.
- Simplest example: A function of two values.



- Any such function can be written as:

$$u(x) = c_1 \mathbf{1}_{\Sigma}(x) + c_2 \mathbf{1}_{\Omega \setminus \Sigma}(x)$$

Piecewise Constant Mumford-Shah Model

$$u(x) = c_1 \mathbf{1}_\Sigma(x) + c_2 \mathbf{1}_{\Omega\setminus\Sigma}(x)$$

Then, we have:

1. $\mathrm{K} = \partial\Sigma,$
2. $\int_{\Omega\setminus \mathrm{K}} |\nabla \mathrm{u}|^2 \,\mathrm{dx} = 0,$
3. $\mathrm{Length}(\mathrm{K}) = \mathrm{Per}(\Sigma),$
4. $\int_\Omega (\mathrm{f}-\mathrm{u})^2 \,\mathrm{dx} = \int_\Sigma (\mathrm{f}-\mathrm{c}_1)^2 \,\mathrm{dx} + \int_{\Omega\setminus\Sigma} (\mathrm{f}-\mathrm{c}_2)^2 \,\mathrm{dx}$

Piecewise Constant Mumford-Shah Model

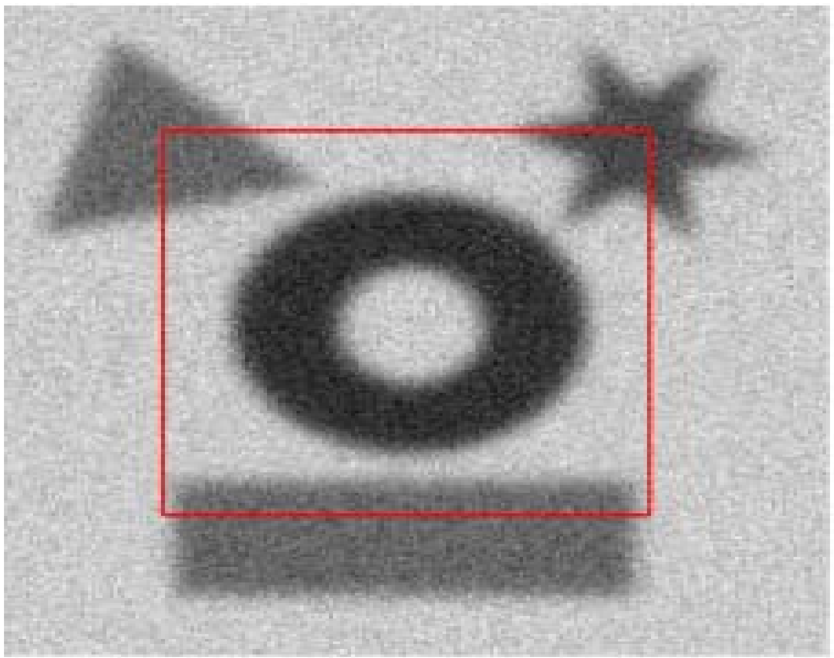
- The energy becomes:

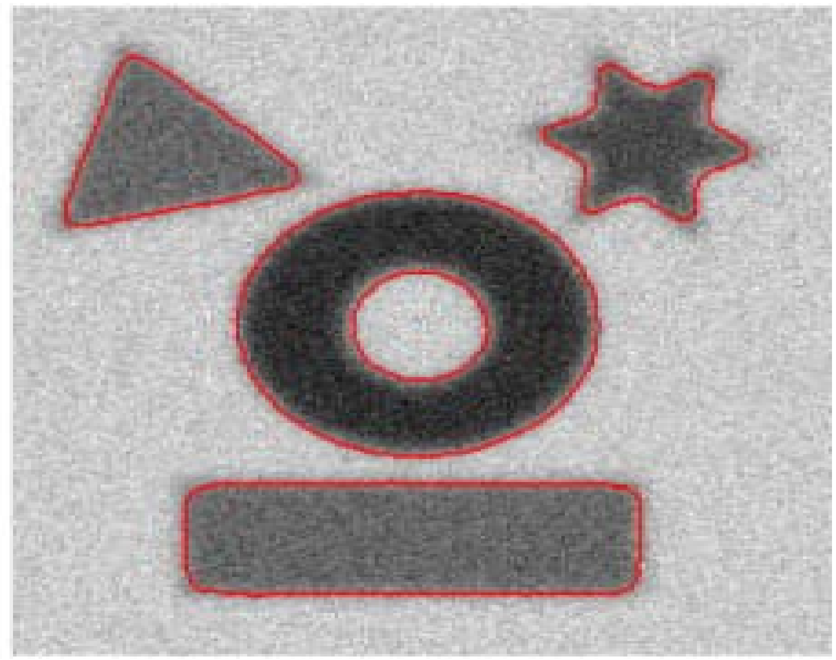
$$E(\Sigma, c_1, c_2) = \mathrm{Per}(\Sigma) + \lambda \left\{ \int_{\Sigma} (f - c_1)^2 \, dx + \int_{\Omega \setminus \Sigma} (f - c_2)^2 \, dx \right\}$$

- How can we minimize such energies?
  - Make an intial guess for $\partial\Sigma$.
  - Update $\partial\Sigma$ so that energy decreases as fast as possible.

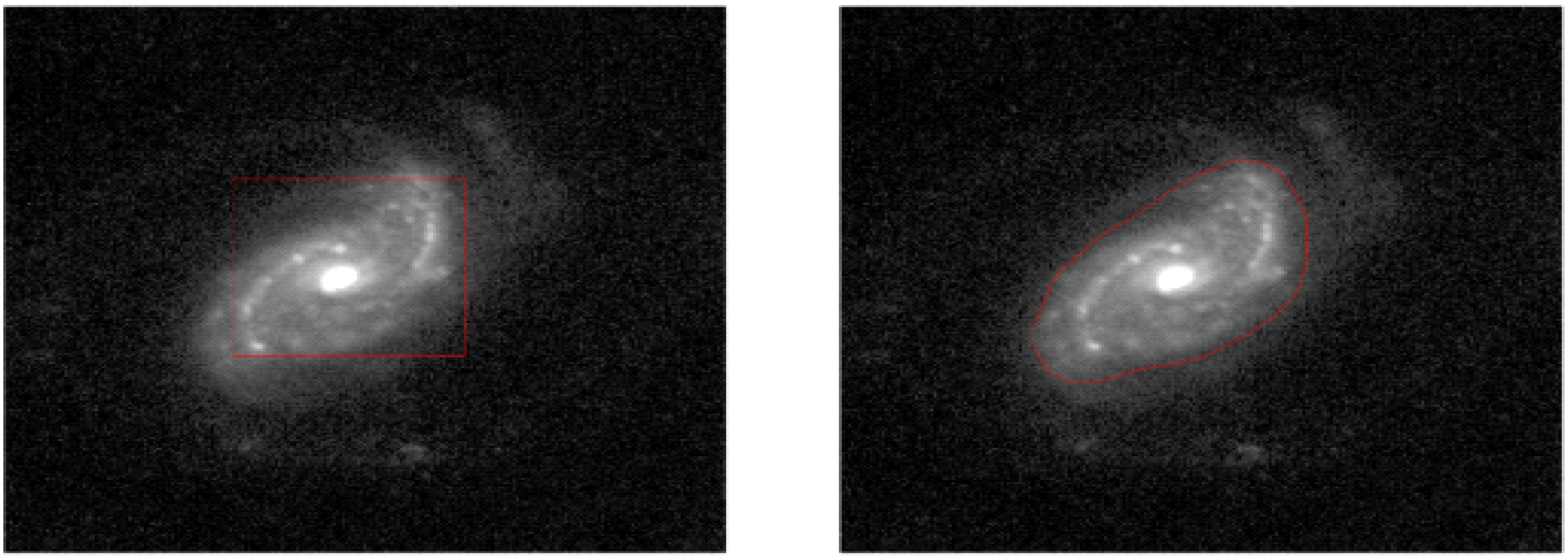
$\Rightarrow$ SOLVE a PDE describing the motion of $\partial\Sigma$.

