On Existence and Regularity for a Cahn-Hilliard Variational Model for Lithium-Ion Batteries

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Abstract

The Cahn-Hilliard reaction model, a nonlinear, evolutionary system of PDEs, was introduced to model phase separation in lithium-ion batteries. Using Butler-Volmer kinetics for electrochemical consistency, this model incorporates a nonlinear Neumann boundary condition $\partial_\nu \mu = R(c, \mu)$ for the chemical potential μ, with c, the lithium-ion density. Importantly, R depends exponentially on μ. In arbitrary dimension, existence of a weak solution for the Cahn-Hilliard reaction model with elasticity is proven using a generalized gradient structure. This approach is, at present, restricted to polynomial growth in R. Working to remove this limitation, fixed point methods are applied to obtain existence of strong solutions of the Cahn-Hilliard reaction model without elasticity in dimensions 2 and 3. This method is then extended to prove existence of higher regularity solutions in dimension 2, allowing for recovery of exponential boundary conditions as in the physical application to lithium-ion batteries.

Key words: Cahn-Hilliard, gradient flows, lithium-ion batteries
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1 Introduction

Modern technology relies heavily on lithium-ion batteries, from mobile phones to hybrid cars. More broadly, for the use of inconsistent renewable energies such as solar, it is imperative to develop effective means of energy storage, and lithium-ion batteries are premiere candidates for such storage [1]. The centrality of the need for a better understanding of batteries was underscored by the 2019 Nobel Prize in Chemistry, which was awarded to Goodenough, Whittingham, and Yoshino for their pioneering works in the development of lithium-ion batteries [1].

A prominent phenomenological behavior of lithium-ion batteries is phase separation, wherein lithium-ions intercalate into the host structure of the cathode inhomogeneously. In contrast to classical fluid-fluid transitions, such as oil and water, the separation of lithium-ions takes place within a solid host. Consequently, phase transitions induce a strain, damaging the cathode's host material, and this leads to a decrease in battery performance and limited life-cycle (see [9], [23], and references therein).

Understanding the onset of phase transitions is, therefore, imperative to improving battery performance, and much work has been done in this direction. Contemporary paradigms for modeling lithium-ion batteries are moving towards the incorporation of phase field models, also known as diffuse interface models (see, e.g., [6], [8], [21], [52], [58]). These phase field models are governed by global energy functionals, which have regular inputs (e.g., Sobolev functions). As noted in [9], the phase field field model is robust, allowing for electrochemically consistent models for the time evolution of lithium-ion batteries. Competing models include the shrinking core model and the sharp interface model. As noted in Burch et al. [15], the shrinking core model fails to capture fundamental qualitative behavior. Furthermore, in [40] it is proposed that the phase field model may provide a more accurate numerical analysis of the problem than the sharp interface model, which seeks to model the evolution of the phase boundary as a free boundary problem (see [16]; see also [3], and references therein, for benefits of the phase field model).

In this paper we study a variational model introduced by Singh et al. in [58], and incorporating elasticity as proposed by Cogswell and Bazant in [21] (see also [9], [14], [60]), to study the evolution of a crystal of the battery's cathode material, such as $LiFePO_4$. For a fixed domain $\Omega \subset \mathbb{R}^N$, the free energy functional associated with the phase field model is given by the Cahn-Hilliard energy coupled with linearized elasticity, as introduced by Cahn and Larché [43],

$$I_{el}[u, c] := \int_\Omega \left(f(c) + \frac{\rho}{2}\|\nabla c\|^2 + \frac{1}{2}\mathbb{C}(e(u) - ce_0) : (e(u) - ce_0) \right) dx. \tag{1.1}$$

