BOUNDS ON THE HEAT TRANSFER RATE VIA PASSIVE ADVECTION

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ABSTRACT. In heat exchangers, an incompressible fluid is heated initially and cooled at the boundary. The goal is to transfer the heat to the boundary as efficiently as possible. In this paper we study a related steady version of this problem where a steadily stirred fluid is uniformly heated in the interior and cooled on the boundary. For a given large Péclet number, how should one stir to minimize some norm of the temperature? This version of the problem was previously studied by Marcotte, Doering et al. (SIAM Appl. Math '18) in a disk, where the authors showed that when the Péclet number, Pe, is sufficiently large one can stir the fluid in a manner that ensures the total heat is $O(1/\mathrm{Pe})$. In this paper we instead study the problem on an infinite strip. By forming standard convection rolls we show that one can stir the fluid in a manner that ensures that the temperature of the hottest point is $O(1/\mathrm{Pe}^{4/7})$, up to a logarithmic factor. The same upper bound is expected to be true for the total heat and other L^p-norms of the temperature. We do not, however, know if this is optimal in a strip and are presently unable to prove a matching lower bound.

1. Introduction

A heat exchanger is a system used to transfer heat between a fluid and a heat source or sink, for either heating or cooling. These are used for both heating and cooling processes and have a broad range applications including combustion engines, sewage treatment, nuclear power plants and cooling CPU's in personal computers. The study of heat exchanger is a vibrant field of research despite having a very long history. The literature in the field is vast, ranging from engineering textbooks to cutting edge research articles. We give here a very non-exhaustive list of representatives [WBZ92, QM02, VP14, SuHS$^+$19, AK18, WWZ$^+$18, LL20]. Mathematically, although there have been some rigorous treaments [DT19, MDTY18], a lot still remain to be explored.

The temperature of the fluid in the heat exchanger evolves according to the advection diffusion equation

$$\partial_t \theta + v \cdot \nabla \theta - \kappa \Delta \theta = 0 \qquad \text{in } \Omega, \tag{1.1}$$

where $\Omega \subseteq \mathbb{R}^d$ is the region occupied by the fluid. Here θ is the temperature of the fluid, κ is the thermal diffusivity and $v = v(x,t)$ is velocity field of the fluid. Throughout this paper we will assume the fluid is incompressible and doesn't flow through the container walls. That is, we require hence require

$$\nabla \cdot v = 0 \quad \text{in } \Omega, \qquad \text{and} \qquad v \cdot \hat{n} = 0 \quad \text{on } \partial\Omega. \tag{1.2}$$

Some portion of the boundary of Ω may be insulated, and some portion may be connected to a heat source/sink maintained at a constant temperature. Denoting these pieces by $\partial_N \Omega$ and $\partial_D \Omega$ respectively, and normalizing so that the temperature of the heat source/sink is 0, we study (1.1) with mixed Dirichlet/Neumann boundary conditions

$$\partial_{\hat{n}} \theta = 0 \quad \text{on } \partial_N \Omega\,, \qquad \text{and} \qquad \theta = 0 \quad \text{on } \partial_D \Omega\,.$$

A problem of practical interest is to minimize some norm of the temperature θ under a constraint on the stirring velocity field v. Note, here we assume (1.1) is a passive scalar equation – the velocity field v is arbitrarily prescribed and is not coupled to the temperature profile. The active scalar case entails coupling v to θ via the Boussinesq system and leads to Rayleigh–Bénard convection which has been extensively studied [Ray16, SG88, Kad01, DOR06].

In order to simplify matters, we set $\kappa = \frac{1}{2}$, assume v is time independent, and assume the initial temperature θ_0 is identically 1. In this case we set $T = \int_0^\infty \theta(x,t)\,dt$ and observe

$$(1.3) \qquad\qquad -\frac{1}{2}\Delta T + v \cdot \nabla T = 1\,,$$

in Ω, with boundary conditions

$$(1.4) \qquad\qquad T = 0 \quad \text{on } \partial_D \Omega\,, \qquad \text{and} \qquad \partial_{\hat{n}} T = 0 \quad \text{on } \partial_N \Omega\,.$$

In this case the analog of the above minimization problem would be to minimize some norm of T under a size constraint on the advecting velocity field v.

In the recent paper [MDTY18], the authors studied this minimization problem when $\Omega \subseteq \mathbb{R}^2$ is a disk of radius 1, and $\partial_N \Omega = \emptyset$. Given $p \in [1, \infty)$ and Pe > 0 let $\mathcal{V}_{\text{Pe}}^p$ be the set of all L^p velocity fields satisfying (1.2) such that

$$(1.5) \qquad\qquad \|v\|_{L^p(\Omega)} \leqslant \text{Pe}\,,$$

and define

$$\mathcal{E}_{p,q}(\text{Pe}) \overset{\text{def}}{=} \inf_{v \in \mathcal{V}_{\text{Pe}}^p} \|T^v\|_q\,.$$

Physically when $p = 2$, the constraint (1.5) limits the kinetic energy of the ambient fluid. The quantity Pe is the Péclet number associated to (1.3) and is a non-dimensional ratio measuring the relative strength of the advection to the diffusion. When the Péclet number is sufficiently large, the authors of [MDTY18] use matched asymptotics to show

$$(1.6) \qquad\qquad \mathcal{E}_{2,1}(\text{Pe}) \leqslant O\Big(\frac{1}{\text{Pe}}\Big)\,,$$

and support their results with numerics (see Remark 1.4). Here T^v is simply the solution to (1.3)–(1.4), and we introduced the superscript v to emphasize the dependence of T on v.

In this paper we revisit this problem and aim to provide mathematically rigorous proofs of the bounds in [MDTY18]. Making matched asymptotics rigorous arises in many situations and has been extensively studied (see for instance [BLP78, Kus84, Ngu89, Eva90, All92, PS08]). In this situation, however, the flow considered in [MDTY18] leads to a degenerate homogenization problem, for which one can not use these standard techniques. Instead we reformulate the problem probabilistically and use asymmetric large deviations estimates handle the degenerate diffusivity.

To simplify the proofs, we study the problem in a horizontal strip instead of the disk. For boundary conditions we cool the top of the strip, insulate the bottom, and impose 1-periodic boundary conditions in the horizontal direction. In this case, a natural ansatz to consider is vertical convection rolls, with height $O(1)$ whose width and amplitude depend on the Péclet number. Moreover, as we will shortly see, it is more efficient to concentrate the entire velocity field in the boundary of these convection rolls, and have large stagnation zones in the interior (see Figure 1, below). Our main result shows that by tuning the width and amplitude of these convection rolls, up to a logarithmic factor, one can ensure $\mathcal{E}_{p,\infty} \leqslant O(1/\operatorname{Pe}^{2p/(4p-1)})$ for all $p \geqslant 1$. For $p = 2$ this gives $\mathcal{E}_{2,\infty} \leqslant O(1/\operatorname{Pe}^{4/7})$ (up to a logarithmic factor).
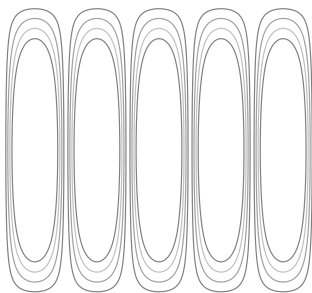

FIGURE 1. Convection rolls with velocity field focussed on the boundary layer.

To formulate our result precisely, let let $S = \mathbb{R} \times (0,1) \subseteq \mathbb{R}^2$ be an infinite horizontal strip and $\partial_D S = \mathbb{R} \times \{1\}$ be the top boundary (where we impose homogeneous Dirichlet boundary conditions), and $\partial_N S = \mathbb{R} \times \{0\}$ the bottom boundary (where we impose homogeneous Neumann boundary conditions). We will impose 1-periodic boundary conditions in the horizontal direction and identify the function spaces $H^1(S)$ and $L^2(S)$ can be identified with 1-periodic functions that are in $H^1(\Omega)$ or $L^2(\Omega)$ respectively, where $\Omega \stackrel{\text{def}}{=} (0,1)^2$ is the unit square.

Theorem 1.1. *Given $v \in \mathcal{V}_{\operatorname{Pe}}^p$ let T^v be the solution to (1.3) in S with $T^v = 0$ on $\partial_D S$, $\partial_2 T^v = 0$ on $\partial_N S$ and 1-periodic boundary conditions in the horizontal direction. Then there exists a finite constant C such that for every $\mu > 0$,*

$$(1.7) \qquad \mathcal{E}_{p,\infty}(\operatorname{Pe}) \leqslant \mathcal{R}_p^\mu(\operatorname{Pe}) \stackrel{\text{def}}{=} C|\ln \operatorname{Pe}|^{13}\left(\frac{1}{\operatorname{Pe}}\right)^{\frac{2p}{4p-1}}$$

for sufficiently large Pe. *The velocity field attaining (1.7) can be chosen to be focussed on the boundary of convection rolls of height 1 and width $O\left(\frac{|\ln \operatorname{Pe}|^{\frac{1}{p}}}{\operatorname{Pe}^{\frac{p}{4p-1}}}\right)$.*

Remark 1.2. Note that the bound in Theorem 1.1 is *weaker* than the bound (1.6) obtained in [MDTY18]. However, the spatial domains in each case are different. The bounds in Theorem 1.1 apply to a strip, where as the bounds in [MDTY18] apply to the unit disk. We do not know whether the optimal bounds are universal (i.e. domain independent), and thus we do not know whether mismatch in powers is intrinsic or a deficiency of the proof.

Remark 1.3. For the velocity field $v \in \mathcal{V}_{\mathrm{Pe}}^p$ that we construct it is not hard to show that for every $q \in [1, \infty]$, there exists a constant $C_{p,q}$ such that

$$(1.8) \qquad \frac{\mathcal{R}_{p,\mu}(\mathrm{Pe})}{C_{p,q,\mu}} \leqslant \|T^v\|_{L^q} \leqslant C_{p,q,\mu}\mathcal{R}_p(\mathrm{Pe}),$$

when Pe is sufficiently large. Moreover, for any non-degenerate velocity field focussed on the boundary of convection rolls with height $O(1)$ and width $O\left(\frac{|\ln \mathrm{Pe}|^{\frac{1}{p}}}{\mathrm{Pe}^{\frac{p}{4p-1}}}\right)$, one can also show that (1.8) holds. The proof of this is similar to that of Theorem 1.1.

Remark 1.4. More generally, one may consider velocity fields with convection rolls with a width and height that vanish as $\mathrm{Pe} \to \infty$. We consider this in Section 6, below, and show that the optimal choice is keep the height $O(1)$ as $\mathrm{Pe} \to \infty$. Moreover, one may also consider convection rolls arising from Hamiltonians with degenerate critical points. One can show that even in this case the bounds for $\|T^v\|_{L^\infty}$ are larger than the bounds obtained in Theorem 1.1. As a result, it appears that standard convection rolls in a strip will never yield the $O(1/\mathrm{Pe})$ bound obtained in [MDTY18] for a disk. As mentioned in Remark 1.2, we do not know if the best bound in a strip is indeed that provided by Theorem 1.1 or not.

Remark 1.5. The study of the effect of such convection rolls also arises in the study of magma flow in the Earth's mantle and other contexts [TS02, KJ03, GHZ11, YVL15, OM17].

Plan of the paper. In Section 2 we use an elementary scaling argument to reduce Theorem 1.1 to obtaining an upper bound on a degenerate cell problem (Proposition 2.3). In Section 3 we prove Proposition 2.3 using probabilistic techniques, modulo two lemmas concerning exit from / the return to the boundary layer. These lemmas are proved in Sections 4 and 5. The proofs of these lemmas rely on certain large deviations estimates, and these are presented in Appendix A. Finally, in Section 6 we confirm that choosing convection rolls whose width and height depend on the Péclet number provides no improvement to Theorem 1.1.

2. **Proof of the Main theorem**

First note that by doubling the domain and using symmetry and rescaling we can reduce the problem to proving (1.7) when $\partial_N S = \emptyset$ and $\partial_D S = (0, 1) \times \{0, 1\}$. In this section we prove Theorem 1.1 by producing a velocity field v (depending on Pe) such that we have

$$(2.1) \qquad \|T^v\|_\infty \leqslant C|\ln \mathrm{Pe}|^{13}\left(\frac{1}{\mathrm{Pe}}\right)^{\frac{2p}{4p-1}},$$

for all Pe sufficiently large. We do this by forming convection rolls with height 1, width ε and amplitude $A_\varepsilon/\varepsilon^2$ for some small ε and large A_ε (see Figure 1). Moreover, as we will see shortly, it is most efficient to to only stir near the boundary of cells, and that ε and A_ε should be chosen according to

$$(2.2) \qquad \frac{A_\varepsilon^{1-\frac{1}{2p}}}{\varepsilon^2} = \mathrm{Pe} .$$

To construct v, consider a Hamiltonian $H \colon \mathbb{R}^2 \to \mathbb{R}$ such that $H(x_1, 0) = H(x_1, 1) = 0$ and

$$H(x_1 + 1, x_2) = H(x_1, x_2) \quad \text{for all } (x_1, x_2) \in \mathbb{R}^2 .$$

To focus the stirring on cell boundaries, we truncate H in cell interiors as follows. Fix $N > 0$ be a large constant, and G^N be a smooth increasing function such that $G^N(0) = 0$, and

$$
(2.3) \qquad (G^N)'(h) = \begin{cases} 1 & h \leqslant \dfrac{N}{\sqrt{A_\varepsilon}} \\[2mm] 0 & h \geqslant \dfrac{2N}{\sqrt{A_\varepsilon}} \,. \end{cases}
$$

Define

$$
H^N(x_1, x_2) = G^N \circ H(x_1, x_2) \,, \qquad v^N = \nabla^\perp H^N = \begin{pmatrix} \partial_2 H^N \\ -\partial_1 H^N \end{pmatrix} \,.
$$

For notational convenience, let

$$
H^0 = H \,, \qquad v^0 = \nabla^\perp H = \begin{pmatrix} \partial_2 H \\ -\partial_1 H \end{pmatrix} \,.
$$

To obtain convection rolls of width ε and height 1, we rescale the horizontal variable. For $k \in \{0, N\}$ define

$$
H^{k,\varepsilon}(x_1, x_2) = H^k\left(\frac{x_1}{\varepsilon}, x_2\right) \,, \quad \text{and} \quad v^{k,\varepsilon} = \frac{A_\varepsilon}{\varepsilon} \nabla^\perp H^{k,\varepsilon} = \frac{A_\varepsilon}{\varepsilon} \begin{pmatrix} \partial_2 H^{k,\varepsilon} \\ -\partial_1 H^{k,\varepsilon} \end{pmatrix} \,,
$$

and let $T_{k,\varepsilon} = T^{v^{k,\varepsilon}}$. By uniqueness of solutions we see that $T_{k,\varepsilon}$ satisfies $T_{k,\varepsilon}(x_1 + \varepsilon, x_2) = T_{k,\varepsilon}(x_1, x_2)$. Thus, we change variables and define

$$
(2.4) \qquad y_1 = \frac{x_1}{\varepsilon} \,, \quad y_2 = x_2 \,, \quad \text{and} \quad v^k = \nabla_y^\perp H^k \,.
$$

In these coordinates we see that $T_{k,\varepsilon}$ satisfies

$$
(2.5) \qquad A_\varepsilon v^k \cdot \nabla_y T_{k,\varepsilon} - \frac{1}{2} \partial_{y_1}^2 T_{k,\varepsilon} - \frac{1}{2} \varepsilon^2 \partial_{y_2}^2 T_{k,\varepsilon} = \varepsilon^2 \,.
$$

Remark 2.1. Throughout the paper, N is a large fixed natural number, and we only consider the Hamiltonians H^N or H^0. All the statements we make below will apply to both Hamiltonians, unless explicitly stated otherwise.

Remark 2.2. As we will see shortly, the main result of this paper, Theorem (1.1), is achieved using the Hamiltonian H^N. We include H^0 here in parallel because there will be a technical step (in the proof of Lemma 4.2) that requires us to compare H^N to H^0.

Examining (2.5) we see that in the horizontal direction the diffusion has strength 1. However, since we impose periodic boundary conditions in this direction, there are no boundaries that provide a cooling effect directly felt by the horizontal diffusion. In the vertical direction, the diffusion coefficient is ε^2, and so the cooling effect from the Dirichlet boundary ∂S will be felt in the domain in time $O(1/\varepsilon^2)$. Since our source (the right hand side of (2.5)) is also ε^2, we expect that the diffusion alone will ensure T_ε is of size $O(1)$ as $\varepsilon \to 0$. This would lead to the bound $\mathcal{E}_\infty(\mathrm{Pe}) \leqslant C$, which is far from optimal.

We claim that the convection term reduces this bound dramatically. Indeed, through convection one can travel an $O(1)$ distance in the vertical direction in time $1/A_\varepsilon$. Do to our no flow requirement $v \cdot \hat{n} = 0$ on ∂S, one can never reach the boundary of S through convection alone. Thus, the cooling effect of the boundary ∂S must propagate into the domain through a combination of the effects of the slow

vertical diffusion $\varepsilon^2 \partial_{y_2}^2$ and the fast convection $A_\varepsilon v \cdot \nabla_y$. Our aim is to estimate how much improvement this can provide over the crude $O(1)$ bound that can be obtained through diffusion alone. This is our next result.

Proposition 2.3. *There exists a smooth Hamiltonian H and a constant C such that for every $\nu > 0$ and A_ε chosen such that $A_\varepsilon \geqslant 1/\varepsilon^\nu$ as $\varepsilon \to 0$, we have, for $k \in \{0, N\}$,*

$$(2.6) \qquad \|T_{k,\varepsilon}\|_{L^\infty} \leqslant C\varepsilon^2 \Big(1 + \frac{|\ln \varepsilon|^{13}}{\varepsilon\sqrt{A_\varepsilon}}\Big)$$

for all sufficiently small ε.

Remark 2.4. We believe the bound (2.6) is true for every smooth, non-degenerate cellular flow v (with a constant C that depends on v), provided $\nu \geqslant 2$. To obtain (2.6) for all $\nu > 0$, our proof requires the velocity field v to be exactly linear near the vertical cell boundaries. We do not know whether (2.6) remains true for $\nu \in (0, 2)$ without this assumption. We note, however, that choosing $\nu \in (0, 2)$ does not lead to an improved bound as in this range the constant term on the right of (2.6) will eliminate any benefit obtained from further increasing the amplitude.

Remark 2.5. For simplicity, the velocity field we construct to prove Proposition 2.3 will be chosen to be exactly linear near cell corners. This assumption is mainly present as it leads to a technical simplification of the proof of Proposition 2.3. Since the proof of Theorem 1.1 only requires us to produce one velocity field v satisfying (2.1), we only state and prove Proposition 2.3 for a specific cellular flow, instead of generic cellular flows.

We prove Proposition 2.3 using probabilistic techniques in the next section. Theorem 1.1 follows immediately from Proposition 2.3 by scaling.

Proof of Theorem 1.1. We only show the calculation for $k = N$ in Proposition 2.3. The calculation for $k = 0$ is easier and gives a worse bound so we omit it here. Fix y, v be as in (2.4), and $\nu > 0$ to be chosen later. By definition, we have

$$v^{N,\varepsilon}(x_1, x_2) = \frac{A_\varepsilon}{\varepsilon} \nabla^\perp H^{N,\varepsilon}(x_1, x_2) = \begin{cases} 0 & H(y_1, y_2) \geqslant \dfrac{2N}{\sqrt{A_\varepsilon}}, \\[2ex] \dfrac{A_\varepsilon}{\varepsilon^2} \begin{pmatrix} \varepsilon v_1^N(y_1, y_2) \\ v_2^N(y_1, y_2) \end{pmatrix} & H(y_1, y_2) \leqslant \dfrac{N}{\sqrt{A_\varepsilon}}. \end{cases}$$

When $1/\varepsilon \in \mathbb{N}$ we note

$$\int_\Omega |v^{N,\varepsilon}|^p \, dx = \frac{A_\varepsilon^p}{\varepsilon^{2p}} \int_\Omega (\varepsilon^2 (v_1^N)^2 + (v_2^N)^2)^{\frac{p}{2}} \, dx_1 dx_2$$

$$= \frac{A_\varepsilon^p}{\varepsilon^{2p}} \int_{\{H \leqslant \frac{N}{\sqrt{A_\varepsilon}}\}} (\varepsilon^2 v_1^2 + v_2^2)^{\frac{p}{2}} \, dy_1 \, dy_2,$$

and so $\mathrm{Pe} = \|v^\varepsilon\|_{L^p} = O\big(\big(A_\varepsilon^{\frac{2p-1}{2p}} |\ln A_\varepsilon|^{\frac{1}{p}}\big)/\varepsilon^2\big)$ as $\varepsilon \to 0$. Choosing $A_\varepsilon = 1/\varepsilon^\gamma$ for some $\gamma \geqslant \nu$, we then have, for large enough Pe,

$$\mathrm{Pe} \leqslant \frac{C\gamma^{\frac{1}{p}} |\ln \varepsilon|^{\frac{1}{p}}}{\varepsilon^{\frac{\gamma(2p-1)}{2p} + 2}}.$$

which means

$$(2.7) \qquad \varepsilon \leqslant C\gamma^{\frac{1}{p}}|\ln\varepsilon|^{\frac{1}{p}}\Big(\frac{1}{\mathrm{Pe}}\Big)^{\frac{1}{\frac{\gamma(2p-1)}{2p}+2}}.$$

Now, choose ν small enough such that

$$\frac{2}{\frac{\nu(2p-1)}{2p}+2} = 1 - \frac{\mu}{2}.$$

Combining this with (2.6) and using the fact that $\gamma \geqslant \nu$,

$$(2.8) \qquad \|T_{N,\varepsilon}\|_{L^\infty} \leqslant C\Big(\varepsilon^2 + \varepsilon^{1+\gamma/2}|\ln\varepsilon|^{13}\Big).$$

The right hand side of the above inequality is minimized when the two terms are equal. For simplicity, we pick $\gamma = 2$, so that

$$\|T_{N,\varepsilon}\|_{L^\infty} \leqslant C2^{\frac{2}{p}}|\ln\varepsilon|^{13+\frac{2}{p}}\Big(\frac{1}{\mathrm{Pe}}\Big)^{\frac{2}{\frac{2(2p-1)}{2p}+2}} \leqslant C|\ln\varepsilon|^{13+\frac{2}{p}}\Big(\frac{1}{\mathrm{Pe}}\Big)^{\frac{2p}{4p-1}}.$$

Substitute this into (2.7), we find the width of the convection rolls

$$\varepsilon = O\Big(\frac{|\ln\mathrm{Pe}|^{\frac{1}{p}}}{\mathrm{Pe}^{\frac{p}{4p-1}}}\Big).$$

Theorem 1.1 follows immediately. $\qquad\square$

Remark 2.6. For the sake of presentation, we slightly abuse the notation and write v, T, A, H for $v^{k,\varepsilon}, T_{k,\varepsilon}, A_\varepsilon, H^N$, respectively. All statements apply to both standard and cut-off Hamiltonians, with the exception of Lemma 4.12, in which we will indicate explicitly that it works only for the standard Hamiltonian H^0.

3. **Proof of Proposition 2.3**

Our aim in this section is to prove Proposition 2.3. Let Z^ε be a solution to the SDE

$$(3.1) \qquad dZ_t^\varepsilon = Av(Z^\varepsilon)\,ds + \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon \end{pmatrix} dB_t\,,$$

where B is a standard two dimensional Brownian motion. For convenience let $Z^\varepsilon = (Z_1^\varepsilon, Z_2^\varepsilon)$, and let

$$\tau^\varepsilon = \inf\{t \mid Z_{2,t}^\varepsilon \notin (0,1)\}$$

be the first exit time of Z^ε from the strip S. By the Dynkin formula we know $T_\varepsilon(z) = \varepsilon^2 \boldsymbol{E}^z \tau^\varepsilon$.

Before delving into the details of the proof of Proposition 2.3, we now briefly explain the main idea. Consider many tracer particles evolving according to (3.1). First, we note that particles near ∂S get convected away from ∂S in time $O(1/A)$. In this time, these particles can travel a distance of $O(\varepsilon/\sqrt{A})$ in the vertical direction through diffusion. Thus, if we can ensure particles get to within a distance of $O(\varepsilon/\sqrt{A})$ from ∂S, then they will exit quickly with probability at least p_0, for some small $p_0 > 0$ that is independent of ε.

We claim that in the boundary layer, every $O(1/\sqrt{A})$ seconds[1] tracer particles will pass within a distance of $O(\varepsilon/\sqrt{A})$ from ∂S. Every pass has an $O(\varepsilon)$ probability

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[1] The reason the time taken is $O(1/\sqrt{A})$ and not the convection time $O(1/A)$ is because the diffusion may carry particles into the interior of the cell before they exit at ∂S.

of being within $\varepsilon/\sqrt{A}$ away from ∂S, and so a probability $O(\varepsilon)$ of exiting from ∂S. This suggests

$$(3.2) \quad \sup_{z \in \Omega} \boldsymbol{E}^z \tau^\varepsilon \leqslant C\Big(1 + \frac{\varepsilon}{\sqrt{A}} + \frac{(1-\varepsilon)2\varepsilon}{\sqrt{A}} + \frac{(1-\varepsilon)^2 3\varepsilon}{\sqrt{A}} + \cdots\Big) = C\Big(1 + \frac{1}{\varepsilon\sqrt{A}}\Big),$$

which is dramatically better than the crude $O(1/\varepsilon^2)$ bound obtained by using diffusion alone.

A second look at the above argument suggests that (3.2) should have a logarithmic correction. Indeed, the flow v has hyperbolic saddles at cell $\{0,1\} \times \mathbb{Z}$ which causes a logarithmic slow down of particles close to it. As a result, we are able to prove the following bound on $\boldsymbol{E}\tau^\varepsilon$.

Proposition 3.1. *Let $\nu > 0$ and $A \geqslant 1/\varepsilon^\nu$. There exists a cellular flow v and a constant C such that*

$$(3.3) \qquad\qquad \sup_{z \in \Omega} \boldsymbol{E}^z \tau^\varepsilon \leqslant C\Big(1 + \frac{|\ln\varepsilon|^{13}}{\varepsilon\sqrt{A}}\Big),$$

holds for all sufficiently small ε.

Of course Proposition 3.1 immediately implies Proposition 2.3.

Proof of Proposition 2.3. Since $T(z) = \varepsilon^2 \boldsymbol{E}^z \tau^\varepsilon$, (3.3) implies

$$\|T\|_{L^\infty(\Omega)} \leqslant \frac{C\varepsilon|\ln\varepsilon|^{13}}{\sqrt{A}},$$

which yields (2.6) as desired. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

We now describe the flow v that will be used in Proposition 3.1. As remarked earlier, we expect Proposition 3.1 to hold for any generic non-degenerate cellular flow. However, the specific form we describe below simplifies many technicalities. For notational convenience, we will now use the domain

$$\Omega' \overset{\text{def}}{=} (0,2) \times (0,1),$$

and assume that all functions are 2-periodic in the horizontal direction.

Assumption 1: The function $H \colon \mathbb{R}^2 \to [-1,1]$ is C^2 with $\|H\|_{C^2} \leqslant 100$ and is 2-periodic x_1, 1-periodic in x_2. The level set $\{H = 0\}$ is precisely $(\mathbb{R} \times \mathbb{Z}) \cup (\mathbb{Z} \times \mathbb{R})$. Moreover, $H(1/2, 1/2) = 1$, $H(3/2, 1/2) = -1$ and these both correspond to non-degenerate critical points of H. All other critical points of H are hyperbolic and like on the integer lattice $\mathbb{Z}^2$.

Assumption 2: There exists $c_0 \in (0, 1/10)$ such that every square of side length $4c_0$ centered at an integer lattice point the Hamiltonian H is quadratic.

Apart from non-degeneracy and normalization, the main content of the first assumption is that H only has one critical point in the interior of every square $(m, m+1) \times (n, n+1)$ with $m, n \in \mathbb{Z}$. The second assumption requires H to be exactly quadratic around all hyperbolic critical points. These are what lead to technical simplification of the proof. For normalization, we will further assume the following:

Assumption 3:

$$(3.4) \qquad H(x_1, x_2) = \begin{cases} x_1 x_2 & (x_1, x_2) \in Q_0 \,, \\ (1 - x_1) x_2 & (x_1, x_2) \in Q_0 + (1, 0) \,, \\ x_1 (1 - x_2) & (x_1, x_2) \in Q_0 + (0, 1) \,, \\ (1 - x_1)(1 - x_2) & (x_1, x_2) \in Q_0 + (1, 1) \,, \end{cases}$$

where $Q_0 = (-2c_0, 2c_0)^2$.

Assumption 4: There exists a constant h_0 such that for $x \in \{|H| < h_0\}$ and $i \in \{1, 2\}$,

$$\operatorname{sign} \partial_i^2 H = - \operatorname{sign} H \,.$$

Finally, we assume that the vertical component of the velocity field doesn't change near the sides of the convection rolls. Without this assumption, the exit time bounds we obtain (Lemma 3.2, below) will only be valid if $A \geqslant 1/\varepsilon^2$, and we elaborate on this in Remark 3.3 below.

Assumption 5: In the region $\{|H| \leqslant h_0\} \cap (i + (-c, c)) \times \mathbb{R}$, where $i \in \mathbb{Z}$,

$$(3.5) \qquad \partial_1 v_2 = -\partial_1^2 H = 0 \,.$$

Now we split Proposition 3.1 into two steps: estimating the time taken to reach the boundary layer, and then estimating the time taken to exit from the boundary layer. Given $c > 0$ define the boundary layer $\mathcal{B}_c$ by

$$\mathcal{B}_c = \mathcal{B}_c^\varepsilon \overset{\text{def}}{=} \left\{ |H| < \frac{c}{\sqrt{A}} \right\} \qquad \text{where} \qquad \delta \overset{\text{def}}{=} \frac{\varepsilon}{\sqrt{A}} \,.$$

Lemma 3.2. *Let $\nu > 0$ and suppose $A \geqslant 1/\varepsilon^\nu$. There exists a constant C such that*

$$(3.6) \qquad \sup_{z \in \bar{\mathcal{B}}_1} \boldsymbol{E}^z \tau^\varepsilon \leqslant \frac{C |\ln \delta|^{13}}{\varepsilon \sqrt{A}} \,.$$

Remark 3.3. In the proof of Lemma 3.2 we will see that if H doesn't satisfy Assumption 5, then Lemma 3.2 is only valid if $\nu \geqslant 2$ (see Remark A.5, below). It turns out that choosing $\nu \leqslant 2$ provides no additional advantage in this paper. This is because when $\nu \leqslant 2$, the constant term on the right of (3.3) dominates we get no improvement on $\boldsymbol{E}^z \tau^\varepsilon$.

Lemma 3.4. *Let $\alpha \in [0, 1)$, and $\eta_\alpha = \eta_\alpha^\varepsilon = \inf\{t > 0 \mid Z_t^\varepsilon \in \partial \mathcal{B}_\alpha\}$ be the first time the process Z_t^ε hits $\partial \mathcal{B}_\alpha$. There exists a constant C, independent of α, such that*

$$\sup_{z \in \mathcal{B}_\alpha^c} \boldsymbol{E}^z \eta_1^\varepsilon \leqslant C$$

for all sufficiently small ε. (Here $\mathcal{B}_\alpha^c$ is the compliment of $\mathcal{B}_\alpha$.)

A proof of Lemma 3.4 using a blow-up argument can be found in [IS12]. Another method is to use the horizontal diffusion alone to reach the vertical boundary $\mathbb{Z} \times (0, 1)$ in time $O(1)$. In Section 5, below, we present a different proof of this fact by constructing a supersolution based on the Freidlin averaging problem [FW12].

Momentarily postponing the proofs of Lemmas 3.2–3.4 we prove Proposition 3.1.

Proof of Proposition 3.1. If $z \notin \mathcal{B}_1$, the strong Markov property and Lemma 3.2 imply

$$\boldsymbol{E}^z \tau^\varepsilon = \boldsymbol{E}^z \eta_1^\varepsilon + (\tau^\varepsilon - \eta_1^\varepsilon) = \boldsymbol{E}^z \big(\eta_1^\varepsilon + (\tau^\varepsilon - \eta_1^\varepsilon) \mid \mathcal{F}_{\eta_1^\varepsilon} \big)$$

$$(3.7) \qquad \leqslant C + \boldsymbol{E}^z \sup_{z' \in \bar{B}_1} \boldsymbol{E}^{z'} \tau^\varepsilon \leqslant C\Big(1 + \frac{|\ln \delta|^{13}}{\varepsilon \sqrt{A}}\Big).$$

If $z \in \mathcal{B}_1$, then Lemma 3.2 directly implies (3.7). Thus in either case we have (3.3), as desired. $\qquad \square$

4. Exit from the Boundary layer (Lemma 3.2)

In this section, we will prove Lemma 3.2. We will fix $\nu > 0$ and suppose $A \geqslant 1/\varepsilon^\nu$ as in the hypothesis of Lemma 3.2 through out this section. Furthermore, for notational convenience, we will now drop the explicit ε dependence from Z^ε and A.

Lemma 4.1. *For every $d \in \mathbb{N}$, there exists a constant $C = C(d)$ such that*

$$(4.1) \qquad \inf_{z \in \mathcal{B}_d} \boldsymbol{P}^z(\tau^\varepsilon < \eta_{d+4}^\varepsilon) \geqslant \frac{C\varepsilon}{|\ln \delta|^{12}}$$

for all sufficiently small ε.

Lemma 4.2. *There exists a constant C such that*

$$(4.2) \qquad \sup_{z \in \mathcal{B}_1} \boldsymbol{E}^z \eta_5^\varepsilon \leqslant \frac{C|\ln \delta|}{A}$$

for all sufficiently small ε.

Lemma 4.3. *There exists a constant C such that there exists an ε_0, where*

$$(4.3) \qquad \sup_{z \in \partial \mathcal{B}_5} \boldsymbol{E}^z \eta_1^\varepsilon \leqslant C \frac{|\ln \delta|}{\sqrt{A}}$$

for all $\varepsilon < \varepsilon_0$.

Intuitively, estimate (4.1) measures the success of exit the domain after each trial when the process Z_t^ε starts inside the layer boundary $\mathcal{B}_1$. Estimate (4.2) measures the expected time for the process Z_t^ε to escape away from he separatix after a failed attempt. Estimate (4.3) measures the expected time for the process Z_t^ε to get back to the boundary layer after a failed attempt of escape.

We can now give a proof for Lemma 3.2.

Proof of Lemma 3.2. In this proof, the constant C may vary from line to line but does not depend on ε. We first define two sequences of barrier stopping times,

$$\sigma_0 = 0, \qquad\qquad \tilde{\sigma}_0 = \inf\{t \geqslant \sigma_0 \mid Z_t^\varepsilon \in \partial \mathcal{B}_5\},$$
$$\sigma_n = \inf\{t \geqslant \tilde{\sigma}_{n-1} \mid Z_t^\varepsilon \in \partial \mathcal{B}_1\}, \qquad \tilde{\sigma}_n = \inf\{t \geqslant \sigma_n \mid Z_t^\varepsilon \in \partial \mathcal{B}_5\}.$$

We have

$$\boldsymbol{E}^z \tau^\varepsilon = \int_0^\infty \boldsymbol{P}^z\big(\tau^\varepsilon \geqslant t\big)\, dt$$

$$= \boldsymbol{E}^z \sum_{n=1}^\infty \int_{\sigma_{n-1}}^{\sigma_n} \mathbf{1}_{\{\tau^\varepsilon \geqslant t\}}\, dt \leqslant \sum_{n=1}^\infty \boldsymbol{E}^z \mathbf{1}_{\{\tau^\varepsilon \geqslant \sigma_{n-1}\}} (\sigma_n - \sigma_{n-1})$$

$$= \sum_{n=1}^\infty \boldsymbol{E}^z \mathbf{1}_{\{\tau^\varepsilon \geqslant \sigma_{n-1}\}} \boldsymbol{E}^{Z^\varepsilon(\sigma_{n-1})} \sigma_1$$

$$(4.4) \qquad \leqslant \sum_{n=1}^\infty \boldsymbol{P}^z(\tau^\varepsilon \geqslant \sigma_{n-1}) \sup_{z' \in \partial \mathcal{B}_1} \boldsymbol{E}^{z'} \sigma_1.$$

We will now estimate each term on the right.

First, by the strong Markov property and Lemmas 4.2–4.3 we have

$$(4.5) \qquad \boldsymbol{E}^z \sigma_1 = \boldsymbol{E}^z \big(\tilde{\sigma}_0 + \boldsymbol{E}^{Z^\varepsilon(\tilde{\sigma}_0)} \eta_1 \big) \leqslant \boldsymbol{E}^z \Big(\eta_5 + \sup_{z' \in \partial \mathcal{B}_5} \boldsymbol{E}^{z'} \eta_1 \Big) \leqslant \frac{C|\ln \delta|}{\sqrt{A}} \,.$$

for every $z \in \partial \mathcal{B}_1$. To estimate $\boldsymbol{P}^z(\tau^\varepsilon \geqslant \sigma_n)$, we use Lemma 4.1 and the fact that $\sigma_1 \geqslant \tilde{\sigma}_0 = \eta_5$ to obtain

$$\sup_{z \in \partial \mathcal{B}_1} \boldsymbol{P}^z(\tau^\varepsilon \geqslant \sigma_1) \leqslant \sup_{z \in \partial \mathcal{B}_1} \boldsymbol{P}^z(\tau^\varepsilon \geqslant \eta_5) = 1 - \inf_{z \in \partial \mathcal{B}_1} \boldsymbol{P}^z(\tau^\varepsilon < \eta_5) \leqslant 1 - \frac{C\varepsilon}{(\ln \delta)^{12}} \,.$$

Now, by the strong Markov property,

$$\sup_{z \in \mathcal{B}_1} \boldsymbol{P}^z \big(\tau^\varepsilon \geqslant \sigma_n \big) = \sup_{z \in \mathcal{B}_1} \boldsymbol{E}^z \big(\mathbf{1}_{\{\tau^\varepsilon \geqslant \sigma_{n-1}\}} \boldsymbol{E}^{Z^\varepsilon(\sigma_{n-1})} \mathbf{1}_{\{\tau^\varepsilon \geqslant \sigma_1\}} \big)$$

$$\leqslant \sup_{z \in \mathcal{B}_1} \boldsymbol{E}^z \mathbf{1}_{\{\tau^\varepsilon \geqslant \sigma_{n-1}\}} \sup_{z' \in \partial \mathcal{B}_1} \boldsymbol{P}^{z'}(\tau^\varepsilon \geqslant \sigma_1)$$

$$\leqslant \Big(1 - \frac{C\varepsilon}{(\ln \delta)^{12}} \Big) \boldsymbol{E}^z \mathbf{1}_{\{\tau^\varepsilon \geqslant \sigma_{n-1}\}} \,.$$

Hence by induction

$$(4.6) \qquad \sup_{z \in \mathcal{B}_1} \boldsymbol{P}^z \big(\tau^\varepsilon \geqslant \sigma_n \big) \leqslant \Big(1 - \frac{C\varepsilon}{|\ln \delta|^{12}} \Big)^n \,,$$

for all $n \in \mathbb{N}$.

Using (4.5) and (4.6) in (4.4) yields

$$\boldsymbol{E}^z \tau^\varepsilon \leqslant \frac{C|\ln \delta|}{\sqrt{A}} \sum_{n=0}^\infty \Big(1 - \frac{C\varepsilon}{|\ln \delta|^{12}} \Big)^n$$

finishing the proof. $\qquad \square$

4.1. **Proof of Lemma 4.1.** In this subsection, we will give the proof of Lemma 4.1. We let the coordinate processes of Z be Z_1 and Z_2 respectively (i.e. $Z = (Z_1, Z_2)$). We also denote $\delta = \varepsilon/\sqrt{A}$ and γ_t the deterministic curve defined by

$$(4.7) \qquad\qquad\qquad \partial_t \gamma_t = A v(\gamma_t) \,,$$

where $v = \nabla^\perp H$ as before. We again need a few results to prove Lemma 4.1.

The first result we state is a "tube lemma" estimating the probability that the process Z stays within a small tube around the deterministic trajectories. This is well studied and many such estimates can be found in the literature (see for instance [FW12]). The standard estimates, however, work well for times of order $1/A$. Due to the degeneracy, and the hyperbolic saddles near cell corners, we need an estimate that works for time scales of order $|\ln \delta|/A$. We state this estimate here.

Lemma 4.4. *Let $z_0 \in Q_0/2 + (j, k)$ where $j, k \in \{-1, 0\}$ and suppose γ satisfies (4.7) with $\gamma_0 = z_0$. Let,*

$$(4.8) \qquad\qquad T = \inf\{t > 0 \mid |\gamma_t^2| \leqslant \delta \ \text{or} \ |\gamma_t^1| = c_0 \ \text{or} \ |\gamma_t^2| = c_0 \}$$

and recall $\sigma_1 = 1$, $\sigma_2 = \varepsilon$. There exists ε_0 so that for every $\varepsilon < \varepsilon_0$,

$$P^{z_0} \Big(\sup_{0 \leqslant t \leqslant T} |Z_{i,t} - \gamma_{i,t}| \leqslant \frac{\sigma_i}{\sqrt{|\ln \delta| A}} \,, \forall i \in \{1, 2\} \Big) \geqslant \frac{C}{|\ln \delta|^2} \,.$$

Remark 4.5. By a direct calculation, we can check that $T \leqslant |\ln \delta|/A$.

The proof of Lemma 4.4 uses the Girsanov theorem and is greatly simplified by the fact that H is exactly quadratic near cell corners. Since it is similar to the standard proofs, we present it in Appendix A.

Once Lemma 4.4 is established it quickly gives an estimate on the probability of getting within a distance of $O(1/\sqrt{A})$ away from cell boundaries.

Lemma 4.6. *Let $z_0 \in \mathcal{B}_1$. There exists constants $C, M > 0$ such that for small enough ε,*

$$(4.9) \qquad \boldsymbol{P}^{z_0}\big(\lambda_0 < \eta_{4M}^\varepsilon\big) \geqslant \frac{C}{|\ln \delta|^2}\,.$$

Here, $\lambda_0 \stackrel{\mathrm{def}}{=} \inf\big\{t > 0 \,\big|\, Z_t \in \{\mathrm{dist}(z, \partial\Omega) \leqslant M/\sqrt{A}\}\big\}$.

Proof. Note first that by Taylor expansion of H, for small ε there exists $M > 0$ such that $\mathrm{dist}(z_0, \partial\Omega) \leqslant M/\sqrt{A}$ for all z_0 outside the corners $Q_0/2 + (j, k)$, where $j, k \in \{-1, 0\}$. So now, we assume $z_0 \in Q_0/2 + (j, k)$ for some $j, k \in \{-1, 0\}$. For brevity, we only present the proof when $z_0 \in Q_0/2$, as the other cases are identical.

If $\mathrm{dist}(z_0, \partial\Omega) \leqslant 1/\sqrt{A}$ we are done, so we now suppose $z_0 \in Q_0/2$ with $\mathrm{dist}(z_0, \partial\Omega) > 1/\sqrt{A}$. Let γ be the deterministic trajectory defined by (4.7) with $\gamma_0 = z_0$, and let T be as in (4.8). Note that since $\mathrm{dist}(z_0, \partial\Omega) > 1/\sqrt{A}$ we can not have $|\gamma_{2,T}| \leqslant \delta$. Thus, either $|\gamma_{1,T}| = c_0$ or $|\gamma_{1,T}| = c_0$. In either case there exists a constant M such that $|\gamma_{2,T}| \leqslant M/\sqrt{A}$ or $|\gamma_{1,T}| \leqslant M/\sqrt{A}$, respectively. Now using Lemma 4.4 we obtain (4.9) as desired. $\qquad\square$

Remark 4.7. For notational convenience, we assume that $M = 1$ for the rest of the paper.

Another consequence of Lemma 4.4 is a lower bound on the probability of reaching $O(\delta)$ away from the top boundary before re-entering the cell interior.

Lemma 4.8. *Let $Q_{top}^\delta = (-2c_0, 2c_0) \times (-4\delta, 0)$ be a box of height 4δ at the top of the cell corner. Let $\lambda \stackrel{\mathrm{def}}{=} \inf\{t \geqslant 0 \mid Z_t \in Q_{top}^\delta\}$. Then, there exists a constant $C > 0$ such that*

$$(4.10) \qquad \inf_{z_0 \in (-\delta, \delta) \times (-c_0, 0)} \boldsymbol{P}^{z_0}\big(\lambda < \eta_4^\varepsilon\big) \geqslant \frac{C}{(\ln \delta)^2}\,.$$

Proof. Let $T = \inf\big\{t > 0 \,\big|\, |\gamma_{2,t}| \leqslant \delta\big\}$ the time the deterministic process hits the top boundary layer with width δ. By Lemma 4.4, there exists a constant $C > 0$ so that

$$P^{z_0}\Big(\sup_{0 \leqslant t \leqslant T} |Z_{i,t} - \gamma_{i,t}| \leqslant \frac{\sigma_i}{\sqrt{|\ln \delta| A}}\,, \forall i \in \{1, 2\}\Big) \geqslant \frac{C}{(\ln \delta)^2}\,.$$

As $z_0 \in (-\delta, \delta) \times (-c_0, 0)$, $\gamma_{1,T} \in (-c_0, 0)$. Therefore,

$$\Big\{\sup_{0 \leqslant t \leqslant T} |Z_{i,t} - \gamma_{i,t}| \leqslant \frac{\sigma_i}{\sqrt{|\ln \delta| A}}\,, \forall i \in \{1, 2\}\Big\} \subseteq \big\{\eta_4^\varepsilon > \lambda\big\}\,,$$

from which (4.10) follows. $\qquad\square$

Next, we bound the probability of exiting from the top when trajectories start in Q_{top}^δ.

Lemma 4.9. *There exists a constant $C > 0$ such that*

$$(4.11) \qquad \inf_{z_0 \in Q_{top}^\delta} \boldsymbol{P}^{z_0}\big(\tau^\varepsilon < \eta_4^\varepsilon\big) \geqslant C\,.$$

Proof. Let $\tilde{T} = 1/A$. When A is sufficiently large, we note that given $X_0 = z_0 \in Q_{\text{top}}^\delta$, there exists $n \geqslant 1$, independent of ε, such that the deterministic flow γ_t starting at z_0 still remains in the top edge of the boundary layer $\{|H| \leqslant n\delta\} \cap [-1, 0] \times [-n\delta, 0]$ for time $\tilde{T}$. Define $\tilde{\gamma}_t$ by
$$\partial_t \tilde{\gamma}_t = Au(\tilde{\gamma}_t),$$
where u is chosen to satisfy the following condition $\tilde{\gamma}_t = (\gamma_{1,t}, \tilde{\gamma}_{2,t})$, where $\gamma_{1,t}$ is the first coordinate of γ, and $\tilde{\gamma}_{2,t}$ is some continuous function such that
$$\tilde{\gamma}_{2,0} = \gamma_{2,0}, \quad |v_2 - u_2| \leqslant 2n\delta \quad \text{and} \quad \tilde{\gamma}_{2,\tilde{T}} \geqslant n\delta.$$
An example of such $\tilde{\gamma}$ is $\tilde{\gamma}_t = (\gamma_{1,t}, \gamma_{2,t} + 2An\delta t)$. By continuity,
$$E_3 \stackrel{\text{def}}{=} \left\{ \sup_{0 \leqslant t \leqslant \tilde{T}} |Z_{2,t} - \tilde{\gamma}_{2,t}| \leqslant \delta \right\} \subset \left\{ \tau^\varepsilon < \eta_4^\varepsilon \right\}.$$

Now a standard large deviation estimate will show that $\boldsymbol{P}^{z_0}(E_3) \geqslant C_\varepsilon$, for some constant C_ε that vanishes as $\varepsilon \to 0$. In order to prove Lemma 4.9, we need to remove this ε dependence. We do this here using the fact that in this box $|\partial_1 v_2| \leqslant O(\varepsilon)$, and $|v_2 - u_2| \leqslant O(\delta)$. We claim that if we go through the standard large deviation estimate with these additional assumptions, the constant C_ε can be made independent of ε. Since the details are not too different from the standard proof, we carry them out in Lemma A.3 in Appendix A, below. Hence, we see that there exists a constant C (independent of z_0, ε) so that
$$\boldsymbol{P}^{z_0}(E_3) \geqslant C,$$
proving (4.11). $\qquad\square$

Lemma 4.10. *Let* $\tilde{\lambda} \stackrel{\text{def}}{=} \inf\{t \geqslant 0 \mid Z_t \in (-\delta, \delta) \times (-c_0, 0)\}$. *There exists a constant* $C > 0$ *such that*
$$(4.12) \qquad \inf_{z_0 \in \{\text{dist}(z, \partial_\Omega) \leqslant 1/\sqrt{A}\}} \boldsymbol{P}^{z_0}\left(\tilde{\lambda} < \eta_4^\varepsilon\right) \geqslant C \frac{\varepsilon}{(\ln \delta)^8}.$$

Proof. Define the regions $\square_1, \ldots, \square_5$ by
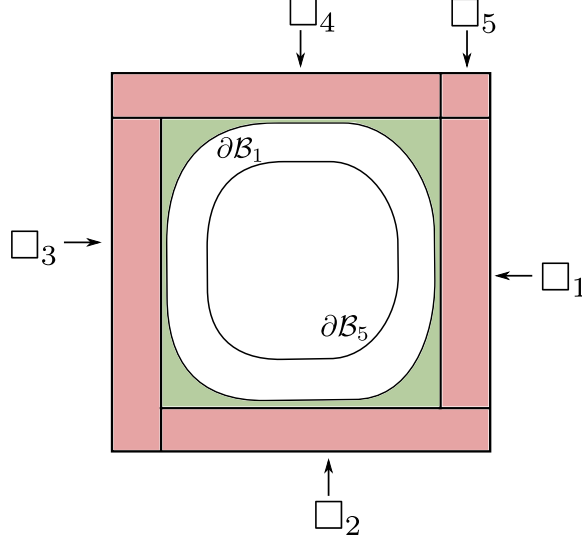$$\square_1 \stackrel{\text{def}}{=} \left(-\frac{1}{\sqrt{A}}, \frac{1}{\sqrt{A}}\right) \times \left(-1 + \frac{1}{\sqrt{A}}, -\frac{1}{\sqrt{A}}\right),$$
$$\square_2 \stackrel{\text{def}}{=} \left(-1 + \frac{1}{\sqrt{A}}, 0\right) \times \left(-1, -1 + \frac{1}{\sqrt{A}}\right),$$
$$\square_3 \stackrel{\text{def}}{=} \left(-1, -1 + \frac{1}{\sqrt{A}}\right) \times \left(-1, -\frac{1}{\sqrt{A}}\right),$$
$$\square_4 \stackrel{\text{def}}{=} \left(-1, -\frac{1}{\sqrt{A}}\right) \times \left(-\frac{1}{\sqrt{A}}, 0\right),$$
$$\square_5 \stackrel{\text{def}}{=} \left(-\frac{1}{\sqrt{A}}, 0\right)^2,$$
as shown in Figure 2. If $\text{dist}(z_0, \partial_\Omega) \leqslant 1/\sqrt{A}$, then z_0 must be in one of the boxes $\square_1$, $\ldots$, $\square_5$. Suppose first $z_0 \in \square_1$. Let $\gamma(t)$ is the deterministic trajectory such that $\gamma_0 = z_0$, $T_0 \stackrel{\text{def}}{=} \inf\{t > 0 : \gamma_{2,t} = -c_0/2\} \leqslant m/A$ for some $m \geqslant 1$, and
$$E_4 \stackrel{\text{def}}{=} \left\{ \sup_{0 \leqslant t \leqslant T_0} |Z_{1,t} - \gamma_{1,t}| \leqslant \frac{2}{\sqrt{A}}, \sup_{0 \leqslant t \leqslant T_0} |Z_{2,t} - \gamma^{2,t}| \leqslant \frac{\varepsilon}{\sqrt{A}}, |Z_{1,T_0}| \leqslant \frac{\varepsilon}{2\sqrt{A}}, \right\}.$$
By continuity, we have that
$$E_4 \subset \left\{ \tilde{\lambda} < \eta_4^\varepsilon \right\}.$$

FIGURE 2. $\partial\mathcal{B}_n$ and $\square_i$.

We claim

$$(4.13) \qquad \boldsymbol{P}^{z_0}\big(\tilde{\lambda} < \eta_4^\varepsilon\big) \geqslant \boldsymbol{P}^{z_0}(E_4) \geqslant C\varepsilon\,,$$

where $C > 0$ independent of z_0. The proof of (4.13) is presented with the other tube lemmas we use in Appendix A. We in fact prove a more general estimate (Lemma A.4 applied to the deterministic flow), from which (4.13) follows.

Now, let $z_0 \in \square_2$, define $\square_{2R} = \square_2 \cap [-c_0, 0] \times [-1, -1 + 2/\sqrt{A}]$, and let $\lambda_1 = \inf\big\{t > 0 \,\big|\, Z_t \in \square_{2R}\big\}$. Proceeding as the case for $\square_1$ with $\gamma(t)$ being the deterministic trajectory so that $\gamma(0) = z_0$, $T_1 = \inf\{t > 0 : \gamma_{1,t} = -1 + c_0/2\}$, we have

$$(4.14) \qquad \boldsymbol{P}^{z_0}\big(\lambda_1 < \eta_4^\varepsilon\big) \geqslant \boldsymbol{P}^{z_0}\Big(\sup_{0 \leqslant t \leqslant T_1} |Z_t - \gamma_t| \leqslant \frac{1}{\sqrt{A}}\Big) \geqslant C\,.$$

To see why the last lower bound is true, we consider by Itô formular,

$$\sup_{0 \leqslant t \leqslant T_1} \boldsymbol{E}^{z_0}|Z_t - \gamma_t|^2 \leqslant 2A\|v\|_{C^1} \int_0^{T_1} \boldsymbol{E}^{z_0} \sup_{0 \leqslant t \leqslant T_1} |Z_t - \gamma_t|^2 + (\varepsilon^2 + 1)T_1,$$

which, by Gronwall's inequality and Assumption 1, implies

$$\sup_{0 \leqslant t \leqslant T_1} \boldsymbol{E}^{z_0}|Z_t - \gamma_t|^2 \leqslant (1 + \varepsilon^2)T_1 e^{200T_1}\,.$$

Inequality (4.14) follows by Chebychev's inequality.

Now let $\lambda' = \inf\big\{t \geqslant 0 \,\big|\, Z_t \in \square_1\big\}$. Using Lemmas 4.4 and Markov property, there exists a constant C (independent of z_0) so that

$$(4.15) \qquad \boldsymbol{P}^{z_0}\big(\lambda' < \eta_4^\varepsilon\big) \geqslant \boldsymbol{P}^{z_0}\big(\lambda_1 < \eta_4^\varepsilon\big) \inf_{z_1 \in \square_{2R}} \boldsymbol{P}^{z_1}\big(\lambda' < \eta_4^\varepsilon\big) \geqslant \frac{C}{(\ln \delta)^2}\,.$$

Combining (4.13), (4.15) and using the Markov property gives

$$\boldsymbol{P}^{z_0}\big(\tilde{\lambda} < \eta_4^\varepsilon\big) \geqslant \boldsymbol{P}^{z_0}\big(\lambda' < \eta_4^\varepsilon\big) \inf_{z_1 \in \square_1} \boldsymbol{P}^{z_1}\big(\tilde{\lambda} < \eta_4^\varepsilon\big) \geqslant \frac{C\varepsilon}{(\ln \delta)^2}\,.$$

Repeating this argument again for $\square_3$, $\dots$, $\square_5$ we see that we obtain an extra $C/|\ln \delta|^2$ factor every time we pass a corner. Combining these estimates gives (4.12) as claimed. $\qquad\square$

We are now ready to give the proof for Lemma 4.1.

Proof of Lemma 4.1. Let $z_0 \in \mathcal{B}_1$ and denote $D_1 \stackrel{\mathrm{def}}{=} \big\{\mathrm{dist}(z, \partial_\Omega) \leqslant 1/\sqrt{A}\big\}$, $D_2 \stackrel{\mathrm{def}}{=}$ $(-\delta, \delta) \times (-c_0, 0)$ and $D_3 \stackrel{\mathrm{def}}{=} (-2c_0, 2c_0) \times (-4\delta, 0)$. As $\eta_4^\varepsilon < \eta_5^\varepsilon$ when $z_0 \in \mathcal{B}_1$, by Lemmas 4.6–4.10 and Markov property, we have that

$$\begin{aligned}
\boldsymbol{P}^{z_0}(\tau^\varepsilon < \eta_5^\varepsilon) &\geqslant \boldsymbol{E}^{z_0} \mathbf{1}_{\{\tau^\varepsilon < \eta_5^\varepsilon\}} \mathbf{1}_{\{\lambda < \eta_5^\varepsilon\}} \mathbf{1}_{\{\lambda_0 < \eta_5^\varepsilon\}} \mathbf{1}_{\{\tilde{\lambda} < \eta_5^\varepsilon\}} \\
&= \boldsymbol{E}^{z_0} \mathbf{1}_{\{\lambda_0 < \eta_5^\varepsilon\}} \boldsymbol{E}^{z_0}\Big(\mathbf{1}_{\{\tau^\varepsilon < \eta_5^\varepsilon\}} \mathbf{1}_{\{\tilde{\lambda} < \eta_5^\varepsilon\}} \mathbf{1}_{\{\lambda < \eta_5^\varepsilon\}} \,\Big|\, \mathcal{F}_{\lambda_0}\Big) \\
&= \boldsymbol{E}^{z_0} \mathbf{1}_{\{\lambda_0 < \eta_5^\varepsilon\}} \boldsymbol{E}^{Z_{\lambda_0}}\Big(\mathbf{1}_{\{\tau^\varepsilon < \eta_5^\varepsilon\}} \mathbf{1}_{\{\tilde{\lambda} < \eta_5^\varepsilon\}} \mathbf{1}_{\{\lambda < \eta_5^\varepsilon\}}\Big) \\
&\geqslant \boldsymbol{E}^{z_0} \mathbf{1}_{\{\lambda_0 < \eta_5^\varepsilon\}} \inf_{z_1 \in D_1} \boldsymbol{E}^{z_1}\Big(\mathbf{1}_{\{\lambda < \eta_5^\varepsilon\}} \mathbf{1}_{\{\tilde{\lambda} < \eta_5^\varepsilon\}} \mathbf{1}_{\{\tau^\varepsilon < \eta_5^\varepsilon\}}\Big) \\
&\geqslant \boldsymbol{E}^{z_0} \mathbf{1}_{\{\lambda_0 < \eta_5^\varepsilon\}} \inf_{z_1 \in D_1} \boldsymbol{E}^{z_1} \mathbf{1}_{\{\tilde{\lambda} < \eta_5^\varepsilon\}} \inf_{z_2 \in D_2} \boldsymbol{E}^{z_2} \mathbf{1}_{\{\lambda < \eta_5^\varepsilon\}} \inf_{z_3 \in D_3} \boldsymbol{E}^{z_3} \mathbf{1}_{\{\tau^\varepsilon < \eta_5^\varepsilon\}} \\
&\geqslant \frac{C\varepsilon}{|\ln \delta|^{12}}\,,
\end{aligned}$$

where C is independent of z_0. Taking the infimum over z_0, we achieve the desired result. $\qquad\square$

4.2. **Proof of Lemma 4.2.** In this subsection, we give a proof of Lemma 4.2. The strategy then will be similar to that of the proof of Lemma 4.1 as will will estimate the probability for a typical particle to successfully enter the inner region after each time it goes around the boundary layer $\mathcal{B}_5$. To do this, we first need a few results.

Lemma 4.11. *Let* $\tilde{\square}_1 = \mathcal{B}_5 \cap \{x_2 \in [-1 + c_0, -c_0]\}$. *There exists a constant* C *such that*

$$(4.16) \qquad\qquad \inf_{z_0 \in \tilde{\square}_1} \boldsymbol{P}^{z_0}\Big(\eta_5^\varepsilon < \frac{1}{A}\Big) \geqslant C\,.$$

Proof. Since we restrict our attention to region of the boundary layer on the sides, for each $\varepsilon > 0$ there exists an interval R_ε with length $|R_\varepsilon| = 1/\sqrt{A}$ such that

$$\mathrm{dist}\big(R_\varepsilon \times [-1 + c_0, -c_0]\,, \mathcal{B}_5 \cap \{x_2 \in [-1 + c_0, -c_0]\}\big) = \frac{1}{\sqrt{A}}\,.$$

Let M be independent of ε such that

$$R_\varepsilon \times [-1 + c_0, -c_0] \cup \big(\mathcal{B}_5 \cap \{x_2 \in [-1 + c_0, -c_0]\}\big) \subseteq \Big(-\frac{M}{\sqrt{A}}, \frac{M}{\sqrt{A}}\Big) \times [-1 + c_0, -c_0]\,,$$

and $z_0 \in \tilde{\square}_1$. By Lemma A.4 applied to the deterministic curve γ (given by (4.7)) with $\gamma_0 = z_0$, we have

$$\boldsymbol{P}^{z_0}\Big(\eta_5^\varepsilon < \frac{1}{A}\Big)$$

$$\geqslant \boldsymbol{P}^{z_0}\left(\sup_{0\leqslant t\leqslant 1/A}|Z_{1,t}-\gamma_t^1|\leqslant \frac{M}{\sqrt{A}}\,,\,\sup_{0\leqslant t\leqslant 1/A}|Z_{2,t}-\gamma_{2,t}|\leqslant \frac{\varepsilon}{\sqrt{A}}\,,\,Z_{1,T_0}\in R_\varepsilon\right)\geqslant C\,,$$

where C is independent of z_0 as desired. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

Lemma 4.12. *Consider only the velocity v^0 coming from the standard Hamiltonian H^0. Let $\tilde{\lambda}_2 = \inf\{t > 0 \mid Z_{2,t} \in \{-1 + c_0, -c_0\}\}$ and $z_0 \in \mathcal{B}_5 - \tilde{\square}_1$. Then*

$$(4.17) \qquad\qquad \lim_{\varepsilon\to 0}\inf_{\mathcal{B}_5-\tilde{\square}_1}\boldsymbol{P}^{z_0}\left(\tilde{\lambda}_2\leqslant \frac{5|\ln\delta|}{A}\right)\to 1\,.$$

Proof. Let $q \geqslant 2$ be some large number to be chosen later, and let $\tilde{z}_0$ be the closest point on $\{H = A^{-1/q}\}$ to z_0. Let $\tilde{d} = A|z_0 - \tilde{z}_0|$ and γ_t be the deterministic curve (defined by (4.7)) with $\gamma_0 = \tilde{z}_0$. Note that, by Assumptions 1–2,

$$(4.18) \qquad\qquad\qquad\qquad \frac{\tilde{d}}{A} \leqslant \frac{C}{A^{1/2q}}\,.$$

By Itô formula, we have

$$\boldsymbol{E}^{z_0}|Z_t-\gamma_t|^2 \leqslant \frac{\tilde{d}^2}{A^2} + 2A\|v\|_{C^1}\int_0^t \boldsymbol{E}^{z_0}|Z_s-\gamma_s|^2\,ds + (1+\varepsilon^2)t\,.$$

By Gronwall's inequality and Assumption 1, it follows that

$$\boldsymbol{E}^{z_0}|Z_t-\gamma_t|^2 \leqslant \left(\frac{\tilde{d}^2}{A^2} + (1+\varepsilon^2)t\right)e^{200At}\,.$$

Now, let $T = \inf\{t > 0 : \gamma_t \in \{-1 + 2c_0, -2c_0\}\}$, and note that $T \leqslant D\ln A/(Aq)$ for some constant $D > 0$. By (4.18), we have

$$\boldsymbol{P}^{z_0}\left(|Z_T-\gamma_T|\geqslant \frac{c_0}{10}\right)\leqslant \frac{100}{c_0^2}\left(\frac{C}{A^{2q}} + (1+\varepsilon^2)\frac{D\ln A}{Aq}\right)e^{200D\ln A/q}$$
$$\leqslant CA^{200D/q-1}\ln A\,.$$

Picking q such that $200D/q - 1 < -1/2$, we have

$$(4.19) \qquad\qquad \boldsymbol{P}^{z_0}\left(|Z_T-\gamma_T| < \frac{c_0}{10}\right)\geqslant 1 - \frac{C\ln A}{A^{1/4}}\,.$$

As $q \geqslant 2$, $T < 5|\ln\delta|/A$. Therefore, by continuity, it follows that

$$\left\{Z_T^2\in[-1+2c_0,-2c_0]\right\}\subseteq\left\{\tilde{\lambda}_2\leqslant\frac{5|\ln\delta|}{A}\right\}\,.$$

Combining this with (4.19), we deduce

$$\lim_{\varepsilon\to 0}\inf_{\mathcal{B}_5-\tilde{\square}_1}\boldsymbol{P}^{z_0}\left(\tilde{\lambda}_2\leqslant \frac{5|\ln\delta|}{A}\right) = 1\,,$$

as desired. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

We are now ready for the proof of Lemma 4.2.

Proof of Lemma 4.2. Step 1: We first claim that for each $z_0 \in \mathcal{B}_5$ and $\varepsilon > 0$, there exists a constant $C > 0$, independent of z_0 and ε, such that

$$(4.20) \qquad\qquad \boldsymbol{P}^{z_0}\left(\sup_{0\leqslant t\leqslant 6|\ln\delta|/A}|H(Z_t)| > \frac{5}{\sqrt{A}}\right)\geqslant C\,.$$

To prove this, suppose for contradiction there exists a subsequence $\{z_n, \varepsilon_n\}_{n=1}^{\infty}$ such that

$$(4.21) \qquad \lim_{n \to \infty} \boldsymbol{P}^{z_n}\left(\sup_{0 \leqslant t \leqslant 6|\ln \delta|/A} |H(Z_t)| > \frac{5}{\sqrt{A}}\right) = 0.$$

Let C_0 be the lower bound in Lemma 4.11 and denote $\tilde{\lambda}_1 = \inf\{t \geqslant 0 \mid Z_t \in \tilde{\square}_1\}$. By Lemma 4.11 and the strong Markov property,

$$\boldsymbol{P}^{z_n}\left(\sup_{0 \leqslant t \leqslant 6|\ln \delta|/A} |H(Z_t)| > \frac{5}{\sqrt{A}}\right)$$

$$\geqslant \boldsymbol{E}^{z_n}\left(\boldsymbol{E}^{z_n}\left(\mathbf{1}_{\left\{\sup_{0 \leqslant t \leqslant \tilde{\lambda}_1} |H(Z_t)| \leqslant \frac{5}{\sqrt{A}}\right\}} \mathbf{1}_{\left\{\tilde{\lambda}_1 \leqslant 5|\ln \delta|/A\right\}} \mathbf{1}_{\left\{\eta_5^{\varepsilon} < \tilde{\lambda}_1 + 1/A\right\}} \mid \mathcal{F}_{\tilde{\lambda}_1}\right)\right)$$

$$= \boldsymbol{E}^{z_n}\left(\mathbf{1}_{\left\{\sup_{0 \leqslant t \leqslant \tilde{\lambda}_1} |H(Z_t)| \leqslant \frac{5}{\sqrt{A}}\right\}} \mathbf{1}_{\left\{\tilde{\lambda}_1 \leqslant 5|\ln \delta|/A\right\}} \boldsymbol{E}^{Z_{\tilde{\lambda}_1}} \mathbf{1}_{\left\{\eta_5^{\varepsilon} < 1/A\right\}}\right)$$

$$\geqslant \boldsymbol{E}^{z_n}\left(\mathbf{1}_{\left\{\sup_{0 \leqslant t \leqslant \tilde{\lambda}_1} |H(Z_t)| \leqslant \frac{5}{\sqrt{A}}\right\}} \mathbf{1}_{\left\{\tilde{\lambda}_1 \leqslant 5|\ln \delta|/A\right\}}\right) \inf_{z \in \tilde{\square}_1} \boldsymbol{E}^{z} \mathbf{1}_{\left\{\eta_5^{\varepsilon} < 1/A\right\}}$$

$$\geqslant C_0 \boldsymbol{P}^{z_n}\left(\sup_{0 \leqslant t \leqslant \tilde{\lambda}_1} |H(Z_t)| \leqslant \frac{5}{\sqrt{A}} \,;\, \tilde{\lambda}_1 \leqslant \frac{5|\ln \delta|}{A}\right).$$

The second equality follows from the fact that $\eta_5^{\varepsilon} > \tilde{\lambda}_1$ under the event

$$\left\{\sup_{0 \leqslant t \leqslant \tilde{\lambda}_1} |H(Z_t)| \leqslant \frac{5}{\sqrt{A}}\right\}.$$

We claim that for large enough n, we have

$$\boldsymbol{P}^{z_n}\left(\sup_{0 \leqslant t \leqslant \tilde{\lambda}_1} |H(Z_t)| \leqslant \frac{5}{\sqrt{A}} \,;\, \tilde{\lambda}_1 \leqslant \frac{5|\ln \delta|}{A}\right) \geqslant \frac{1}{2},$$

which contradicts our assumption (4.21). To see that this lower bound is true, we first note that $z_i \notin \tilde{\square}_1$ by Lemma 4.11. Thus, we only consider the case $z_n \in \mathcal{B}_5 - \tilde{\square}_1$.

Recall $\tilde{\lambda}_2 = \inf\{t > 0 \mid Z_t^2 \in \{-1 + c_0, -c_0\}\}$. We need to be careful here as we want to apply Lemma 4.12 but it is only valid when $v = v^0$. Therefore, denote $\tilde{\lambda}_2^k = \tilde{\lambda}_2$ when $v = v^k$, $k \in \{0, N\}$.

Observe that, if $v = v^0$,

$$\mathbf{1}_{\{\sup_{0 \leqslant t \leqslant \tilde{\lambda}_1^0} |H^0(Z_t)| \leqslant \frac{5}{\sqrt{A}}\}} \mathbf{1}_{\{\tilde{\lambda}_1^0 \leqslant 5|\ln \delta|/A\}}$$

$$= \mathbf{1}_{\{\sup_{0 \leqslant t \leqslant \tilde{\lambda}_1^0} |H^0(Z_t)| \leqslant \frac{5}{\sqrt{A}}\}} \mathbf{1}_{\{\tilde{\lambda}_2^0 \leqslant 5|\ln \delta|/A\}}.$$

By (4.17) and (4.21) and, we can pick n large enough such that

$$\boldsymbol{P}^{z_n}\left(\sup_{0 \leqslant t \leqslant \tilde{\lambda}_1^0} |H^0(Z_t)| \leqslant \frac{5}{\sqrt{A}} \,;\, \tilde{\lambda}_1^0 \leqslant 5|\ln \delta|/A\right)$$

$$\geqslant \boldsymbol{P}^{z_n}\left(\sup_{0 \leqslant t \leqslant 6|\ln \delta|/A} |H^0(Z_t)| \leqslant \frac{5}{\sqrt{A}} \,;\, \tilde{\lambda}_2^0 \leqslant 5|\ln \delta|/A\right) \geqslant \frac{1}{2}.$$

On the other hand, if $v = v^N$, since $H^0 = H^N$ when $h \leqslant N/\sqrt{A}$ (recall (2.3)) and $z_n \in \mathcal{B}_5$, we have

$$\mathbf{1}_{\{\sup_{0 \leqslant t \leqslant \tilde{\lambda}_1^N} |H^N(Z_t)| \leqslant \frac{5}{\sqrt{A}}\}} \mathbf{1}_{\{\tilde{\lambda}_1^N \leqslant 5|\ln \delta|/A\}}$$

$$= \mathbf{1}_{\{\sup_{0 \leqslant t \leqslant \tilde{\lambda}_1^0} |H^0(Z_t)| \leqslant \frac{5}{\sqrt{A}}\}} \mathbf{1}_{\{\tilde{\lambda}_1^0 \leqslant 5|\ln \delta|/A\}},$$

which implies

$$\boldsymbol{P}^{z_n}\Big(\sup_{0 \leqslant t \leqslant \tilde{\lambda}_1^N} |H^N(Z_t)| \leqslant \frac{5}{\sqrt{A}}\Big) \geqslant \frac{1}{2}\,.$$

In both cases, there is a contradiction, proving (4.20) as desired.

Step 2: Once (4.20) is established, we can estimate $\boldsymbol{E}\eta_5^\varepsilon$ as the expected time to success of a Bernoulli trial using a similar argument as in the proof of Lemma 3.2. Explicitly, let $\Delta t = 6|\ln \delta|/A$, and observe that by (4.20),

$$\boldsymbol{P}^{z_0}\big(\eta_5^\varepsilon < \Delta t\big) = \boldsymbol{P}^{z_0}\Big(\sup_{0 \leqslant t \leqslant 6|\ln \delta|/A} |H(Z_t)| > \frac{5}{\sqrt{A}}\Big) \geqslant C\,.$$

By the strong Markov property and estimate (4.20), we have that for $i > 1$,

$$\begin{aligned}
\boldsymbol{P}^{z_0}\big(\eta_5^\varepsilon \geqslant i\Delta t\big) &= \boldsymbol{E}^{z_0}\boldsymbol{E}^{z_0}\big(\mathbf{1}_{\{\eta_5^\varepsilon \geqslant i\Delta t\}}\mathbf{1}_{\{\eta_5^\varepsilon \geqslant (i-1)\Delta t\}} \mid \mathcal{F}_{(i-1)\Delta t}\big) \\
&= \boldsymbol{E}^{z_0}\mathbf{1}_{\{\eta_5^\varepsilon \geqslant (i-1)\Delta t\}}\boldsymbol{E}^{Z_{(i-1)\Delta t}}\mathbf{1}_{\{\eta_5^\varepsilon \geqslant \Delta t\}} \\
&\leqslant \boldsymbol{E}^{z_0}\mathbf{1}_{\{\eta_5^\varepsilon \geqslant (i-1)\Delta t\}}\sup_{z \in \mathcal{B}_5}\boldsymbol{E}^z\mathbf{1}_{\{\eta_5^\varepsilon \geqslant \Delta t\}} \\
&= \boldsymbol{E}^{z_0}\mathbf{1}_{\{\eta_5^\varepsilon \geqslant (i-1)\Delta t\}}\big(1 - \inf_{z \in \mathcal{B}_5}\boldsymbol{P}^z\big(\eta_5^\varepsilon < \Delta t\big)\big) \\
&= \boldsymbol{E}^{z_0}\mathbf{1}_{\{\eta_5^\varepsilon \geqslant (i-1)\Delta t\}}(1 - C) \leqslant (1 - C)^i\,,
\end{aligned}$$

where C is the constant in (4.20). Therefore,

$$\begin{aligned}
\boldsymbol{E}^{z_0}\eta_5^\varepsilon &= \int_0^\infty \boldsymbol{P}^{z_0}\big(\eta_5^\varepsilon \geqslant t\big)\,dt \leqslant \sum_{i=1}^\infty \int_{(i-1)\Delta t}^{i\Delta t} \boldsymbol{P}^{z_0}\big(\eta_5^\varepsilon \geqslant t\big)\,dt \\
&\leqslant \Delta t \sum_{i=0}^\infty \boldsymbol{P}^{z_0}\big(\eta_5^\varepsilon \geqslant i\Delta t\big) \leqslant \Delta t \sum_{i=0}^\infty (1 - C)^i \leqslant \frac{6|\ln \delta|}{(1 - C)A}\,,
\end{aligned}$$

from which (4.2) follows immediately. $\square$

4.3. **Proof of Lemma 4.3.** In this subsection, we restrcit our attention to a particular cell and therefore assume for simplicity that $|H| = H$. By Assumption 4, $\partial_i^2 H \leqslant 0$ for $i \in \{1, 2\}$. Let $z \in \overline{\mathcal{B}}_1^c$ and denote $U_\varepsilon(z) = \boldsymbol{E}^z\eta_1^\varepsilon$. Then, U_ε solves the following equation
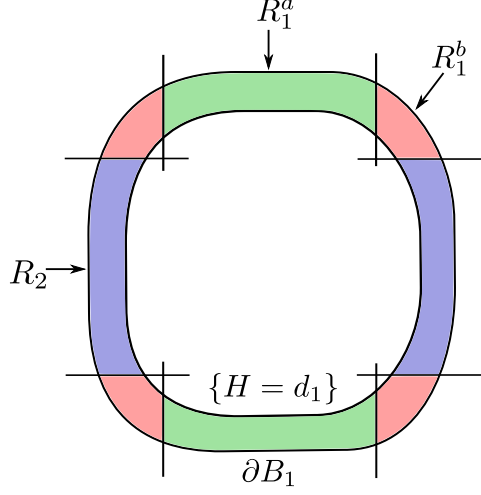
$$(4.22) \qquad \begin{cases} -\partial_1^2 U_\varepsilon - \varepsilon^2 \partial_2^2 U_\varepsilon + Av \cdot \nabla U_\varepsilon = 1 & \text{in } \overline{\mathcal{B}}_1^c\,, \\ U_\varepsilon = 0 & \text{on } \partial\mathcal{B}_1\,. \end{cases}$$

In order to prove Lemma 4.3, we construct an explicit supersolution to (4.22), independent of ε. Recall by Lemma 3.4,

$$S \stackrel{\text{def}}{=} \sup_{\varepsilon > 0}\|U_\varepsilon\|_\infty < \infty\,.$$

Let $d_1 \ll 1$ be a small constant that will be chosen later, and define

$$\Lambda = \Big\{\frac{1}{\sqrt{A}} \leqslant |H| \leqslant d_1\Big\}$$

$$R_2 = \Lambda \cap \{y \in [-1 + c_0, -c_0]\} \quad \text{and} \quad R_1 = \Lambda - R_2\,.$$

FIGURE 3. Λ, R_1^a (green), R_1^b (red) and R_2 (blue).

Denote by (θ, h) the curvilinear coordinate, where $\theta = \Theta(x_1, x_2)$ is the "angle" and $h = H(x_1, x_2)$ the level of the Hamiltonian H (See Appendix 5). Let f (to be specified later) be a smooth periodic function of Θ that satisfies

$$0 < \inf f < \sup f < \infty \,,$$

(4.23) $$-\infty < \inf f'(\Theta) \leqslant \sup f'(\Theta) < -1 \quad \text{on } R_1 \,,$$

$$\text{and} \quad \sup|f''| < \infty \,.$$

Then, consider the function

$$\phi = \chi_1 + \chi_2 \,,$$

where

$$\chi_1 = -\frac{S}{d_1} H \ln H \quad \text{and} \quad \chi_2 = -\frac{f(\Theta)}{AH} + \frac{\|f\|_\infty}{\sqrt{A}} \,.$$

By construction, $\phi(\Theta, H) \geqslant 0$ on Λ. We claim that for an appropriate f, ϕ is a desired supersolution.

Lemma 4.13. *Let U_ε be the solution to equation* (4.22)*. Then, there exists a function f that satisfies the requirement* (4.23) *so that for small enough d_1,*

$$\phi \geqslant U_\varepsilon \quad \text{on } \Lambda \,.$$

Postponing the proof of this lemma, we now give the proof of Lemma 4.3.

Proof of Lemma 4.3. By construction, on $\overline{\mathcal{B}}_5 - \mathcal{B}_1$ and for small enough ε, we have $\frac{5}{\sqrt{A}} \leqslant d_1$. Therefore, when $H = 5/\sqrt{A}$,

$$\phi \leqslant -\frac{S}{d_1} \frac{5}{\sqrt{A}} \ln\left(\frac{5}{\sqrt{A}}\right) + \frac{\|f\|_\infty}{\sqrt{A}} \leqslant \frac{|\ln \delta|}{\sqrt{A}} \,.$$

It follows that

$$\boldsymbol{E}^z \eta_1^\varepsilon = U(z) \leqslant \phi(z) \leqslant \frac{|\ln \delta|}{\sqrt{A}} \,,$$

for every $z \in \partial \mathcal{B}_5$, as desired. $\square$

Proof of Lemma 4.13. Step 1: Recall that $v = \nabla^\perp H$ and $H \geqslant 1/\sqrt{A}$. We have that

$$\nabla \chi_2 = -\frac{f'(\Theta)}{AH}\nabla\Theta + \frac{f(\Theta)}{AH^2}\nabla H \,,$$

$$-\partial_1^2\chi_2 = \frac{1}{A}\Big(\frac{f''(\Theta)}{H}(\partial_1\Theta)^2 - 2\frac{f'(\Theta)}{H^2}\partial_1\Theta\partial_1 H + \frac{f'(\Theta)}{H}\partial_1^2\Theta\Big)$$
$$+ \frac{1}{A}\Big(\frac{2f(\Theta)}{H^3}(\partial_1 H)^2 - \frac{f(\Theta)}{H^2}\partial_1^2 H\Big)$$
$$\geqslant \frac{1}{A}\Big(\frac{f''(\Theta)}{H}(\partial_1\Theta)^2 - 2\frac{f'(\Theta)}{H^2}\partial_1\Theta\partial_1 H + \frac{f'(\Theta)}{H}\partial_1^2\Theta\Big)\,,$$

and

$$-\partial_2^2\chi_2 \geqslant \frac{1}{A}\Big(\frac{f''(\Theta)}{H}(\partial_2\Theta)^2 - 2\frac{f'(\Theta)}{H^2}\partial_2\Theta\partial_2 H + \frac{f'(\Theta)}{H}\partial_2^2\Theta\Big)\,.$$

Therefore, by (4.23) and $H \geqslant 1/\sqrt{A}$,

$$(4.24) \qquad -(\partial_1^2 + \varepsilon\partial_2^2)\chi_2 \geqslant -\frac{2}{A}\Big(\frac{f'(\Theta)}{H^2}\big(\partial_1\Theta\partial_1 H + \varepsilon\partial_2\Theta\partial_2 H\big)\Big) - \frac{C}{\sqrt{A}}\,.$$

Step 2: On the other hand,

$$\nabla\chi_1 = -\frac{S}{d_1}(1 + \ln H)\nabla H$$

and

$$-\partial_1^2\chi_1 = \frac{S}{d_1}\partial_1^2 H(\ln H + 1) + \frac{S}{d_1}\frac{(\partial_1 H)^2}{H}$$

We note that there exists a function $\rho = \rho(x) > 0$ that

$$\nabla\Theta = \rho(x)\nabla^\perp H = \rho(x)v(x)\,,$$

and $\lambda_1 \leqslant \rho \leqslant \lambda_2$ on $\{|H| \leqslant c_0\}$ for some $0 < \lambda_1 < \lambda_2$. Therefore, by (4.24) and $H \geqslant 1/\sqrt{A}$,

$$-\partial_1^2\phi - \varepsilon\partial_2^2\phi + Av\cdot\nabla\phi$$
$$(4.25) \qquad \geqslant \frac{S}{d_1}\partial_1^2 H(\ln H + 1) + \frac{S}{d_1}\frac{(\partial_1 H)^2}{H} - \frac{f'(\Theta)|\nabla H|^2}{H}\rho$$
$$- \frac{2}{A}\Big(\frac{f'(\Theta)}{H^2}\big(\partial_1\Theta\partial_1 H + \varepsilon\partial_2\Theta\partial_2 H\big)\Big) - \frac{C}{\sqrt{A}}\,.$$

Recall
$$R_2 = \Lambda \cap \{z_2 \in [-1 + c_0, -c_0]\} \quad \text{and} \quad R_1 = \Lambda - R_2\,.$$

We would like to estimate the above quantity in R_1 and R_2.

Step 3: For R_1, we decompose this set further

$$R_1^a = R_1 \cap \{-1 + c_0 \leqslant z_1 \leqslant -c_0\} \quad \text{and} \quad R_1^b = R_1 - R_1^a\,.$$

In R_1^a, there exists a constant $\tilde{C}$ such that $|\nabla H|^2 \geqslant \tilde{C}$. Therefore, by (4.23), (4.25) and $H \geqslant 1/\sqrt{A}$,

$$-\partial_1^2\phi - \varepsilon\partial_2^2\phi + Av\cdot\nabla\phi \geqslant -\frac{f'(\Theta)|\nabla H|^2}{H}\rho - C\|f'\|_{L^\infty}$$
$$\geqslant \frac{\lambda_1\tilde{C}\inf_{R_1}|f'(\Theta)|}{d_1} - C\|f'\|_{L^\infty}\,.$$

By (4.23), we could then pick d_1 small, independent of ε, to make the following hold
$$-\partial_1^2\phi - \varepsilon\partial_2^2\phi + Av\cdot\nabla\phi > 1$$
in R_1^a.

On the other hand, in R_1^b, we have $|\nabla H(z_1, z_2)|^2 = z_1^2 + z_2^2$. Therefore, by Cauchy-Schwarz inequality,
$$(4.26) \qquad \left|f'(\Theta)\frac{|\nabla H|^2}{H}\right| = -f'(\Theta)\frac{|\nabla H|^2}{H} = -f'(\Theta)\frac{z_1^2 + z_2^2}{z_1 z_2} \geqslant 2\inf_{R_1}|f'|\,.$$

Also, note that in R_1^b it holds that $|\partial_i\Theta\partial_i H| = (\partial_i H)^2$ for $i = 1, 2$. Thus, by (4.23)–(4.26) and $H \geqslant 1/\sqrt{A}$, we choose f such that $\lambda_1\inf_{R_1}|f'| > 2$ and ε small enough to get
$$\begin{aligned}
&-\partial_1^2\phi - \varepsilon\partial_2^2\phi + Av\cdot\nabla\phi \\
&\geqslant -\frac{f'(\Theta)|\nabla H|^2}{H}\rho - \frac{2}{A}\Big(\frac{f'(\Theta)}{H^2}\big(\partial_1\Theta\partial_1 H + \varepsilon\partial_2\Theta\partial_2 H\big)\Big) - \frac{C}{\sqrt{A}} \\
&\geqslant -\frac{f'(\Theta)|\nabla H|^2}{H}\rho - \frac{2}{A}\Big|\frac{f'(\Theta)|\nabla H|^2}{H^2}\Big| - \frac{C}{\sqrt{A}} \\
&= \Big|\frac{f'(\Theta)|\nabla H|^2}{H}\Big|\Big(\rho - \frac{2}{AH}\Big) - \frac{C}{\sqrt{A}} \\
&\geqslant \lambda_1\inf_{R_1}|f'| - \frac{C}{\sqrt{A}} > 1\,.
\end{aligned}$$

Thus, we have just shown that there exists a function f that satisfies (4.23) so that in R_1,
$$-\partial_1^2\phi - \varepsilon\partial_2^2\phi + Av\cdot\nabla\phi > 1\,.$$

Step 4: In R_2, there exist constants C_1, C_2 so that
$$0 < C_2 \leqslant C_1|\nabla H|^2 \leqslant (\partial_1 H)^2\,.$$

We then look at
$$\begin{aligned}
&-\partial_1^2\phi - \varepsilon\partial_2^2\phi + Av\cdot\nabla\phi \\
&\geqslant \frac{S}{d_1}\partial_1^2 H(\ln H + 1) + \frac{S}{d_1}\frac{(\partial_1 H)^2}{H} - \frac{f'(\Theta)|\nabla H|^2}{H}\rho - C \\
&\geqslant \frac{S}{d_1}\partial_1^2 H(\ln H + 1) + \frac{S}{d_1}\frac{C_1|\nabla H|^2}{H} - \lambda_2\|f'\|_{L^\infty(R_2)}\frac{|\nabla H|^2}{H} - C \\
&\geqslant \frac{C_2}{C_1 d_1}\Big(\frac{SC_1}{d_1} - \lambda_2\|f'\|_{L^\infty(R_2)}\Big) - C\,.
\end{aligned}$$
Pick d_1 smaller if needed to get
$$-\partial_2\phi - \varepsilon\partial_2^2\phi + Av\cdot\nabla\phi > 1 \quad \text{in } R_2\,.$$

Step 5: Combining Steps 3 and 4, we have shown that there exists a function f such that
$$-\partial_2\phi - \varepsilon\partial_2^2\phi + Av\cdot\nabla\phi > 1 \quad \text{in } \Lambda\,.$$
By construction, $\phi \geqslant U_\varepsilon$ on $\{H = d_1\} \cup \{H = \frac{1}{\sqrt{A}}\}$. The comparison principle then tells us that
$$\phi \geqslant U_\varepsilon \quad \text{in } \Lambda$$
as desired. $\qquad\square$

5. **Proof of Lemma 3.4**

In this section, we give the proof of Lemma 3.4. We only treat the case for the standard Hamiltonian H^0 as the case for cut-off Hamiltonian H^N could be treated similarly by a slight modification. This fact has been obtained in more generality by PDE method by Ishii and Souganidis [IS12]. Our method proof, still PDE-based, is different than that in [IS12]. Although the argument is new for our particular situation, it is an adaptation of the method in [Kum18], where the author studies the Freidlin problem for first order Hamilton-Jacobi equations.

It is convenient to work in the so-called curvilinear coordinates (h, θ), in one cell. Let $\mathcal{Q}_0^* = (0, 1)^2 - \Gamma_0$, where Γ_0 is the closure of one trajectory of the gradient flow of H starting on the boundary of the unit square. On $\mathcal{Q}_0^*$ we define the curvilinear coordinates by setting $h = H(x)$, $\theta = \Theta(x)$, where Θ solves

$$\nabla \Theta \cdot \nabla H = 0 \, ,$$

in $\mathcal{Q}_0^*$, normalized so that the range of Θ is $(0, 2\pi)$. In this coordinate system, $h(x)$ determines the level set of the Hamiltonian to which x belongs and θ describes the position of x on this level set. Since $\nabla \Theta$ and $\nabla^\perp H$ are parallel, there must exist a non-zero function ρ such that

$$\nabla \Theta = \rho \nabla^\perp H \, .$$

By reversing the orientation of Θ if needed, we may assume, without loss of generality, that $\rho > 0$. Let $J = \partial_1 H \partial_2 \Theta - \partial_2 H \partial_1 \Theta$ be the Jacobian of the coordinate transformation, and note

$$J = \rho |\nabla H|^2 \, , \quad |\nabla \Theta| = \rho |\nabla H| \, .$$

Let γ be the solution to (4.7) with $\gamma_0 = x$, and T be the time period of γ. Note T only depends on $h = H(x)$, and is given by

$$(5.1) \qquad T(h) \overset{\text{def}}{=} \inf\{t > 0 : \gamma(t, x) = x\} = \oint_{\{H=h\}} \frac{1}{|\nabla H|} \, |d\ell| \, ,$$

where $|d\ell|$ denotes the arc-length integral along the curve $\{H = h\}$.

Let $S(x) \overset{\text{def}}{=} \inf\{t : \gamma(t, x) \in \Gamma_0\}$ be the amount of time γ takes to to reach Γ_0 starting from x. This time is not a continuous function of x. Therefore, in order to make it continuous, we modify it to the following continuous function

$$(5.2) \qquad \tilde{S}(x) := \begin{cases} S(x) & \text{if } S(x) > \Gamma(H(x))/2, \\ -S(x) + \Gamma(H(x)) & \text{if } S(x) < \Gamma(H(x))/2. \end{cases}$$

As we have restricted our attention to one cell, we can assume $H \in [0, 1]$. Define the coefficients D_1 and D_2 on $[0, 1]$ as follows

$$(5.3a) \qquad D_1(h) = \frac{1}{T(h)} \oint_{\{H=h\}} \frac{|\partial_1 H|^2}{|\nabla H|} \, |d\ell| \, ,$$

$$(5.3b) \qquad D_2(h) = \frac{1}{T(h)} \oint_{\{H=h\}} \frac{\partial_1^2 H}{|\nabla H|} \, |d\ell| \, .$$

Note that by Gauss–Green theorem, we have

$$T(h) D_1(h) = -\int_{\{H \geqslant h\}} \partial_1^2 H(x) \, dx = \int_1^h \oint_{\{H=h\}} \frac{\partial_1^2 H}{|\nabla H|} \, |d\ell| \, dh \, .$$

Therefore,

$$\frac{d}{dh}(T(h)D_1(h)) = T(h)D_2(h).$$ (5.4)

We are now ready to show the proof of Lemma 3.4.

Proof of Lemma 3.4. Step 1: Let $U_\varepsilon(x) \stackrel{\text{def}}{=} \boldsymbol{E}^x \tau_0^\varepsilon$ and $\Omega_\varepsilon \stackrel{\text{def}}{=} \mathcal{B}_\alpha^c$. Then, U_ε is the solution to the equation

$$-\frac{1}{2}\partial_1^2 U_\varepsilon - \frac{\varepsilon^2}{2}\partial_2^2 U_\varepsilon + Av \cdot \nabla U_\varepsilon = 1 \quad \text{on } \Omega_\varepsilon,$$

with boundary condition

$$U_\varepsilon = 0 \quad \text{on } \partial\Omega_\varepsilon.$$

Lemma 3.4 will follow immediately from the uniform bound

$$\sup_\varepsilon \|U_\varepsilon\|_{L^\infty(\Omega_1^\varepsilon)} \leqslant C.$$

To see why this bound is true, let us consider the solution $\bar{U}$ to the ODE

$$\begin{cases} -D_1(h)\partial_h^2 \bar{U} - D_2(h)\partial_h \bar{U} &= 4, \\ U(0) &= 4. \end{cases}$$

Note that $\bar{U}$ is bounded. To see this, we use (5.4) to rewrite the equation

$$-\frac{1}{T(h)}\partial_h\Big(T(h)D_1(h)\partial_h\bar{U}\Big) = 4.$$

Observe that $T(h)D_1(h) \approx O(1-h)$ and $T(h) \to T_0 > 0$ as $h \to 1$; $T(h) \approx O(|\ln h|)$ and $D_1(h) \approx O(1/|\ln h|)$ as $h \to 0$ (see Chapter 8.2 in [FW12]). Using these asymptotics, we deduce

$$\partial_h\bar{U}(h) = \frac{4}{T(h)D_1(h)}\int_h^1 T(s)\,ds, \qquad \bar{U}(h) = \int_0^h \frac{4}{T(s)D_1(s)}\int_s^1 T(r)\,drds,$$

and

$$\|\bar{U}\|_{W^{1,\infty}} \leqslant C.$$

Step 2: Note that $\bar{U} \circ H$ is a function on Ω. Let

$$g = \partial_1^2(\bar{U} \circ H),$$

and we see that

$$\bar{g}(x) \stackrel{\text{def}}{=} \frac{1}{T(H(x))}\int_0^{T(H(x))} g(\gamma(t,x))\,dt = -4,$$

where T is defined in (5.1). Define

$$\varphi(x) = \int_0^{\tilde{S}(x)} (\bar{g}(x) - g(\gamma_t(x)))\,dt,$$

where $\tilde{S}$ is defined in (5.2). Note that

$$v(x) \cdot \nabla\varphi(x) = g(x) - \bar{g}(x) = g(x) + 4.$$ (5.5)

To see this, consider

$$\varphi(\gamma(s,x)) = -\int_0^{\tilde{S}(\gamma(s,x))} \Big(g(\gamma(t,\gamma(s,x))) - \bar{g}(\gamma(s,x))\Big)\,dt$$

$$= -\int_s^{\tilde{S}(x)} \Big(g(\gamma(t,x)) - \bar{g}(x)\Big)\, dt\,.$$

Differentiate in s and evaluate at $s = 0$, we get (5.5).

Step 3: Let

$$G_\varepsilon \stackrel{\text{def}}{=} \bar{U} \circ H + \frac{1}{A}\varphi\,, \qquad L_\varepsilon = -\frac{1}{2}\partial_1^2 - \frac{\varepsilon^2}{2}\partial_2^2 + Av\cdot\nabla\,,$$

and note

$$L_\varepsilon G_\varepsilon = -\frac{1}{2}\partial_1^2(\bar{U}\circ H) - \frac{1}{2A}\partial_1^2\varphi - \frac{\varepsilon^2}{2}\partial_2^2(\bar{U}\circ H) - \frac{\varepsilon^2}{2A}\partial_2^2\varphi + g(x) + 4$$

$$= -\frac{1}{2A}\partial_1^2\varphi - \frac{\varepsilon^2}{2}\partial_2^2(\bar{U}\circ H) - \frac{\varepsilon^2}{2A}\partial_2^2\varphi + 4 = e_\varepsilon + 4\,,$$

where $e_\varepsilon \stackrel{\text{def}}{=} -\frac{1}{2A}\partial_1^2\varphi - \frac{\varepsilon^2}{2}\partial_2^2(\bar{U}\circ H) - \frac{\varepsilon^2}{2A}\partial_2^2\varphi$. Since U is smooth and e_ε converge uniformly to 0 as $\varepsilon \to 0$, there exists an ε_0 such that for all $\varepsilon \leqslant \varepsilon_0$, $L_\varepsilon G_\varepsilon \geqslant 1$ and $G_\varepsilon \geqslant U_\varepsilon$ on $\partial\Omega_\varepsilon$. By the maximum principle, $G_\varepsilon \geqslant U_\varepsilon$ on Ω_ε. Finally, observe that $\sup_\varepsilon \|G_\varepsilon\|_\infty < \infty$, which implies what we want. $\qquad\square$

6. **Other Scalings**

In this section we study the behavior of $\|T^v\|_{L^\infty}$ when the velocity field v has convection rolls with varying width and height. Through scaling we can reduce the study of these to the case we have already considered.

Let $\alpha \geqslant 0$, and define

$$H^\varepsilon(x_1, x_2) = H\Big(\frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon^\alpha}\Big), \qquad \text{and} \qquad v^\varepsilon = \frac{A}{\varepsilon^{1-\alpha}}\nabla^\perp H^\varepsilon = \frac{A}{\varepsilon^{1-\alpha}}\begin{pmatrix} \partial_2 H^\varepsilon \\ -\partial_1 H^\varepsilon \end{pmatrix}\,.$$

Let $T^\varepsilon = T^{v^\varepsilon}$ solve (1.3) on S with 1-periodic boundary conditions in the horizontal direction and homogeneous Dirichlet boundary conditions on ∂S. In this case note that

$$(6.1) \qquad\qquad \mathrm{Pe} = \|v^\varepsilon\|_{L^2} = \begin{cases} O\Big(\dfrac{A}{\varepsilon^{2-\alpha}}\Big) & \alpha < 1\,, \\[2mm] O\Big(\dfrac{A}{\varepsilon}\Big) & \alpha \geqslant 1\,. \end{cases}$$

Proposition 6.1. *If $A = 1/\varepsilon^2$, then*

$$(6.2) \qquad\qquad \|T^\varepsilon\|_{L^\infty(S)} \leqslant \frac{C\varepsilon^{1-2\alpha}|\ln\varepsilon|^{13}}{\sqrt{A}}\,.$$

Rewriting this in terms of the Péclet number, we see

$$(6.3) \qquad\qquad \|T^\varepsilon\|_{L^\infty(S)} \leqslant \begin{cases} C\,\mathrm{Pe}^{-\frac{1}{2}+\frac{3\alpha}{8-2\alpha}}|\ln\mathrm{Pe}|^{13} & \alpha < 1\,, \\[2mm] C\,\mathrm{Pe}^{-\frac{1}{2}+\frac{4\alpha-1}{6}}|\ln\mathrm{Pe}|^{13} & \alpha \geqslant 1\,. \end{cases}$$

From the above we see that the large α is, the worse the bound (6.3) becomes, and so choosing $\alpha = 0$ is optimal. This corresponds to having cells of constant height.

Since the case when $\alpha > 0$ gives a sub-optimal bound, we only sketch the proof of Proposition 6.1 and show how it can be deduced quickly from Proposition 3.1.

Proof sketch of Proposition 6.1. Let X^ε be the diffusion defined by

$$dX_t^\varepsilon = \frac{A}{\varepsilon^{1-\alpha}} v^\varepsilon(X^\varepsilon)\, dt + dW_t\,,$$

and observe that T^ε is simply the expected exit time of X from the strip S. Define

$$Y_t^\varepsilon = \begin{pmatrix} \frac{1}{\varepsilon} X^{\varepsilon,1}(\varepsilon^2 t) \\ \frac{1}{\varepsilon^\alpha} X^{\varepsilon,2}(\varepsilon^2 t) \end{pmatrix}\,,$$

and observe

$$dY_t^\varepsilon = Av(Y_t^\varepsilon) + \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon \end{pmatrix} d\tilde{W}_t\,,$$

where $v = \nabla^\perp H$ and $\tilde{W}$ is a Brownian motion. Now we know (from Proposition 3.1) that the expected exit time of Y from either the top or bottom of one cell (the unit square) is bounded above by $C|\ln \varepsilon|^{13}/(\varepsilon A^{1/2})$. Since our original domain S is composed of $\varepsilon^{-\alpha}$ such cells, and since the time for the process X is ε^2 times the time for the process Y we obtain (6.2) as claimed. Using (6.1) in (6.2) immediately yields (6.3), finishing the proof of Proposition 6.1. □

Appendix A. Tube Lemmas

In this appendix, we are interested in estimating probabilities of short-time events of the stochastic process satisfying the following equation

$$(A.1) \qquad\qquad dZ_t = Av(Z_t)\, dt + \sigma dB_t\,,$$

where

$$(A.2) \qquad\qquad \|v\|_\infty \leqslant 1\,, \quad \|Dv\|_\infty \leqslant 1\,,$$

$$(A.3) \qquad\qquad \sigma = \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon \end{pmatrix}\,.$$

By abusing of notation, we denote $\sigma_i = \sigma_{ii}$. Thus, $\sigma_1 = 1$ and $\sigma_2 = \varepsilon$.

The following lemma is useful when we are at the top boundary layer.

Lemma A.1. *Fix $\lambda, \beta > 0$, and define $T = T_{\beta,A}$ and $R = R_{A,\lambda}$ by*

$$T \stackrel{\text{def}}{=} \frac{\beta}{A}\,, \qquad R \stackrel{\text{def}}{=} \frac{\lambda}{\sqrt{A}}(-1, 1) \times (-\varepsilon, \varepsilon)\,.$$

Let $z_0 \in R$, $u \in C^1(\mathbb{R}^2)$ and let γ be the solution to the ODE

$$\partial_t \tilde{\gamma}_t = Au(\tilde{\gamma}_t)\, dt\,, \qquad with \qquad \tilde{\gamma}_0 = z_0\,,$$

and $\Gamma = \{\tilde{\gamma}(t) \mid t \in [0,T]\}$ be the image of γ. Denote

$$L_T = -\frac{A^2}{2} \int_0^T \sum_{i=1,2} \Big(\frac{|u_i(\gamma(t)) - v_i(\gamma(t))|}{\sigma_i} + \sum_{j=1}^2 \frac{\sigma_j \|\partial_j v_i\|_{L^\infty(R+\Gamma)}}{\sigma_i \sqrt{A}} \Big)^2 dt\,.$$

Then for some $\alpha > 0$ we have

$$\boldsymbol{P}^{z_0}\Big(\sup_{0 \leqslant t \leqslant T} |\sigma^{-1}(Z_t - \tilde{\gamma}_t)|_\infty \leqslant \frac{\lambda}{\sqrt{A}} \Big) \geqslant \boldsymbol{P}\Big(\sup_{t \leqslant T} |B_t|_\infty \leqslant \frac{\lambda}{\sqrt{A}} \Big) \exp\Big(-\alpha\sqrt{L_T} - \frac{1}{2} L_T \Big)$$

for all sufficiently large A. Here the notation $|z|_\infty$ denotes $\max_i |z_i|$.

Remark A.2. A similar upper bound also holds, but is not needed for purposes of this paper.

Proof. Define the process $\tilde{Z}$ by

$$d\tilde{Z}_t = Au(\tilde{\gamma}_t)\,dt + \sigma\,dB_t\,, \qquad \text{with} \qquad \tilde{Z}_0 = z_0\,.$$

Define

$$h(t) \stackrel{\text{def}}{=} A(u(\tilde{\gamma}_t) - v(\tilde{Z}_t))\,,$$

$$\hat{h}(t) \stackrel{\text{def}}{=} \sigma^{-1}h(t)\,,$$

$$(A.4) \qquad M_t \stackrel{\text{def}}{=} \exp\Big(-\int_0^t \hat{h}(s)\,dB_s - \frac{1}{2}\int_0^t \hat{h}(s)^2\,ds\Big)$$

and a measure $\hat{\boldsymbol{P}}$ so that

$$d\hat{\boldsymbol{P}} = M_T\,d\boldsymbol{P}\,.$$

By the Girsanov theorem (see, for example, Theorem 8.6.6 in [Øks03]), the process

$$\hat{B}_t \stackrel{\text{def}}{=} \int_0^t \hat{h}(s)\,ds + B_t$$

is a Brownian motion with respect to the measure $\hat{P}$ up to time T. Since

$$d\tilde{Z} = Av(\tilde{Z})\,dt + \sigma\,d\hat{B}_t\,,$$

by weak uniqueness we have

$$\boldsymbol{E}^{z_0} f(Z_t) = \hat{\boldsymbol{E}}^{z_0} f(\tilde{Z}_t) = \hat{\boldsymbol{E}}^{z_0} f(\tilde{\gamma}_t + \sigma^\varepsilon B_t) = \boldsymbol{E}^{z_0} f(\tilde{\gamma}_t + \sigma^\varepsilon B_t) M_t\,,$$

for any test function f. Thus

$$\boldsymbol{P}^{z_0}\Big(\sup_{t\leqslant T}|\sigma^{-1}(Z_t - \tilde{\gamma}_t)|_\infty \leqslant \frac{\lambda}{\sqrt{A}}\Big) = \boldsymbol{E}_{\boldsymbol{P}}^{z_0}\Big(\mathbf{1}_K M_T\Big)\,.$$

where

$$K \stackrel{\text{def}}{=} \Big\{\sup_{t\leqslant T}|B_t|_\infty \leqslant \frac{\lambda}{\sqrt{A}}\Big\}\,.$$

Now let $\alpha = (2/\boldsymbol{P}(K))^{1/2}$, and $\hat{K}$ be the event

$$\hat{K} \stackrel{\text{def}}{=} \Big\{\Big(\int_0^T \hat{h}(t)\,dB_t\Big)^2 < \alpha^2 \int_0^T \hat{h}(t)^2\,dt\Big\}\,.$$

By Chebychev's inequality and the Itô isometry, we see

$$\boldsymbol{P}^{z_0}(\hat{K}^c) \leqslant \frac{1}{\alpha^2} = \frac{\boldsymbol{P}^{z_0}(K)}{2}\,.$$

Thus

$$\boldsymbol{E}^{z_0}(\mathbf{1}_K M_T) \geqslant \boldsymbol{E}^{z_0}\Big(\mathbf{1}_{K\cap\hat{K}} \exp\Big(-\alpha\Big(\int_0^T \hat{h}(t)^2\,dt\Big)^{1/2} - \frac{1}{2}\int_0^T \hat{h}(t)^2\,dt\Big)\Big)$$

$$(A.5) \qquad \geqslant \frac{\boldsymbol{P}^{z_0}(K)}{2} \exp\Big(-\alpha\Big(\int_0^T \hat{h}(t)^2\,dt\Big)^{1/2} - \frac{1}{2}\int_0^T \hat{h}(t)^2\,dt\Big)\,.$$

To estimate the exponential, note that under the event K we have

$$|\hat{h}_i(t)| = \frac{|h_i(t)|}{\sigma_i} = \frac{A}{\sigma_i}\Big|v_i(\tilde{\gamma}_t + \sigma B_t) - v_i(\tilde{\gamma}_t) + v_i(\tilde{\gamma}_t) - u_i(\tilde{\gamma}_t)\Big|$$

$$(A.6) \qquad \leqslant \frac{\lambda\sqrt{A}}{\sigma_i}\sum_j \sigma_j\|\partial_j v_i\|_{L^\infty(\Gamma+R)} + \frac{A|u_i(\tilde{\gamma}_t) - v_i(\tilde{\gamma}_t)|}{\sigma_i}\,,$$

for every $i = 1, 2$. Combining (A.6) with (A.5), the proof is finished. $\square$

Lemma A.3. *Using the same notation as Lemma A.1, we now additionally assume*

$$(A.7) \qquad \max_{i \in \{1,2\}} \sum_{j=1,2} \frac{\sigma_j \|\partial_j v_i\|_{L^\infty(R+\Gamma)}}{\sigma_i} \leqslant C_0$$

$$(A.8) \qquad \sum_{i=1,2} \int_0^T \frac{A^2 |u_i(\gamma_t) - v_i(\gamma_t)|^2}{\sigma_i^2} \, dt \leqslant C_0^2 .$$

Then there exists $C_1 = C_1(C_0, \lambda, \beta) > 0$ such that

$$\boldsymbol{P}^{z_0} \Big(\sup_{0 \leqslant t \leqslant T} |\sigma^{-1}(Z_t - \gamma_t)|_\infty \leqslant \frac{\lambda}{\sqrt{A}} \Big) \geqslant C_1$$

Proof. Following the proof of Lemma A.1, and using (A.7)–(A.8) in (A.6) gives

$$\int_0^T |\hat{h}(t)|^2 \, dt \leqslant 2C_0^2 (1 + \lambda \beta d) .$$

Combined with (A.5) the lemma follows. $\square$

Next, we show the following estimate for the side boundary layer.

Lemma A.4. *Let $z_0 \in \tilde{\mathcal{B}}_n \stackrel{\text{def}}{=} \mathcal{B}_n - [-1 + c_0, -c_0] \times [-1, 0]$ and $n \in \mathbb{N}$; Z_t be a stochastic process satisfying (A.1)–(A.3) and $\tilde{\gamma}_t^{z_0}$ be a deterministic process satisfying (4.7) and starting at z_0, i.e.,*

$$\partial_t \gamma_t = Av(\gamma_t) \quad \text{and} \quad \gamma_0 = z_0 .$$

For $M \geqslant 1$, let also $\tilde{R}_\varepsilon \subseteq [-M/\sqrt{A}, M/\sqrt{A}]$ be a Borel set, $\{d_\varepsilon \mid 0 < d_\varepsilon \leqslant 1\}$ be a sequence, and $T = m/A$ for some $m \in \mathbb{N}$. Then, there exists a constant $C = C_{m,M}$ and $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$,

$$\boldsymbol{P}^{z_0} \Big(\sup_{0 \leqslant t \leqslant T} |Z_t^1 - \gamma_t^1| \leqslant \frac{2M}{\sqrt{A}}, \sup_{0 \leqslant t \leqslant T} |Z_t^2 - \gamma_t^2| \leqslant \frac{\varepsilon}{\sqrt{A}}, Z_T^1 - \gamma_T^1 \in \tilde{R} \Big)$$

$$(A.9) \qquad \geqslant C_{m,n} \boldsymbol{P} \Big(|B_t| \leqslant \frac{2M}{\sqrt{A}}, B_T^1 \in \tilde{R} \Big)$$

Proof. We follow the proof of Lemma A.1, and explicitly substitute $\sigma_1 = 1$ and $\sigma_2 = \varepsilon$. Our conclusion (A.9) will follow provided we can show

$$(A.10) \qquad \int_0^T \hat{h}(t)^2 \, dt \leqslant C ,$$

for some finite constant C, independent of ε. To bound this, we use the upper bound (A.6), and observe that the second term on the right hand side is identically 0. For the first term, the only term that may grow faster than $\sqrt{A}$ is when $i = 2$ and $j = 1$. In this case, assumption 5 guarantees that this term is identically 0. Now squaring and integrating from 0 to $T = m/A$ proves (A.10) as desired. $\square$

Remark A.5. If the velocity field v does not satisfy Assumption 5, then Lemma A.4 still holds provided A is chosen so that $A \geqslant 1/\varepsilon^2$. To see this we note that (A.6) implies $\int_0^T \hat{h}(t)^2 \, dt \leqslant Cm/(A\varepsilon^2)$. If $A \geqslant 1/\varepsilon^2$ the right hand side of this is bounded independent of ε, and so the remainder of the proof of Lemma A.4 remains unchanged.

Proof of Lemma 4.4. We only consider the case where $z_0 \in Q_0/2$. The other cases are similar. First, recall that, by a direct calculation, we can check $T \leqslant |\ln \delta|/A$. Therefore, for small enough ε, under the event $\{|Z_t^i - \gamma_t^i| \leqslant \sigma_i(|\ln \delta|A)^{-1/2} \, , \ \forall t \leqslant T \, , i = 1, 2\}$, we must have $Z_t \in Q_0$ for $t \leqslant T$. Thus,

$$(A.11) \qquad\qquad v_1(Z_t) = Z_t^1 \quad \text{and} \quad v_2(Z_t) = -Z_t^2 \, .$$

Now define

$$d\tilde{Z}_t = A \begin{pmatrix} v_1(\gamma_t) \\ v_2(\gamma_t) \end{pmatrix} dt + \sigma \, dB_t$$

and write

$$(A.12) \qquad h(t) \stackrel{\text{def}}{=} A \begin{pmatrix} v_1(\gamma_t) - v_1(\tilde{Z}_t) \\ v_2(\gamma_t) - v_2(\tilde{Z}_t) \end{pmatrix} = A \begin{pmatrix} \gamma_t^1 - \tilde{Z}_t^1 \\ -\gamma_t^2 + \tilde{Z}_t^2 \end{pmatrix} = A \begin{pmatrix} -B_t^1 \\ \varepsilon B_t^2 \end{pmatrix} \, .$$

As before, we define $\hat{h}$ and a new measure $\hat{\boldsymbol{P}}$ by

$$\hat{h}(t) \stackrel{\text{def}}{=} \sigma^{-1} h(t) = \begin{pmatrix} 1 & 0 \\ 0 & 1/\varepsilon \end{pmatrix} h(t) = A \begin{pmatrix} -B_t^1 \\ B_t^2 \end{pmatrix} \, ,$$

$$d\hat{\boldsymbol{P}} = M_T \, d\boldsymbol{P} \, ,$$

where

$$M_t \stackrel{\text{def}}{=} \exp\left(-\int_0^t \hat{h}(s) \, dB_s - \frac{1}{2} \int_0^t \hat{h}(s)^2 \, ds \right) \, ,$$

for $0 \leqslant t \leqslant T$. By the Girsanov theorem, the process

$$\hat{B}_t \stackrel{\text{def}}{=} \int_0^t \hat{h}(s) \, ds + B_t$$

is a Brownian motion with respect to the measure $\hat{\boldsymbol{P}}$. Therefore, by uniqueness of weak solutions of SDEs, we have

$$\boldsymbol{E}(f(Z_t)) = \hat{\boldsymbol{E}}(f(\tilde{Z}_t)) = \hat{\boldsymbol{E}}(f(\gamma_t^1 + B_t^1, \gamma_t^2 + \varepsilon B_t^2))$$
$$= \boldsymbol{E}(f(\gamma_t^1 + B_t^1, \gamma_t^2 + \varepsilon B_t^2) M_t) \, .$$

Hence

$$\boldsymbol{P}^x\left(|Z_t^i - \gamma_t^i| \leqslant \frac{\sigma_i}{\sqrt{|\ln \delta|A}} \, , \ \forall t \leqslant T \, , i = 1, 2 \right)$$

$$= \boldsymbol{E}^x\left(\mathbf{1}_{\left\{ |B_t|_\infty \leqslant (|\ln \delta|A)^{-1/2} \, , \ \forall t \leqslant T \right\}} M_T \right) \, .$$

Now, we have that, by Itô formula,

$$\int_0^t \hat{h}(s) \, dB_s = -A \int_0^t B_s^1 \, dB_s^1 + A \int_0^t B_s^2 \, dB_s^2$$

$$= \frac{A}{2} (-(B_t^1)^2 + (B_t^2)^2) \, .$$

Therefore,

$$M_t \geqslant \exp\left(-\frac{A}{2}((B_t^1)^2 + (B_t^2)^2) - A^2 \int_0^t ((B_s^1)^2 + (B_s^2)^2) \, ds \right) \, .$$

Therefore, as $T \leqslant |\ln \delta|/A$, under the event

$$K \stackrel{\text{def}}{=} \left\{ |B_t|_\infty \leqslant \frac{1}{\sqrt{|\ln \delta|A}} \, , \ \forall t \leqslant T \right\} \, ,$$

we must have

$$M_T \geqslant \exp\left(-\frac{1}{2|\ln \delta|} - 2\right) \geqslant C\,.$$

Since $\boldsymbol{P}(K) \approx 1/|\ln \delta|^2$, this finishes the proof. $\qquad \square$

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