Piecewise Constant Mumford-Shah Model

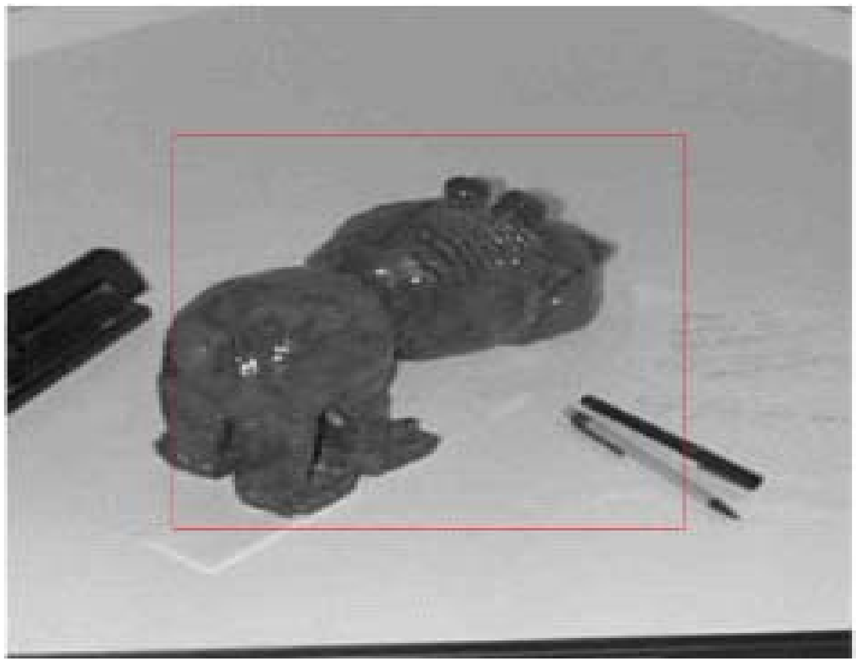


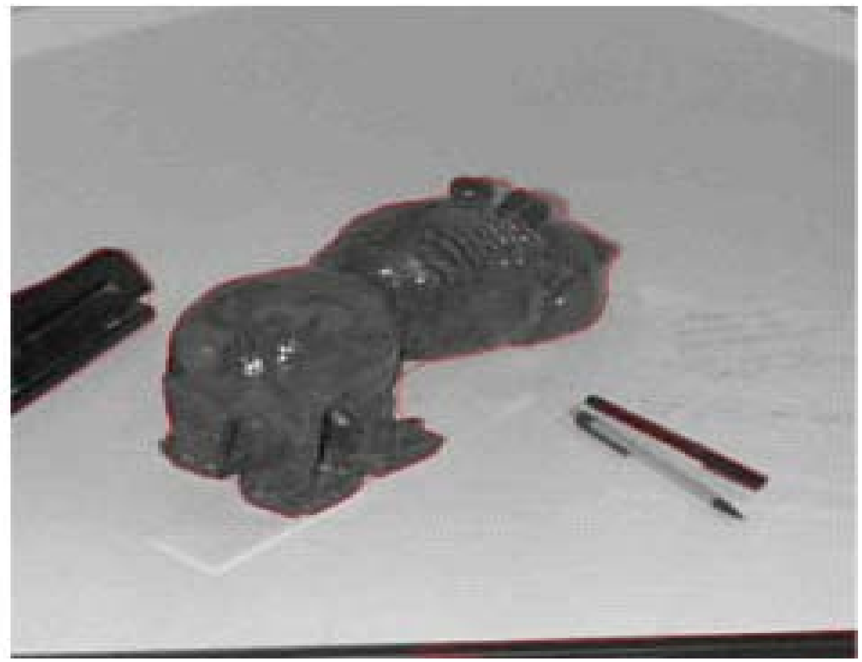


Piecewise Constant Mumford-Shah Model

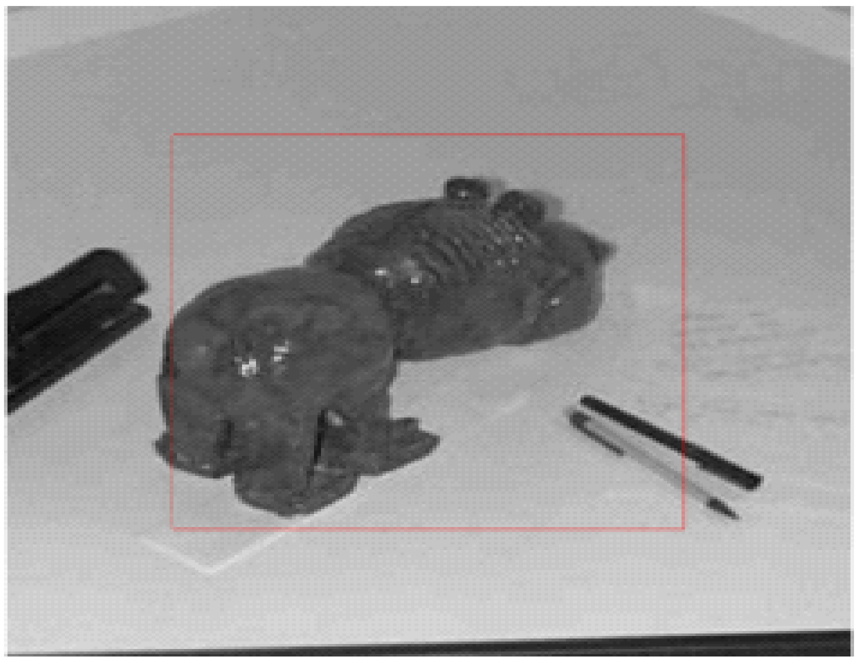


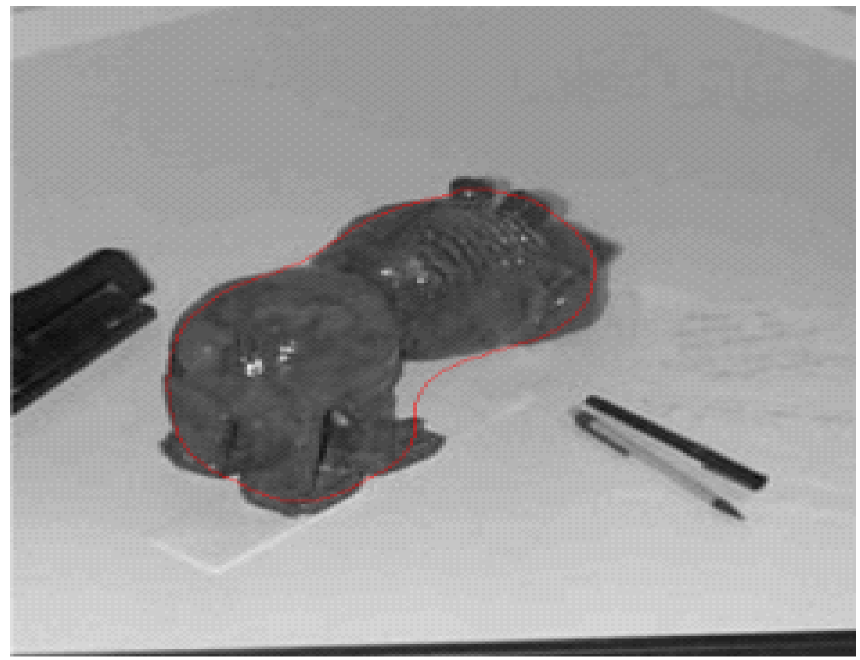
Piecewise Constant Mumford-Shah Model





Piecewise Constant Mumford-Shah Model





Piecewise Constant Mumford-Shah Model

Piecewise Constant Mumford-Shah Model

Gradient Descent: Bulk Energy

- Take variation of terms of the form:

$$A(\Sigma) = \int_{\Sigma} b(x)\, dx$$

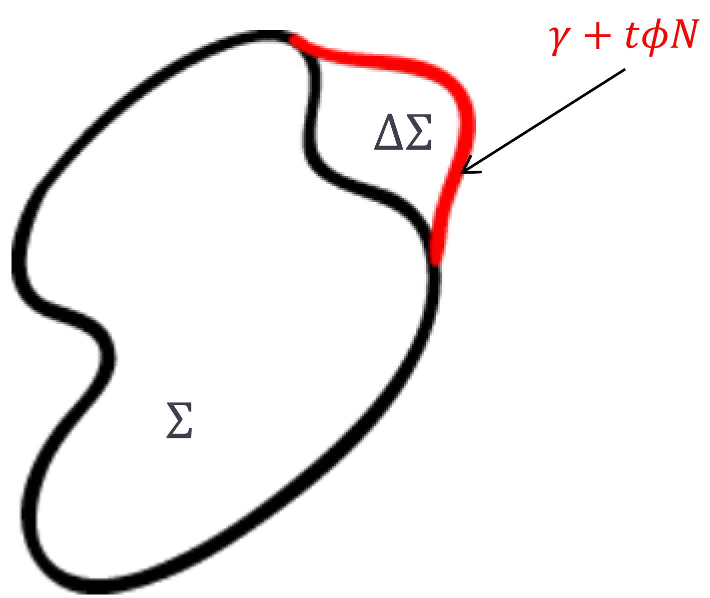
where $f(x)$ is a given function.

