Replication under Price Impact and Martingale Representation Property

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Classical model for a "small" agent

Input: price process $S = (S_t)$ for traded stock. Usually, S is a solution of SDE:

$$dS_t = \mu(t, S_t)dt + \sigma(t, S_t)dB_t, \quad S_0 = x,$$

where B is a Brownian motion.

Key assumption: trader's actions do not affect S.

Strategy: predictable S-integrable process $\gamma = (\gamma_t)$ of the number of stocks.

Gain from strategy:

$$G_t = \int_0^t \gamma dS = (\gamma \cdot S)_t.$$

Model with price impact

Price impact: $\gamma \to S(\gamma)$.

Practice & Mathematical Finance: the price impact is *postulated* (*exogenous*).

Financial Economics: the price impact is an *output* of equilibrium (*endogenous*).

Main idea: Let ψ be stock's dividend paid at maturity T, so that, $S_T(\gamma) = \psi$. Recall that for the "small" agent model,

$$(NA) \quad \Leftrightarrow \quad \exists\, \mathbb{Q}(0) \sim \mathbb{P} : \quad S_t(0) = \mathbb{E}^{\mathbb{Q}(0)}\left[\psi | \mathcal{F}_t\right].$$

For $\gamma \neq 0$ we similarly expect to have

$$S_t(\gamma) = \mathbb{E}^{\mathbb{Q}(\gamma)}\left[\psi | \mathcal{F}_t\right],$$

where the measure $\mathbb{Q}(\gamma) \sim \mathbb{P}$ is obtained from an equilibrium.

Optimal investment

Input: financial market and investor's preferences

1. $S = (S_t)$: stocks' prices;
2. $U(x) = -\frac{1}{a}e^{-ax}$, $x \in \mathbf{R}$: utility function; $a > 0$ is investor's risk-aversion.

Output: optimal process $\gamma = (\gamma_t)$ of the number of stocks:

$$\gamma = \arg\max_{\zeta} \mathbb{E}\left[U(\int_0^T \zeta dS)\right] = \arg\min_{\zeta} \mathbb{E}\left[\exp(-a\int_0^T \zeta dS)\right].$$

Martingale characterization: $\gamma = (\gamma_t)$ is optimal $\Leftrightarrow$ S is a local martingale and $\gamma \cdot S$ is a UI martingale under $\mathbb{Q}$ given by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \text{const}\, U'(\int_0^T \gamma dS) = \text{const} \exp(-a\int_0^T \gamma dS).$$

Price impact model

Input: dividends, market makers' preferences, demand

1. ψ: stocks' dividends paid at maturity T;
2. $U(x) = -\frac{1}{a}e^{-ax}$, $x \in \mathbf{R}$: representative utility; $a > 0$ is aggregate risk-aversion (harmonic mean),

$$a \downarrow 0 \quad \Leftrightarrow \quad \text{market's liquidity} \quad \uparrow \infty.$$

3. $\gamma = (\gamma_t)$: demand process (number of stocks owned by the market).

Output: stocks' prices $S = (S_t)$ such that $S_T = \psi$ and

$$\gamma = \arg\max_{\zeta} \mathbb{E}\left[U(\int_0^T \zeta dS)\right] = \arg\min_{\zeta} \mathbb{E}\left[\exp(-a\int_0^T \zeta dS)\right]$$

$\Leftrightarrow S_t = \mathbb{E}^{\mathbb{Q}}[\psi | \mathcal{F}_t]$ with $\frac{d\mathbb{Q}}{d\mathbb{P}} = \text{const}\exp(-a\int_0^T \gamma dS)$.

References

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Brownian framework

Assumption

The filtration is generated by a Brownian motion $B = (B_t)$:

$$\mathcal{F}_t = \mathcal{F}_t^B, \quad t \in [0, T].$$

Stocks' prices evolve as

$$dS = \sigma \lambda dt + \sigma dB, \quad S_T = \psi,$$

where

$\lambda = (\lambda_t)$: the market price of risk;
$\sigma = (\sigma_t)$: the volatility.

