Global generalized solutions for Maxwell-alpha and Euler-alpha equations

Dmitry Vorotnikov

Universidade de Coimbra

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Overview

Recently, so-called alpha-models of fluid mechanics have attracted attention of many researchers. These models turned out to be rather ubiquitous, being relevant in such issues as turbulence and large eddy simulations, and, on the other hand, being related to the second grade fluids. We study initial-boundary value problems for the Lagrangian averaged alpha models for the equations of motion for the corotational Maxwell and inviscid fluids in 2D and 3D. We show existence of (global in time) dissipative solutions to these problems.

Euler-alpha

$$\frac{\partial v}{\partial t} + \sum_{i=1}^{n} u_i \frac{\partial v}{\partial x_i} + \sum_{i=1}^{n} v_i \nabla u_i + \nabla p = 0, \tag{1}$$

$$v = u - \alpha^2 \Delta u, \tag{2}$$

$$\operatorname{div} u = 0, \tag{3}$$

$$u\Big|_{\partial\Omega} = 0, \tag{4}$$

$$u|_{t=0} = a. \tag{5}$$

Corotational Maxwell-alpha

$$\frac{\partial v}{\partial t} + \sum_{i=1}^{n} u_i \frac{\partial v}{\partial x_i} + \sum_{i=1}^{n} v_i \nabla u_i + \nabla p = \operatorname{Div} \sigma, \tag{6}$$

$$\sigma + \lambda \left(\frac{\partial \sigma}{\partial t} + \sum_{i=1}^{n} u_i \frac{\partial \sigma}{\partial x_i} + \sigma W - W \sigma \right) = 2\eta \mathcal{E}, \tag{7}$$

$$v = u - \alpha^2 \Delta u, \tag{8}$$

$$\operatorname{div} u = 0, \tag{9}$$

$$u\Big|_{\partial\Omega} = 0, \tag{10}$$

$$u|_{t=0} = a, \ \ \sigma|_{t=0} = \sigma_0. \tag{11}$$

Comments

Here, Ω is a domain in the space $\mathbb{R}^n$, $n = 2, 3$, u is an unknown velocity vector, p is an unknown modified pressure, σ is an unknown deviatoric stress tensor, v is an auxiliary variable (all of them depend on points x in the domain Ω, and on time t);

$$\mathcal{E} = \mathcal{E}(u) = (\mathcal{E}_{ij}(u)),\ \mathcal{E}_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right),$$

is the strain velocity tensor,

$$W = W(u) = (W_{ij}(u)),\ W_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right),$$

is the vorticity tensor, $\eta > 0$ is the viscosity of the fluid, $\lambda > 0$ is the relaxation time, $\alpha = 0$ (Euler) or > 0 (Maxwell) is a scalar parameter, a and σ_0 are given functions. The external force is, for simplicity, assumed to be zero.

Dissipative solutions

- *introduced for 3D Euler equations (P.-L. Lions, 1996)*
- *Boltzmann's equation (Lions, 1994, Gamba–Panferov–Villani, 2009)*
- *ideal MHD (Wu, 2000)*
- *viscoelastic diffusion in polymers (Vorotnikov, 2008)*
- *Navier-Stokes-Maxwell eqs. (Arsénio–Saint-Raymond, announced in 2011)*
- *image denoising (Surya Prasath–Vorotnikov, announced in 2011)*

A Gronwall-like lemma

Let $f, \chi, L, M : [0, T] \to \mathbb{R}$ be scalar functions, $\chi, L, M \in L_1(0, T)$, and $f \in W_1^1(0, T)$ (i.e. f is absolutely continuous). If

$$\chi(t) \geq 0, L(t) \geq 0 \tag{12}$$

and

$$f'(t) + \chi(t) \leq L(t)f(t) + M(t) \tag{13}$$

for almost all $t \in (0, T)$, then

$$f(t) + \int_0^t \chi(s)\, ds \leq$$

$$\exp\left(\int_0^t L(s)ds\right)\left[f(0) + \int_0^t \exp\left(\int_s^0 L(\xi)d\xi\right) M(s)\, ds\right] \tag{14}$$

for all $t \in [0, T]$.

Some notation

Let $\mathcal{V}$ be the set of smooth, divergence-free, compactly supported in Ω functions with values in $\mathbb{R}^n$. The symbols H and V denote the closures of $\mathcal{V}$ in L_2 and H^1, respectively. We also use the spaces $V_i = V \cap H^i$, $i = 2, 3$, with the scalar product inherited from H^i. In the space V, we use the scalar product $(u, v)_V = (u, v) + (\alpha \nabla u, \alpha \nabla v)$, and the corresponding Euclidean norm $\|\cdot\|_V$. Let

$$P : L_2(\Omega, \mathbb{R}^3) \to H \text{ be the Leray projection,}$$

$$\mu = \eta/\lambda,$$

$$\Delta_\alpha = I - \alpha^2 \Delta,$$

$$E_1(w, \tau) = -\frac{\partial \Delta_\alpha w}{\partial t} - P \sum_{i=1}^{n} w_i \frac{\partial \Delta_\alpha w}{\partial x_i} - P \sum_{i=1}^{n} (\Delta_\alpha w)_i \nabla w_i + P \operatorname{Div} \tau,$$

$$E_2(w, \tau) = -\frac{\tau}{\lambda} - \frac{\partial \tau}{\partial t} - \sum_{i=1}^{n} w_i \frac{\partial \tau}{\partial x_i} - \tau W(w) + W(w)\tau + 2\mu \mathcal{E}(w).$$