- Again, perturb $\gamma(s)$ as $\gamma(s) + t\phi(s)N(s)$. Assume $\phi(s) \geq 0\ \forall s$.

A parametrization for $\Delta\Sigma$:

$$\psi(s, \xi) = \gamma(s) + \xi\phi(s)N(s)$$

where $s \in [0, L]$ and $\xi \in [0, t]$.



Gradient Descent: Bulk Energy

- We have:

$$\iint_{\Delta\Sigma} b\, dx = \int_0^t \int_0^L b(\psi)|D\psi| ds\, d\xi$$

- Also,

$$|D\psi| = |\partial_s \psi \times \partial_\xi \psi|$$

- Furthermore,

$$\partial_s \psi = T + t\phi' N - t\phi\kappa T$$

and

$$\partial_\xi \psi = \phi N$$

- That gives:

$$|D\psi| = \phi - t\phi^2\kappa.$$

Gradient Descent: Bulk Energy

$$\frac{1}{t}\iint_{\Delta\Sigma} b\, dx = \frac{1}{t}\int_0^t \int_0^L b(\psi)(\phi - t\phi^2\kappa)ds\ \ d\xi$$
$$= \int_0^L b\big(\gamma(s)\big)\phi(s)\, ds$$
$$= \int_{\partial\Sigma} b\phi ds\,.$$

- We thus see that

$$\boxed{\nabla A = b}$$

Gradient Descent: Mumford-Shah

- P.C. Mumford-Shah:

$$E(\Sigma, c_1, c_2) = \mathrm{Per}(\Sigma) + \lambda \left\{ \int_{\Sigma} (f - c_1)^2 \, dx + \int_{\Omega \setminus \Sigma} (f - c_2)^2 \, dx \right\}$$

- Rewrite as:

$$E(\Sigma, c_1, c_2) = \mathrm{Per}(\Sigma) + \lambda \left\{ \int_{\Sigma} (f - c_1)^2 \, dx - \int_{\Sigma} (f - c_2)^2 \, dx + \int_{\Omega} (f - c_2)^2 \, dx \right\}$$

- Normal speed:

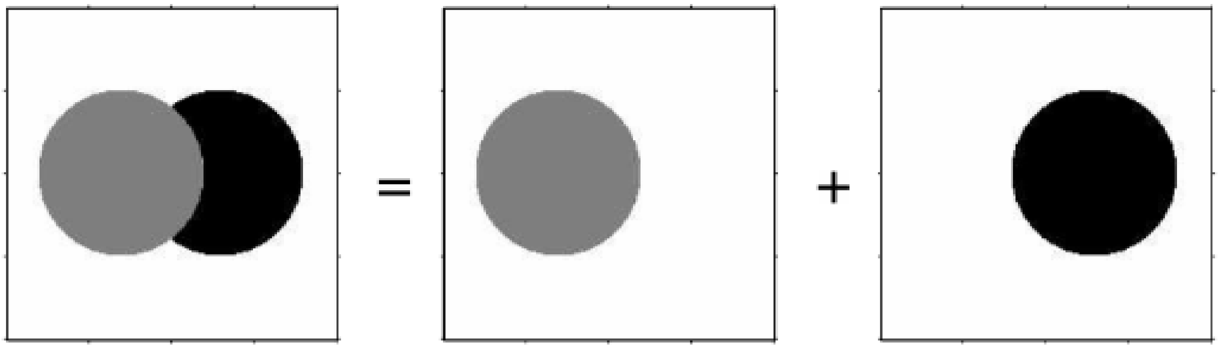
$$\boxed{v_n = \kappa + \lambda\left((f - c_2)^2 - (f - c_1)^2\right)}$$

Segmentation with Depth

- Given: A 2D picture of a scene with various objects in it:

$$f(x): \Omega \rightarrow [0,1]$$

- Goal: Determine automatically the shapes and relative nearness of the objects to the observer through their occlusion relations:

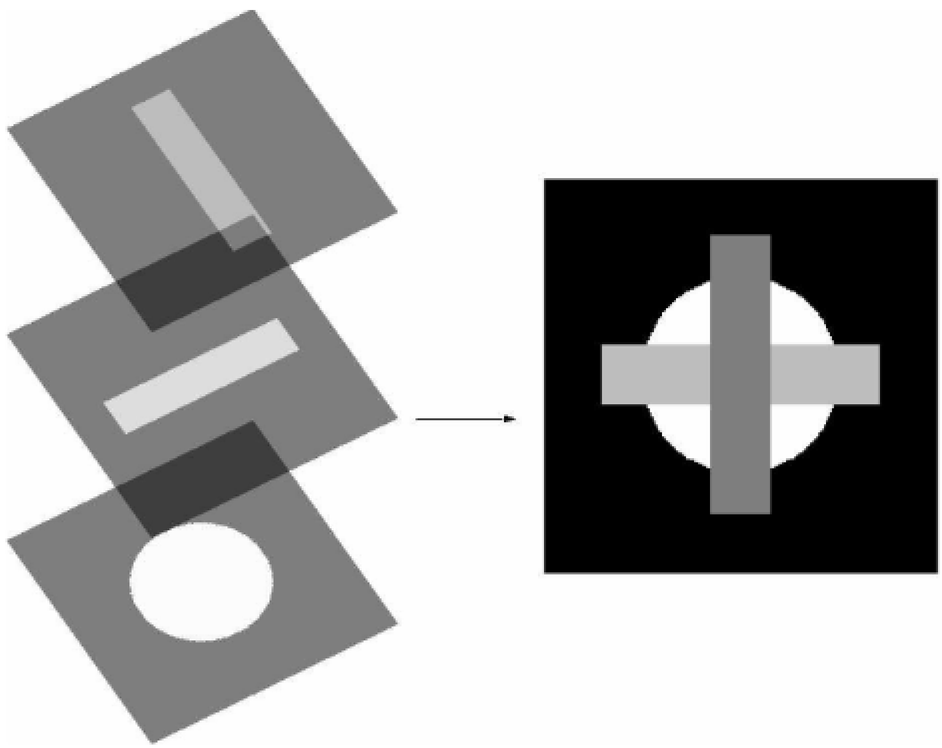


- We have prejudices: Prefer low curvature, connect T-junctions.
- Find an algorithm that mimics our prejudices:

$\Rightarrow$ CURVATURE DEPENDENT FUNCTIONALS

Nitzberg-Mumford-Shiota Model

- Each object lives in a plane perp. to line of sight:
  - No self occlusions
  - No entanglements.
- Each object may occlude parts of objects behind it:



Nitzberg-Mumford-Shiota Model

- Strategy: Exhibit the solution as minimizer of an energy.
- Unknowns:
  1. Number of objects n,
  2. Regions $\Sigma_1, \dots, \Sigma_n$ that they occupy.
  3. Approximate "color" of each object: $c_1, \dots, c_n$.
- Notation:
  1. $\mathrm{j} > \mathrm{i}$ means Σ_j is in front of Σ_i.
  2. Σ_i' is the visible part of Σ_i.

$$\Sigma_i' := \Sigma_i - \bigcup_{j>i} \Sigma_j$$

  3. κ_i = Curvature of $\partial\Sigma_i$.