Here $c : \Omega \to [0, 1]$ stands for the normalized density of lithium-ions, $u : \Omega \to \mathbb{R}^N$ represents the material displacement with symmetrized gradient $e(u) := \frac{\nabla u + \nabla u^T}{2}$, $e_0 \in \mathbb{R}^{N\times N}$ is the lattice misfit, $\rho > 0$ is a constant associated with interfacial energy scaling with interface width (see [7], [11], [41], [51], and references therein), and $\mathbb{C}$ is a symmetric, positive definite, fourth order tensor that captures the material constants (stiffness) and satisfies

$$\mathbb{C} : \mathbb{R}^{N\times N} \to \mathbb{R}^{N\times N}_{\text{sym}}, \quad \mathbb{C}(\xi) : \xi > 0 \text{ for all } \xi \in \mathbb{R}^{N\times N}_{\text{sym}} \text{ with } \xi \neq 0. \tag{1.2}$$

In the cases studied by Bazant et al. (see, e.g., [21], [58]), f is the physically relevant regular solution free energy with

$$f(s) := \omega s(1 - s) + KT_{\text{abs}}(s \log(s) + (1 - s) \log(1 - s)), \qquad s \in [0, 1], \tag{1.3}$$

where $\omega \in \mathbb{R}$ is a regular solution parameter (enthalpy of mixing), $K > 0$ is the Boltzman constant, $T_{\text{abs}} > 0$ is the absolute temperature. As is standard, the chemical potential μ is given by the first variation with respect to c of the energy potential,

$$\mu := \delta_c I_{el} = -\rho\Delta c + f'(c) + \mathbb{C}(ce_0 - e(u)) : e_0. \tag{1.4}$$

Assuming a quasi-static equilibrium, Singh et al., in [58], and Cogswell and Bazant, in [21], use Butler-Volmer kinetics to derive the electrochemically consistent model for the evolution of a crystal of the cathode material given by

$$\begin{cases} \partial_t c = \mathrm{div}(\mathrm{M(c)}\nabla\mu) & \text{in } \Omega, \\ \mathrm{div}(\mathbb{C}(\mathrm{e(u)} - \mathrm{ce_0})) = 0 & \text{in } \Omega, \\ \partial_\nu c = 0 & \text{on } \Gamma, \\ (M(c)\nabla\mu)\cdot\nu = R(c,\mu) & \text{on } \Gamma, \\ \mathbb{C}(e(u) - ce_0)\nu = 0 & \text{on } \Gamma, \end{cases} \tag{1.5}$$

where $\Gamma := \partial\Omega$, M is the mobility tensor (degenerate at 0), and

$$R(s,w) := R_{\text{ins}} - R_{\text{ext}} = k_{\text{ins}} \exp(\beta(\mu_e - w)) - k_{\text{ext}} s \exp(\beta(w-\mu_e)), \qquad s \in (0,1),\ w \in \mathbb{R}, \tag{1.6}$$

with constant μ_e and positive constants k_{ext}, k_{ins}, β. The first equality of (1.6) emphasizes that R is a reaction rate determined by the insertion rate R_{ins} minus the extraction rate R_{ext} of lithium-ions. The system of PDEs (1.5) is referred to as the *Cahn-Hilliard reaction model* or the CHR model. Furthermore, looking to the classical free energy proposed by Cahn and Hilliard [17],

$$I[c] := \int_\Omega \left(f(c) + \frac{\rho}{2}\|\nabla c\|^2 \right) dx, \tag{1.7}$$

where ρ and f are as in (1.1), we may define μ as the first variation of (1.7) to consider (1.5) without elasticity as was first done by Singh et al. [58]. The primary purpose of this paper is to examine existence of solutions of the system of PDEs (1.5) with and without elasticity. The methods developed in this paper are inspired by the vast literature on the subject.

In 1958, Cahn and Hilliard proposed the free energy (1.7) to model isotropic systems of varying density [17]. Considering the mass balance equation of the free energy (1.7) in context of a constitutive equation similar to Fick's law, one obtains the equation

$$\partial_t c = -\mathrm{div}(h), \qquad h = -M\nabla\mu, \tag{1.8}$$

where M is a mobility function and μ is as before the first variation of (1.7) [49]. In many applications, M is dependent on c and degenerate at the wells of f. Thermodynamic consistency and conservation of mass require equation (1.8) be equipped with Neumann boundary conditions $\partial_\nu c = 0$ and $(M(c)\nabla\mu)\cdot\nu = 0$, respectively. Altogether, we have the *Cahn-Hilliard equation*

$$\begin{cases} \partial_t c = \mathrm{div}(M(c)\nabla\mu) & \text{in } \Omega, \\ \partial_\nu c = 0 & \text{on } \Gamma, \\ (M(c)\nabla\mu)\cdot\nu = 0 & \text{on } \Gamma. \end{cases} \tag{1.9}$$