Dissipative solution for Euler-alpha

A function $u \in C_w([0,\infty);V)$ is called a dissipative solution to problem (1) – (5) if, for all test functions $\zeta \in C^1([0,\infty);V_3)$, and all non-negative moments of time t, one has

$$\|u(t)-\zeta(t)\|_V^2 \leq \exp\left(\int_0^t \Gamma(s)ds\right)\Big\{\|a-\zeta(0)\|_V^2$$

$$+2\int_0^t \exp\left(\int_s^0 \Gamma(\xi)d\xi\right)\left(E_1(\zeta,0)(s), u(s)-\zeta(s)\right)ds\Big\} \tag{15}$$

where

$$\Gamma(t)=\gamma\max\{1,1/\alpha^2\}\left(\|\Delta_\alpha\zeta(t)\|_1+\|\zeta(t)\|_1+\alpha^2\|\zeta(t)\|_3\right),$$

and γ is a certain constant depending only on Ω.

Dissipative solution for Maxwell-alpha

Let $a \in V$, $\sigma_0 \in L_2(\Omega, \mathbb{R}_S^{n\times n})$. A pair of functions (u, σ) from the class

$$u \in C_w([0,\infty); V), \ \ \sigma \in C_w([0,\infty); L_2(\Omega, \mathbb{R}_S^{n\times n})), \tag{16}$$

is called a *dissipative* solution to problem (6) – (11) if, for all test functions $\zeta \in C^1([0,\infty); V_3)$, $\theta \in C^1([0,\infty); H^2(\Omega, \mathbb{R}_S^{n\times n}))$ and all non-negative moments of time t, one has

$$2\mu\|u(t)-\zeta(t)\|_V^2 + \|\sigma(t)-\theta(t)\|^2 \leq \exp\left(\int_0^t \Gamma(s)ds\right)\Big\{2\mu\|a-\zeta(0)\|_V^2 + \|\sigma_0-\theta(0)\|^2$$

$$+\int_0^t \exp\left(\int_s^0 \Gamma(\xi)d\xi\right)\left[4\mu\big(E_1(\zeta,\theta)(s), u(s)-\zeta(s)\big) + 2\big(E_2(\zeta,\theta)(s), \sigma(s)-\theta(s)\big)\right] ds\Big\} \tag{17}$$

where

$$\Gamma(t) = \gamma \max\{1, 1/\alpha^2\}\left(\|\Delta_\alpha\zeta(t)\|_1 + \|\zeta(t)\|_1 + \alpha^2\|\zeta(t)\|_3 + \frac{(1+\mu)\|\theta(t)\|_2}{\mu}\right), \tag{18}$$

and γ is a certain constant depending only on the properties of the domain Ω.

Straightforward properties

The dissipative solutions satisfy the initial condition (11). Indeed, at $t = 0$, inequality (17) becomes

$$2\mu\|u(0)-\zeta(0)\|_V^2+\|\sigma(0)-\theta(0)\|^2 \leq 2\mu\|a-\zeta(0)\|_V^2+\|\sigma_0-\theta(0)\|^2 \tag{19}$$

This easily yields (11). Moreover, these solutions obey the following "dissipative" estimate

$$2\mu\|u(t)\|_V^2 + \|\sigma(t)\|^2 \leq 2\mu\|u(0)\|_V^2 + \|\sigma(0)\|^2,\ \forall t > 0. \tag{20}$$

It follows from (11) and (17) with identically zero ζ and θ. In the Euler-alpha case, (20) reads as

$$\|u(t)\|_V^2 \leq \|u(0)\|_V^2,\ \forall t > 0. \tag{21}$$

The main result

Let Ω be a bounded domain having the cone property.

a) Given $a \in V$, $\sigma_0 \in L_2$, there is a dissipative solution to problem (6) – (11).

b) If, for some $a \in V$, $\sigma_0 \in L_2$, there exist $T > 0$ and a strong solution $(u_T, \sigma_T) \in C^1([0,T]; V_3) \times C^1([0,T]; H^2)$ to problem (6) – (11), then the restriction of any dissipative solution (with the same initial data) to $[0,T]$ coincides with (u_T, σ_T).

c) Every strong solution $(u, \sigma) \in C^1([0,\infty); V_3) \times C^1([0,\infty); H^2)$ is a (unique) dissipative solution.

Thank you!!!