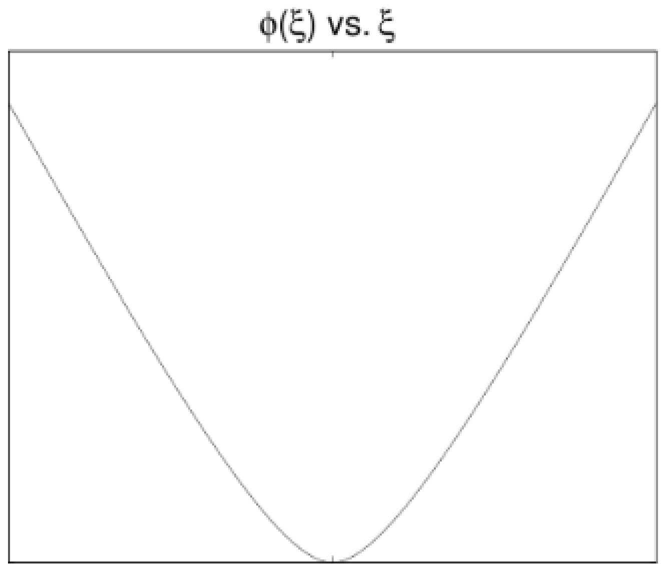
Nitzberg-Mumford-Shiota Model

- The Energy:

$$E_{2.1} = \sum_{i=1}^{n} \left\{ \int_{\partial \Sigma_i} 1 + \phi(\kappa_i)\, ds + \int_{\Sigma_i'} (f(x) - c_i)^2\, dx \right\}$$

where the function $\phi(\xi)$ is:

1. $\approx \xi^2$ for $|\xi|$ small,
2. $\approx |\xi|$ for $|\xi|$ large,
3. +ve, even, C^2, and convex.



Explanation of Terms

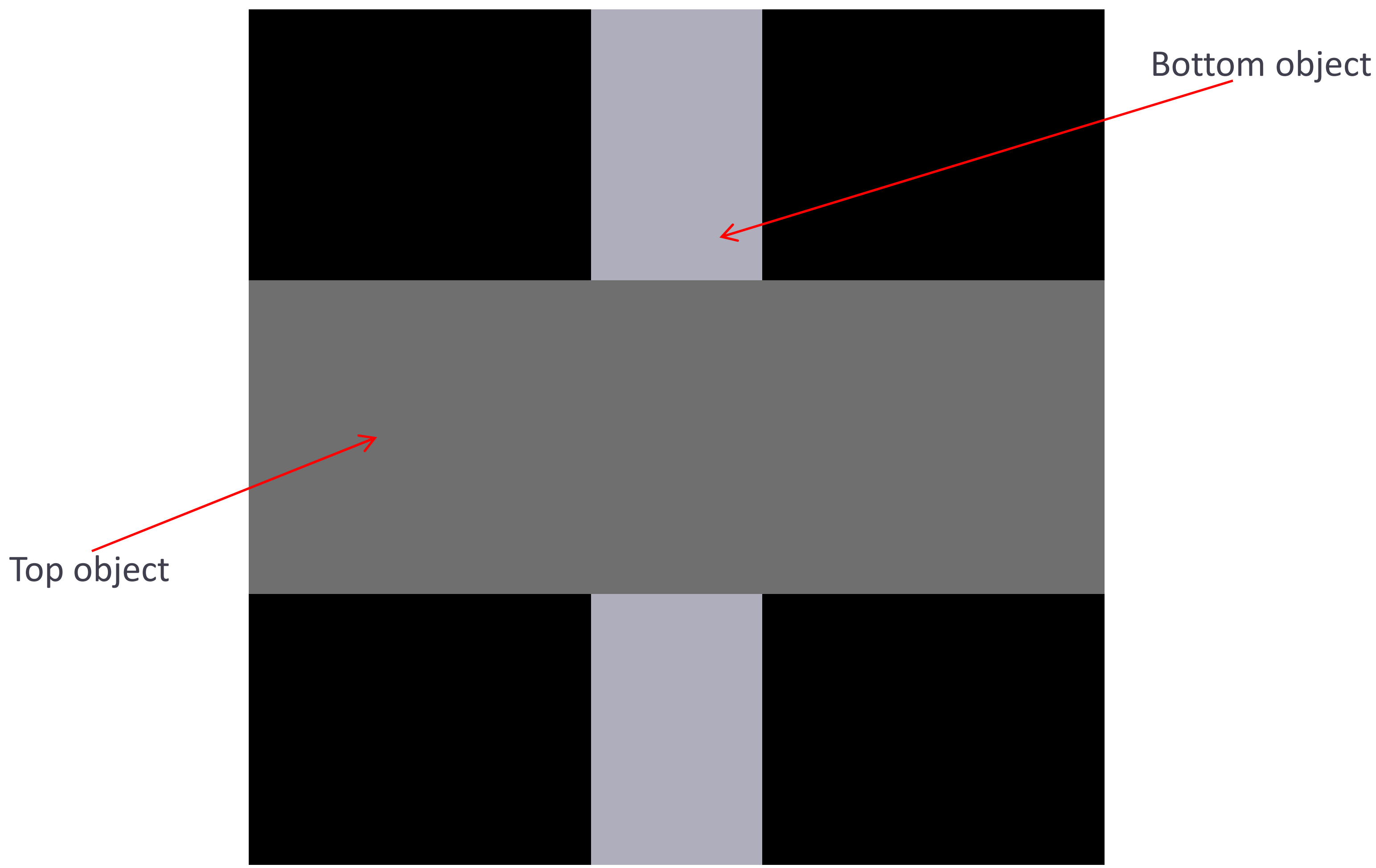
- Length term: $\int_{\partial\Sigma_i} ds$: Regions should be simple.
- Curvature term: $\int_{\partial\Sigma_i} \phi(\kappa_i)\, ds$: Edge contours should have a *tendency to continue straight*, not make sharp turns.
- Fidelity term: Approximate scene $u(x)$ (assembled from regions Σ_i and the constants c_i), given by

$$u(x) = \sum_{i=1}^{n} c_i \mathbf{1}_{\Sigma_i'}(x)$$

should be faithful to the original image $f(x)$:

$$\int \big(u(x) - f(x)\big)^2\, dx = \sum_{i=1}^{n} \int_{\Sigma_i'} (f(x) - c_i)^2\, dx$$