Many works on the Cahn-Hilliard equation (1.9) make simplifying assumptions dependent on the motivation. In 1986, Elliott and Somgmu [28] used Galerkin methods and a priori estimates to prove global existence of strong solutions of (1.9) in dimensions up to 3, with a classical solution in 1-dimension. This analysis assumes constant scalar mobility and places restrictions on the type of well function f used, but includes the prototypical double-well function $f(c) := (c^2-1)^2$. Addressing problems of integrability, Elliott and Luckhaus [27] proved existence of solutions to (1.9) with the thermodynamically relevant regular solution free energy (1.3). For the analysis of solutions to the Cahn-Hilliard equation in the case of degenerate mobility, we direct the reader to [22] and [26].

In the results of a workshop in 1990 not published till much later, Fife [31] showed that the Cahn-Hilliard equation (1.9) is, in fact, the gradient flow of the energy functional I in the dual topology of $H^1(\Omega) \cap \{\xi : \int_\Omega \xi \, dx = 0\}$, thereby providing a fundamental variational perspective. Making use of such gradient flow structures, Garcke proved existence of a unique, weak solution to the Cahn-Hilliard equation with elasticity [35]. However, this result was limited in that the potential f could not be logarithmic. This restriction was lifted in [36], where Garcke treated weak existence and uniqueness with the inclusion of elasticity and f given by (1.3).

A topic of recent interest has been that of the Cahn-Hilliard equation equipped with dynamic boundary conditions, for example

$$\partial_\nu c = -\partial_t c + \kappa \Delta_\Gamma c - g'(c) \qquad \text{on } \Gamma \times (0,T)$$

where Δ_Γ is the Laplace-Beltrami operator and g is a surface potential (see [37] and references therein). We also note that there is a variety of work on sharp interface models for both the static and evolutionary problems associated with Cahn-Hilliard type energies (see, e.g., [2], [5], [19], [50], [59]).

Recently, in [42], Kraus and Roggensack proved existence for a variant of the CHR model (1.5). They assumed constant scalar mobility M, and allow for anisotropy in the interfacial energy via a positive definite diagonal tensor $\mathcal{K}$, i.e., in (1.1) $\frac{\rho}{2}\|\nabla c\|^2$ is replaced by $\mathcal{K}(\nabla c) \cdot \nabla c$. They proposed a generalized gradient structure (see [47]) which allows for the inclusion of higher order nonlinear boundary conditions in a gradient flow type framework, and proved weak existence of a solution for finite time intervals. Their variant of the CHR model also includes damage effects, but is limited by the inclusion of a viscosity term in the chemical potential, which helps to simplify the mathematical analysis. Explicitly, they define μ in (1.11) by

$$\mu := -\rho \Delta c + f'(c) + \mathbb{C}(ce_0 - e(u)) : e_0 + \epsilon \partial_t c, \tag{1.10}$$

for some $\epsilon > 0$. Though not used by Bazant et al. (see, e.g., [9], [58]), Kraus and Roggensack note that the viscosity term $\epsilon \partial_t c$ can be viewed as a microforce. Lastly, as proposed by Bazant et al. (see, e.g., [21], [58]) the reaction rate R in (1.6) is exponential in μ. The work of Kraus and Roggensack is limited to a truncation of the function R which has polynomial growth, and f is also restricted to having polynomial growth.