Denote also

$$R_t = -\frac{1}{a} \log \mathbb{E}\left[\exp(-a \int_t^T \gamma dS) \middle| \mathcal{F}_t \right], \quad t \in [0, T],$$

the certainty equivalence value (CEV) of remaining gain.

BSDE for $S = S(\psi, a, \gamma)$

Theorem (K.,Pulido AAP-16)

For dividends ψ, risk-aversion $a > 0$, and demand γ the following items are equivalent:

1. $S = S(\psi, a, \gamma)$ *is a price process,* σ *is the volatility,* λ *is the market price of risk, and* R *is the CEV process.*
2. (R, S, η, θ) *with* $\eta = \lambda - a\sigma\gamma$ *and* $\theta = a\sigma$ *solves the BSDE:*

$$aR_t = \frac{1}{2}\int_t^T (|\theta\gamma|^2 - |\eta|^2)ds - \int_t^T \eta dB,$$
$$aS_t = a\psi - \int_t^T \theta(\eta + \theta\gamma)ds - \int_t^T \theta dB,$$

and the products $Z(\gamma \cdot S)$ *and* ZS *are UI martingales, where*

$$Z \triangleq \mathcal{E}(-\lambda \cdot B) = \mathcal{E}(-(\eta + \theta\gamma) \cdot B).$$

BMO norms

- For a continuous martingale M with $M_0 = 0$,
$$\|M\|_{\mathrm{BMO}} \triangleq \operatorname*{ess\,sup}_{\tau}\{\mathbb{E}\left[|M_T - M_\tau|^2 \middle| \mathcal{F}_\tau\right]\}^{1/2},$$
where the supremum is taken with respect to all stopping times τ.
- For an integrable random variable ξ with $\mathbb{E}[\xi] = 0$ set
$$\|\xi\|_{\mathrm{BMO}} \triangleq \|(\mathbb{E}\left[\xi | \mathcal{F}_t\right])_{t\in[0,T]}\|_{\mathrm{BMO}}$$
- For a predictable process $\zeta = (\zeta_t)$ set
$$\|\zeta\|_{\mathrm{BMO}} \triangleq \operatorname*{ess\,sup}_{\tau}\left(\mathbb{E}\left[\int_\tau^T |\zeta_s|^2 ds \middle| \mathcal{F}_\tau\right]\right)^{1/2},$$
where the supremum is taken with respect to all stopping times τ. By Ito's isometry,
$$\|\zeta\|_{\mathrm{BMO}} = \|\int \zeta dB\|_{\mathrm{BMO}}.$$

Existence and uniqueness

Theorem (K.,Pulido AAP-16)

There is a constant $c > 0$ such that if

$$a\|\gamma\|_\infty \|\psi - \mathbb{E}[\psi]\|_{\mathrm{BMO}} \leq c,$$

then the prices $S = S(\psi, a, \gamma)$ exist and unique.

Proposition (K.,Pulido AAP-16)

There are bounded γ and ψ such that

$$a\|\gamma\|_\infty \|\psi - \mathbb{E}[\psi]\|_\infty = 1$$

and such that the prices $S = S(\psi, a, \gamma)$ either do not exist or are not unique.

Asymptotic expansion

Theorem (K.,Pulido JFM-16)

Assume that

$$\|\gamma\|_\infty \|\psi - \mathbb{E}[\psi]\|_{\mathrm{BMO}} < \infty.$$

Then there is a constant $K = K(\psi, \gamma)$ *such that*

$$\|S(a,\gamma) - (S(0) + aS^{(1)}(\gamma))\|_{\mathrm{BMO}} \leq Ka^2, \quad a \to 0,$$

where

$$S_t(0) = \mathbb{E}[\psi | \mathcal{F}_t] = \mathbb{E}[\psi] + \int_0^t \sigma(0) dB \quad \textit{("small" agent's model)}$$

$$S_t^{(1)}(\gamma) = -\mathbb{E}\left[\int_t^T \sigma^2(0)\gamma ds \middle| \mathcal{F}_t\right] \quad \textit{(first-order correction)}$$

Replication problem

Denote

$$Exp \triangleq \{\xi : \mathbb{E}\left[e^{t|\xi|}\right] < \infty, \quad t > 0\}.$$