Role of Curvature Dependence

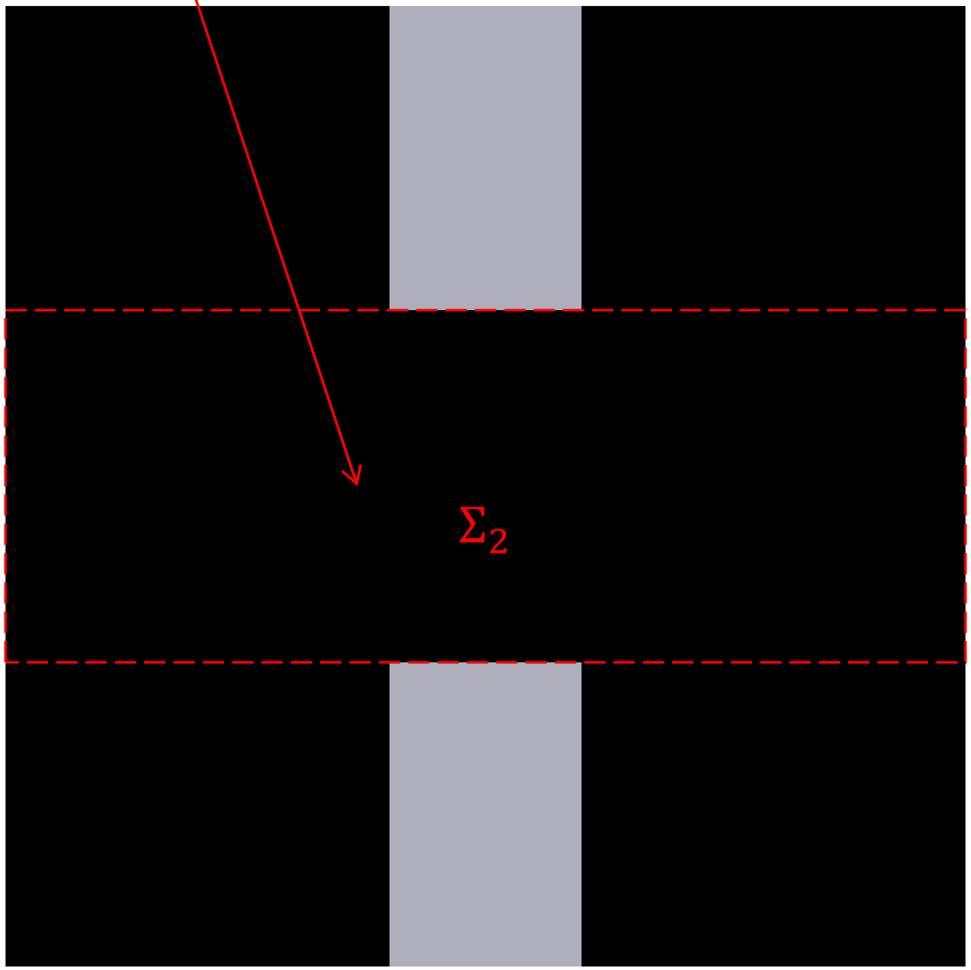


Role of Curvature Dependence

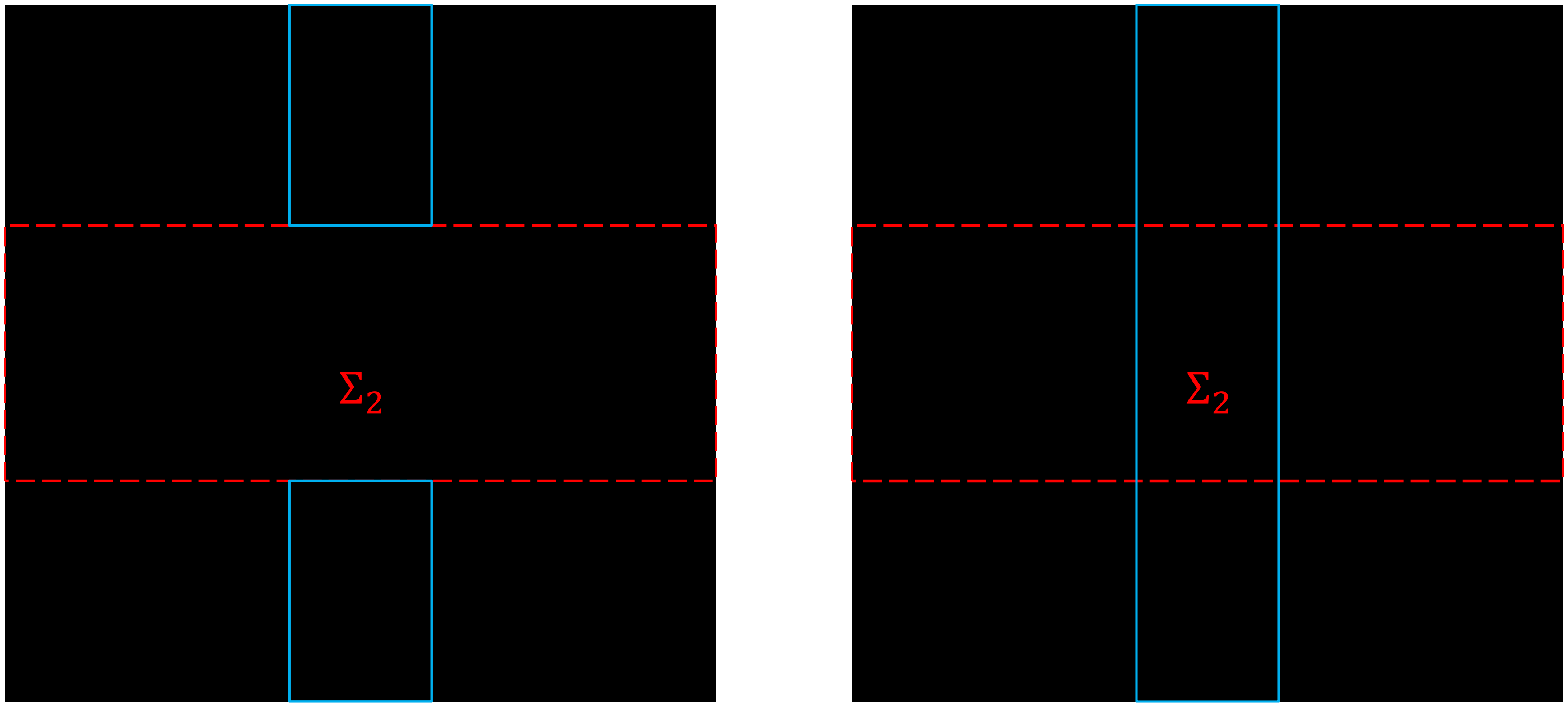
Minimization w.r.t. Σ_1:

$$\int 1 + \phi(\kappa)\, d\sigma + \int_{\Sigma_2^c} \left(c_1 1_{\Sigma_1} - f\right)^2 dx$$
$$+ \int_{\Sigma_2^c} \left(c_0 1_{\Sigma_1^c} - f\right)^2 dx$$

Fidelity term inactive
in this region.



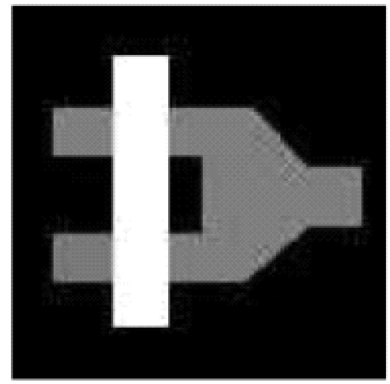
Role of Curvature Dependence



Curvature term will prefer the completion on the right; fidelity term is indifferent between the two.

Nitzberg-Mumford-Shiota Model

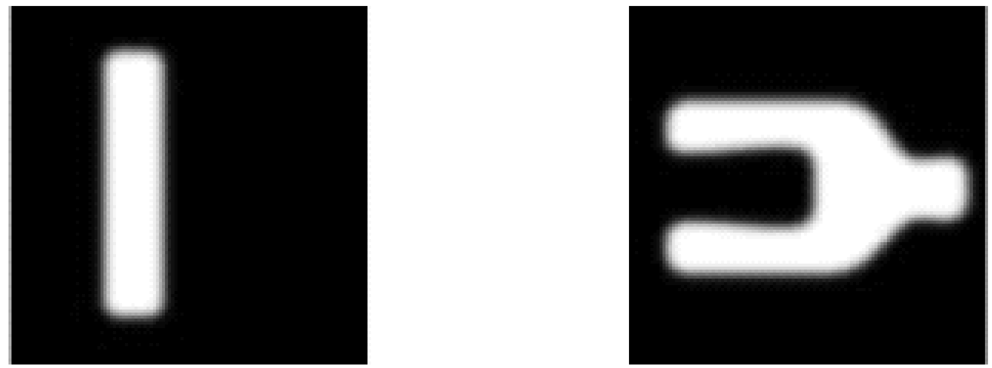
Original image:



Regions taken as initial guess:

Nitzberg-Mumford-Shiota Model

Order guess = AB.
Energy = 29.



Order guess = BA.
Energy = 41.

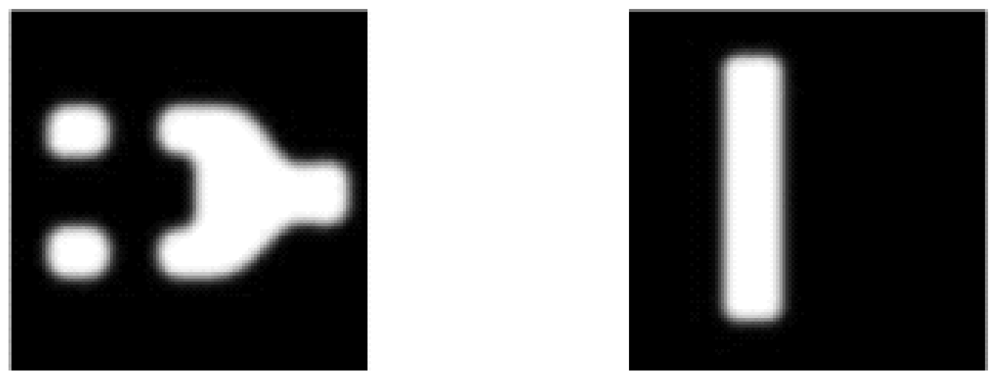


Image Denoising: Heat Equation

- GOAL: Remove oscillations from a noisy image.
- Simplest method: Filtering.

$$f(x) \to (G_\sigma * f)(x)$$

- Equivalent to solving the heat equation:

$$\begin{aligned} u_t &= \Delta u \\ u(x,0) &= f(x) \end{aligned}$$

- Oscillations are suppressed: Good.
- Edges are blurred: Bad.
- Generates a one-parameter family of gradually simplifying images.