This paper is directed by a motivation to understand the CHR model in dimension $N = 3$. We begin by extending the work of Kraus and Roggensack and remove the assumption of a viscosity term $\epsilon \partial_t c$ in μ; these results hold in arbitrary dimension. Departing from the variational perspective, we apply fixed point methods to recover strong solutions of the CHR model in dimensions $N = 2$ and 3 for f and R of polynomial growth. Our arguments culminate by showing, in dimension $N = 2$, one can recover strong solutions of the CHR model for short time with f and R defined by (1.3) and (1.6), respectively. The respective question for existence with $N = 3$ is a work in progress.

In order to state our results, we write down the complete CHR models for which we prove existence. For $T > 0$, define $\Omega_T := \Omega \times (0,T)$ and $\Sigma_T := \Gamma \times (0,T)$, where as before $\Gamma := \partial\Omega$. Assuming constant scalar mobility and, without loss of generality, that $\rho = 1$, the CHR model with elasticity is given by

$$\text{CHR model with elasticity} \begin{cases} \partial_t c = \Delta \mu & \text{in } \Omega_T, \\ \mu = -\Delta c + f'(c) + \mathbb{C}(ce_0 - e(u)) : e_0 & \text{in } \Omega_T, \\ \operatorname{div}(\mathbb{C}(e(u) - ce_0)) = 0 & \text{in } \Omega_T, \\ \partial_\nu c = 0 & \text{on } \Sigma_T, \\ \partial_\nu \mu = R(c, \mu) & \text{on } \Sigma_T, \\ \mathbb{C}(e(u) - ce_0)\nu = 0 & \text{on } \Sigma_T, \\ c(0) = c_0 & \text{in } \Omega. \end{cases} \tag{1.11}$$

Likewise, the CHR model without elasticity is given by

$$\text{CHR model}\begin{cases}\partial_t c = \Delta\mu & \text{in } \Omega_T,\\ \mu = -\Delta c + f'(c) & \text{in } \Omega_T,\\ \partial_\nu c = 0 & \text{on } \Sigma_T,\\ \partial_\nu \mu = R(c,\mu) & \text{on } \Sigma_T,\\ c(0) = c_0 & \text{in } \Omega.\end{cases} \tag{1.12}$$

We obtain existence of weak, strong, and regular solutions using the following notion of weak solution (for notation we refer the reader to the preliminaries in Section 2).

Definition 1.1. *We say that* (c,u) *is a* weak solution of the CHR model with elasticity *(1.11) in* Ω_T*, if for some* $\delta > 0$

$$\begin{aligned}&c \in L^{(2^\# - \delta)'}(0,T;H^3(\Omega)) \cap C([0,T);L^2(\Omega)),\\ &\partial_t c \in L^{(2^\# - \delta)'}(0,T;H^1(\Omega)^*),\\ &u \in L^{(2^\# - \delta)'}(0,T;\dot{H}^2(\Omega;\mathbb{R}^N)),\\ &c(0) = c_0 \in H^1(\Omega),\end{aligned}$$

and for t-a.e. in $(0,T)$ *the following equations are satisfied for all* $\xi \in H^1(\Omega)$ *and* $\psi \in H^1(\Omega;\mathbb{R}^N)$*:*

$$\begin{aligned}-\langle \partial_t c(t), \xi\rangle_{H^1(\Omega)^*,H^1(\Omega)} &= \int_\Omega \nabla\mu(t)\cdot\nabla\xi \, dx - \int_\Gamma R(c(t),\mu(t))\xi \, d\mathcal{H}^{N-1},\\ \int_\Omega \mathbb{C}(e(u(t)) - c(t)e_0) : e(\psi)\, dx &= 0,\end{aligned} \tag{1.13}$$

where for t-a.e., $\mu(t) \in H^1(\Omega) \subset L^2(\Omega)$ *is defined by duality as*

$$(\mu(t),\xi)_{L^2(\Omega)} := \int_\Omega (\nabla c(t)\cdot\nabla\xi + f'(c(t))\xi + \mathbb{C}(c(t)e_0 - e(u(t))) : e_0\xi)\, dx, \tag{1.14}$$

which holds for all $\xi \in H^1(\Omega)$*.*

Remark 1.2. *In the above definition of a weak solution,* μ *is defined by its action on* $\xi \in H^1(\Omega)$ *versus directly setting* $\mu := -\Delta c + f'(c) + \mathbb{C}(ce_0 - e(u)) : e_0$*. This is to guarantee that the boundary condition* $\partial_\nu c = 0$ *is satisfied.*

