HIGHER ORDER AMBROSIO-TORTORELLI SCHEME WITH NON-NEGATIVE SPATIALLY DEPENDENT PARAMETERS

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Abstract. The Ambrosio-Tortorelli approximation scheme with weighted underlying metric is investigated. It is shown that it Γ-converges to a Mumford-Shah image segmentation functional depending on the weight $\omega\,dx$, where ω is a special function of bounded variation, and on its values at the jumps.

1. Introduction and Main Results

One of the most succesful methods for image denoising involves minimizing an energy of the form

$$MS_\alpha(u) + \|u - u_0\|^2_{L^2(\Omega)}\,,$$

where Ω is a given domain, u_0 is a (given) corrupted image, the argument of the minimization $u \in SBV(\Omega)$ is a *special function of bounded variation*, encoding an image, with its jump set S_u representing the edges of such image. The functional MS_α is the so-called *Mumford-Shah image segmentation functional*, defined as

$$MS_\alpha(u) := \alpha\int_\Omega |\nabla u|^2\,dx + \beta\mathcal{H}^{N-1}(S_u),\ \ \alpha, \beta \in \mathbb{R}^+. \tag{1.1}$$ mumfordshahori

Extensive literature is available, from both computational (see e.g. MR1488299, MR2049779 [16, 18]) and theoretical points of view (see e.g. ambrosio1989variational, de1989existence [1, 10]).

In the following, we will always take $\alpha = \beta$ in (mumfordshahori 1.1). The scalar tuning parameter $\alpha \in \mathbb{R}^+$ in (mumfordshahori 1.1), which uniformly determines the regularization strength over the entire image, plays an important role. The problem of finding a "good" tuning parameter $\alpha \in \mathbb{R}^+$ is still open, and widely discussed (see e.g., reyes2015structure, kunisch2013bilevel [11, 19]). However, the uniform regularization strength provided by a scalar tuning parameter $\alpha \in \mathbb{R}^+$ is undesirable when both fine details and large flat areas are present in the same image, which is often the case in image denoising problems. Ideally, one should impose a weaker regularization strength in regions with fine details, so to preserve them, and a greater regularization

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strength over large flat areas, so to remove the noise.

To this aim, the following Mumford-Shah functional, coupled with a spatially dependent parameter function ω: $\Omega \to [0, +\infty]$, was introduced in [14]:

$$MS_\omega(u) := \int_\Omega |\nabla u|^2 \, \omega \, dx + \int_{S_u} \omega^- \, d\mathcal{H}^{N-1}, \tag{1.2}$$

mumfordshahoriom

where $\omega \in SBV(\Omega) \cap L^\infty(\Omega)$ is positive and bounded away from 0, and ω^- is defined in (2.1) below. The minimization problem (1.2) can be viewed as a weighted version of the minimizing problem (1.1), with underlying metric $\omega\mathcal{L}^N \lfloor \Omega$ instead of $\mathcal{L}^N \lfloor \Omega$, where $\mathcal{L}^N$ denotes the N dimensional Lebesgue measure. However, it is well known that the minimization problem (1.1) is numerically difficult to solve in an efficient and robust way, and hence we would expect (1.2) to inherit similar issues. To overcome this drawback, an alternative approach has been proposed in [14], by adopting the approximation scheme of Ambrosio and Tortorelli from [4], and by changing the underlying metric in an appropriate manner. To be precise, in [14] the authors introduced the family of elliptic functionals with a spatially dependent parameter function ω

$$AT_{\omega,\varepsilon}(u,v) := \int_\Omega |\nabla u|^2 \, v^2 \omega \, dx + \int_\Omega \left[\varepsilon \, |\nabla v|^2 + \frac{1}{4\varepsilon}(v-1)^2 \right] \omega \, dx, \tag{1.3}$$

mumfordshahoriom

where $(u,v) \in W^{1,2}(\Omega) \times W^{1,2}(\Omega)$, and a rigorous analysis of properties of the functional (1.3) was undertaken. It turns out that, for a parameter function $\omega \in SBV(\Omega)$ satisfying $\mathcal{H}^{N-1}(S_\omega) < +\infty$ and

$$0 < l_1 \le \operatorname{ess\,inf} \{\omega(x) : \; x \in \Omega\} \le \operatorname{ess\,sup} \{\omega(x) : \; x \in \Omega\} \le l_2 < +\infty, \tag{1.4}$$

pos_lower_ass2

the functionals $AT_{\omega,\varepsilon}$ Γ-converge ([5]) to the functional $MS_\omega(u)$ in the $L^1 \times L^1$ topology.

At this point, how to construct a "good" parameter function ω becomes relevant. In [21], the author proposed to construct ω via a spatially dependent bilevel learning scheme (see also [9, 19]). Although the parameter function ω suggested in [21] does belong to $SBV(\Omega) \cap L^\infty(\Omega)$, i.e., the upper bound l_2 in (1.4) exists, there is no guarantee that the positive lower bound l_1 exists too. In fact, the analysis in both [17, 21] suggests that in certain situation a vanishing parameter function can yield a better denoising result, and in particular, mitigate the so called staircasing effect. Hence, it is necessary to improve the method proposed in [14] so that the positive lower bounded requirement can be removed, and this is the main topic of this article.

In addition, we remark (1.3) is one among many that approximation schemes to (1.2). Indeed, recalling that the original Ambrosio and Tortorelli approximation introduced in [4] ($\omega \equiv 1$ in (1.3)) is the reminiscent of the "first order" Cahn-Hilliard approximation, we may also consider an approximation by using the "second order" Cahn-Hilliard approximation or even higher order Cahn-Hilliard approximations (see [15]).

In view of this, in this article we will consider a family of approximation schemes defined by, for $k = 1, 2, 3, \ldots$,

$$AT^k_{\omega,\varepsilon}(u,v) := \int_\Omega |\nabla u|^2 \, v^2 \omega \, dx + \frac{1}{2c_k} \int_\Omega \left[\varepsilon^{2k-1} \left| \nabla^{(k)} v \right|^2 + \frac{1}{4^k \varepsilon}(1-v)^2 \right] \omega \, dx,$$

where

$$c_k := \inf \Bigg\{ \int_0^{+\infty} \left| v^{(k)} \right|^2 + \frac{1}{4^k}(1-v)^2 dx : \; v \in W^{k,2}_{\mathrm{loc}}(0, +\infty)$$
$$v(0) = v'(0) = \cdots v^{(k-1)}(0) = 0, \; v(t) = 1 \text{ if } t > K_k \text{ for some } K_k > 0 \text{ depends on } k \Bigg\}.$$

It has been observed in [8] that, for $\omega(x) \equiv \alpha \in \mathbb{R}^+$ and $k = 2$, the second order Ambrosio and Tortorelli approximation, i.e., $AT^2_\alpha(u,v)$, shows several advantages. For example, certain structure that are larger than a typical noise, but still not relevant for the segmentation (edge), can be suppressed. Hence, we should expect that the weighted version of AT^2_α, i.e., AT^2_ω, to inherit similar advantages.

In order to state the main result of our paper, we first introduce some notations.

haishiyoudehao2 **Notation 1.1.** Let $\Omega \subset \mathbb{R}^N$ be an open, bounded, Lipschitz regular domain, and let $\omega \in SBV(\Omega)$ be a non-negative function.

1. We say that $S \in \mathcal{R}(\Omega)$ if $\bar{S}$ is $\mathcal{H}^{N-1}$-rectifiable and $\mathcal{H}^{N-1}(\bar{S} \setminus S) = 0$ (note that $\bar{S}$ is $\mathcal{H}^{N-1}$-rectifiable implies that S is $\mathcal{H}^{N-1}$-rectifiable. See [3], Proposition 2.76).
2. Set $F^t(\omega) := \{x \in \Omega : \ \omega(x) > t\}$, for $t > 0$, and

$$P^\infty(\omega) := \bigcap_{t>0} F^t(\omega) \text{ and } P^0(\omega) := \bigcap_{t>0} \left(\Omega \setminus F^t(\omega)\right). \tag{1.5}$$

3. Define $A_\delta := \{x \in \Omega : \operatorname{dist}(x, A) < \delta\}$ for $A \subset \Omega$ and $\delta > 0$.

We can now introduce the parameter functions used in our main theorem.

Definition 1.2 (The spatially dependent parameter function)**.** *Let* $\omega\colon \Omega \to [0, +\infty]$ *belong to* $SBV(\Omega)$*.*

1. *We say that* $\omega \in \mathcal{P}(\Omega)$ *if* $\mathcal{H}^{N-1}(S_\omega) < +\infty$*, and* $P^0(\omega) \in \mathcal{R}(\Omega)$*.*
2. *We say that* $\omega \in \mathcal{P}_r(\Omega)$ *if* $\omega \in \mathcal{P}(\Omega)$*, and*

$$\lim_{\delta \to 0} \int_{\partial((P^\infty(\omega))_\delta)} \omega \, d\mathcal{H}^{N-1} + \int_{\partial((P^0(\omega))_\delta)} \omega \, d\mathcal{H}^{N-1} = 0. \tag{1.6}$$

We remark that any positive, bounded, and continuous function ω *satisfies* (1.6)*.*

3. *We say that* $\omega \in \mathcal{P}_b(\Omega)$ *if* $\omega \in \mathcal{P}(\Omega)$*, and satisfies* (1.4)*.*

Our main result is the following:

Theorem 1.3. *Let* $\Omega \subset \mathbb{R}^N$ *be an open, bounded, Lipschitz regular domain, let* $\omega \in \mathcal{P}_r(\Omega)$*, and for* $k \in \mathbb{N}$*,* $\varepsilon > 0$*, let* $\mathcal{AT}^k_{\omega,\varepsilon}\colon L^1(\Omega) \times L^1(\Omega) \to [0, +\infty]$ *be given by*

$$\mathcal{AT}^k_{\omega,\varepsilon}(u, v) := \begin{cases} AT^k_{\omega,\varepsilon}(u, v) & \text{if } (u, v) \in W^{1,2}(\Omega) \times W^{1,2}(\Omega),\ 0 \le v \le 1, \\ +\infty & \text{otherwise.} \end{cases}$$

Then the functionals $\mathcal{AT}^k_{\omega,\varepsilon}$ Γ*-converge, with respect to the* $L^1 \times L^1$ *topology, to*

$$\mathcal{MS}_\omega(u, v) := \begin{cases} MS_\omega(u) & \text{if } u \in GSBV_\omega(\Omega) \text{ and } v = 1 \text{ a.e.}, \\ +\infty & \text{otherwise,} \end{cases}$$

where $GSBV_\omega(\Omega)$ *is defined in Definition* 2.3*.*

Although the parameter function proposed in [21] belongs to L^∞, here we allow ω to be unbounded, although the structure of the set $P^\infty(\Omega)$ has to satisfy the restrictive requirement (1.6).

The proof of the Γ-$\liminf$ requires only $\omega \in \mathcal{P}(\Omega)$. To this aim, we first restrict our analysis to the domain $\Omega \setminus (P^0(\omega))_\delta$, with $\delta > 0$. Hence ω is bounded away from zero in $\Omega \setminus (P^0(\omega))_\delta$. Together with a truncation argument on ω, we have $\omega_K := \omega \wedge K \in \mathcal{P}_b(\Omega \setminus (P^0(\omega))_\delta)$, and hence the Γ-$\liminf$ result obtained in [21] can be applied. Second, we take the limit $\delta \to 0$, and using the assumption $\partial(P^0(\omega)) \in \mathcal{R}(\Omega)$, we can obtain the lower bound in $\Omega \setminus \overline{P^0(\omega)}$. Finally, by using the definition of $P^0(\omega)$, we recover the Γ-$\liminf$ inequality in the entire domain Ω.

The proof of the Γ-$\limsup$ is more delicate, requiring the extra assumption $\omega \in \mathcal{P}_r(\Omega)$. Still, similarly to the $\Gamma - \liminf$ inequality, we first restrict our analysis to the subset Ω' of Ω such that $\omega \in \mathcal{P}_b(\Omega')$, and apply the construction from [21]. Then, using (1.6), we can construct the recovery sequence in the entire domain Ω. To conclude this section, we state a lower semicontinuity result, which will be used in Section 4, which can be viewed as the weighted version of the main theorem of [2].

Theorem 1.4 (Theorem 4.2)**.** *Assume that* $\omega \in SBV(\Omega)$ *has a positive lower bound, and define*

$$F_\omega(u) := \int_\Omega f(x, u, \nabla u)\, \omega \, dx + \int_{S_u} \omega^- \, d\mathcal{H}^{N-1},$$

where $f(x, s, p)$ *is integrable in* x*, continuous in* s*, convex with respect to* p*, and satisfies*

$$|p|^2 \le f(x, s, p) \le a(x) + \Phi(|s|)(1 + |p|^2) \text{ for all } (x, s, p) \in \Omega \times \mathbb{R} \times \mathbb{R}^N$$

for some $a \in L^1(\Omega)$*, and some continuous function* $\Phi\colon [0, +\infty) \to [0, +\infty)$*. Then the functional* F_ω *is* $L^1_{\rm loc}(\Omega)$ *is lower semicontinuous in* $SBV(\Omega) \cap L^\infty(\Omega)$*.*

This article is organized as follows: in Section 2 we will introduce the main definitions, and recall several preliminary results. In Section 3, we prove the Γ-$\liminf$ inequality, and in Section 4 we construct the recovery sequence by using fine properties of SBV functions, and we prove Theorem 1.4.

2. Definitions and Preliminary Results

Def_Preres

Throughout this paper, $\Omega \subset \mathbb{R}^N$ is an open, bounded set with Lipschitz boundary, and $I := (-1, 1)$.

polyhedral

Definition 2.1. *We say that a subset $P \subset \Omega$ is $(N-1)$* polyhedral *if it is the intersection of Ω with finitely many $(N-1)$-dimensional simplexes of $\mathbb{R}^N$.*

SBV_SBV2_GSBV

Definition 2.2. *We say that $u \in BV(\Omega)$ is a* special function of bounded variation, *and we write $u \in SBV(\Omega)$, if the Cantor part of its derivative, $D^c u$, is zero, so that (see* ambrosio2000functions *[3, equation (3.89)])*

$$Du = D^a u + D^j u = \nabla u \mathcal{L}^N \lfloor \Omega + (u^+ - u^-)\nu_u \mathcal{H}^{N-1} \lfloor S_u.$$ absolutely_cont_

Moreover, we say that

1. *$u \in SBV^2(\Omega)$ if $u \in SBV(\Omega)$ and $\nabla u \in L^2(\Omega)$;*
2. *$u \in GSBV(\Omega)$ if $K \wedge u \vee -K \in SBV(\Omega)$ for all $K \in \mathbb{N}$.*

Here we always identify $u \in SBV(\Omega)$ with its representative $\bar{u}$, where $\bar{u}(x) := \left(u^+(x) + u^-(x)\right)/2$, with

$$u^+(x) := \inf\left\{ t \in \mathbb{R} : \lim_{r\to 0} \frac{\mathcal{L}^N(B(x,r) \cap \{u > t\})}{r^N} = 0 \right\},$$

and

$$u^-(x) := \sup\left\{ t \in \mathbb{R} : \lim_{r\to 0} \frac{\mathcal{L}^N(B(x,r) \cap \{u < t\})}{r^N} = 0 \right\}. \tag{2.1}$$ mm_negative_part

We note that u^-, u^+, and $\bar{u}$ are all Borel measurable (see evans2015measure [12, Lemma 1]).

parameter_space

Definition 2.3. *Let $\omega \in \mathcal{P}(\Omega)$ be given. We say that $u \in SBV_\omega(\Omega)$ if $u \in L^1(\Omega)$, $u \in SBV(\Omega \setminus (P^0(\omega))_\delta)$ for every $\delta > 0$, and*

$$\int_\Omega |\nabla u|^2 \,\omega\, dx + \int_{S_u^0} \left|u^+ - u^-\right| \omega \, d\mathcal{H}^{N-1} < +\infty, \tag{2.2}$$ define_zero_int

where the jump set S_u^0 of $u \in SBV_\omega(\Omega)$, with a vanishing parameter ω, is defined by

$$S_u^0 := \left(\bigcup_{\delta > 0} S_u^\delta \right) \cup P^0(\omega).$$

Here S_u^δ denotes the jump set of u in $SBV(\Omega \setminus (P^0(\omega))_\delta)$. Moreover, we say that $u \in GSBV_\omega(\Omega)$ if $K \wedge u \vee -K \in SBV_\omega(\Omega)$ for all $K \in \mathbb{N}$.

Remark 2.4. Since $u \in SBV(\Omega \setminus (P^0(\omega))_\delta)$ for every $\delta > 0$, ∇u is defined $\mathcal{L}^N$ a.e. in $\Omega \setminus (P^0(\omega))_\delta$, and hence $\mathcal{L}^N$ a.e. in $\Omega \setminus \overline{P^0(\omega)}$. Recalling that $P^0(\omega) \in \mathcal{R}(\Omega)$, which implies that $\mathcal{H}^{N-1}\left(\overline{P^0(\omega)}\right) < +\infty$, we have that ∇u is defined $\mathcal{L}^N$ a.e., hence the first integral in define_zero_int (2.2) is well defined. Similarly, $u^\pm$ is well defined for $\mathcal{H}^{N-1}$-a.e. $x \in \Omega \setminus \overline{P^0(\omega)}$, hence the second integral in define_zero_int (2.2) us also well defined. Finally, it is clear that if ω has a positive lower bounded, then $P^0(\omega) = \varnothing$ and $S_u^0 = S_u$.

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Notation 2.5. Let $\Gamma \subset \Omega$ be a $\mathcal{H}^{N-1}$-rectifiable set, and let $x \in \Gamma$ be given.

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1. We denote by $\nu_\Gamma(x)$ the normal vector at x with respect to Γ, and by $Q_{\nu_\Gamma}(x, r)$ the cube centered at x with side length r and two faces normal to $\nu_\Gamma(x)$;

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2. T_{x,ν_Γ} denotes the hyperplane through x and normal to $\nu_\Gamma(x)$, and $\mathbb{P}_{x,\nu_\Gamma}$ denotes the projection operator from Γ onto T_{x,ν_Γ};

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3. we define, for $t \in \mathbb{R}$, the hyperplane $T_{x,\nu_\Gamma}(t) := T_{x,\nu_\Gamma} + t\nu_\Gamma(x)$;
4. we define the half-spaces and half-cubes by,

$$H_{\nu_\Gamma}(x)^{+(-)} := \left\{ y \in \mathbb{R}^N : \ \nu_\Gamma(x) \cdot (y - x) \geq (\leq) 0 \right\}$$

and

$$Q_{\nu_\Gamma}^\pm(x, r) := Q_{\nu_\Gamma}(x, r) \cap H_{\nu_\Gamma}(x)^\pm,$$

respectively;

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5. for given $\tau > 0$, we denote by $R_{\tau,\nu_\Gamma}(x, r)$ the part of $Q_{\nu_\Gamma}(x, r)$ which lies strictly between the two hyperplanes $T_{x,\nu_\Gamma}(-\tau r)$ and $T_{x,\nu_\Gamma}(\tau r)$.

ne_properties_BV

Theorem 2.6 ([12], Theorem 3, page 213)**.** *Assume that $u \in BV(\Omega)$. Then, for $\mathcal{H}^{N-1}$-a.e. $x_0 \in S_u$,*

$$\lim_{r\to 0} \fint_{B(x_0,r)\cap H_{\nu_{S_u}}(x_0)^{\pm}} \left|u(x) - u^{\pm}(x_0)\right|^{\frac{N}{N-1}} dx = 0,$$

and

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^{N-1}} \int_{S_u \cap Q_{\nu_{S_u}}(x_0,\varepsilon)} \left|u^+(x) - u^-(x)\right| d\mathcal{H}^{N-1}(x) = \left|u^+(x_0) - u^-(x_0)\right|.$$

equivalent_norm

Theorem 2.7 ([7], Remark 8)**.** *Let $\Omega \subset \mathbb{R}^N$ be an open and bounded domain. Then, for any $\delta < \delta_0$,*

$$C(\delta_0, k, \Omega)\, \delta \left\|\nabla u\right\|^2_{L^2(\Omega)} \le \delta^{2k-1} \left\|D^{\alpha} u\right\|^2_{L^2(\Omega)} + \frac{1}{2^k \delta} \left\|1 - u\right\|^2_{L^2(\Omega)},$$

where $|\alpha| = k$, and $C(\delta_0, k, \Omega) > 0$ is some constant depending on δ_0, $k \in \mathbb{N}$, and Ω.

0_D_Toy_Sec

3. The Γ-lim inf inequality

In this section we will prove the Γ-lim inf inequality.

liminf_part_c

Proposition 3.1. (Γ-lim inf) *Given $u \in L^1(\Omega)$, let $\omega \in \mathcal{P}(\Omega)$, and*

$$MS^-_{\omega}(u) := \inf \Big\{ \liminf_{\varepsilon\to 0} AT^k_{\omega,\varepsilon}(u_\varepsilon, v_\varepsilon) : (u_\varepsilon, v_\varepsilon) \in W^{1,2}(\Omega) \times W^{1,2}(\Omega),\ u_\varepsilon \to u \text{ in } L^1,\ v_\varepsilon \to 1 \text{ in } L^1,\ 0 \le v_\varepsilon \le 1 \Big\}.$$

Then $MS^-_{\omega}(u) \ge MS_{\omega}(u)$.

bdd_away_zero

3.1. Special case: $\omega \in \mathcal{P}_b(\Omega)$.

In Section 3.1 we prove Proposition 3.1 when

$$0 < l_1 \le \operatorname{ess\,inf}\{\omega(x) :\ x \in \Omega\} \le \operatorname{ess\,sup}\{\omega(x) :\ x \in \Omega\} \le l_2 < +\infty, \tag{3.1}$$

easy_way_out

and, without loss of generality, $l_1 = 1$.

liminf_part_ref

Proposition 3.2 ([21], Proposition 3.1)**.** *Given $\omega \in \mathcal{P}_b(\Omega)$ and $u \in L^1(\Omega)$, let*

$$MS^-_{\omega}(u) := \inf \Big\{ \liminf_{\varepsilon\to 0} AT^1_{\omega,\varepsilon}(u_\varepsilon, v_\varepsilon) : (u_\varepsilon, v_\varepsilon) \in W^{1,2}(\Omega) \times W^{1,2}(\Omega),\ u_\varepsilon \to u \text{ in } L^1,\ v_\varepsilon \to 1 \text{ in } L^1,\ 0 \le v_\varepsilon \le 1 \text{ a.e.} \Big\}.$$

Then $MS^-_{\omega}(u) \ge MS_{\omega}(u)$.

3.2. General case: $\omega \in \mathcal{P}(\Omega)$.

Now we are ready to prove Proposition 3.1. In the following, we set

$$L_\delta := \{x \in \Omega :\ \omega(x) > \delta\} \cap \left(\Omega \setminus (P^0(\omega))_\delta\right),$$

where $P^0(\omega)$ is from Definition 1.2, and

$$\omega_l := l \wedge \omega, \qquad l > 0.$$

We recall from [12, Theorem 1] that, for $\mathcal{L}^1$ a.e. $\delta > 0$, L_δ has finite perimeter.

Proof of Proposition 3.1. Without loss of generality, assume that $M := MS^-_{\omega}(u) < \infty$. Let $\{(u_\varepsilon, v_\varepsilon)\}_{\varepsilon>0} \subset W^{1,2}(\Omega) \times W^{1,2}(\Omega)$ be such that $u_\varepsilon \to u$ in $L^1(\Omega)$, $v_\varepsilon \to 1$ in $L^1(\Omega)$, and $\lim_{\varepsilon\to 0} AT^k_{\omega,\varepsilon}(u_\varepsilon, v_\varepsilon) = MS^-_{\omega}(u)$. Fix $\delta > 0$ and $l > 0$, and note that

$$\begin{aligned}
&\liminf_{\varepsilon\to 0} AT^k_{\omega,\varepsilon}(u_\varepsilon, v_\varepsilon)\\
&\quad\ge \liminf_{\varepsilon\to 0} \int_{L_\delta} |\nabla u_\varepsilon|^2 v_\varepsilon^2\, \omega_l\, dx + \frac{1}{c_k} \int_{L_\delta} \left[\varepsilon^{2k-1} \left|\nabla^{(k)} v_\varepsilon\right|^2 + \frac{1}{\varepsilon 4^k}(1 - v_\varepsilon)^2\right] \omega_l\, dx\\
&\quad\ge \int_{L_\delta} |\nabla u|^2\, \omega_l\, dx + \int_{S_u^\delta \cap L_\delta} \omega_l^-\, d\mathcal{H}^{N-1},
\end{aligned}$$

where in the last inequality we used Proposition 3.2. Letting $l \nearrow +\infty$ on the right hand side, we have

$$\liminf_{\varepsilon\to 0} AT^k_{\omega,\varepsilon}(u_\varepsilon, v_\varepsilon) \ge \int_{L_\delta} |\nabla u|^2\, \omega\, dx + \int_{S_u^\delta \cap L_\delta} \omega^-\, d\mathcal{H}^{N-1}$$

for $\mathcal{L}^1$ a.e. $\delta > 0$.

Finally, taking the limit $\delta \searrow 0$ on the right hand side, in view of (1.5), and the fact that $S_u^\delta \cap L_\delta \subset S_u^{\delta'} \cap L_{\delta'}$ for $\delta > \delta'$, by the Monotone Convergence Theorem we infer

$$\begin{aligned}\liminf_{\varepsilon\to 0} AT^k_{\omega,\varepsilon}(u_\varepsilon, v_\varepsilon) &\geq \int_{\Omega\setminus\overline{P^0(\omega)}} |\nabla u|^2\,\omega\,dx + \int_{S_u^0\setminus\overline{P^0(\omega)}} \omega^-\,d\mathcal{H}^{N-1}\\ &= \int_\Omega |\nabla u|^2\,\omega\,dx + \int_{S_u^0\setminus P^0(\omega)} \omega^-\,d\mathcal{H}^{N-1} = \int_\Omega |\nabla u|^2\,\omega\,dx + \int_{S_u^0} \omega^-\,d\mathcal{H}^{N-1},\end{aligned}$$

where in the last equality we used the fact that $\omega^-(x) \leq \omega(0) = 0$ in $P^0(\omega)$. □

t_n_D_C

4. The Γ-lim sup Inequality

This section is devoted to the proof of the Γ-lim sup inequality, and Theorem 1.3, under the additional assumption $\omega \in \mathcal{P}_r(\Omega)$.

The main goal is to prove the following proposition.

limsup_n_c

Proposition 4.1. (Γ-lim sup) *Given $u \in L^1(\Omega) \cap L^\infty(\Omega)$, let $\omega \in \mathcal{P}_r(\Omega)$, and*

$$MS^+_\omega(u) := \inf\Big\{ \limsup_{\varepsilon\to 0} AT^k_{\omega,\varepsilon}(u_\varepsilon, v_\varepsilon) : (u_\varepsilon, v_\varepsilon) \in W^{1,2}(\Omega)\times W^{1,2}(\Omega), u_\varepsilon \to u \text{ in } L^1(\Omega),\, v_\varepsilon \to 1 \text{ in } L^1(\Omega),\, 0 \leq v_\varepsilon \leq 1\Big\}.$$

Then $E^+_\omega(u) \leq E_\omega(u)$.

To prove this result, we will establish some preliminary results on the lower semicontinuity of convex integrals in the space $SBV_\omega(\Omega) \cap L^\infty(\Omega)$, under the condition that $\omega \in \mathcal{P}(\Omega)$ has a positive lower bound.

lsc_SBV_om

4.1. **Lower semicontinuity results in the space $SBV_\omega(\Omega) \cap L^\infty$ with a positive lower bounded ω.** In this section we study the lower semicontinuity of integral functionals defined in $SBV_\omega(\Omega)$, with respect to the $L^\infty(\Omega)$ topology. Consider

$$F_\omega(u) := \int_\Omega f(x, u, \nabla u)\,\omega\,dx + \int_{S_u} \omega^-\,d\mathcal{H}^{N-1},$$

SBV_om_lsc_F

where $f(x, s, p)$ is a nonnegative Carathéodory function in x, and continuous in (s, p), and the parameter function $\omega \in \mathcal{P}(\Omega)$ is assumed to be bounded from below by a constant $l > 0$, i.e.

$$\operatorname{ess\,inf}\Big\{\omega^-(x) :\ x \in \Omega\Big\} = l > 0. \tag{4.1}$$

llll_om_lsc_F

Without loss of generality, we take $l = 1$. This condition implies that the space SBV_ω is embedded in $SBV(\Omega)$, and hence we may apply results concerning $SBV(\Omega)$.

The main result is the following.

lsc_SBV_thm

Theorem 4.2. *Given $\omega \in \mathcal{P}(\Omega)$ satisfying (4.1), assume that $f(x, s, p)$ is convex with respect to p, and satisfies the condition*

$$|p|^2 \leq f(x, s, p) \leq a(x) + \Phi(|s|)(1 + |p|^2) \text{ for all } (x, s, p) \in \Omega\times\mathbb{R}\times\mathbb{R}^N$$

for some $a \in L^1(\Omega)$, and some continuous function Φ: $[0, +\infty) \to [0, +\infty)$. Then, for any sequence $\{u_\varepsilon\}_{\varepsilon>0} \subset L^\infty(\Omega)$ such that $u_\varepsilon \to u$ in $L^1(\Omega)$, and

$$\sup\Big\{\|u_\varepsilon\|_{L^\infty(\Omega)} :\ \varepsilon > 0\Big\} < +\infty, \tag{4.2}$$

buqingyuandebdd

we have

$$\liminf_{\varepsilon\to 0} F_\omega(u_\varepsilon) \geq F_\omega(u).$$

Proof. Without loss of generality, we may assume that $M := \liminf_{\varepsilon\to 0} F(u_\varepsilon) < +\infty$. Hence,

$$F_1(u_\varepsilon) \leq F_\omega(u_\varepsilon) \leq M + 1 \tag{4.3}$$

finite_jump_num_

for all sufficiently small $\varepsilon > 0$. Therefore, by (4.2) and [3, Theorem 4.7], there exists $u \in SBV(\Omega) \cap L^\infty(\Omega)$ such that $u_\varepsilon \rightharpoonup u$ in $BV(\Omega)$. Fix $K \in \mathbb{N}$, and define

$$f_K(x,s,p) := f(x,s,p)(\omega \wedge K),$$

and by [2, Theorem 0.1], we have

$$\liminf_{\varepsilon\to 0} \int_\Omega f(x,u_\varepsilon,\nabla u_\varepsilon)\,\omega\,dx \geq \liminf_{\varepsilon\to 0} \int_\Omega f_K(x,u_\varepsilon,\nabla u_\varepsilon)\,dx \geq \int_\Omega f_K(x,u,\nabla u)\,dx.$$

Letting $K \nearrow +\infty$, we recover

$$\liminf_{\varepsilon\to 0} \int_\Omega f(x,u_\varepsilon,\nabla u_\varepsilon)\,\omega\,dx \geq \int_\Omega f(x,u,\nabla u)\,\omega\,dx.$$

We next show that

$$\liminf_{\varepsilon\to 0} \int_{S_{u_\varepsilon}} \omega^-\,d\mathcal{H}^{N-1} \geq \int_{S_u} \omega^-\,d\mathcal{H}^{N-1}. \tag{4.4}$$ lower_bdd_3dnd

To this aim, we first prove it in the case $N = 1$, and then recover the general case $N > 1$ using the slicing argument from [21, Lemma 3.9].

In the case $N = 1$, we need to show that

$$\liminf_{\varepsilon\to 0} \sum_{x\in S_{u_\varepsilon}} \omega^-(x) \geq \sum_{x\in S_u} \omega^-(x). \tag{4.5}$$ lower_bdd_2dnd

Recalling (4.3), note that

$$\sup_{\varepsilon>0} \mathcal{H}^0(S_{u_\varepsilon}) < +\infty \text{ and } \mathcal{H}^0(S_u) < +\infty,$$

and, without loss of generality, we may assume that $S_{u_\varepsilon} = \{x_\varepsilon\}$, and $S_u = \{x\}$. Hence, the convergence $u \rightharpoonup u$ in $BV(\Omega)$ implies that $x_\varepsilon \to x$. We claim that

$$\liminf_{\varepsilon\to 0} \omega^-(x_\varepsilon) \geq \omega^-(x). \tag{4.6}$$ lower_bdd_1dnd

If $x \notin S_\omega$, then there exists $\delta > 0$ such that

$$S_\omega \cap (x-\delta, x+\delta) = \varnothing,$$

so ω is absolutely continuous in $(x-\delta, x+\delta)$, and (4.6) is trivially satisfied with $\omega(x) = \omega^-(x)$, with the inequality in (4.6) being actually an equality.

Suppose that $x \in S_\omega$ and, without loss of generality, assume that $x = 0$. Since $\mathcal{H}^0(S_\omega) < \infty$, choose $\bar r > 0$ such that

$$S_\omega \cap (0-\bar r, 0+\bar r) = 0.$$

As ω is absolutely continuous in $(-\bar r, 0)$ and $(0, \bar r)$, we may extend ω uniquely to $x = 0$ to the left and right (see [20, Exercise 3.7, (1)]), which allows us to define

$$\omega(0^+) := \lim_{x\searrow 0^+} \omega(x) \text{ and } \omega(0^-) := \lim_{x\nearrow 0^-} \omega(x).$$ lemma_jump_jiuehuishi

This gives immediately

$$\liminf_{\varepsilon\to 0} \omega^-(x_\varepsilon) \geq \omega(0^-).$$

We next claim that $\omega(0^-) = \omega^-(0)$. By part 2 of Theorem 2.6, we have

$$\omega^-(0) = \lim_{r\to 0} \frac{1}{r}\int_{-r}^0 \omega(t)\,dt = \omega(0^-),$$

where in the last equality we used basic properties of absolutely continuous functions, and the definition of $\omega(0^-)$. Thus (4.6) holds, hence (4.5) holds too.

We now claim (4.4). Define $\omega_K := \omega \wedge K$, and by Lemma A.6, we obtain a set $S \subset S_u$ such that Lemmas A.6, a-c, are satisfied. Fixed one such $Q \in \mathcal{Q}$, and observe that, due to $\omega^-_{x,\nu}(t) = \omega^-(x + t\nu)$ (see [3, Remark 3.109]), we have

$$\liminf_{\varepsilon\to 0} \int_{S_{u_\varepsilon}\cap Q} \omega_K^-\,d\mathcal{H}^{N-1}$$

$$\begin{aligned}
&= \liminf_{\varepsilon\to 0} \int_{\left[Q_{\nu_S(x_0,r_0)}\right]_{\nu_S(x_0)}} \left(\sum_{t\in S_{(u_\varepsilon)_{x,\nu_S(x_0)}}\cap\left[Q_{\nu_S}(x_0,r_0)\right]_{x,\nu_S(x_0)}} (\omega_K^-)_{x,\nu_S(x_0)}(t) \right) d\mathcal{H}^{N-1}(x) \\
&\geq \liminf_{\varepsilon\to 0} \int_{T_g(x_0,r_0)} \left(\sum_{t\in S_{(u_\varepsilon)_{x,\nu_S(x_0)}}\cap\left[Q_{\nu_S}(x_0,r_0)\right]_{x,\nu_S(x_0)}\cap S} (\omega_K^-)_{x,\nu_S(x_0)}(t) \right) d\mathcal{H}^{N-1}(x) \\
&\geq \int_{T_g(x_0,r_0)} \liminf_{\varepsilon\to 0} \left(\sum_{t\in S_{(u_\varepsilon)_{x,\nu_S(x_0)}}\cap\left[Q_{\nu_S}(x_0,r_0)\right]_{x,\nu_S(x_0)}\cap S} (\omega_K^-)_{x,\nu_S(x_0)}(t) \right) d\mathcal{H}^{N-1}(x) \\
&= \int_{T_g(x_0,r_0)} \omega_K^-(x)\, d\mathcal{H}^{N-1}(x) = \int_{T_g(x_0,r_0)} \omega_K^-(x', l_{x_0}(x'))d\mathcal{L}^{N-1}(x').
\end{aligned}$$

Using a similar argument as in [14, equation (3.33)], we have, for any $\tau, \eta > 0$,

$$\begin{aligned}
\liminf_{\varepsilon\to 0} \int_{S_{u_\varepsilon}} \omega^-\, d\mathcal{H}^{N-1} &\geq \liminf_{\varepsilon\to 0} \int_{S_{u_\varepsilon}} \omega_K^-\, d\mathcal{H}^{N-1} \geq \liminf_{\varepsilon\to 0} \sum_{Q\in\mathcal{Q}} \int_{S_{u_\varepsilon}\cap Q} \omega_K^-\, d\mathcal{H}^{N-1} \\
&\geq \frac{1}{\sqrt{1+\tau^2}} \sum_{Q\in\mathcal{Q}} \int_{S\cap Q} \omega_K^-\, d\mathcal{H}^{N-1} \geq \frac{1}{\sqrt{1+\tau^2}} \left(\int_{S_u} \omega_K^-\, d\mathcal{H}^{N-1} - \|\omega_K\|_{L^\infty}\, \eta \right).
\end{aligned}$$

Taking first the limits $\tau \searrow 0$ and $\eta \searrow 0$, and then $K \nearrow +\infty$, gives (4.4). □

Lemma 4.3. *Given $u \in SBV_\omega(\Omega) \cap L^\infty(\Omega)$ satisfying $MS_\omega(u) < +\infty$, where $\omega \in \mathcal{P}_r(\Omega)$ and satisfies (4.1), there exists a sequence $\{u_\varepsilon\}_{\varepsilon>0} \subset SBV^2_\omega(\Omega) \cap L^\infty(\Omega)$ such that the following assertions hold:*

1. $\|u_\varepsilon\|_{L^\infty} \leq \|u\|_{L^\infty}$;
2. $S_{u_\varepsilon} \subset \Omega \setminus (P^\infty(\omega))_{o(\varepsilon)}$ *(note that* $\operatorname{ess\,sup}\left\{\omega(x) :\ x \in \Omega \setminus (P^\infty(\omega))_{o(\varepsilon)}\right\} < +\infty$*);*
3. $$\lim_{\varepsilon\to 0} \int_\Omega |\nabla u_\varepsilon|^2\, \omega\, dx + \int_{S_{u_\varepsilon}} \omega^-\, d\mathcal{H}^{N-1} = \int_\Omega |\nabla u|^2\, \omega\, dx + \int_{S_u} \omega^-\, d\mathcal{H}^{N-1}.$$

Proof. Let $\varepsilon > 0$ be sufficiently small, so that

$$\int_{(P^\infty(\omega))_{o(\varepsilon)}} |\nabla u|^2\, \omega dx < o(\varepsilon). \tag{4.7}$$

Let K_ε be a compact subset of $S_u \setminus (P^\infty(\omega))_{o(\varepsilon)}$ such that

$$\mathcal{H}^{N-1}\left(S_u \setminus K_\varepsilon\right) \leq \varepsilon \qquad \text{and} \qquad \int_{S_u\setminus K_\varepsilon} \omega^- \leq \varepsilon.$$

Consider the minimization problem

$$\begin{aligned}
\min \Bigg\{ &\int_\Omega |\nabla v|^2\, \omega\, dx + \int_{S_v\setminus K_\varepsilon} \omega^-\, d\mathcal{H}^{N-1} + \int_{K_\varepsilon} \omega^-\, d\mathcal{H}^{N-1} \\
&+ \frac{1}{\varepsilon}\int_\Omega |u-v|^2\, \omega\, dx :\ v \in SBV^2_\omega(\Omega) \text{ and } S_v \subset \Omega \setminus (P^\infty(\omega))_{o(\varepsilon)} \Bigg\}.
\end{aligned} \tag{4.8}$$

By a truncation argument, we may impose the restriction that v satisfies $\|v\|_{L^\infty(\Omega)} \leq \|u\|_{L^\infty(\Omega)}$. Let $\{v_n\}_{n=1}^\infty$ be a minimizing sequence. Then,

$$\begin{aligned}
&\int_\Omega |\nabla v_n|^2\, dx + \mathcal{H}^{N-1}\left(S_{v_n} \setminus K_\varepsilon\right) + \mathcal{H}^{N-1}\left(K_\varepsilon\right) + \frac{1}{\varepsilon}\int_\Omega |u - v_n|^2\, dx \\
&\quad \leq \frac{1}{l}\left[\int_\Omega |\nabla v_n|^2\, \omega\, dx + \int_{S_{v_n}\setminus K_\varepsilon} \omega^-\, d\mathcal{H}^{N-1} + \int_{K_\varepsilon} \omega^-\, d\mathcal{H}^{N-1} + \frac{1}{\varepsilon}\int_\Omega |u-v_n|^2\, \omega\, dx \right] < \frac{1}{l}(M_\varepsilon + 1),
\end{aligned}$$

where M_ε is defined as the minimum of (4.8), with $\varepsilon > 0$ fixed.

Assume first that $M_\varepsilon < +\infty$. By [3, Theorem 4.7], there exists $u_\varepsilon \in SBV(\Omega)$ such that $v_n \rightharpoonup u_\varepsilon$ in $BV(\Omega)$, and for $\mathcal{H}^{N-1}$-a.e. $x \in S_{u_\varepsilon}$, there exists $x_n \in S_{v_n}$ such that $x_n \to x$, which implies that $S_{u_\varepsilon} \subset \Omega \setminus (P^\infty(\omega))_{o(\varepsilon)}$. Moreover, by Theorem 4.2, we have $u_\varepsilon \in SBV_\omega \cap L^\infty(\Omega)$, and

$$\int_\Omega |\nabla u_\varepsilon|^2 \, dx + \int_{S_{u_\varepsilon} \setminus K_\varepsilon} \omega^- d\mathcal{H}^{N-1} + \int_{K_\varepsilon} \omega^- d\mathcal{H}^{N-1} + \frac{1}{\varepsilon}\int_\Omega |u - u_\varepsilon|^2 \, \omega \, dx$$
$$\leq \liminf_{n\to\infty} \int_\Omega |\nabla v_n|^2 \, \omega \, dx + \int_{S_{v_n} \setminus K_\varepsilon} \omega^- d\mathcal{H}^{N-1} + \int_{K_\varepsilon} \omega^- d\mathcal{H}^{N-1} + \frac{1}{\varepsilon}\int_\Omega |u - v_n|^2 \, \omega \, dx. \quad (4.9)$$

yi_limit2

Define

$$\bar{u}_\varepsilon := \begin{cases} u(x) & \text{if } x \in \Omega \setminus (P^\infty(\omega))_{2o(\varepsilon)}, \\ 0 & \text{otherwise.} \end{cases}$$

Then $S_{\bar{u}_\varepsilon} \subset \Omega \setminus (P^\infty(\omega))_{o(\varepsilon)}$, and

$$\int_\Omega |\nabla \bar{u}_\varepsilon|^2 \, \omega \, dx \leq \int_\Omega |\nabla u|^2 \, \omega \, dx.$$

In view of (1.6),

$$\int_{S_{\bar{u}_\varepsilon}} \omega^- d\mathcal{H}^{N-1} \leq \int_{S_u} \omega^- d\mathcal{H}^{N-1} + \int_{\partial((P^\infty(\omega))_\varepsilon)} \omega^- d\mathcal{H}^{N-1} \leq \int_{S_u} \omega^- d\mathcal{H}^{N-1} + O(\varepsilon),$$

hence $M_\varepsilon < +\infty$. Let $v = \bar{u}_\varepsilon$ in (4.8), by (4.7) and (4.9), we have

$$\int_\Omega |\nabla u_\varepsilon|^2 \, \omega \, dx + \int_{S_{u_\varepsilon} \setminus K_\varepsilon} \omega^- d\mathcal{H}^{N-1} + \int_{K_\varepsilon} \omega^- d\mathcal{H}^{N-1} + \frac{1}{\varepsilon}\int_\Omega |u - u_\varepsilon|^2 \, \omega \, dx$$
$$\leq \int_\Omega |\nabla \bar{u}_\varepsilon|^2 \, \omega dx + \int_{S_{\bar{u}_\varepsilon} \setminus K_\varepsilon} \omega^- d\mathcal{H}^{N-1} + \int_{K_\varepsilon} \omega^- d\mathcal{H}^{N-1} + \frac{1}{\varepsilon}\int_\Omega |u - \bar{u}_\varepsilon|^2 \, \omega \, dx$$
$$\leq \int_\Omega |\nabla u|^2 \, \omega \, dx + \int_{S_u} \omega^- \, d\mathcal{H}^{N-1} + O(\varepsilon) + o(\varepsilon)/O(\varepsilon) \leq C < +\infty. \quad (4.10)$$

yi_limit3

In particular,

$$\int_\Omega |u - u_\varepsilon|^2 \, \omega \, dx \leq C\varepsilon \to 0.$$

By (4.9), Theorem 4.2, and $\operatorname{ess\,inf} \omega > l > 0$, up to a subsequence it holds

$$\lim_{\varepsilon\to 0} \int_\Omega |\nabla u_\varepsilon|^2 \, \omega \, dx \geq \int_\Omega |\nabla u|^2 \, \omega \, dx, \text{ and } \lim_{\varepsilon\to 0} \int_{S_{u_\varepsilon}} \omega^- \, d\mathcal{H}^{N-1} \geq \int_{S_u} \omega^- \, d\mathcal{H}^{N-1}.$$

Hence, in view of (4.10),

$$\int_\Omega |\nabla u|^2 \, \omega \, dx + \int_{S_u} \omega^- \, d\mathcal{H}^{N-1} \leq \liminf_{\varepsilon\to 0} \int_\Omega |\nabla u_\varepsilon|^2 \, \omega \, dx + \int_{S_{u_\varepsilon}} \omega^- \, d\mathcal{H}^{N-1}$$
$$\leq \limsup_{\varepsilon\to 0} \int_\Omega |\nabla u_\varepsilon|^2 \, \omega \, dx + \int_{S_{u_\varepsilon} \setminus K_\varepsilon} \omega^- d\mathcal{H}^{N-1} + \int_{K_\varepsilon} \omega^- d\mathcal{H}^{N-1} + \frac{1}{\varepsilon}\int_\Omega |u - u_\varepsilon|^2 \, \omega \, dx$$
$$\leq \int_\Omega |\nabla u|^2 \, \omega \, dx + \int_{S_u} \omega^- \, d\mathcal{H}^{N-1} + \limsup_{\varepsilon\to 0}(O(\varepsilon) + o(\varepsilon)/O(\varepsilon))$$
$$= \int_\Omega |\nabla u|^2 \, \omega \, dx + \int_{S_u} \omega^- \, d\mathcal{H}^{N-1}.$$

Finally, we obtain

$$\lim_{\varepsilon\to 0} \int_\Omega |\nabla u_\varepsilon|^2 \, \omega \, dx + \int_{S_{u_\varepsilon}} \omega^- \, d\mathcal{H}^{N-1} = \int_\Omega |\nabla u|^2 \, \omega \, dx + \int_{S_u} \omega^- \, d\mathcal{H}^{N-1},$$

and

$$\lim_{\varepsilon\to 0} \int_{S_{u_\varepsilon} \setminus K_\varepsilon} \omega^- = 0, \text{ and } \lim_{\varepsilon\to 0} \int_{K_\varepsilon} \omega^- = \int_{S_u} \omega^-, \quad (4.11)$$

yi_limit4

concluding the proof □

Remark 4.4. We note that u_ε is a local minimum for the function

$$\int_\Omega |\nabla v|^2\, \omega\, dx + \int_{S_v} \omega^- \, d\mathcal{H}^{N-1} + \frac{1}{\varepsilon}\int_\Omega |u-v|^2\, \omega\, dx$$

in $\Omega \setminus K_\varepsilon$. It can be shown that

$$\lim_{\varepsilon\to 0}\int_{(\overline{S_{u_\varepsilon}}\setminus S_{u_\varepsilon})\cap(\Omega\setminus K_\varepsilon)} \omega^- = 0,$$

which, together with (4.11), yields

$$\lim_{\varepsilon\to 0}\int_{\overline{S_{u_\varepsilon}}\setminus S_{u_\varepsilon}} \omega^- = 0. \tag{4.12}$$ yi_limit5

Although (4.12) could simplify the argument used in Section 4.3, and relax the assumptions on $P^\infty(\omega)$, to keep this article self contained, we refrain from using this fact.

first_step_finites

4.2. **Construction of recovery sequence with $\omega \in \mathcal{P}_b(\Omega)$.**

limsup_n_ref

Proposition 4.5 ([21, Proposition 4.1]). *Given $\omega \in \mathcal{P}_b(\Omega)$ and $u \in L^1(\Omega)\cap L^\infty(\Omega)$, set*

$$MS^+_\omega(u) := \inf\Big\{ \limsup_{\varepsilon\to 0} AT^1_{\omega,\varepsilon}(u_\varepsilon, v_\varepsilon) : \\ (u_\varepsilon, v_\varepsilon) \in W^{1,2}(\Omega)\times W^{1,2}(\Omega), u_\varepsilon \to u \text{ in } L^1,\, v_\varepsilon \to 1 \text{ in } L^1,\, 0\le v_\varepsilon \le 1\Big\}.$$

Then $MS^+_\omega(u) \le MS_\omega(u)$.

yi_limit_sec

4.3. **Proof of Proposition 4.1.** We are now ready to prove the main result of this section. To do so, we define localized versions of MS_ω and $AT^k_{\omega,\varepsilon}$ by

$$MS_\omega(u)(A) := \int_A |\nabla u|^2\, \omega\, dx + \int_{S_u\cap A} \omega^- \, d\mathcal{H}^{N-1},$$

and

$$AT^k_{\omega,\varepsilon}(u,v)(A) := \int_A |\nabla u|^2\, v^2 \omega\, dx + \frac{1}{2c_k}\int_A \left[\varepsilon^{2k-1}\left|\nabla^{(k)} v\right|^2 + \frac{1}{4^k\varepsilon}(1-v)^2\right]\omega\, dx,$$

respectively. Here $A\subset\Omega$ is an open set.

Proof of Proposition 4.1. Let $\omega \in \mathcal{P}_r(\Omega)$ be given. By Definition 1.2, we have for any $\tau > 0$,

$$\operatorname{ess\,inf}\left\{\omega(x) :\ x \in \Omega\setminus (P^0(\omega))_\eta\right\} > 0.$$

Define

$$u_\eta := \begin{cases} 0 & \text{if } x \in (P^0(\omega))_{\eta/3},\\ u(x) & \text{otherwise.}\end{cases}$$

Then we have $S_{u_\eta} \subset \Omega\setminus (P^0(\omega))_{\eta/4}$, and observe that

$$\begin{aligned} MS_\omega(u_\tau) &= \int_\Omega |\nabla u_\eta|^2\, \omega\, dx + \int_{S_{u_\eta}} \omega^- dx \le \int_\Omega |\nabla u|^2\, \omega\, dx + \int_{S^0_u}\omega^- dx + \int_{\partial((p^0(\omega))_\eta)} \omega^- dx \\ &\le \int_\Omega |\nabla u|^2\, \omega\, dx + \int_{S^0_u} \omega^- dx + O(\eta) = MS_\omega(u) + O(\eta). \end{aligned}$$

Applying Lemma 4.3 on u_η, inside $\Omega\setminus(p^0(\omega))_{\eta/4}$, gives a sequence $u_{\eta,\tau}$ such that

$$MS(u_{\eta,\tau})(\Omega\setminus(p^0(\omega))_{\eta/4}) \le MS(u_\tau) + O(\tau) \le MS(u) + O(\tau) + O(\eta),$$

and

$$S_{u_{\eta,\tau}} \subset \Omega\setminus\Big((P^0(\omega))_{\eta/4}\cup(P^\infty(\omega))_\tau\Big) \text{ and } \omega\in\mathcal{P}_b\Big(\Omega\setminus\Big((P^0(\omega))_{\eta/4}\cup(P^\infty(\omega))_\tau\Big)\Big).$$

Then, by Proposition 4.5, there exists

$$\{\tilde u_{\eta,\tau,\varepsilon}, \tilde v_{\eta,\tau,\varepsilon}\}_{\varepsilon>0} \subset W^{1,2}\Big(\Omega\setminus\Big((P^0(\omega))_{\eta/4}\cup(P^\infty(\omega))_\tau\Big)\Big)\times W^{1,2}\Big(\Omega\setminus\Big((P^0(\omega))_{\eta/4}\cup(P^\infty(\omega))_\tau\Big)\Big)$$

such that either

$$\limsup_{\varepsilon\to 0} AT^1_{\omega,\varepsilon}(\tilde u_{\eta,\tau,\varepsilon}, \tilde v_{\eta,\tau,\varepsilon})\Big(\Omega\setminus\Big((P^0(\omega))_{\eta/4}\cup(P^\infty(\omega))_\tau\Big)\Big) \le MS(u_{\eta,\tau})(\Omega\setminus(p^0(\omega))_{\eta/4}),$$

or

$$AT^1_{\omega,\varepsilon}(\tilde{u}_{\eta,\tau,\varepsilon},\tilde{v}_{\eta,\tau,\varepsilon})\left(\Omega\setminus\left((P^0(\omega))_{\eta/4}\cup(P^\infty(\omega))_\tau\right)\right)\le MS(u_{\eta,\tau})(\Omega\setminus(p^0(\omega))_{\eta/4})+O(\varepsilon)$$

holds.

Let $\varphi_{\eta,\tau}$ to be a cut off function such that $\varphi_{\eta,\tau}\in C^\infty(\Omega)$,

$$\varphi_{\eta,\tau}(x)\equiv 1 \text{ in } \Omega\setminus\left((P^0(\omega))_{\eta/3}\cup(P^\infty(\omega))_{2\tau}\right) \text{ and } \varphi_\eta(x)\equiv 0 \text{ in } \left((P^0(\omega))_{\eta/4}\cup(P^\infty(\omega))_\tau\right).$$

Define

$$u_{\eta,\tau,\varepsilon}:=(1-\varphi_{\eta,\tau})\,\tilde{u}_{\eta,\tau,\varepsilon},$$

and

$$v_{\eta,\tau,\varepsilon}:=\tilde{v}_{\eta,\tau,\varepsilon}\wedge\tilde{v}_\varepsilon(\operatorname{dist}(\partial\left[(P^0(\omega))_{\eta/4}\cup(P^\infty(\omega))_\tau\right])),$$

with $\tilde{v}_\varepsilon$ from [14, equation (4.29)].

Hence, we have $\{u_{\eta,\tau,\varepsilon},v_{\eta,\tau,\varepsilon}\}_{\varepsilon>0}\subset W^{1,2}(\Omega)\times W^{1,2}(\Omega)$, and

$$\limsup_{\varepsilon\to 0}A^1_{\omega,\eta}(u_{\eta,\tau,\varepsilon},v_{\eta,\tau,\varepsilon})\le MS_\omega(u_{\eta,\tau})+\int_{\partial\left((P^0(\omega))_{\eta/4}\right)}\omega^+d\mathcal{H}^{N-1}+\int_{\partial((P^\infty(\omega))_\tau)}\omega^+d\mathcal{H}^{N-1}. \tag{4.13}$$

last_need_1111

We claim that the last term on the right hand side of (4.13) vanishes. Indeed, we have

$$\limsup_{\eta\to 0}\int_{\partial\left((P^0(\omega))_{\eta/4}\right)}\omega^+d\mathcal{H}^{N-1}=\limsup_{\eta\to 0}\int_{\partial\left((P^0(\omega))_{\eta/4}\right)}(2\omega-\omega^-)\,d\mathcal{H}^{N-1}\le\limsup_{\eta\to 0}\int_{\partial\left((P^0(\omega))_{\eta/4}\right)}2\omega\,d\mathcal{H}^{N-1}=0,$$

where in the last equality we used (1.6). This, together with (4.13), concludes the proof by letting $\eta\to 0$. □

Proof of Theorem 1.3. The lim inf inequality follows from Proposition 3.1. On the other hand, for any given $u\in GSBV(\Omega)$ such that $MS_\omega(u)<\infty$, we have, by Lebesgue Monotone Convergence Theorem,

$$MS_\omega(u)=\lim_{K\to\infty}MS_\omega(K\wedge u\vee -K).$$

Using a diagonal argument, together with Proposition 4.1, concludes the proof. □

Appendix

We consider the one-dimensional case $N=1$ first, and then extend to the general case $N>1$ via the slicing argument introduced in [14]. To avoid confusion, when $N=1$, we define the approximating functional, with a spatially dependent parameter $\omega\in\mathcal{P}(I)$, as

$$T^k_{\omega,\varepsilon}(u,v)=\int_I\left|u'\right|^2v^2\omega\,dx+\frac{1}{2c_k}\int_I\left[\varepsilon^{2k-1}\left|\nabla v\right|^2+\frac{1}{2^k\varepsilon}(1-v)^2\right]\omega\,dx,$$

and the one-dimensional Mumford-Shah functional, with a spatially dependent parameter $\omega\in\mathcal{P}(I)$, by

$$T_\omega(u)=\int_I\left|u'\right|^2\omega\,dx+\sum_{x\in S_u}\omega^-(x).$$

We recall that $\omega\in\mathcal{P}(I)$ implies $\mathcal{H}^0(S_\omega)<\infty$. Also, we note that ω^- is defined $\mathcal{H}^0$-a.e, hence everywhere in I. We begin with an auxiliary result.

sl_small_contral

Proposition A.1. *Let $\{v_\varepsilon\}_{\varepsilon>0}\subset W^{1,2}(I)$ be such that $0\le v_\varepsilon\le 1$, $v_\varepsilon\to 1$ in $L^1(I)$ and a.e., and*

$$\limsup_{\varepsilon\to 0}\int_I\left[\varepsilon^{2k-1}\left|v_\varepsilon^{(k)}\right|^2+\frac{1}{4^k\varepsilon}(1-v_\varepsilon)^2\right]dx<\infty.$$

Then, for any $0<\eta<1$, there exists an open set $H_\eta\subset I$ such that $I\setminus H_\eta$ is a collection of finitely many points in I, and for every set $T\subset\subset H_\eta$, we have $T\subset B^\eta_\varepsilon$ for all sufficiently small $\varepsilon>0$, where

$$B^\eta_\varepsilon:=\left\{x\in I:\,v^2_\varepsilon(x)\ge\eta\right\}.$$

Proof. Using Theorem 2.7, we have there exists $C:=C(\varepsilon_0,k,\Omega)>0$ such that

$$\limsup_{\varepsilon\to 0}\int_I\left[\varepsilon\left|v'_\varepsilon\right|^2+\frac{1}{4\varepsilon}(1-v_\varepsilon)^2\right]dx\le C\limsup_{\varepsilon\to 0}\int_I\left[\varepsilon^{2k-1}\left|v_\varepsilon^{(k)}\right|^2+\frac{1}{4^k\varepsilon}(1-v_\varepsilon)^2\right]dx<\infty.$$

Hence, by the arguments from [4, pages 1020-1021], we conclude the proof. □

We next study the minimization problem

$$c_k := \inf \left\{ \int_0^{+\infty} \left|v^{(k)}\right|^2 + \frac{1}{4^k}(1-v)^2 dx : \ v \in W^{k,2}_{\text{loc}}(0,+\infty) \right.$$
$$\left. v(0) = v'(0) = \cdots v^{(k-1)}(0) = 0, \ v(t) = 1 \text{ if } t > K_k \text{ for some } K_k > 0 \text{ depends on } k \right\}.$$

represent_constant_k

Lemma A.2. *The constant c_k is positive and*

$$c_k = \inf \left\{ \int_0^{+\infty} \left|v^{(k)}(x)\right|^2 + \frac{1}{4^k}(1-v(x))^2 dx : \ v \in W^{k,2}_{\text{loc}}(0,+\infty), \right.$$
$$\left. v(0) = v'(0) = \cdots v^{(k-1)}(0) = 0, \ \lim_{x\to\infty} v(x) = 1 \right\}.$$

Proof. The proof employs the arguments used in [15, Lemma 2.5]. Moreover, by solving the associated Euler-Lagrange equation, we have also

$$c_1 = \frac{1}{2}, \quad c_2 = \frac{1}{8}\sqrt{2}, \quad c_3 = \frac{1}{16}.$$

□

liminf_part_1d_c

Proposition A.3. (Γ-lim inf) *Given $u \in L^1(I)$, let $\omega \in \mathcal{P}(I)$ satisfying (3.1), and*

$$T^-_\omega(u) := \inf \left\{ \liminf_{\varepsilon\to 0} T^k_{\omega,\varepsilon}(u_\varepsilon, v_\varepsilon) : \right.$$
$$\left. (u_\varepsilon, v_\varepsilon) \in W^{1,2}(I) \times W^{1,2}(I), u_\varepsilon \to u \text{ in } L^1, \ v_\varepsilon \to 1 \text{ in } L^1, \ 0 \le v_\varepsilon \le 1 \right\}.$$

Then $T^-_\omega(u) \ge T_\omega(u)$.

Proof. Assume that $M := T^-_\omega(u) < \infty$, and choose u_ε and v_ε that are admissible for $T^-_\omega(u)$, such that $\lim_{\varepsilon\to 0} T^k_{\omega,\varepsilon}(u_\varepsilon, v_\varepsilon) = T^-_\omega(u)$. Since $\inf_{x\in I}\omega(x) \ge 1$, we have $\liminf_{\varepsilon\to 0} T^k_{1,\varepsilon}(u_\varepsilon, v_\varepsilon) \le \liminf_{\varepsilon\to 0} T^k_{\omega,\varepsilon}(u_\varepsilon, v_\varepsilon) < +\infty$. By Theorem 2.7 we have

$$T^1_{1,\varepsilon}(u_\varepsilon, v_\varepsilon) \le C_k T^k_{1,\varepsilon}(u_\varepsilon, v_\varepsilon) \le C_k T^k_{\omega,\varepsilon}(u_\varepsilon, v_\varepsilon) \le M + 1,$$

and by [4], we get also

$$u \in GSBV(I) \text{ and } \mathcal{H}^0(S_u) < +\infty. \tag{A.1}$$ use_la_nonsm0

The proof would be complete provided we show the following inequalities:

$$\int_I \left|u'\right|^2 \omega\, dx \le \liminf_{\varepsilon\to 0} \int_I \left|u'_\varepsilon\right|^2 v_\varepsilon^2\, \omega\, dx < +\infty, \tag{A.2}$$ use_la_nonsm1

and

$$\sum_{x\in S_u} \omega^-(x) \le \liminf_{\varepsilon\to 0} \frac{1}{c_k} \int_I \left[\varepsilon^{2k-1} \left|v_\varepsilon^{(k)}\right|^2 + \frac{1}{2^k \varepsilon}(1-v_\varepsilon)^2 \right] \omega\, dx < +\infty. \tag{A.3}$$ use_la_nonsm2

Up to a (not relabeled) subsequence, we have $u_\varepsilon \to u$ and $v_\varepsilon \to 1$ *a.e.* in I, with

$$\limsup_{\varepsilon\to 0} \frac{1}{2c_k} \int_I \left[\varepsilon^{2k-1} \left|v_\varepsilon^{(k)}\right|^2 + \frac{1}{2^k \varepsilon}(1-v_\varepsilon)^2 \right] dx \le \limsup_{\varepsilon\to 0} \frac{1}{2c_k} \int_I \left[\varepsilon^{2k-1} \left|v_\varepsilon^{(k)}\right|^2 + \frac{1}{2^k \varepsilon}(1-v_\varepsilon)^2 \right] \omega\, dx < +\infty.$$

By Proposition A.1, we deduce that, for a fixed $\eta \in (1/2, 1)$, there exists a set H_η such that for every $T \subset\subset H_\eta$, it holds

$$\int_T \left|u'\right|^2 \omega\, dx \le \liminf_{\varepsilon\to 0} \int_T \left|u'_\varepsilon\right|^2 \omega\, dx \le \frac{1}{\eta} \liminf_{\varepsilon\to 0} \int_I v_\varepsilon^2 \left|u'_\varepsilon\right|^2 \omega\, dx. \tag{A.4}$$ liminf_2

Here we used [13, Theorem 6.3.7] in the first inequality. By taking the limit $T \nearrow H_\eta$ on the left hand side of (A.4) first, and then the limit $\eta \nearrow 1$ on the right hand side, we get (A.2).

We next show (A.3). Let $t \in S_u$ be given, and for simplicity, assume that $t = 0$ and $t \in S_\omega$. By the same arguments in [4, page 1021], we can prove that there exist $\left\{t_n^1\right\}_{n=1}^\infty$, $\left\{t_n^2\right\}_{n=1}^\infty$, and $\{s_n\}_{n=1}^\infty$ such that

$$-1 < t_n^1 < s_n < t_n^2 < 1, \text{ and } \lim_{n\to\infty} t_n^1 = \lim_{n\to\infty} t_n^2 = \lim_{n\to\infty} s_n = 0,$$

and, up to a subsequence, also

$$\lim_{n\to\infty} v_{\varepsilon(n)}(t_n^1) = \lim_{n\to\infty} v_{\varepsilon(n)}(t_n^2) = 1, \text{ and } \lim_{n\to\infty} v_{\varepsilon(n)}(s_n) = 0.$$

We conclude, using Lemma A.2, that

$$\liminf_{n\to\infty} \frac{1}{2c_k}\int_{t_n^1}^{s_n}\left[\varepsilon(n)^{2k-1}\left|(v_{\varepsilon(n)})^{(k)}\right|^2 + \frac{1}{4^k\varepsilon(n)}(1-v_{\varepsilon(n)})^2\right]dx \ge \frac{c_k}{2c_k} = \frac{1}{2},$$

and, since ω is positive,

$$\begin{aligned}
&\liminf_{n\to\infty} \frac{1}{2c_k}\int_{t_n^1}^{t_n^2}\left[\varepsilon(n)^{2k-1}\left|(v_{\varepsilon(n)})^{(k)}\right|^2 + \frac{1}{4^k\varepsilon(n)}(1-v_{\varepsilon(n)})^2\right]\omega(x)\,dx \\
&\quad\ge \left(\liminf_{n\to\infty}\ \operatorname*{ess\,inf}_{r\in(t_n^1,t_n^2)} \omega(r)\right)\liminf_{n\to\infty}\frac{1}{2c_k}\left\{\int_{t_n^1}^{s_n}\left[\varepsilon(n)^{2k-1}\left|(v_{\varepsilon(n)})^{(k)}\right|^2 + \frac{1}{4^k\varepsilon(n)}(1-v_{\varepsilon(n)})^2\right]dx\right. \\
&\qquad\left. + \int_{s_n}^{t_n^2}\left[\varepsilon(n)^{2k-1}\left|(v_{\varepsilon(n)})^{(k)}\right|^2 + \frac{1}{4^k\varepsilon(n)}(1-v_{\varepsilon(n)})^2\right]dx\right\} \ge \left(\frac{1}{2}+\frac{1}{2}\right)\omega^-(0) = \omega^-(0).
\end{aligned} \tag{A.5}$$

Moreover, if $t \in S_u \setminus S_\omega$, we may use the above arguments to infer that (A.5) holds also with $\omega^-(0)$ replaced by $\omega(0)$, since $t = 0 \notin S_\omega$ implies $\omega^-(0) = \omega(0)$.

Finally, since $S_u \subset I \setminus H_\eta$, by (A.1) we have that S_u is a finite collection of points, and we may repeat the above arguments for all $t \in S_u$ by partitioning I into disjoint intervals, each of which containing at most one single point of S_u, to deduce (A.3). □

We next recall some notations and results from [14], and prove Proposition 3.1 with $N > 1$, under the assumption that $\omega \in \mathcal{W}(I)$ satisfies (3.1).

Let $\mathcal{S}^{N-1}$ be the unit sphere in $\mathbb{R}^N$, and let $\nu \in \mathcal{S}^{N-1}$ be a fixed direction. We set

$$\begin{cases}
\Pi_\nu := \left\{x \in \mathbb{R}^N : \langle x, \nu\rangle = 0\right\},\ \Omega_\nu := \{x \in \Pi_\nu : \Omega_{x,\nu} \neq \emptyset\}, \\
\Omega_{x,\nu}^1 := \{t \in \mathbb{R} : x + t\nu \in \Omega\} \quad \text{for } x \in \Pi_\nu, \\
\Omega_{x,\nu} := \{y = x + t\nu : \ t \in \mathbb{R}\} \cap \Omega, \\
u_{x,\nu}(t) := u(x + t\nu),\ x \in \Omega_\nu,\ t \in \Omega_{x,\nu}^1.
\end{cases} \tag{A.6}$$

Set $x = (x', x_N) \in \mathbb{R}^N$, where $x' \in \mathbb{R}^{N-1}$ denotes the first $N-1$ components of $x \in \mathbb{R}^N$, and given $l\colon \mathbb{R}^{N-1} \to \mathbb{R}$ and $G \subset \mathbb{R}^{N-1}$, we define the graph of l over G as

$$F(l; G) := \left\{(x', x_N) \in \mathbb{R}^N : x' \in G,\ x_N = l(x')\right\}.$$

If l is Lipschitz regular, then we call $F(l; G)$ a Lipschitz - $(N-1)$ - graph.

Theorem A.4 ([4], Theorem 3.3)**.** *Let $\nu \in \mathcal{S}^{N-1}$ be given, and assume that $u \in W^{1,2}(\Omega)$. Then, for $\mathcal{H}^{N-1}$-a.e. $x \in \Omega_\nu$, $u_{x,\nu}$ belongs to $W^{1,2}(\Omega_{x,\nu})$ and $u'_{x,\nu}(t) = \langle \nabla u(x + t\nu), \nu\rangle$.*

Proposition A.5 ([14, Proposition 3.6])**.** *Let $\nu \in \mathcal{S}^{N-1}$ be a fixed direction, $\Gamma \subset \mathbb{R}^N$ be such that $\mathcal{H}^{N-1}(\Gamma) < \infty$, and $\mathbb{P}_\nu\colon \mathbb{R}^N \to \Pi_\nu$ be a projection operator, where by (A.6), $\Pi_\nu \subset \mathbb{R}^N$ is a hyperplane in $\mathbb{R}^{N-1}$. Then*

$$\mathcal{H}^{N-1}(\mathbb{P}_\nu(\Gamma)) \le \mathcal{H}^{N-1}(\Gamma),$$

and, for $\mathcal{H}^{N-1}$-a.e. $x \in \Pi_\nu$,

$$\mathcal{H}^0(\Omega_{x,\nu} \cap \Gamma) < +\infty. \tag{A.7}$$

Lemma A.6 ([14, Lemma 3.9])**.** *Let $\tau > 0$ and $\eta > 0$ be given. Let $u \in SBV(\Omega)$ and assume that $\mathcal{H}^{N-1}(S_u) < \infty$. The following statements hold:*

a. there exist a set $S \subset S_u$ with $\mathcal{H}^{N-1}(S_u \setminus S) < \eta$, and a countable collection $\mathcal{Q}$ of mutually disjoint, open cubes centered on elements of S_u, such that $\bigcup_{Q\in\mathcal{Q}} Q \subset \Omega$, and $\mathcal{H}^{N-1}\left(S \setminus \bigcup_{Q\in\mathcal{Q}} Q\right) = 0$;

b. for every $Q \in \mathcal{Q}$ there exists a direction vector $\nu_Q \in \mathcal{S}^{N-1}$ such that $\mathcal{H}^0(S \cap Q_{x,\nu_Q}) = 1$ for $\mathcal{H}^{N-1}$ a.e. $x \in Q \cap S$;

c. $S \cap Q$ is contained in a Lipschitz $(N-1)$- graph Γ_Q, with Lipschitz constant not exceeding τ.

Now we are ready to prove the main result of this Section.

Proof of Proposition 3.1, with ω satisfying (3.1). Assume that $M := MS^-_\omega(u) < \infty$. Let $\{(u_\varepsilon, v_\varepsilon)\}_{\varepsilon>0} \subset W^{1,2}(\Omega)\times W^{1,2}(\Omega)$ be such that $u_\varepsilon \to u$ in L^1, $v_\varepsilon \to 1$ in $L^1(\Omega)$, and $\lim_{\varepsilon\to 0} AT^k_{\omega,\varepsilon}(u_\varepsilon, v_\varepsilon) = MS^-_\omega(u)$. Since $\inf_{x\in\Omega}\omega(x) \geq 1$, we have

$$\liminf_{\varepsilon\to 0} AT^k_{1,\varepsilon}(u_\varepsilon, v_\varepsilon) \leq \liminf_{\varepsilon\to 0} AT^k_{\omega,\varepsilon}(u_\varepsilon, v_\varepsilon) < \infty,$$

and by [4], we deduce that

$$u \in GSBV(\Omega) \text{ and } \mathcal{H}^{N-1}(S_u) < \infty.$$

We show separately that

$$\liminf_{\varepsilon\to 0} \int_\Omega |\nabla u_\varepsilon|^2\, v_\varepsilon\, \omega\, dx \geq \int_\Omega |\nabla u|^2\, \omega\, dx, \tag{A.8}$$

and

$$\liminf_{\varepsilon\to 0} \frac{1}{2c_k}\int_\Omega \left(\varepsilon^{2k-1}\left|\nabla^{(k)} v_\varepsilon\right|^2 + \frac{1}{4^k\varepsilon}(1-v_\varepsilon)^2\right)\omega\, dx \geq \int_{S_u} \omega^-\, d\mathcal{H}^{N-1}. \tag{A.9}$$

Let A be an open subset of Ω. Fix $\nu \in \mathcal{S}^{N-1}$, and define $A_{x,\nu}$, $A^1_{x,\nu}$, and A_ν as in (A.6). For $K \in \mathbb{R}^+$, set $u_K := K \wedge u \vee -K$, $K \in \mathbb{N}$, and we observe, by Fubini's Theorem, Fatou's Lemma, Theorem A.4, equation (A.2), and Theorem 2.3 in [4], that

$$\begin{aligned}\liminf_{\varepsilon\to 0} \int_A |\nabla u_\varepsilon|^2\, v_\varepsilon^2\, \omega\, dx &\geq \int_{A_\nu} \liminf_{\varepsilon\to 0} \int_{A^1_{x,\nu}} \left|(u_\varepsilon)'_{x,\nu}\right|^2 (v_\varepsilon)^2_{x,\nu}\, \omega_{x,\nu}\, dt\, d\mathcal{H}^{N-1}(x)\\ &\geq \int_{A_\nu}\int_{A^1_{x,\nu}} \left|(u_K)'_{x,\nu}\right|^2 \omega_{x,\nu}\, dt\, d\mathcal{H}^{N-1}(x) \geq \int_A |\langle \nabla u_K(x), \nu\rangle|^2\, \omega\, dx.\end{aligned} \tag{A.10}$$

Taking the limit $K \to \infty$, and using Dominated Convergence Theorem, we have

$$\liminf_{\varepsilon\to 0} \int_A |\nabla u_\varepsilon|^2\, v_\varepsilon^2\, \omega\, dx \geq \int_A |\langle \nabla u(x), \nu\rangle|^2\, \omega\, dx. \tag{A.11}$$

Let $\phi_n(x) := |\langle \nabla u(x), \nu_n\rangle|^2\, \omega$ for $\mathcal{L}^N$-a.e. $x \in \Omega$, where $\{\nu_n\}_{n=1}^\infty$ is a dense subset of $\mathcal{S}^{N-1}$, and let

$$\mu(A) := \liminf_{\varepsilon\to 0} \int_A |\nabla u_\varepsilon|^2\, v_\varepsilon^2\, \omega\, dx.$$

Then μ is positive, super-additive on any pair of open sets A and B with disjoint closures, and, by [6, Lemma 15.2] and (A.11), we conclude (A.8). Now we prove (A.9). By Fubini's Theorem, Fatou's Lemma, (A.7), and (A.3), and using similar arguments as in (A.10), we have

$$\liminf_{\varepsilon\to 0} \frac{1}{2c_k}\int_A \left(\varepsilon^{2k-1}\left|\nabla^{(k)} v_\varepsilon\right|^2 + \frac{1}{4^k\varepsilon}(1-v_\varepsilon)^2\right)\omega\, dx \quad\geq\quad \int_{A_\nu}\left[\sum_{t\in S_{u_{x,\nu}}\cap A^1_{x,\nu}} \omega^-_{x,\nu}(t)\right] d\mathcal{H}^{N-1}(x). \tag{A.12}$$

Next, given arbitrary $\tau > 0$ and $\eta > 0$, we choose a set $S \subset S_u$ and a collection $\mathcal{Q}$ of mutually disjoint cubes according to Lemma A.6 with respect to S_u. Fix one such cube $Q_{\nu_S}(x_0, r_0) \in \mathcal{Q}$. By Lemma A.6, we have, up to rigid motions,

$$\Gamma_{x_0} = \left\{(y', l_{x_0}(y')):\ y \in T_{x_0,\nu_S} \cap Q_{\nu_S}(x_0, r_0)\right\} \text{ and } \|\nabla l_{x_0}\|_{L^\infty} < \tau.$$

In (A.12), set $A = Q_{\nu_S}(x_0, r_0)$ and $\nu = \nu_S(x_0)$. Using the same notation from the proof of Lemma A.6, we obtain

$$\begin{aligned}&\int_{[Q_{\nu_S}(x_0,r_0)]_{\nu_S(x_0)}} \left(\sum_{t\in S_{u_{x,\nu_S(x_0)}}\cap[Q_{\nu_S}(x_0,r_0)]_{x,\nu_S(x_0)}} \omega^-_{x,\nu_S(x_0)}(t)\right) d\mathcal{H}^{N-1}(x)\\ &\qquad\geq \int_{T_g(x_0,r_0)} \omega^-(x)\, d\mathcal{H}^{N-1}(x) = \int_{T_g(x_0,r_0)} \omega^-(x', l_{x_0}(x'))\, d\mathcal{L}^{N-1}(x').\end{aligned} \tag{A.13}$$

Next, considering that $\omega^-_{x,\nu}(t) = \omega^-(x+t\nu)$ (see [3, Remark 3.109]), we have that

$$\begin{aligned}\int_{Q_{\nu_S}(x_0,r_0)\cap S} \omega^-\, d\mathcal{H}^{N-1} &= \int_{T_{x_0,\nu_S}\cap Q_{\nu_S}(x_0,r_0)} \omega^-(x', l_{x_0}(x'))\sqrt{1+|\nabla l_{x_0}(x')|^2}dx' \\ &\le \sqrt{1+\tau^2}\int_{T_{x_0,\nu_S}\cap Q_{\nu_S}(x_0,r_0)} \omega^-(x', l_{x_0}(x'))dx',\end{aligned}$$

which, together with (A.12) and (A.13), yields

$$\begin{aligned}&\liminf_{\varepsilon\to 0}\int_\Omega \left(\varepsilon^{2k-1}\left|\nabla^{(k)} v_\varepsilon\right|^2 + \frac{1}{4^k\varepsilon}(1-v_\varepsilon)^2\right)\omega\, dx \\ &\ge \liminf_{\varepsilon\to 0}\int_{\cup_{Q\in\mathcal{Q}}Q} \left(\varepsilon^{2k-1}\left|\nabla^{(k)} v_\varepsilon\right|^2 + \frac{1}{4^k\varepsilon}(1-v_\varepsilon)^2\right)\omega\, dx \\ &\ge \frac{1}{\sqrt{1+\tau^2}}\sum_{Q\in\mathcal{Q}}\int_{S\cap Q}\omega^-\, d\mathcal{H}^{N-1} \ge \frac{1}{\sqrt{1+\tau^2}}\left(\int_{S_u}\omega^-\, d\mathcal{H}^{N-1} - \|\omega\|_{L^\infty}\eta\right).\end{aligned}$$

Finally, (A.9) follows by the arbitrariness of η and τ. □

We recall $Q_{\nu_{S_\omega}}(x_0,r)$ and $T_{x_0,\nu_{S_\omega}}(l)$ from Notation 2.5, 1, and 2, and define $I(t_0,t) := (t_0-t, t_0+t)\subset\mathbb{R}$ for $t_0\in\mathbb{R}$ and $t\in\mathbb{R}^+$.

Proposition A.7. *Let $\omega\in\mathcal{W}(\Omega)$ and $\tau\in(0,1/4)$ be given. Then, there exist a set $S\subset S_\omega$, and a countable family of disjoint cubes $\mathcal{F} = \left\{Q_{\nu_{S_\omega}}(x_n,r_n)\right\}_{n=1}^\infty$, with $r_n<\tau$, such that the following assertions hold:*

a. $\mathcal{H}^{N-1}(S_\omega\setminus S) < \tau$ *and* $S\subset\bigcup_{n=1}^\infty Q_{\nu_{S_\omega}}(x_n,r_n)\subset\Omega$;

b. $\operatorname{dist}(Q_{\nu_{S_\omega}}(x_n,r_n), Q_{\nu_{S_\omega}}(x_{n'},r_{n'})) > 0$ *for* $n\neq n'$;

c. $S\cap Q_{\nu_{S_\omega}}(x_n,r_n)\subset R_{\tau/2,\nu_{S_\omega}}(x_n,r_n)$;

d. $(1+\tau^2)^{-1}r_n^{N-1}\le\mathcal{H}^{N-1}\left(S\cap Q_{\nu_{S_\omega}}(x_n,r_n)\right)\le(1+\tau^2)r_n^{N-1}$;

e. $\sum_{n=1}^\infty r_n^{N-1}\le 4\mathcal{H}^{N-1}(S_\omega)$;

f. for each $n\in\mathbb{N}$, there exists $t_n\in(2.5\tau r_n, 3.5\tau r_n)$ and $0<t_{x_n,r_n}<t_n$, depending on τ, r_n, and x_n, such that $T_{x_n,\nu_{S_\omega}}(-t_n\pm t_{x_n,r_n})\cap Q_{\nu_{S_\omega}}(x_n,r_n)\subset Q^-_{\nu_{S_\omega}}(x_n,r_n)\setminus R_{\tau/2,\nu_{S_\omega}}(x_n,r_n)$ and, where we recall $I(t_n,t) := (-t_n-t, -t_n+t)$,

$$\begin{aligned}&\sup_{0<t\le t_{x_n,r_n}}\frac{1}{|I(t_n,t)|}\int_{I(t_n,t)}\int_{Q_{\nu_{S_\omega}}(x_n,r_n)\cap T_{x_n,\nu_{S_\omega}}(-l)}\omega^-(x)d\mathcal{H}^{N-1}dl \\ &\le\int_{S\cap Q_{\nu_{S_\omega}}(x_n,r_n)}\omega^-\, d\mathcal{H}^{N-1} + O(\tau)r^{N-1}.\end{aligned}\tag{A.14}$$

Proof. The proof uses similar arguments as in [14, Proposition 4.4]. □

Since this proof is quite lengthy, we summarize the main ideas. We modify the bulk part of S_u by replacing it with $(N-1)$ polyhedra located in the $-\nu_{S_\omega}$ direction of S_ω, and note that both the L^1-norm of u and the L^2-norm of ∇u do not change much. This will be done via a reflection argument around suitable hyperplanes. For the remaining part of S_u, we shall cover them using a finite collection of cubes, and change the value of u to 0 over such cubes. Hence, in this way, we transfer the jump set of S_u to a finite union of polyhedra.

Proposition A.8. *Let $u\in SBV^2(\Omega)\cap L^\infty(\Omega)$ be given, satisfying $\mathcal{H}^{N-1}(\overline{S_u})<+\infty$ and $\omega\in\mathcal{W}(\Omega)$. Then there exists a sequence $\{(u_\varepsilon,v_\varepsilon)\}_{\varepsilon>0}\subset W^{1,2}(\Omega)\times W^{1,2}(\Omega)$ such that*

$$\limsup_{\varepsilon\to 0}E_{\omega,\varepsilon}(u_\varepsilon,v_\varepsilon)\le E_\omega(u).$$

Proof. Without loss of generality, we assume that $E_\omega(u)<+\infty$, which implies $\mathcal{H}^{N-1}(S_u)<+\infty$.
Step 1: Assume $\mathcal{H}^{N-1}((S_\omega\setminus S_u)\cup(S_u\setminus S_\omega)) = 0$. Fix $\tau\in(0,1/4)$. Applying Proposition A.7 to ω, we obtain a set S_τ, a collection $\mathcal{F}_\tau = \left\{Q_{\nu_{S_\omega}}(x_n,r_n)\right\}_{n=1}^\infty$, and corresponding $t_n\in(2.5\tau r_n, 3.5\tau r_n)$ and t_{x_n,r_n}, for which (A.14)

holds. Extract a finite collection $\mathcal{T}_\tau = \left\{Q_{\nu_{S_\omega}}(x_n, r_n)\right\}_{n=1}^{M_\tau}$ from $\mathcal{F}_\tau$ with $M_\tau > 0$, large enough such that

$$\mathcal{H}^{N-1}\left[S_\tau \setminus \bigcup_{n=1}^{M_\tau} Q_{\nu_{S_\omega}}(x_n, r_n)\right] < \tau,$$

ext_fint_part

and set $F_\tau := S_\tau \cap \left[\bigcup_{n=1}^{M_\tau} Q_{\nu_{S_\omega}}(x_n, r_n)\right]$. Note that

$$\mathcal{H}^{N-1}(S_u \setminus F_\tau) \le \mathcal{H}^{N-1}(S_u \setminus S_\tau) + \mathcal{H}^{N-1}(S_\tau \setminus F_\tau) < 2\tau. \tag{A.15}$$

qu_left_F_cont

We observe that

$$\mathcal{L}^N(\{x \in \Omega,\, \bar{u}(x) \neq \bar{u}_\tau(x)\}) = \mathcal{L}^N\left(\bigcup_{n=1}^{M_\tau} U_n\right) \le \sum_{n=1}^{M_\tau} \mathcal{L}^N(U_n) \le 7\tau^2 \sum_{n=1}^{M_\tau} r_n^{N-1} \le O(\tau),$$

where in the last inequality we used Propositions A.7 and e. We note that

a. $\bar{u}_\tau$ is a reflection of $\bar{u}$ within the set with measure less than $O(\tau)$;
b. $\mathcal{L}^N(\{\bar{u} \neq u\}) \le \sum_{m=1}^{Y_\tau} \mathcal{L}^N(Q_m) \le O(\tau)$;
c. $u \in SBV^2(\Omega) \cap L^\infty(\Omega)$.

Then,

$$\lim_{\tau\to 0} \int_\Omega |\bar{u}_\tau - u|\, dx = 0 \text{ and } \lim_{\tau\to 0} \int_\Omega |\nabla \bar{u}_\tau - \nabla u|^2\, dx = 0. \tag{A.16}$$

L1omcont

For brevity, in the rest of the proof we abbreviate $Q_{\nu_{S_\omega}}(x_n, r_n)$ by Q_n, $T_{x_n,\nu_{S_u}}$ by T_{x_n}, and $T_{x_n,\nu_{S_u}}(-t_n)$ by $T_{x_n}(-t_n)$. Note that the jump set of $\bar{u}_\tau$ is contained in

$$P_\tau := \bigcup_{n=1}^{M_\tau} [T_{x_n}(-t_n) \cap Q_n] \cup \bigcup_{n=1}^{M_\tau} \partial Q_n \cap \overline{U_n} \cup \bigcup_{m=1}^{Y_\tau} \partial Q_m \cup \bigcup_{m=1}^{Y_\tau} \partial R_m,$$

poly_p_tau

and $S_{\bar{u}_\tau} \subset P_\tau$ and P_τ are both union of finitely many polyhedra. We also observe that, denoting by $\mathrm{cl}(\cdot)$ the closure of a set,

$$\begin{aligned}
&\mathcal{H}^{N-1}\left[\mathrm{cl}\left(\left(\bigcup_{n=1}^{M_\tau} \partial Q_n \cap \overline{U_n}\right) \cup \left(\bigcup_{m=1}^{Y_\tau} \partial Q_m\right) \cup \left(\bigcup_{m=1}^{Y_\tau} \partial R_m\right)\right)\right] \\
&\le \sum_{n=1}^{M_\tau} \mathcal{H}^{N-1}(\partial Q_n \cap \overline{U_n}) + \sum_{m=1}^{Y_\tau} \mathcal{H}^{N-1}(\partial Q_m) + \sum_{m=1}^{Y_\tau} \mathcal{H}^{N-1}(\partial R_m) \qquad (A.17) \\
&\le 2\tau + C\tau \sum_{n=1}^{\infty} r_n^{N-1}\tau + 2\mathcal{H}^{N-1}(\overline{S_u} \setminus S_u) \le O(\tau) + 2\mathcal{H}^{N-1}(\overline{S_u} \setminus S_u) < +\infty,
\end{aligned}$$

finite_size_rede

where we used Proposition A.7 e, (A.15), and the assumption that $\mathcal{H}^{N-1}(\overline{S_u}) < +\infty$.
Let $\varepsilon > 0$ be such that

$$\varepsilon^2 + \sqrt{\varepsilon} \ll \min\{a_\tau,\, t_{x_n,r_n} \text{ for } 1 \le n \le M_\tau\}.$$

finite_small_pos

Hence, by Propositions A.7 and f, we have

$$\varepsilon^2 + \sqrt{\varepsilon} < t_{x_n,r_n} < |t_n| < \frac{1}{4}\tau r_n < r_n.$$

finite_small_pos

We set $u_{\tau,\varepsilon} := (1 - \varphi_\varepsilon)\bar{u}_\tau$, where φ_ε is such that $\varphi_\varepsilon \in C_c^\infty(\Omega; [0,1])$, $\varphi_\varepsilon \equiv 1$ on $(\overline{S_{\bar{u}_\tau}})_{\varepsilon^2/4}$, and $\varphi_\varepsilon \equiv 0$ in $\Omega \setminus (\overline{S_{\bar{u}_\tau}})_{\varepsilon^2/2}$. By (A.17) we have $\mathcal{H}^{N-1}(\overline{S_{\bar{u}_\tau}}) < +\infty$, hence $\{u_{\tau,\varepsilon}\}_{\varepsilon>0} \subset W^{1,2}(\Omega)$. Moreover, by the Dominated Convergence Theorem, and (A.16), we conclude that $u_{\tau,\varepsilon} \to u$ in $L^1(\Omega)$.

Consider the sequence $\{v_{\tau,\varepsilon}\}_{\varepsilon>0} \subset W^{1,2}(\Omega)$ given by $v_{\tau,\varepsilon}(x) := \tilde{v}_\varepsilon(d_\tau(x))$, where $d_\tau(x) := \operatorname{dist}(x, P_\tau)$ and $\tilde{v}_\varepsilon$ are defined by

$$\tilde{v}_\varepsilon(t) := \begin{cases} 0 & \text{if } t \le \varepsilon^2, \\ -e^{-\frac{1}{2}\frac{t-\varepsilon^2}{\varepsilon}} + 1 & \text{if } \varepsilon^2 \le t \le \sqrt{\varepsilon} + \varepsilon^2, \\ 1 - e^{-\frac{1}{2\sqrt{\varepsilon}}} & \text{if } t > \sqrt{\varepsilon} + \varepsilon^2. \end{cases}$$ benshen

An explicit computation shows that

$$\tilde{v}_\varepsilon'(t) = \frac{1}{2\varepsilon}(1 - \tilde{v}_\varepsilon(t))$$ benzun

for $\varepsilon^2 \le t \le \sqrt{\varepsilon} + \varepsilon^2$ and $\tilde{v}_\varepsilon \in W^{1,2}_{\text{loc}}(\mathbb{R})$, and we remark that

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} e^{-\frac{1}{2\sqrt{\varepsilon}}} = 0,$$ ele_multi_ago1

and

$$-\frac{d}{dt}\left(\frac{1}{2}(1 - \tilde{v}_\varepsilon(t))^2\right) = (1 - \tilde{v}_\varepsilon(t))\,\tilde{v}_\varepsilon'(t) \ge 0.$$ ele_multi_ago2

Moreover, since $S_{u_\tau} \subset P_\tau$ and by (L1omcont A.16), we conclude that

$$\int_\Omega |\nabla u_{\tau,\varepsilon}|^2\, v_{\tau,\varepsilon}^2\, \omega\, dx \le \int_\Omega |\nabla \bar{u}_\tau|^2\, \omega\, dx \le \int_\Omega |\nabla u|^2\, \omega\, dx + O(\tau).$$ cont_naj_est2

Step 2: For the general case $\mathcal{H}^{N-1}(S_u \setminus S_\omega) > 0$, the proof follows by applying the same construction in Step 1 on S_u, and noticing that $\omega^-(x) = \omega(x)$ if $x \in S_u \setminus S_\omega$. □

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References

o1989variational [1] L. Ambrosio. Variational problems in SBV and image segmentation. *Acta Appl. Math.*, 17(1):1–40, 1989.

mbrosio1994lower [2] L. Ambrosio. On the lower semicontinuity of quasiconvex integrals in $\mathrm{SBV}(\Omega, \mathbb{R}^k)$. *Nonlinear Anal.*, 23(3):405–425, 1994.

sio2000functions [3] L. Ambrosio, N. Fusco, and D. Pallara. *Functions of bounded variation and free discontinuity problems.* Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 2000.

990approximation [4] L. Ambrosio and V. M. Tortorelli. Approximation of functionals depending on jumps by elliptic functionals via Γ-convergence. *Comm. Pure Appl. Math.*, 43(8):999–1036, 1990.

h1984variational [5] H. Attouch. *Variational convergence for functions and operators.* Applicable Mathematics Series. Pitman (Advanced Publishing Program), Boston, MA, 1984.

braides2002gamma [6] A. Braides. *Γ-convergence for beginners*, volume 22 of *Oxford Lecture Series in Mathematics and its Applications.* Oxford University Press, Oxford, 2002.

is2010functional [7] H. Brezis. *Functional analysis, Sobolev spaces and partial differential equations.* Universitext. Springer, New York, 2011.

MR3429728 [8] M. Burger, T. Esposito, and C. I. Zeppieri. Second-order edge-penalization in the Ambrosio-Tortorelli functional. *Multiscale Model. Simul.*, 13(4):1354–1389, 2015.

troni2015bilevel [9] L. Calatroni, C. Chung, J. C. D. L. Reyes, C.-B. Schönlieb, and T. Valkonen. Bilevel approaches for learning of variational imaging models. *arXiv preprint arXiv:1505.02120*, 2015.

de1989existence [10] E. De Giorgi, M. Carriero, and A. Leaci. Existence theorem for a minimum problem with free discontinuity set. *Arch. Rational Mech. Anal.*, 108(3):195–218, 1989.

reyes2015structure [11] J. C. De Los Reyes, C.-B. Schönlieb, and T. Valkonen. The structure of optimal parameters for image restoration problems. *J. Math. Anal. Appl.*, 434(1):464–500, 2016.

evans2015measure [12] L. C. Evans and R. F. Gariepy. *Measure theory and fine properties of functions*. Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1992.

fonseca2015modern [13] I. Fonseca and G. Leoni. *Modern methods in the calculus of variations: Sobolev spaces*. Springer Monographs in Mathematics. Unpublished yet, 2015.

liu2016weightedMS [14] I. Fonseca and P. Liu. The Ambrosio - Tortorelli Approximation with spatially dependent parameters. *arXiv:1608.03878*, Aug. 2016.

fonseca2000second [15] I. Fonseca and C. Mantegazza. Second order singular perturbation models for phase transitions. *SIAM J. Math. Anal.*, 31(5):1121–1143 (electronic), 2000.

MR1488299 [16] M. Gobbino. Finite difference approximation of the Mumford-Shah functional. *Comm. Pure Appl. Math.*, 51(2):197–228, 1998.

2016arXiv160901074H [17] M. Hintermüller, K. Papafitsoros, and C. N. Rautenberg. Analytical aspects of spatially adapted total variation regularisation. *ArXiv e-prints*, Sept. 2016.

MR2049779 [18] M. Hintermüller and W. Ring. An inexact Newton-CG-type active contour approach for the minimization of the Mumford-Shah functional. *J. Math. Imaging Vision*, 20(1-2):19–42, 2004. Special issue on mathematics and image analysis.

kunisch2013bilevel [19] K. Kunisch and T. Pock. A bilevel optimization approach for parameter learning in variational models. *SIAM J. Imaging Sci.*, 6(2):938–983, 2013.

leoni2009first [20] G. Leoni. *A first course in Sobolev spaces*, volume 105 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2009.

liu2016optimal [21] P. Liu. The spatially dependent bi-level learning scheme for image reconstruction problem. *Submitted*.