$\Rightarrow$ Scale space

Image Denoising: Heat Equation

Heat equation:

Image Denoising: Heat Equation

Image Denoising: Mean Curvature Motion

- IDEA: Suppress diffusion across edges:

$$\Delta u = \langle (D^2\, u)\xi, \xi \rangle + \langle (D^2 u)\eta, \eta \rangle$$

where $|\xi|, |\eta| = 1$ and $\xi \perp \eta$.

Choose:

$$\xi = \frac{\nabla u}{|\nabla u|} \quad \text{and} \quad \eta = \frac{\nabla^{\perp} u}{|\nabla^{\perp} u|}$$

Suppress diffusion in ∇u direction:

$$\boxed{u_t = \langle (D^2 u)\eta, \eta \rangle = |\nabla u| \nabla \cdot \left(\frac{\nabla u}{|\nabla u|} \right)}$$

$\Rightarrow$ Motion by mean curvature of level sets of u.

Image Denoising: Mean Curvature Motion

- Curvature: Let $\gamma(s)$ be an arc-length parametrization of the 0-level set:
$$\{\gamma(s): s \in \boldsymbol{R}\} = \{x: u(x) = 0\}$$
- Since $u\big(\gamma(s)\big) = 0$ for all s, we have
$$0 = \frac{d}{ds} u(\gamma(s)) = (\nabla u)\Big|_{\gamma(s)} \cdot \dot{\gamma}(s)$$

and

$$\begin{aligned} 0 = \frac{d^2}{ds^2} u\big(\gamma(s)\big) &= \langle (D^2 u)\Big|_{\gamma(s)} \dot{\gamma}(s), \dot{\gamma}(s)\rangle \\ &\quad + \langle (\nabla u)\Big|_{\gamma(s)}, \ddot{\gamma}(s)\rangle \\ &= \langle (D^2 u)\Big|_{\gamma(s)} \eta, \eta \rangle + \kappa |\nabla u| \end{aligned}$$

so that

$$\kappa = -\frac{1}{|\nabla u|} \langle (D^2 u)\Big|_{\gamma(s)} \eta, \eta \rangle$$

Image Denoising: Curvature Motion

- In addition,

$$\frac{\partial}{\partial t} u(\gamma(s,t),t) = u_t(\gamma(s,t),t) + \nabla u \cdot \gamma_t = 0$$

- But,

$$\nabla u \cdot \gamma_t = |\nabla u| v_n$$

so that

$$u_t = -|\nabla u| v_n$$

- Thus, combining:

$$\begin{aligned} u_t &= \langle (D^2 u)\eta, \eta \rangle \\ &= -\kappa |\nabla u| \\ &= -v_n |\nabla u| \end{aligned}$$

which means

$$\boxed{v_n = \kappa.}$$

Image Denoising: Mean Curvature Motion

Mean curvature motion:

Image Denoising: Mean Curvature Motion

Image Denoising: Ill-Posed Equations

- Go even further: Reverse diffusion in ∇u direction.

$$u_t = \nabla \cdot (g(|\nabla u|^2)\nabla u)$$

where

$$\text{e.g.} \quad g(\xi) = \frac{1}{1+x}$$

Expand:

$$\nabla \cdot (g(|\nabla u|^2)\nabla u) = |\nabla u|^2 g(|\nabla u|^2)\langle (D^2 u)\eta, \eta\rangle$$

$$+ \ |\nabla u|^2 \boxed{[g(|\nabla u|^2) + 2g'(|\nabla u|^2)]}\langle (D^2 u)\xi, \xi\rangle$$

+ve for $|\nabla u| < 1$, -ve for $|\nabla u| > 1$

Image Denoising: Perona-Malik Model

- Perona-Malik Model:

$$u_t = \nabla \cdot (g(|\nabla u|^2)\nabla u)$$

Typically implemented as

$$u_t = \left(R(u_x)\right)_x + \left(R(u_y)\right)_y$$

where

$$R(\xi) = \xi g(\xi^2) \quad \text{e.g.} \quad R(\xi) = \frac{\xi}{1+\xi^2}$$

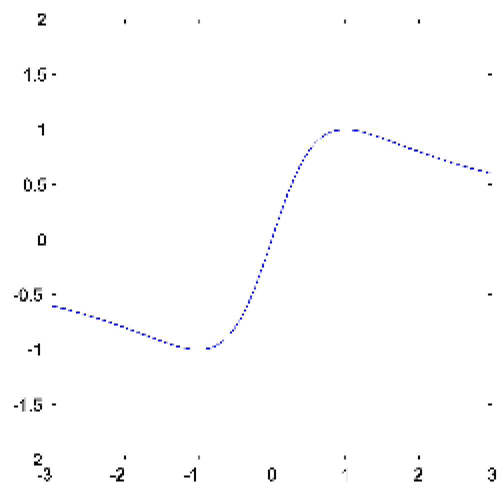


Image Denoising: Perona-Malik Model

Perona-Malik Evolution

Image Denoising: Perona-Malik Model

- Perona-Malik Scheme

$$u_t = \big(R(u_x)\big)_x + \big(R(u_y)\big)_y$$

is L^2 gradient descent for the energy

$$E(u) = \int \psi(u_x) + \psi(u_y)\, dxdy$$

where density $\psi(\xi)$ is:

- Convex for $|\xi|$ small,
- Concave for $|\xi|$ large.

E.g. for $R(\xi) = \frac{\xi}{1+\xi^2}$,

$$\psi(\xi) = \log(1 + \xi^2)\,.$$

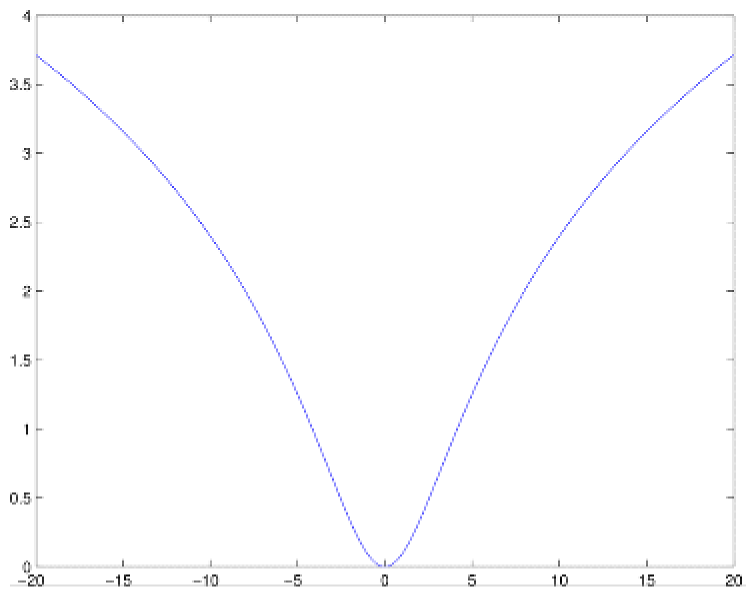


Image Denoising: Perona-Malik Model

- Perona-Malik is intimately related to Mumford-Shah.
- Consider energy densities of the form

$$\psi_h(\xi) = \begin{cases} \xi^2 & \text{if } |\xi| < \dfrac{1}{\sqrt{h}} \\ \dfrac{1}{h} & \text{if } |\xi| \geq \dfrac{1}{\sqrt{h}} \end{cases}$$

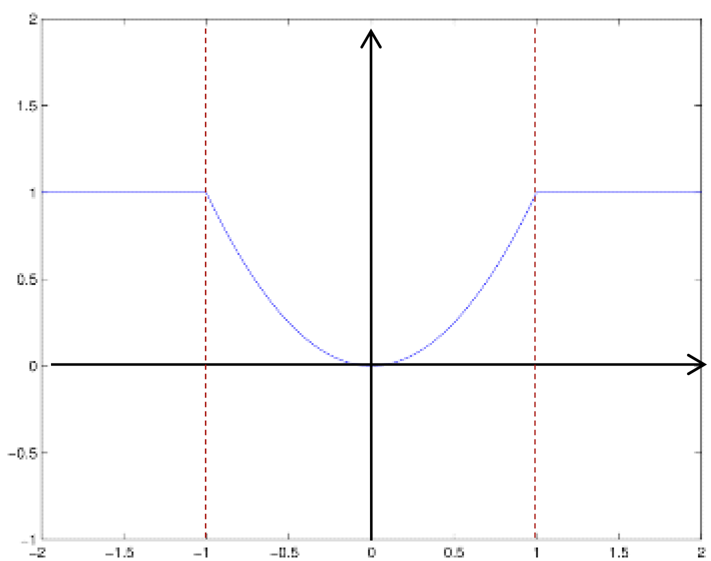


Image Denoising: Perona-Malik Model

- Corresponding discrete energies:

$$E_h(u) = \sum_{i,j} \{\psi_h(D_x^+ u) + \psi_h(D_y^+ u)\} h$$

- Weak spring model of Geman & Geman (Blake & Zisserman):