Remark 1.3. *One could alternatively define the weak solution as the triple* (c,u,μ)*, where* $\mu \in L^{(2^\# - \delta)'}(0,T;H^1(\Omega))$*, such that (1.14) is satisfied, i.e., (1.14) is no longer a definition but instead a relation between c, u, and* μ*. In Kraus and Roggensack [42], the introduction of viscosity renders this approach necessary.*

We now state the existence of weak and strong solution under hypotheses that may be found in Subsection 2.2

Theorem 1.4. *Let* $\Omega \subset \mathbb{R}^N$ *be a bounded, open domain with* C^3 *boundary and* $T > 0$*. Suppose that* f *and* R *satisfy assumptions (2.1), (2.3), (2.4), and (2.5). Then for any* $c_0 \in H^1(\Omega)$*, a weak solution of the CHR model with elasticity (1.11) exists in* Ω_T*.*

The following results are stated for domains with C^∞ or smooth boundary. This assumption simplifies liftings of boundary conditions, and we speculate that the next two results hold for domains with C^4 boundaries.

Theorem 1.5. *Let $\Omega \subset \mathbb{R}^N$, where $N = 2$ or 3, be a bounded, open domain with smooth boundary and $T > 0$. Suppose f and R satisfy assumptions (2.8), (2.9), (2.10), and (2.11). Then for any $c_0 \in H^2(\Omega)$ such that $\partial_\nu c_0 = 0$ on Γ, there is a strong solution, given by $c \in L^2(0,T;H^4(\Omega)) \cap H^1(0,T;L^2(\Omega))$, of the CHR model (1.12) (or (4.1)) in Ω_T.*

Given the Sobolev embedding theorem in dimensions $N = 2$ or 3, we directly have the following corollary. This allows us to remove a plethora of restrictive hypotheses on f for short time strong existence.

Corollary 1.6. *Let $\Omega \subset \mathbb{R}^N$, where $N = 2$ or 3, be a bounded, open domain with smooth boundary. Suppose $f \in C^3(\mathbb{R})$ and R satisfies assumptions (2.10) and (2.11). For any $c_0 \in H^2(\Omega)$ such that $\partial_\nu c_0 = 0$ on Γ, there exists $T > 0$ such that a strong solution of the CHR model (1.12) (or (4.1)) exists in Ω_T.*

Remark 1.7. *We note that for existence of strong solutions, it is a necessity that the initial data c_0 satisfy $\partial_\nu c_0 = 0$ on Γ. This is because a strong solution $c \in H^{4,1}(\Omega_T)$ (see Subsection 2.5) belongs to the space of bounded uniformly continuous functions on the interval $[0,T]$ with values in $H^2(\Omega)$ (see [45]). By definition of a solution of the CHR model (1.11), $\partial_\nu c = 0$ on Σ_T, and by continuity, we have $0 = \partial_\nu c(\cdot, 0) = \partial_\nu c_0$ on Γ.*

Lastly, we address the regularity of solutions. In the case of a constant scalar mobility tensor, we prove that there is a solution to the CHR model (1.5) as proposed by Singh, et. al. [58] with R as in (1.6). This result makes sharp use of the growth provided by the Gagliardo-Nirenberg inequality (see Theorem 2.5), and therefore critically relies on smallness estimates developed in the analysis of strong solutions (see Theorem 4.1). Furthermore, reasoning similar to Remark 1.7 requires that the initial data c_0 satisfies an additional compatibility condition.

