Isoperimetric inequalities and cavity interaction

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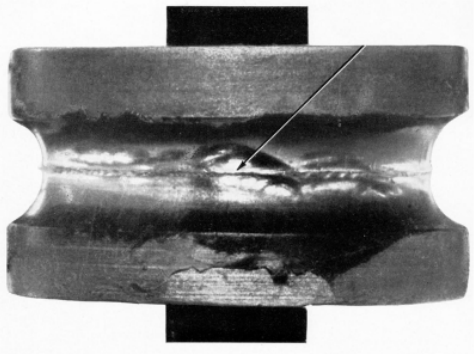
Motivation

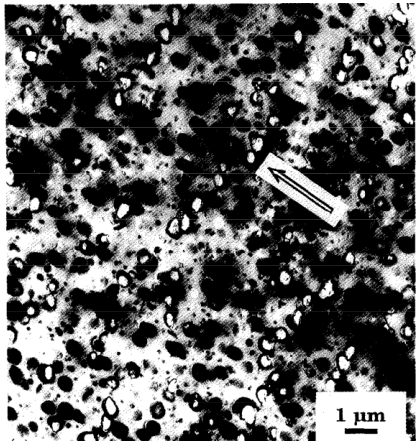
[Gent & Lindley '59]



Internal rupture of rubber under hydrostatic tension

Gent & Lindley '59
Oberth & Bruenner '65
Gent & Park '84
Dorfmann '03
Bayraktar *et al.* '08
Cristiano *et al.* '10

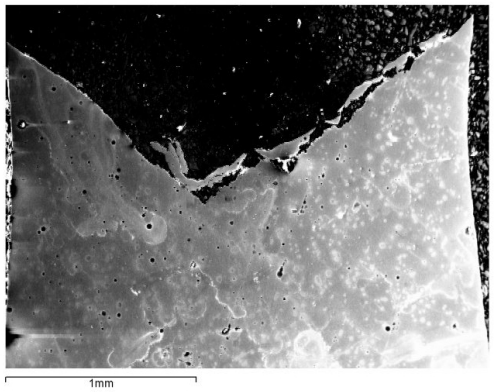
[Lazzeri & Bucknall '95
Dijkstra & Gaymans '93]



Rubber toughening of plastics (polystyrene, ABS, PMMA)

Lazzeri & Bucknall '95
Cheng *et al.* '95
Steenbrink & Van der Giessen '99
Liang & Li '00

[Petrinic *et al.* '06]



Ductile fracture by void growth and coalescence

Goods & Brown '79
Tvergaard '90
Petrinic *et al.* '06

Mathematical modelling (Gent & Lindley '59, Ball '82, Stuart '85, Horgan & Abeyaratne '86, Sivaloganathan '86, Ertan '88, Meynard '92, Horgan '92, . . .)

Stored-energy density $W : M^{n\times n}_1 \to \mathbb{R}$

Incompressible elastic ball

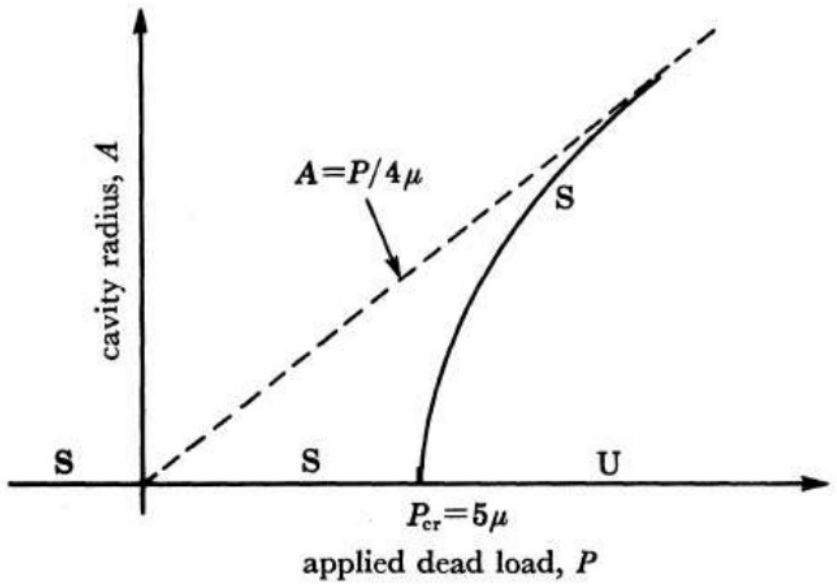
$$\mathbf{T}(\mathbf{x}) = DW\big(D\mathbf{u}\big)\,D\mathbf{u}^T - p(\mathbf{x})\mathbf{1}$$
$$\mathbf{T}_R(\mathbf{x}) = DW\big(D\mathbf{u}\big) - p(\mathbf{x})\,\mathrm{cof}\,D\mathbf{u}(\mathbf{x}),$$
$$\det D\mathbf{u}(\mathbf{x}) = 1 \quad \text{for} \quad \mathbf{x} \in B(\mathbf{0},1)\setminus\{\mathbf{0}\}$$

Radial symmetry: $\mathbf{u}(\mathbf{x}) = u(|\mathbf{x}|)\mathbf{x}/|\mathbf{x}|$

Elastostatics, traction-free cavity surface

$$\mathrm{Div}\,\mathbf{T}_R(\mathbf{x}) = \mathbf{0}, \quad \mathbf{x} \in B(\mathbf{0},1)\setminus\{\mathbf{0}\}$$
$$\mathbf{T}_R(\mathbf{x})\,\boldsymbol{\nu}(\mathbf{x}) = P\boldsymbol{\nu}(\mathbf{x}), \quad \mathbf{x} \in \partial B(\mathbf{0},1)$$
$$\mathbf{T}(\varepsilon\boldsymbol{\theta})\,\boldsymbol{\theta} \overset{\varepsilon\to 0}{\longrightarrow} \mathbf{0}, \quad |\boldsymbol{\theta}| = 1,\ \boldsymbol{\theta} \in \mathbb{R}^n$$

Ball '82



extensions allowing for compressibility, Blatz-Ko and Varga materials, anisotropic loading, material anisotropy, elastodynamics, plasticity, elastic membrane theory, material inhomogeneity, . . .

Mathematical problem

$$\min \int_\Omega |D\mathbf{u}|^p \, \mathrm{d}\mathbf{x}, \quad n-1 < p \leq n$$

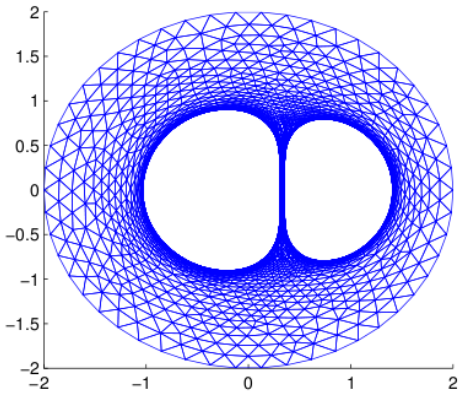
subject to

- $\mathbf{u} : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$
- $\mathbf{u} \in W^{1,p}(\Omega, \mathbb{R}^n)$
- $\det D\mathbf{u} = 1$ a.e.
- $\mathbf{u}(\mathbf{x}) = \lambda \mathbf{x}$ on $\partial\Omega$
- $(\mathrm{Det}\, D\mathbf{u})^s = v_1 \delta_{\mathbf{a}_1} + v_2 \delta_{\mathbf{a}_2}$.