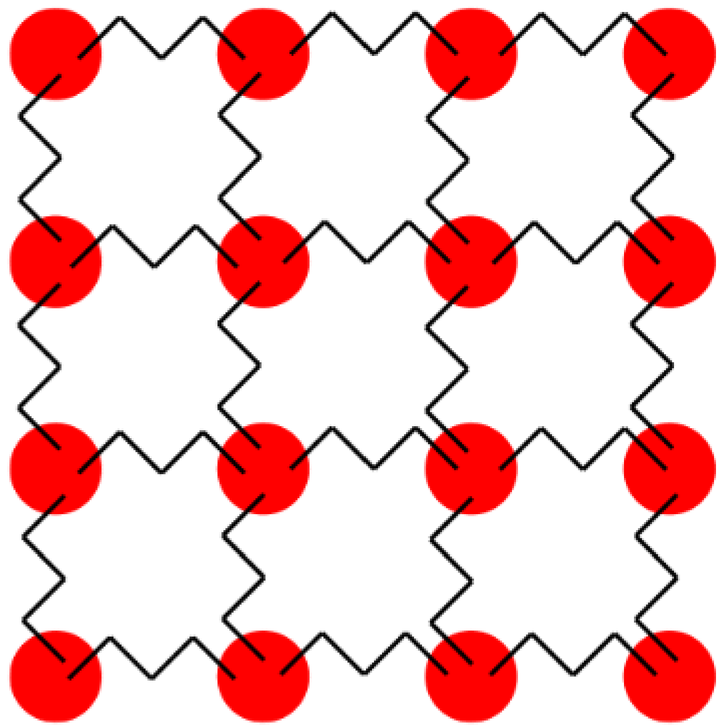


Image Denoising: Perona-Malik Model

- Send $h \to 0^+$:

$$\lim_{h \to 0^+} E_h = \int_{\Omega \setminus K} |\nabla u|^2 \, dx + \int_K d\tilde{\sigma} \approx \text{Mumford–Shah}$$

Explanation:

1. If u is differentiable near $(x, y) = (hi, hj)$, then

$$D_x^+ u_{i,j} = O(1) \text{ and } D_y^+ u_{i,j} = O(1) \text{ as } h \to 0^+.$$

Hence,

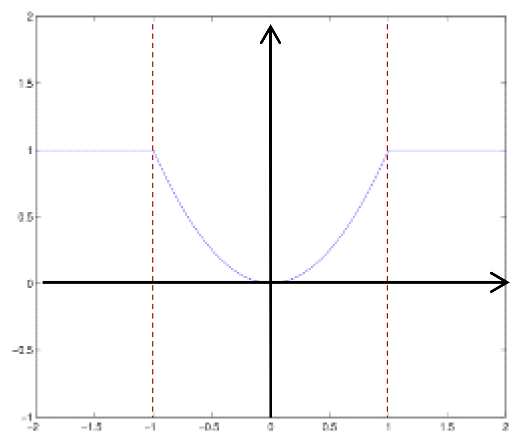
$$\psi_h(D_x^+ u) + \psi_h\left(D_y^+ u\right) \to |\nabla u|^2 \text{ as } h \to 0^+.$$

2. If u has a vertical edge at $(x, y) = (hi, hj)$, then

$$D_x^+ u_{i,j} = O\left(\frac{1}{h}\right) \text{ as } h \to 0^+.$$

Hence,

$$\psi_h(D_x^+ u) \to \frac{1}{h} \text{ as } h \to 0^+.$$



Rigorous result: A. Chambolle, SIAM J. Appl. Math (1996)

Image Denoising: Perona-Malik Model

- For energy densities of the form

$$\psi(\xi) = (|\xi|^2 + 1)^{\frac{p}{2}} - 1 \text{ where } p \in (0,1)$$

If scaled correctly w.r.t. h:

$$\lim_{h \to 0^+} E_h = \int_{\Omega \setminus K} |\nabla u|^2 \, dx + \int_K |u^+ - u^-|^p \, d\tilde{\sigma}$$

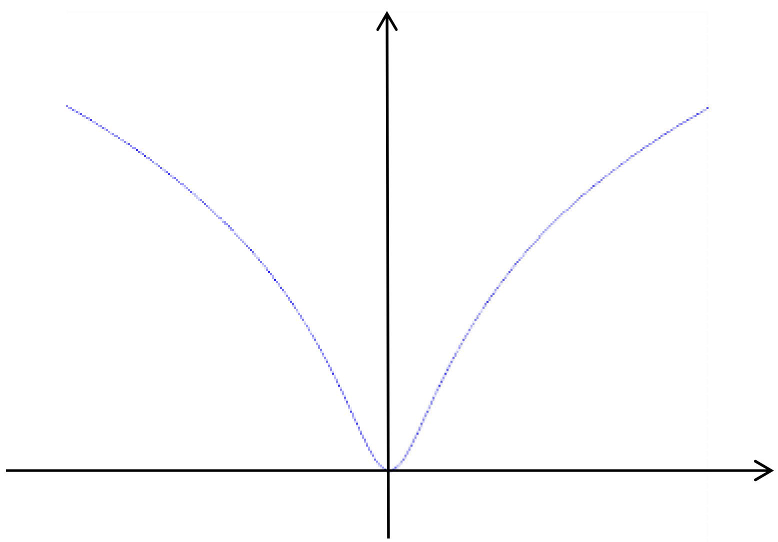


Image Denoising: Perona-Malik Model

- An old method for minimizing Mumford-Shah.
- Graduated non-convexity of Blake & Zisserman.
- Gradient descent (Perona-Malik) for energies:

$$E(u) = \sum_{i,j} \{\psi(D_x^+ u) + \psi(D_y^+ u)\} h$$

with gradually less convex ψ:

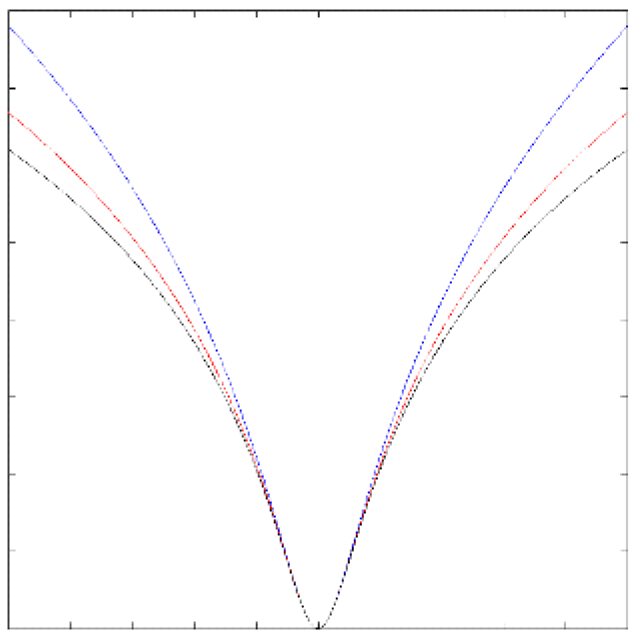


Image Denoising: Total Variation Model

- Rudin, Osher, and Fatemi (1992):

$$E(u) = \int |\nabla u| + \lambda(f - u)^2 \, dx$$

- Preserves sharp edges.
- Many advantages over Perona-Malik:
  - Strictly convex functional: Existence, uniqueness of solutions.
  - Continuous dependence on data (i.e. f), and parameters (i.e. λ).
  - Only one parameter to be chosen by user: λ
- Still some difficulties:
  - Non-differentiable.
  - Naïve numerical methods slow to converge.
- Intimately related to: Piecewise constant Mumford-Shah.