Theorem 1.8. *Suppose $\Omega \subset \mathbb{R}^2$ is an open bounded set with smooth boundary. Let f and R be defined as in (1.3) and (1.6), respectively. There exists $\lambda = \lambda(\Omega, f, R) > 0$ such that if $c_0 \in H^4(\Omega)$ such that $\partial_\nu c_0 = 0$, $\partial_\nu(\Delta c_0) = -R(c_0, -\Delta c_0 + f'(c))$, $\epsilon \leq c_0(\Omega) \leq 1-\epsilon$ in Ω for some $\epsilon > 0$, and $\|\nabla^2 c_0\|_{L^2(\Omega)} \leq \lambda$, then there are $T > 0$ and $c \in L^2(0,T;H^6(\Omega)) \cap H^{1+1/2}(0,T;L^2(\Omega))$ for which c is strong solution of the CHR model (1.12) in Ω_T.*

This paper is organized as follows: In the preliminary Section 2, we recall some elliptic estimates to be used throughout the paper. We introduce fractional Sobolev spaces and a class of anisotropic Sobolev spaces, which will be employed in Sections 4 and 5. In Section 3, we prove weak existence of a solution to the CHR model with (1.11) and without elasticity (1.12), using the generalized gradient structure proposed by Kraus and Roggensack [42]. Our analysis shows how to remove the assumption of a viscosity term introduced in (1.10). This result holds in arbitrary dimensions (see Theorem 1.4). In Section 4, restricted to $N = 2$ and 3, we argue via fixed point and interpolation methods to prove existence of a strong solution to the CHR model (1.12) (see Theorem 1.5 and Corollary 1.6). Finally, in Section 5, and making a priori estimates derived in the previous section, we show that in dimension $N = 2$, for sufficiently small intervals of time and initial data close to a small energy state, we have a strong solution of the CHR model (1.12) for R with exponential growth in μ (see Theorem 1.8).

We note that Section 3 and Sections 4 and 5 may be read independently. We also note that the appendix, in Section 6, analyzes the regularity of a fourth order PDE via a gradient flow in the dual of H^1. Hence, for those unfamiliar with gradient flows or differential inclusions, it may be of use to read Theorem 6.2 before Section 3.

2 Preliminaries

In the first subsection, we list notation used throughout the paper. In Subsection 2.2, we state assumptions used in Theorems 1.4 and 1.5. In Subsection 2.3, we highlight some elliptic and embedding estimates we will use in the following sections. We further introduce function spaces in Subsections 2.4 and 2.5, which will be critical in Sections 4 and 5; these results will not be needed in Section 3. Here, we remind the reader of fractional Sobolev spaces in the 1-dimensional setting. We derive an extension result for such spaces in Corollary 2.8. We then recall a class of anisotropic Sobolev spaces used ubiquitously by Lions and Magenes [46]. Integrating our knowledge of the two spaces, we propose a new semi-norm for the anisotropic Sobolev spaces to be used in the later sections.

2.1 Notation

We enumerate the variety of notation used throughout the paper.