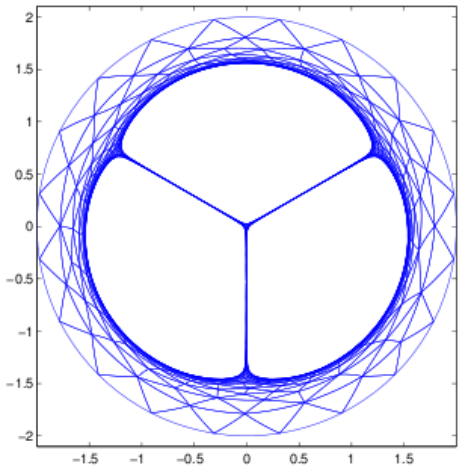
Goal: to determine

- optimal location of singularities
- for $v_1 + v_2$ given, optimal values of v_1 and v_2
- optimal shape of the cavities
- interaction between cavities; dependence on loading conditions, domain geometry, material parameters; void coalescence, alignment of cavities, crack formation

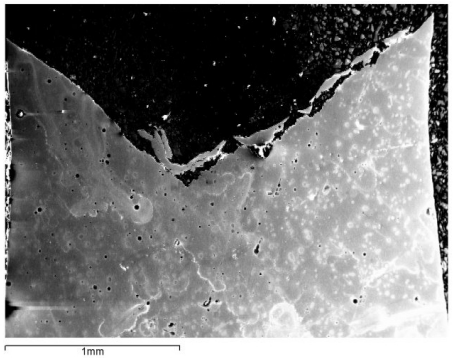
Motivation



[Xu & H. '11]



[Lian & Li, '11]



[Petrinic *et al.* '06]

$$\min \int_{B(\mathbf{0},1)} \left(|D\mathbf{u}|^{\frac{3}{2}} + (\det \mathbf{F} - 1)^2 - c \log \det \mathbf{F} \right) \, \mathrm{d}\mathbf{x}$$

subject to $\mathbf{u}(\mathbf{x}) = \lambda \mathbf{x}$ on $\partial\Omega$
(model of Sivaloganathan & Spector, 2000, 2006)

Motivation (Ball '77, Ball & Murat '84, Müller & Spector '95, Sivaloganathan & Spector '00)

- Weak continuity of $\det D\mathbf{u}$

$$\det D\mathbf{u} = \frac{1}{n}\,\mathrm{Div}\,\big((\mathrm{adj}\,D\mathbf{u})\mathbf{u}\big), \qquad \mathbf{u} \in W^{1,n}$$

- lower semicontinuity of $\int W(D\mathbf{u})$; quasiconvexity
- invertibility
- Ginzburg-Landau (Béthuel, Brezis & Hélein '94, Sandier & Serfaty '07):
  - $|D\mathbf{u}| \sim A/r$ compared to $|D\mathbf{u}| \sim d/r$
  - $(\mathrm{Det}\,D\mathbf{u})^s = \sum v_i \delta_{\mathbf{a}_i}$ compared to $(\mathrm{Det}\,D\mathbf{u})^s = \sum d_i \delta_{\mathbf{a}_i}$.

Main result

- $\Omega_\varepsilon = \Omega \setminus (B_{\mathbf{a}_{1,\varepsilon},\varepsilon}) \cup B_{\mathbf{a}_{2,\varepsilon},\varepsilon}))$
- $(\mathbf{u}_\varepsilon, \mathbf{a}_{1,\varepsilon}, \mathbf{a}_{2,\varepsilon})$ is a minimizer of $\int_{\Omega_\varepsilon} |D\mathbf{u}|^n$ subject to
  - Ω contains a ball $B\left(\frac{\mathbf{a}_{1,\varepsilon}+a_{2,\varepsilon}}{2}, R\right)$, $v_{1,\varepsilon} + v_{2,\varepsilon} < \frac{1}{4^{n+1}n}\omega_n R^n$
  - Det $D\mathbf{u} = \mathcal{L}^n + v_{1,\varepsilon}\delta_{\mathbf{a}_{1,\varepsilon}} + v_{2,\varepsilon}\delta_{\mathbf{a}_{2,\varepsilon}}$ (with $v_{1,\varepsilon}$, $v_{2,\varepsilon}$ fixed)
  - $\mathbf{u}(\mathbf{x}) = \lambda\mathbf{x}$ on $\partial\Omega$, $\lambda^n|\Omega| = |\Omega| + v_{1,\varepsilon} + v_{2,\varepsilon}$
  - condition INV

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$$\pi|\mathbf{a}_2 - \mathbf{a}_1|^2 \geq C_1(v_1+v_2)\exp\left(\frac{-C_2(1+\frac{|\Omega|}{v_1+v_2})}{\left(\frac{v_2}{v_1+v_2}\right)^{\frac{n}{n-1}}}\right)$$

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$$\liminf_{\varepsilon\to 0} \frac{v_1 D(E(\mathbf{a}_{1,\varepsilon},\varepsilon))^{\frac{n}{n-1}} + v_2 D(E(\mathbf{a}_{2,\varepsilon},\varepsilon))^{\frac{n}{n-1}}}{v_1+v_2} > C\left(\frac{v_2}{v_1+v_2}\right)^{\frac{n}{n-1}}$$

Infinite energy

- $\Omega_\varepsilon := \Omega \setminus \bigcup_{i=1}^M B_{\varepsilon_i}(\mathbf{a}_i)$
- $\mathbf{u} \in H^1(\Omega_\varepsilon, \mathbb{R}^2)$, incompressible, $\mathrm{Det}\, D\mathbf{u} = \mathcal{L}^2 + \sum_{i=1}^M (v_i + \pi\varepsilon_i^2)\delta_{\mathbf{a}_i}$

If $B(\mathbf{a}_i, R) \subset \Omega$ then

$$\int_{\Omega_\epsilon} \frac{|D\mathbf{u}|^2 - 1}{2}\, \mathrm{d}\mathbf{x} \geq (v_1 + \cdots + v_M) \log \frac{R}{2(\varepsilon_1 + \cdots + \varepsilon_M)}$$

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Case $p < n$: cavities feel each other much more rapidly (in terms of $\frac{\pi d^2}{v_1 + v_2}$, cf. Ball & Murat '84)

Radial symmetry, isoperimetric inequality

Incompressible neo-Hookean material; $\Omega_\varepsilon := \Omega \setminus \bigcup_{i=1}^{M} B_\varepsilon(\mathbf{a}_i) \subset \mathbb{R}^2$:

$$\int_{\partial B(\mathbf{a},r)} \frac{|D\mathbf{u}|^2}{2}\,\mathrm{d}s \geq \int_{\partial B(\mathbf{a},r)} \frac{|\partial_{\boldsymbol{\tau}}\mathbf{u}|^2}{2}\,\mathrm{d}s$$

$$\geq \frac{1}{4\pi r}\left(\int_{\partial B(\mathbf{a},r)} |\partial_{\boldsymbol{\tau}}\mathbf{u}|\,\mathrm{d}s\right)^2 = \frac{\big(\operatorname{Per} E(\mathbf{a},r)\big)^2}{4\pi r}$$