Total Variation: Basic Facts

- To define total variation for possibly discontinuous functions:
- Given a vector $x \in \mathbb{R}^n$, we can write:

$$|x| = \max_{|y| \leq 1} x \cdot y$$

- Apply this to

$$\int |\nabla u| dx = \int \max_{|g| \leq 1} g \cdot \nabla u \ dx$$

- When $u(x)$ is smooth, can take $g(x)$ to be smooth and compactly supported, and move “max” outside:

$$\int |\nabla u| dx = \sup_{\substack{|g(x)| \leq 1 \\ g \in C_c^1}} \int g \cdot \nabla u \, dx$$

Total Variation: Basic Facts

- Integrate by parts:

$$\int |\nabla u| dx = \sup_{\substack{|g(x)|\le 1 \\ g\in C_c^1}} \int u\, \nabla \cdot g \; dx$$

- Right hand side can be finite even for discontinuous u.
- It is taken to be the definition of total variation:

$$\boxed{\int |\nabla u| = \sup_{\substack{|g(x)|\le 1 \\ g\in C_c^1}} \int u\, \nabla \cdot g \; dx}$$

Total Variation: Basic Facts

- Example:

$$u(x) = \mathbf{1}_\Sigma(x)$$

where Σ is a compact set with smooth boundary $\partial\Sigma$.

- First of all,

$$\int u\, \nabla \cdot g\, dx = \int_\Sigma \nabla \cdot g\, dx = \int_{\partial\Sigma} g \cdot n\, d\sigma \leq \int_{\partial\Sigma} d\sigma = \mathrm{Length}(\partial\Sigma)$$

- Second: There exists a vector field $\psi(x)$ s.t.

  1. ψ is smooth, compactly supported,
  2. $|\psi(x)| \leq 1$ for all x,
  3. $\psi(x) = n(x)$ for all $x \in \partial\Sigma$.

- We have:

$$\int u\, \nabla \cdot g\, dx = \int_{\partial\Sigma} n \cdot n\, d\sigma = \mathrm{Length}(\partial\Sigma)$$

Total Variation: Basic Facts

- Hence, we see that

$$\int |\nabla \mathbf{1}_\Sigma(x)| = \mathrm{Length}(\partial\Sigma) \coloneqq \mathrm{Per}(\Sigma)$$

  when $\partial\Sigma$ is smooth.

- If $u(x)$ is piecewise constant:

$$u(x) = \sum_{j=1}^{N} c_j \mathbf{1}_{\Sigma_j}(x)$$

with $c_{j+1} > c_j > 0$, $\Sigma_j \subset \Sigma_{j+1}$, and $\partial\Sigma_j$ smooth for all j, then:

$$\int |\nabla u| = \sum_{j=1}^{N-1} (c_{j+1} - c_j) \mathrm{Per}(\Sigma_j)$$

Total Variation: Basic Facts

- Given any function $u \in L^1$, approximate by such piecewise constant functions.
- Use our formula.
- Pass to the limit.
- You get the co-area formula

$$\int |\nabla u| = \int_{\mathbb{R}} \mathrm{Per}(\{x : u(x) > \mu\})\, d\mu$$

Image Denoising: Total Variation Model

Image Denoising: Total Variation Model

Image Inpainting

- Image information missing (scratches, holes) in $D \subset \Omega$.
- Interpolate image into D.
- Nonstandard requirement: Propagate sharp edges into D.

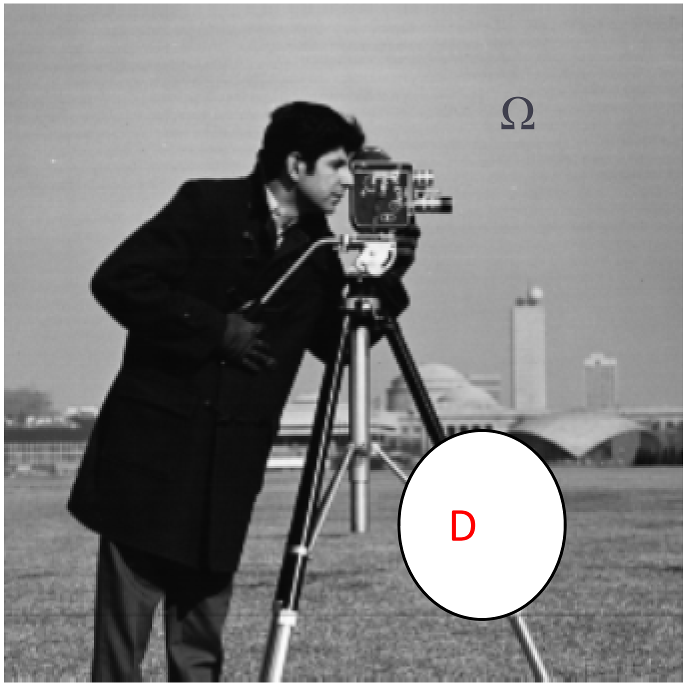


Image Inpainting

- PDE approach: Originates in the work of Bertalmio, Caselles, Sapiro, et. al.

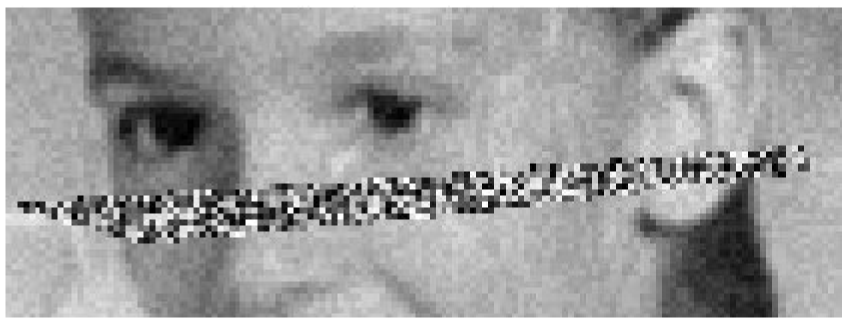
Image Inpainting

- More examples (from Bertalmio, Caselles, Sapiro, Ballester):

Image Inpainting

- Inpainting via the Total Variation Model: Work of T. F. Chan and J. Shen.
- Very robust, variational model.

$$\int_{\Omega} |\nabla u| + \lambda \int_{\Omega \setminus D} (f-u)^2 \, dx$$



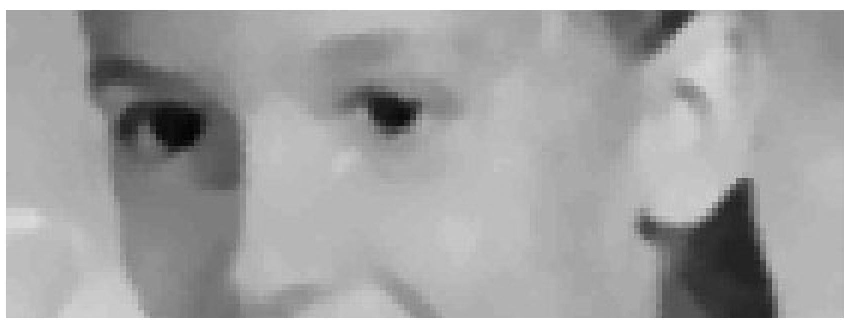
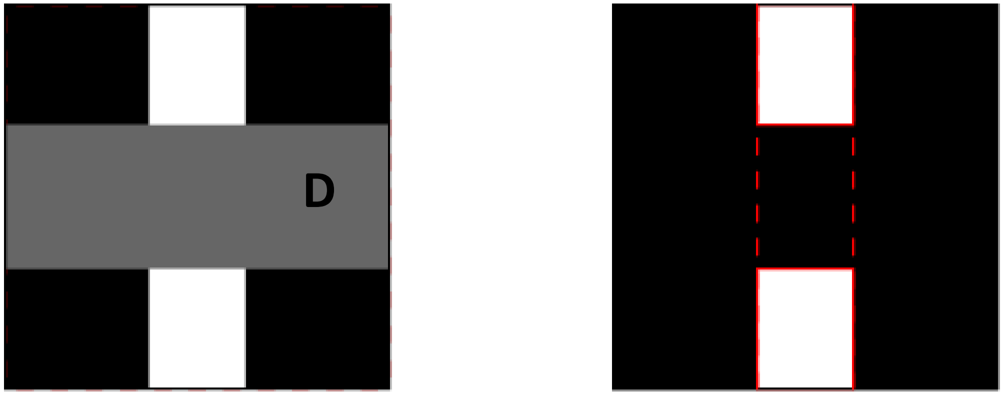


Image Inpainting

Caveats of 2^{nd} order inpainting models:

- No long distance connections between contours:



- Non-smooth, unnatural contour extensions: