**The Kepler Problem in hyperbolic space**

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- Step1: deduction of the potential: Let $U(r,\theta,\varphi)$ be the potential of the Kepler problem. For $U$ to be physically realistic, it must satisfy:
  - *i)* $U$ is a function of $r$ in normal coordinates $U = U(r)$
  - *ii)* The gravitational flow through spheres is constant

$$\int_{S_r} \langle \nabla U \cdot \mathbf{n} \rangle dA_r = cte. \tag{1}$$

- Using the hypotheses,

$$U(r) = \int \frac{c}{4\pi \sinh^2(r)} dr = k \coth(r). \tag{2}$$

- Step 2: The movement of a particle with initial conditions $x_o$, $\dot{x}_0$ is contained in a submanifold isometric to $\mathbf{H}^2$.
- Final Statement of the problem: The potential $U(r) = k \coth(r)$ in $\mathbf{H}^2$. The corresponding Lagrangian is

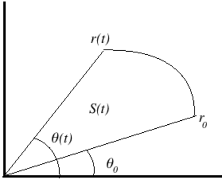
$$L(r,\dot{r},\theta,\dot{\theta}) = \frac{1}{2}(\dot{r}^2 + \sinh^2(r)\dot{\theta}^2) + k\coth(r). \tag{3}$$

**The Kepler laws**

- First law: bounded orbits are ellipses with the origin in one of the focci.
- Second law: Throughout the solutions, $\frac{dA}{dt} = cte$ where

$$A(t) = \int_{S(t)} d\mu,$$

and $d\mu = \cosh(r) d\mathcal{A}$




- Third Law: $T = f(a)$,

$$T = \frac{1}{\sqrt{k}}\pi \left( \frac{1}{\sqrt{\frac{2}{x} - 2}} - \frac{1}{\sqrt{\frac{2}{x} + 2}} \right), \tag{4}$$

where $x = \tanh(2a)$,

the variable $a$ is the major semi axis.