Isoperimetric inequality: $(\operatorname{Per} E)^2 \geq 4\pi|E|$, implies

$$\int_{\varepsilon<|x|<R} \frac{|D\mathbf{u}|^2 - 1}{2}\,\mathrm{d}x \geq (\text{cavity volume}) \cdot \log\frac{R}{\varepsilon}.$$

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$\Rightarrow$ round cavities (Sivaloganathan & Spector 2010)

Quantitiative isoperimetric inequality

$$(\mathrm{Per}\, E)^2 \geq 4\pi |E|$$

Quantitiative isoperimetric inequality

$$(\text{Per}\, E)^2 \geq 4\pi|E|\big(1 + CD(E)^2\big)$$

where

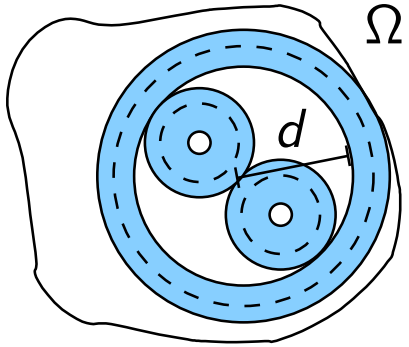
$$D(E) := \min\left\{\frac{|E\triangle B|}{|E|} \,:\, B \text{ ball}, |B| = |E|\right\}, \quad \text{(Fraenkel asymmetry of } E\text{)}$$

which depends on the shape of E only (not on its size).

Bernstein 1905; Bonnesen '24; Fuglede '89; Hall, Hayman & Weitsman '91; Hall '92; Fusco, Maggi & Pratelli '08

Two cavities

$$\int_{\Omega_\varepsilon \cap B_R} \frac{|D\mathbf{u}|^2 - 1}{2}\,\mathrm{d}\mathbf{x} \geq (v_1 + v_2)\log\frac{R}{2\epsilon}$$
$$+ C\int_{\varepsilon}^{d/2}\left(v_1 D(E(\mathbf{a}_1, r))^2 + v_2 D(E(\mathbf{a}_2, r))^2\right)\frac{\mathrm{d}r}{r}$$
$$+ C(v_1 + v_2)\int_{d}^{R} D(E(\mathbf{a}, r))^2\frac{\mathrm{d}r}{r}$$



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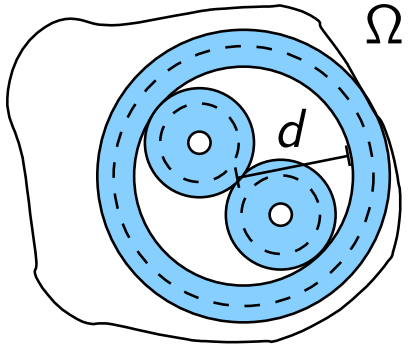

Energy is minimized if 'circles go to circles'

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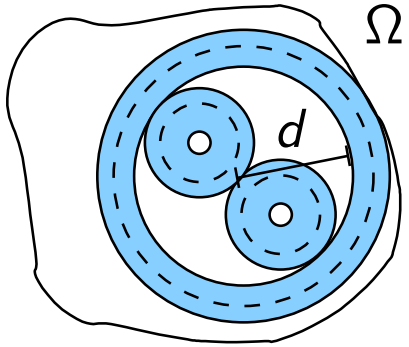
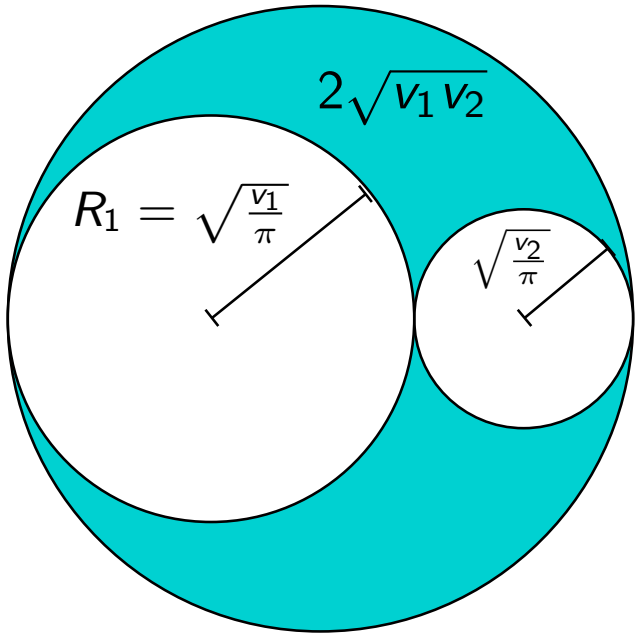


Energy is minimized if 'circles go to circles'

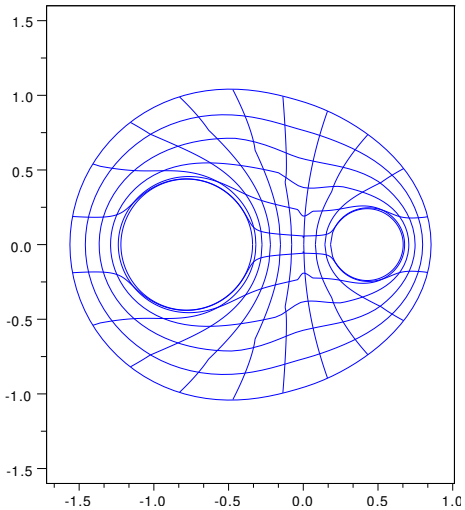
This is not always possible:

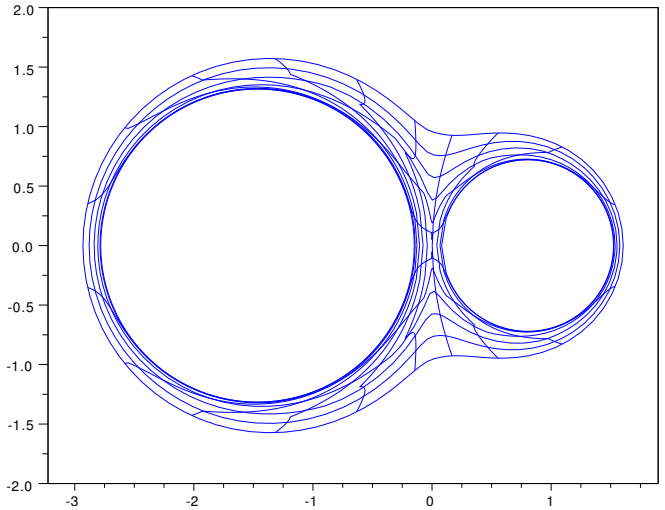
$$\begin{aligned}&\pi(R_1 + R_2)^2 - (\pi R_1^2 + \pi R_2^2)\\ &= (\sqrt{v_1} + \sqrt{v_2})^2 - v_1 - v_2\\ &= 2\sqrt{v_1 v_2}\end{aligned}$$

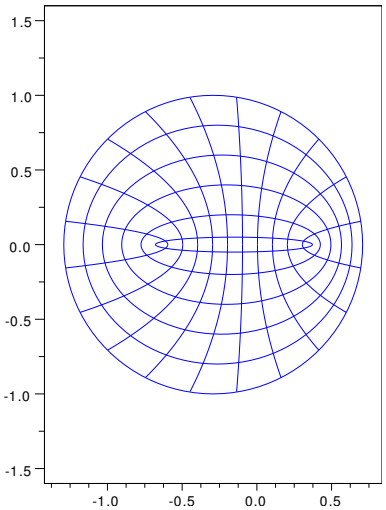
Possible only if $\boldsymbol{\pi d^2 > 2\sqrt{v_1 v_2}}$

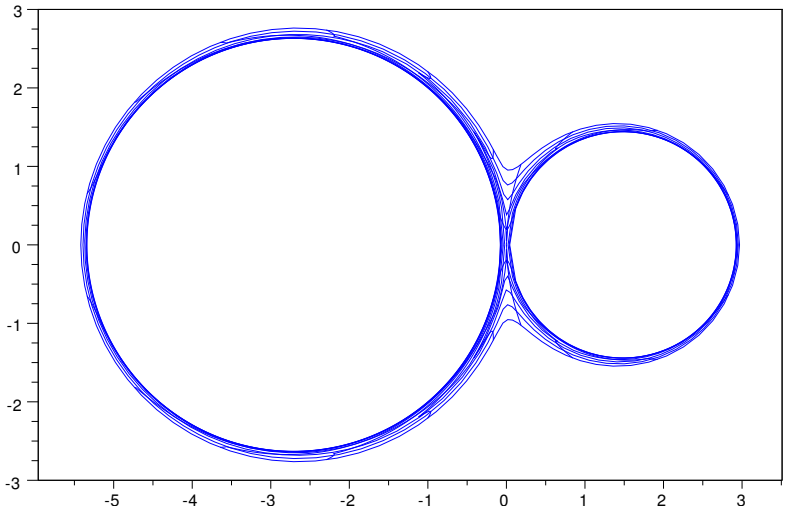


Circular cavities ($\pi d^2 \geq 2\sqrt{v_1 v_2}$)

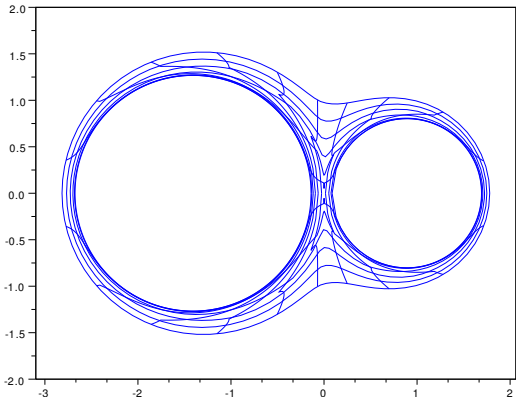




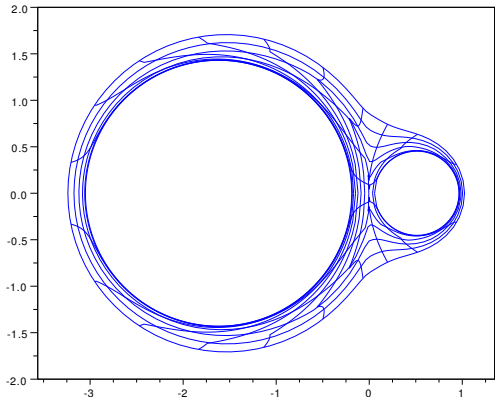




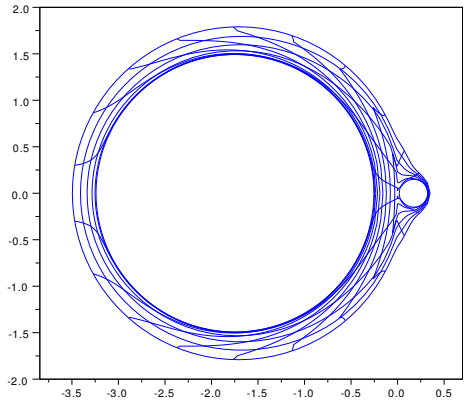
Vanishing volume ratio ($\pi d^2 > 2\sqrt{v_1 v_2}$)



$v_1 = 2.5 v_2$

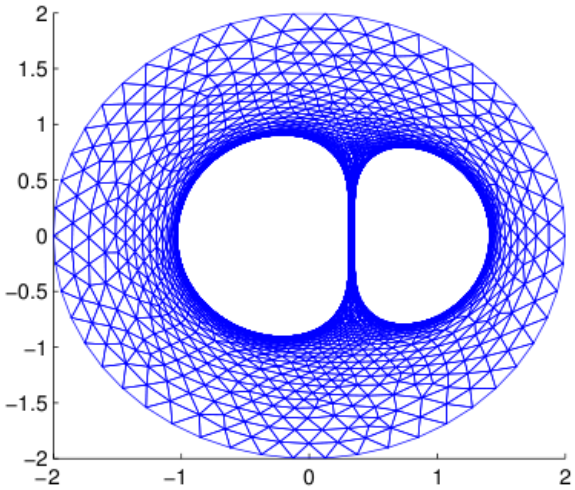


$v_1 = 10 v_2$

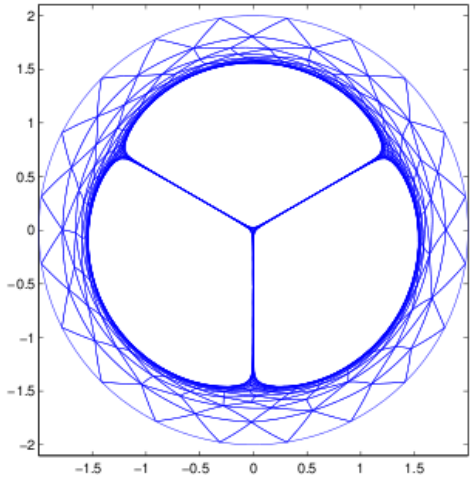


$v_1 = 100 v_2$

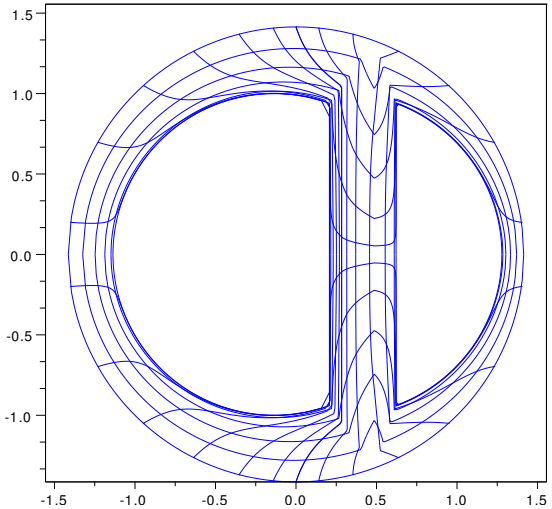
Distorted cavities



Xu & H. '11



Lian & Li '11



H. & Serfaty '11

Estimate on the distortions

$$\int_{\Omega_\varepsilon \cap B_R} \frac{|D\mathbf{u}|^2 - 1}{2}\,\mathrm{d}\mathbf{x} - (v_1+v_2)\log\frac{R}{2\epsilon}$$

$$\geq C\int_\varepsilon^{d/2}\left(|E_{\mathbf{a}_1,r}|D(E_{\mathbf{a}_1,r})^2 + |E_{\mathbf{a}_2,r}|D(E_{\mathbf{a}_2,r})^2\right)\frac{\mathrm{d}r}{r} + C\int_d^R |E_{\mathbf{a},r}|D(E_{\mathbf{a},r})^2\frac{\mathrm{d}r}{r}$$

$$\geq C\min_{\substack{\varepsilon<r<d/2\\ d<r'<R}}\left(|E_{\mathbf{a}_1,r}|D(E_{\mathbf{a}_1,r})^2 + |E_{\mathbf{a}_2,r}|D(E_{\mathbf{a}_2,r})^2 + |E_{\mathbf{a},r'}|D(E_{\mathbf{a},r'})^2\right)\min\left\{\log\frac{d}{2\varepsilon}, \log\frac{R}{d}\right\}$$

Proposition: $E_1 \cup E_2 \subset E$, $E_1 \cap E_2 = \varnothing$, $|E_1| \geq |E_2|$. Then

$$\frac{|E|D(E)^2 + |E_1|D(E_1)^2 + |E_2|D(E_2)^2}{|E| + |E_1 \cup E_2|} \geq C\left(\frac{|E_2|}{|E_1|+|E_2|}\right)^2\left(1 - \frac{|E\setminus(E_1\cup E_2)|}{2\sqrt{|E_1||E_2|}}\right)^3$$

Lower bound

Theorem: $\Omega_\varepsilon := \Omega \setminus (\overline{B}_\varepsilon(\mathbf{a}_1) \cup \overline{B}_\varepsilon(\mathbf{a}_2))$, $\mathbf{u} \in H^1(\Omega_\varepsilon, \mathbb{R}^2)$ incompressible

$$\int_{\Omega_\varepsilon} \frac{|D\mathbf{u}|^2 - 1}{2} \geq (v_1 + v_2) \log \frac{R}{2\varepsilon}$$

$$+ C(v_1 + v_2) \left(\frac{\min\{v_1^2, v_2^2\}}{(v_1 + v_2)^2} - \frac{\pi d^2}{v_1 + v_2} \right) \log \left(\min \left\{ \sqrt[4]{\frac{v_1 + v_2}{4\pi d^2}}, \frac{R}{d}, \frac{d}{\varepsilon} \right\} \right)$$

provided $B(\frac{\mathbf{a}_1 + \mathbf{a}_2}{2}, R) \subset \Omega$

Upper bound

Theorem: $\mathbf{a}_1, \mathbf{a}_2 \in \mathbb{R}^2$, $v_1 \geq v_2$. For all $\delta \in [0,1]$ there exists $\mathbf{a}^* \in [\mathbf{a}_1, \mathbf{a}_2]$ and a piecewise smooth homeomorphism $\mathbf{u} : \mathbb{R}^2 \setminus \{\mathbf{a}_1, \mathbf{a}_2\} \to \mathbb{R}^2$ such that $\operatorname{Det} D\mathbf{u} = 1 \cdot \mathcal{L}^2 + v_1 \delta_{\mathbf{a}_1} + v_2 \delta_{\mathbf{a}_2}$ and for all $R > 0$

$$\int_{B(\mathbf{a}^*,R)\setminus(B_\varepsilon(\mathbf{a}_1)\cup B_\varepsilon(\mathbf{a}_2))} \frac{|D\mathbf{u}|^2}{2} \leq C(v_1+v_2+\pi R^2) + (v_1+v_2)\log\frac{R}{\varepsilon}$$
$$+ C(v_1+v_2)\left((1-\delta)\left(\log\frac{R}{d}\right)_+ + \delta\sqrt[4]{\frac{v_2}{v_1+v_2}}\log\frac{d}{\varepsilon}\right)$$

Upper bound

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$$\int_{B(\mathbf{a}^*,R)\setminus(B_\varepsilon(\mathbf{a}_1)\cup B_\varepsilon(\mathbf{a}_2))} \frac{|D\mathbf{u}|^2}{2} \leq C(v_1+v_2+\pi R^2) + (v_1+v_2)\log\frac{R}{\varepsilon}$$
$$+ C(v_1+v_2)\left((1-\delta)\left(\log\frac{R}{d}\right)_+ + \delta\sqrt[4]{\frac{v_2}{v_1+v_2}}\log\frac{d}{\varepsilon}\right)$$

Terms in lower bound: $C(v_1+v_2)\left(\frac{\min\{v_1^2,v_2^2\}}{(v_1+v_2)^2} - \frac{\pi d^2}{v_1+v_2}\right)\log\left(\min\left\{\sqrt[4]{\frac{v_1+v_2}{4\pi d^2}}, \frac{R}{d}, \frac{d}{\varepsilon}\right\}\right)$

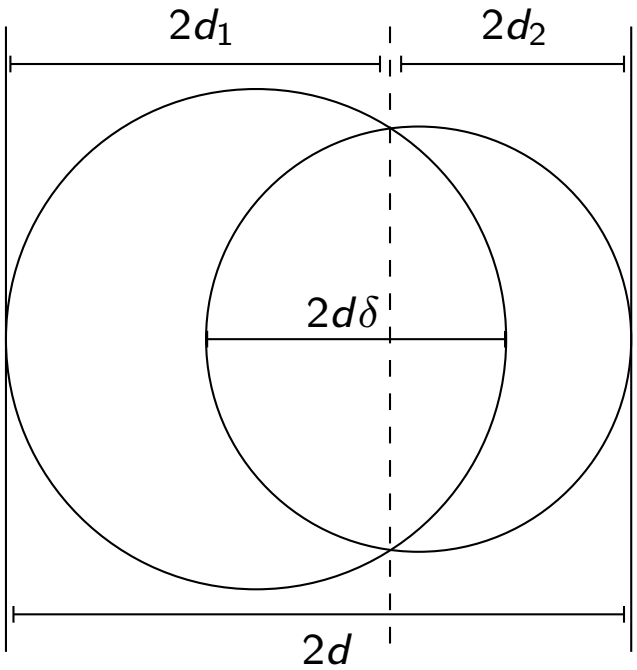
Other difficulties

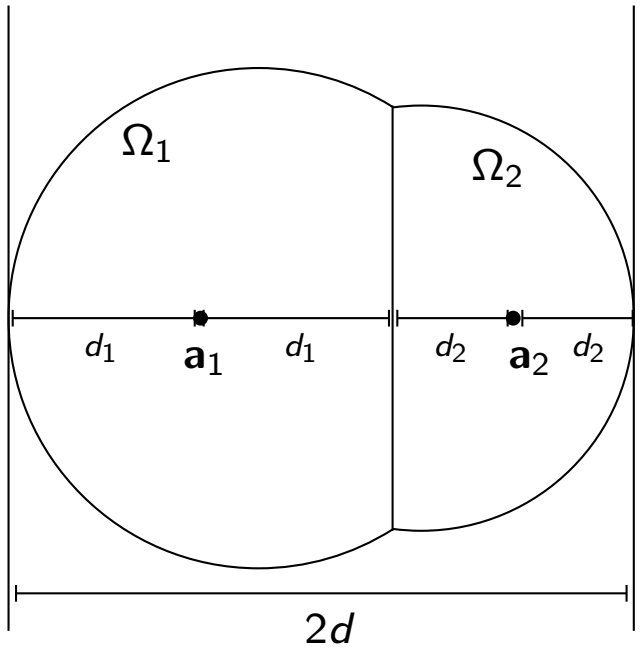
- Explicit test maps (angle-preserving)
- Dirichlet conditions (Dacorogna-Moser flow; Rivière-Ye)
- Compactness (Struwe '94)

Summarizing

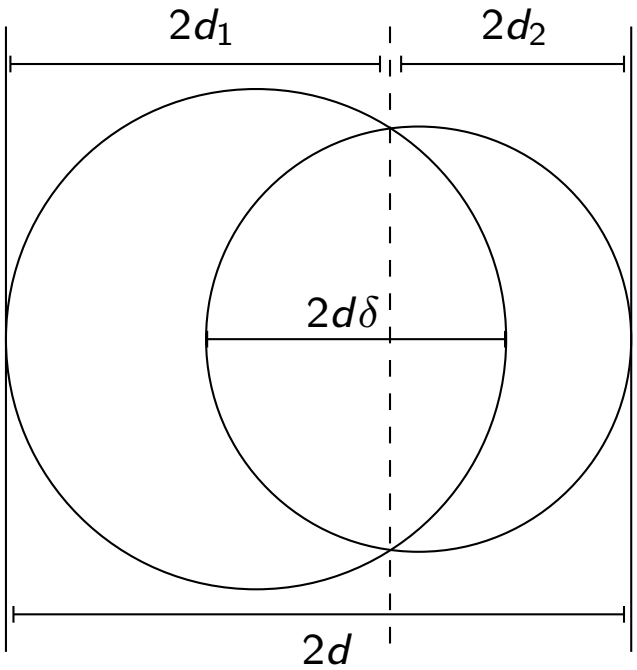
- Connection between cavitation and Ginzburg-Landau theory
- Role of isoperimetric inequalities in elasticity (c.f. Müller '90)
- Relation between quantities in the reference and deformed configuration (c.f. Ball & Murat '84; surface energy)
- Repulsion effect, role of incompressibility
- Void coalescence

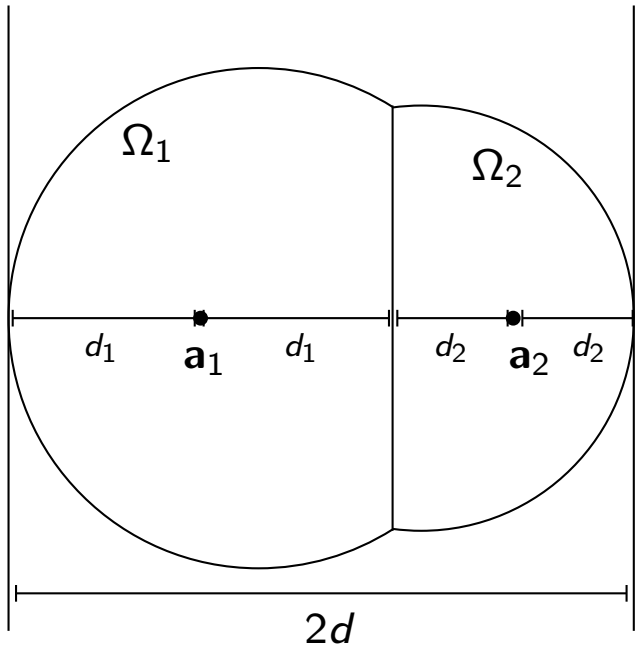
Geometric construction





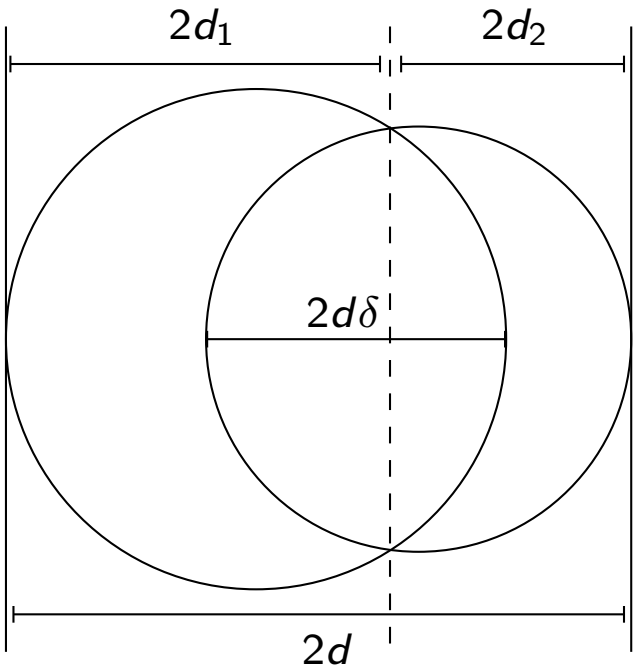
Geometric construction

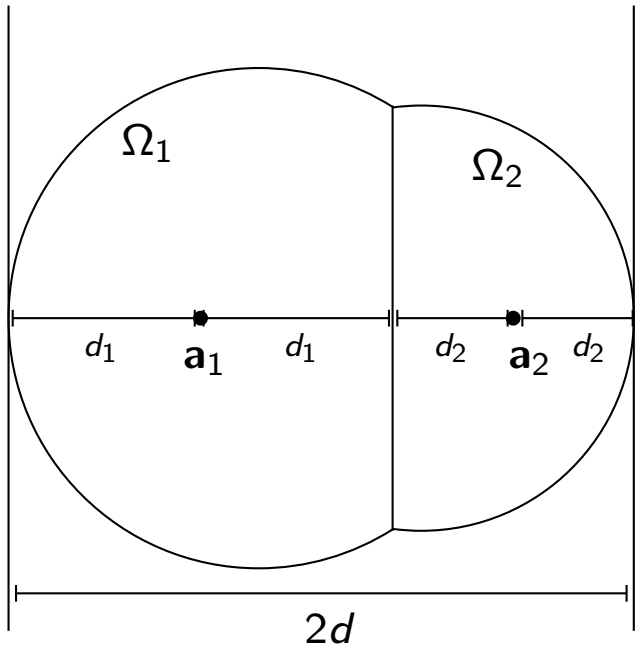




Ratio $\dfrac{|\Omega_1|}{|\Omega_2|} = \dfrac{v_1}{v_2}$

Geometric construction

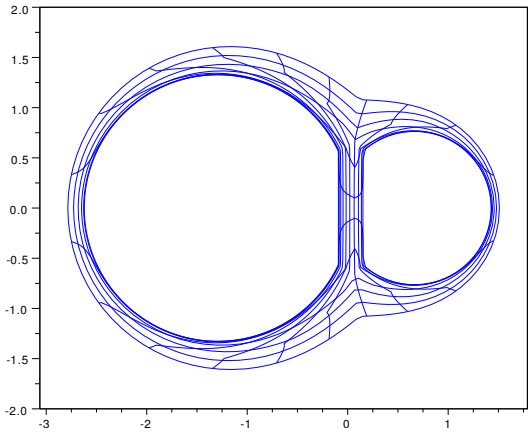




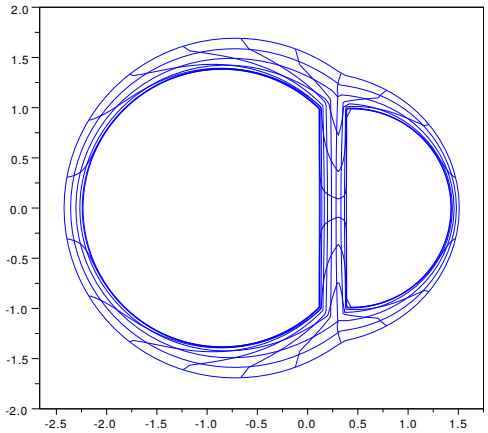
Ratio $\dfrac{|\Omega_1|}{|\Omega_2|} = \dfrac{v_1}{v_2}$; $\qquad \mathbf{u}(\mathbf{x}) \equiv \lambda \mathbf{x}$ on $\partial\Omega_1 \cup \partial\Omega_2$,

$$\lambda^2 - 1 := \frac{v_1 + v_2}{|\Omega_1 \cup \Omega_2|} = \frac{v_1}{|\Omega_1|} = \frac{v_2}{|\Omega_2|}$$

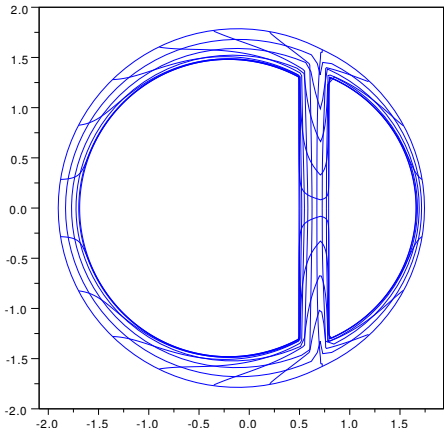
Cavity shapes



$\delta = 0.1$

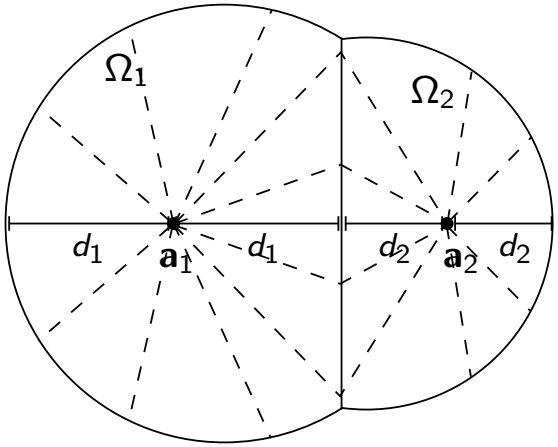


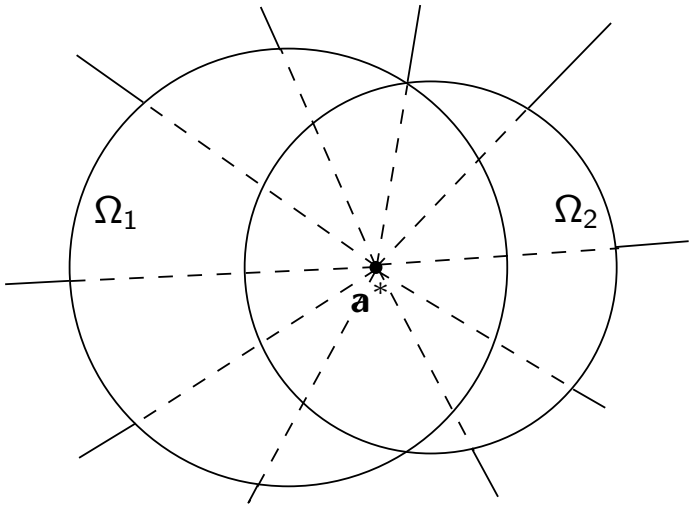
$\delta = 0.4$



$\delta = 0.9$

Angle-preserving maps



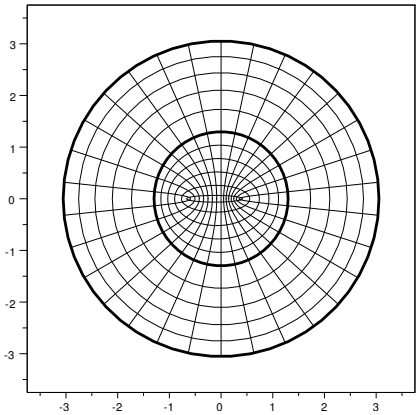


$$\mathbf{u}(\mathbf{x}) = \lambda \mathbf{a}^* + f(\mathbf{x}) \frac{\mathbf{x} - \mathbf{a}^*}{|\mathbf{x} - \mathbf{a}^*|}, \qquad \lambda^n - 1 := \frac{v_1 + v_2}{|\Omega_1 \cup \Omega_2|} = \frac{v_1}{|\Omega_1|} = \frac{v_2}{|\Omega_2|}.$$

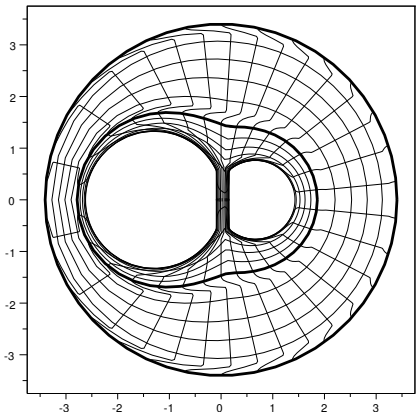
$$\det D\mathbf{u}(\mathbf{x}) = \frac{f^{n-1}(\mathbf{x}) \frac{\partial f}{\partial r}(\mathbf{x})}{r^{n-1}} \equiv 1 \;\Leftrightarrow\; f^n(\mathbf{x}) = |\mathbf{x} - \mathbf{a}^*|^n + A \left(\frac{\mathbf{x} - \mathbf{a}^*}{|\mathbf{x} - \mathbf{a}^*|} \right)^n$$

Dirichlet conditions

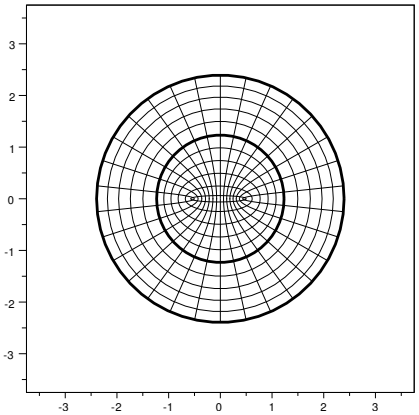
$\delta = 0.1$



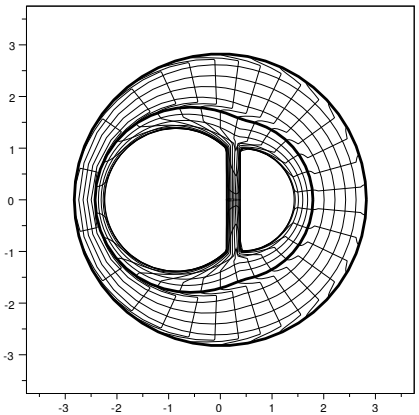
$\pi(R_2^2 - R_1^2) = 3.06(v_1+v_2)(1-\delta)$



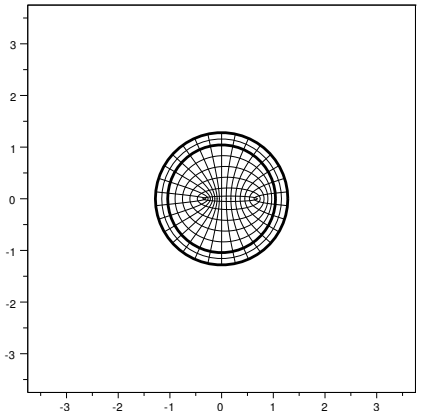
$\delta = 0.4$



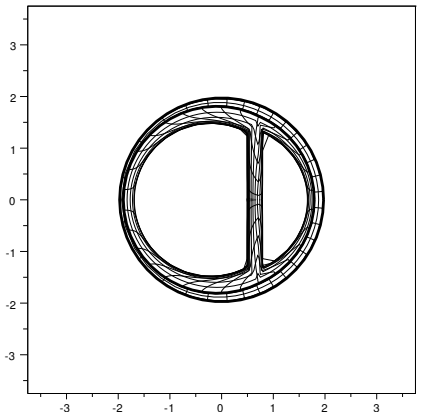
$\pi(R_2^2 - R_1^2) = 3.12(v_1+v_2)(1-\delta)$



$\delta = 0.4$



$\pi(R_2^2 - R_1^2) = 2.46(v_1+v_2)(1-\delta)$



Necessary condition: $\pi(R_2^2 - R_1^2) \geq \frac{2\pi-3}{18\pi}(1-\delta)(v_1+v_2)$.