Replication problem: for a contingent claim $\xi \in Exp$ find the initial wealth $p \in \mathbf{R}$ and a demand $\gamma = (\gamma_t)$ such that

$$p - \int_0^T \gamma dS(\gamma) = \xi. \tag{1}$$

Lemma (uniqueness, easy)

If p *and* γ *satisfy* (1), *then*

$$p = \mathbb{E}^{\mathbb{Q}}[\xi], \quad S_t(\gamma) = \mathbb{E}^{\mathbb{Q}}[\psi | \mathcal{F}_t],$$

where

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \text{const}\, e^{a\xi}.$$

Completeness assumption for $a = 0$

Assumption ($S(0)$-model is complete)

The "small" agent model with price process $S_t(0) = \mathbb{E}\left[\psi | \mathcal{F}_t\right]$ is complete, that is, for for every contingent claim ξ with $\mathbb{E}\left[|\xi|\right] < \infty$ the martingale

$$P_t(0) = \mathbb{E}\left[\xi | \mathcal{F}_t\right],$$

admits integral representation:

$$P_t(0) = \mathbb{E}\left[\xi\right] + \int_0^t \gamma dS(0),$$

for some predictable $S(0)$-integrable process γ.

Example (No existence)

Even if $S(0)$-model is complete, for any $a > 0$ one can find *not replicable* contingent claims ξ such that $|\xi| \leq 1$.

Approximate replication

Theorem (Approximate replication for fixed $a > 0$)

Suppose that $S(0)$-model is complete, that $\psi \in \mathcal{L}_p$ for some $p > 1$ and $\xi \in Exp$. Then for every $\epsilon \in (0, \frac{1}{2}]$ there are $p(\epsilon) \in \mathbf{R}$ and a demand $\gamma(\epsilon)$ such that

$$a\|\xi - (p(\epsilon) - \int_0^T \gamma(\epsilon) dS^{\gamma(\epsilon)})\|_\infty \leq \epsilon.$$

Moreover, in this case,

$$|p(\epsilon) - p| \leq 2\epsilon p, \quad |S^{\gamma(\epsilon)} - S| \leq 2\epsilon |S|,$$

where

$$p = \mathbb{E}^{\mathbb{Q}}[\xi], \quad S_t = \mathbb{E}^{\mathbb{Q}}[\psi | \mathcal{F}_t], \quad \frac{d\mathbb{Q}}{d\mathbb{P}} = \text{const}\, e^{a\xi}.$$

Generic replication

Theorem (Generic replication for variable $a > 0$)

Suppose that $S(0)$-model is complete, that $\psi \in \mathcal{L}_p$ for some $p > 1$ and $\xi \in Exp$. Then there is at most countable set $\mathcal{I} \subset (0, \infty)$ such that for risk-aversions $a \notin \mathcal{I}$ the contingent claim ξ is replicable:

$$\xi = p(a) - \int_0^T \gamma(a) dS^{\gamma(a)}.$$

Moreover, in this case,

$$p(a) = \mathbb{E}^{\mathbb{Q}(a)}[\xi], \quad S_t^{\gamma(a)} = S_t(a) = \mathbb{E}^{\mathbb{Q}(a)}[\psi | \mathcal{F}_t],$$

where

$$\frac{d\mathbb{Q}(a)}{d\mathbb{P}} = \text{const}\, e^{a\xi}.$$

The Martingale Representation Property

We work on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ satisfying the usual conditions.

Definition

Let $\mathbb{Q} \sim \mathbb{P}$ and S be a d-dimensional local martingale under $\mathbb{Q}$. We say that S *has the Martingale Representation Property (MRP)* if every local martingale M under $\mathbb{Q}$ is of the form

$$M = M_0 + \gamma \cdot S = M_0 + \int \gamma \, dS,$$

where γ is a predictable S-integrable process with values in $\mathbf{R}^d$.

Classical (forward) results

Let $\mathbb{Q} \sim \mathbb{P}$ and S be a d-dimensional local martingale under $\mathbb{Q}$.

Jacod's theorem (2FTAP): S has MRP if and only if $\mathbb{Q}$ is the *only* equivalent local martingale measure for S.