(Parameters: $\sqrt{(v_1+v_2)/(\pi d^2)} = 1.5$, $v_2/v_1 = 0.3$, $d = 1$, $R_1 \approx d$.)

Dirichlet conditions

Theorem: Suppose $\pi R_1^2, \pi(R_2^2 - R_1^2) \geq C(v_1+v_2)(1-\delta)$; $R_1 \geq 2d$. Then

$$\int_{B(\mathbf{a}^*,R)\setminus(B_\varepsilon(\mathbf{a}_1)\cup B_\varepsilon(\mathbf{a}_2))} \frac{|D\mathbf{u}|^2}{2} \leq C(v_1+v_2+\pi R^2) + (v_1+v_2)\log\frac{R}{\varepsilon}$$
$$+ C(v_1+v_2)\left((1-\delta)\left(\log\frac{(v_1+v_2)(1-\delta)}{\pi d^2}\right)_+ + \delta\sqrt[4]{\frac{v_2}{v_1+v_2}}\log\frac{d}{\varepsilon}\right),$$

with $\mathbf{u}|_{\partial B(\mathbf{a}^*,R_2)}$ radially symmetric.

Dirichlet conditions

Theorem: Suppose $\pi R_1^2, \pi(R_2^2 - R_1^2) \geq C(v_1+v_2)(1-\delta)$; $R_1 \geq 2d$. Then

$$\int_{B(\mathbf{a}^*,R)\setminus(B_\varepsilon(\mathbf{a}_1)\cup B_\varepsilon(\mathbf{a}_2))} \frac{|D\mathbf{u}|^2}{2} \leq C(v_1+v_2+\pi R^2) + (v_1+v_2)\log\frac{R}{\varepsilon}$$
$$+ C(v_1+v_2)\left((1-\delta)\left(\log\frac{(v_1+v_2)(1-\delta)}{\pi d^2}\right)_+ + \delta\sqrt[4]{\frac{v_2}{v_1+v_2}}\log\frac{d}{\varepsilon}\right),$$

with $\mathbf{u}|_{\partial B(\mathbf{a}^*,R_2)}$ radially symmetric.

Previous upper bound: $C(v_1+v_2)\left((1-\delta)\left(\log\frac{\pi R^2}{\pi d^2}\right)_+ + \delta\sqrt[4]{\frac{v_2}{v_1+v_2}}\log\frac{d}{\varepsilon}\right)$

Dirichlet conditions

Corollary: $\Omega = B_R$, $R \geq 2d$. For every $v_1 \geq v_2$ there exist $\mathbf{a}_1, \mathbf{a}_2 \in \Omega$ with $|\mathbf{a}_2 - \mathbf{a}_1| = d$ and a Lipschitz homeomorphism $\mathbf{u} : \Omega \setminus \{\mathbf{a}_1, \mathbf{a}_2\} \to \mathbb{R}^2$ such that $\operatorname{Det} D\mathbf{u} = \mathcal{L}^2 + v_1\delta_{\mathbf{a}_1} + v_2\delta_{\mathbf{a}_2}$, $\mathbf{u}|_{\partial\Omega} \equiv \lambda \mathbf{id}$, and

$$\int_{B(\mathbf{a}^*,R)\setminus(B_\varepsilon(\mathbf{a}_1)\cup B_\varepsilon(\mathbf{a}_2))} \frac{|D\mathbf{u}|^2}{2} \leq C(v_1+v_2+\pi R^2) + (v_1+v_2)\log\frac{R}{\varepsilon}$$

$$+ C(v_1+v_2)\min_{\delta\in[\delta_0,1]}\left((1-\delta)\left(\log\frac{(v_1+v_2)(1-\delta)}{\pi d^2}\right)_+ + \delta\sqrt[4]{\frac{v_2}{v_1+v_2}}\log\frac{d}{\varepsilon}\right),$$

with $\delta_0 := \max\{0, 1 - \frac{|\Omega|-4\pi d^2}{C\pi d^2}\}$.

Compactness ($\varepsilon \to 0$)

Theorem: $\Omega_\varepsilon = \Omega \setminus (B_\varepsilon(\mathbf{a}_{1,\varepsilon}) \cup B_\varepsilon(\mathbf{a}_{2,\varepsilon}))$.

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$$\int_{\Omega_\varepsilon} \frac{|D\mathbf{u}|^2}{2} \leq (v_{1,\varepsilon} + v_{2,\varepsilon}) \log \frac{\operatorname{diam} \Omega}{\varepsilon} + C(|\Omega| + v_{1,\varepsilon} + v_{2,\varepsilon}).$$

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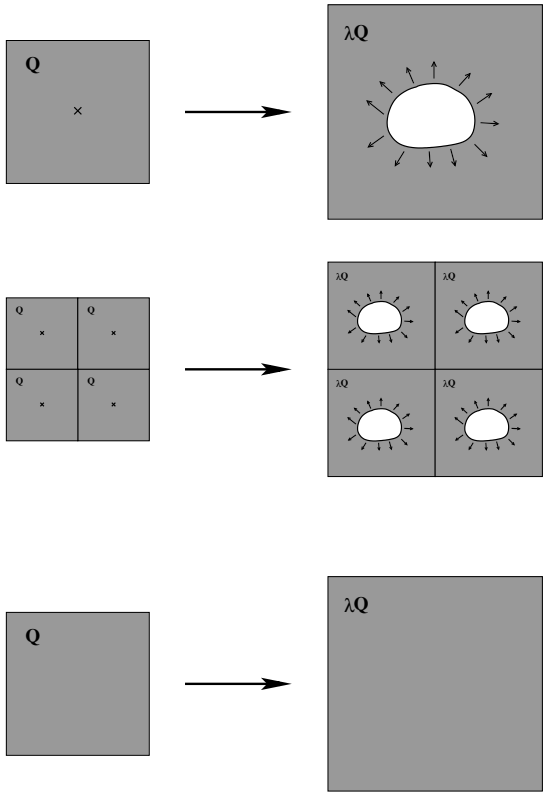
Then there exists $\mathbf{u} \in \cap_{1 \leq p < 2} W^{1,p}(\Omega) \cap H^1_{\text{loc}}(\Omega)$ and convergent subsequences.

Scale invariance in elasticity (Ball & Murat, 1984):

The condition $\pi d^2 > 2\sqrt{v_1 v_2}$ is to be compared with:



Related works: Ortiz & Reina (2010), Lopez-Pamies, Idiart & Nakamura (2011)