Forward dynamics. Suppose that $B^{\mathbb{Q}}$ is a n-dimensional Brownian motion under $\mathbb{Q}$, $\mathcal{F}_t = \mathcal{F}_t^{B^{\mathbb{Q}}}$ and

$$S_t = S_0 + \int_0^t \sigma_s dB_s^{\mathbb{Q}},$$

where $\sigma = (\sigma_t^{ij})$ is a predictable process with values in $\mathbf{R}^{n \times d}$. Then S has the MRP if and only if

$$\operatorname{rank} \sigma_t(\omega) = n, \quad dt \times d\mathbb{P}(\omega) - a.s..$$

Density of probability measures with MRP

Let $\psi = (\psi^i)_{i=1,\dots,d}$ be a d-dimensional random variable. We denote by $\mathcal{Q}(\psi)$ the family of probability measures $\mathbb{Q} \sim \mathbb{P}$ such that

1. $\mathbb{E}^{\mathbb{Q}}[|\psi|] < \infty$,
2. $S_t^{\mathbb{Q}} = \mathbb{E}^{\mathbb{Q}}[\psi | \mathcal{F}_t]$, $t \geq 0$, has the MRP.

Theorem

Suppose that $\mathcal{Q}(\psi) \neq \emptyset$. Then for every $\mathbb{R} \sim \mathbb{P}$ such that $\mathbb{E}^{\mathbb{R}}[|\psi|] < \infty$ and every $\epsilon > 0$ there is $\mathbb{Q} \in \mathcal{Q}(\psi)$ such that

$$\left| \frac{d\mathbb{Q}}{d\mathbb{R}} - 1 \right| \leq \epsilon.$$

Analytic maps with values in a Banach space

Let $\mathbf{X}$ be a Banach space and U be an open connected set in $\mathbf{R}^l$. We recall that a map $x \mapsto X(x)$ of U to $\mathbf{X}$ is *analytic* if for every $y \in U$ there exists $\epsilon = \epsilon(y) > 0$ and $(Y_n = Y_n(y))_{n \geq 0}$ in $\mathbf{X}$ such that

$$X(x) = \sum_{n=0}^{\infty} Y_n (x-y)^n, \quad |y - x| < \epsilon$$

where the series converges in $\mathbf{X}$.

Generic property

Theorem
Let U *be an open connected set in* $\mathbf{R}^l$, $x_0 \in \overline{U}$, *and* $x \mapsto \zeta(x)$ *and* $x \mapsto \xi(x)$ *be continuous maps of* $U \cup \{x_0\}$ *to* $\mathcal{L}_1(\mathbf{R})$ *and* $\mathcal{L}_1(\mathbf{R}^d)$, *respectively, whose restrictions to* U *are analytic. For every* $x \in U \cup \{x_0\}$ *we assume that* $\zeta(x) > 0$ *and define a probability measure* $\mathbb{Q}(x)$ *and a* $\mathbb{Q}(x)$*-martingale* $S(x)$ *by*

$$\frac{d\mathbb{Q}(x)}{d\mathbb{P}} = \frac{\zeta(x)}{\mathbb{E}[\zeta(x)]}, \quad S_t(x) = \mathbb{E}_t^{\mathbb{Q}(x)}\left[\frac{\xi(x)}{\zeta(x)}\right].$$

If $S(x_0)$ *has the MRP, then the exception set*

$$\mathcal{I} = \{x \in U : \; S(x) \text{ does not have the MRP}\}$$

has the Lebesgue measure zero. If, in addition, U *is an interval in* $\mathbf{R}$, *then the set* $\mathcal{I}$ *is at most countable*

Summary

- We study the continuous-time version of a price impact model, which goes back to Grossman and Miller (1986); inverse to optimal investment.
- Stock price $S(\gamma)$ depend on demand γ through a solution to a to multi-dimensional quadratic BSDE.
- While exact replication may not be possible, the model has *approximate* and *generic* completeness properties.
- Prices for contingent claims are quite explicit ($=$ utility-based prices).
- The model is supported by general results on the existence of MRP in "backward" setup. Other applications of these results are forthcoming.