1. Given a Banach space $\mathcal{B}$, we let $\mathcal{B}^*$ denotes the dual space of $\mathcal{B}$. We denote duality between these spaces by $\langle \cdot, \cdot \rangle_{\mathcal{B}^*,\mathcal{B}}$.
2. Given Banach spaces $(\mathcal{B}_0, \|\cdot\|_0)$ and $(\mathcal{B}_1, \|\cdot\|_1)$, we denote the continuous embedding of $\mathcal{B}_0$ into $\mathcal{B}_1$ by $\mathcal{B}_0 \hookrightarrow \mathcal{B}_1$, and the compact embedding of $\mathcal{B}_0$ into $\mathcal{B}_1$ by $\mathcal{B}_0 \hookrightarrow\hookrightarrow \mathcal{B}_1$
3. In a Hilbert space H, we denote the inner product by $(\cdot, \cdot)_H$.
4. $\mathcal{H}^N$ is the Hausdorff measure of dimension N. See Evans and Gariepy [30] for more information.
5. Given a domain $\Omega \subset \mathbb{R}^N$ specified by context, we define $\Gamma := \partial\Omega$. We let ν denote the outward normal of Γ.
6. For $\Omega \subset \mathbb{R}^N$ specified by context and $T > 0$, we define $\Omega_T := \Omega \times (0, T)$. Furthermore, we set $\Sigma_T := \Gamma \times (0, T)$.
7. We interchangably use ∇c as a row vector and a column vector, to be understood from context (though most often a row vector).
8. Pos(N) denotes the set of positive definite matrices acting on $\mathbb{R}^N$.
9. Given a Banach space $\mathcal{B}$ and $a < b \in \mathbb{R}$, with $J := (a, b)$, let $L^p(a, b; \mathcal{B}) = L^p(J; \mathcal{B})$ denote the space of Bocher $p-$integrable functions on J with values in $\mathcal{B}$. For a good resource on such spaces, we refer the reader to [44].
10. Given a Banach space $\mathcal{B}$ and $T > 0$, let $BUC(0, T; \mathcal{B})$ denote the space of bounded uniformly continuous functions with values in $\mathcal{B}$ on the closed interval $[0, T]$.
11. The space of $k-$differentiable continuous functions on $\Omega \subset \mathbb{R}^N$ with values in a Banach space $\mathcal{B}$ will be denoted by $C^k(\Omega; \mathcal{B})$. If $\mathcal{B} = \mathbb{R}$, we abbreviate it as $C^k(\Omega)$. If we additionally restrict ourselves to functions such that derivatives up to (and including) the kth order are bounded and the kth order derivatives are Hölder continuous with parameter α, the space is denoted by $C^{k,\alpha}(\Omega; \mathcal{B})$, and we set
$$\|g\|_{C^{k,\alpha}} := \sum_{i\leq k} \|\nabla_i g\|_\infty + |\nabla_k g|_{C^{0,\alpha}}.$$
12. We say that a set $\Omega \subset \mathbb{R}^N$ has C^k boundary if, for every $x \in \Gamma$ there is some $r > 0$, such that up to rotation, $B(x, r) \cap \Omega$ coincides with the epigraph of a $C^k(\mathbb{R}^{N-1})$ function. In the case that $k = \infty$, we say that Ω has smooth boundary.
13. We let $\mathrm{Tr} : \mathrm{H}^k(\Omega) \to \mathrm{H}^{k-1/2}(\Gamma)$ denote the trace operator. See [44] for more information.

14. As is standard, we let $p^* := \frac{Np}{N-p}$ be the critical exponent for the Sobolev embedding in dimension $N > p$. We further let $p^\#$ be the critical value of q for which the trace operator, Tr, continuously maps $W^{1,p}(\Omega)$ into $L^q(\Gamma)$. Note for $q < p^\#$, the embedding is compact. We specifically note
$$2^\# := \begin{cases} \frac{2N-2}{N-2} & \text{if } N \geq 3, \\ \text{any } q > 2 & \text{if } N = 2, \\ \infty & \text{if } N = 1. \end{cases}$$
We refer the reader to [44].

15. We use C to denote a generic constant, which can change from line to line. If dependence of C on parameter a is emphasized, we will denote this by either C_a or $C(a)$.

16. $\dot{H}^k(\Omega; \mathbb{R}^N)$ is the Sobolev space quotiented by skew affine functions.

2.2 Assumptions

We will make use of a collection of assumptions to prove weak existence.

- We assume that the chemical energy density is governed by a function $f \in C^2(\mathbb{R})$ such that, for some $C > 0$,
$$f \geq -C \quad \text{and} \quad |f''(s)| \leq C(|s|^{2^*/2-1} + 1) \tag{2.1}$$
for all $s \in \mathbb{R}$, where 2^* is the dimension dependent Sobolev exponent, $2^* = \frac{2N}{N-2}$ if $N \geq 3$ and is any fixed constant greater than 2 if $N \leq 2$.

- For the reaction rate R, we assume that there is $G \in C^1(\mathbb{R}^2)$ such that
$$\partial_w G(s, w) = R(s, w) \tag{2.2}$$
for all $s \in \mathbb{R}$ and $w \in \mathbb{R}$. We suppose that the reaction rate is strictly decreasing in the second variable, i.e., there is $C > 0$ such that
$$(R(s, w_2) - R(s, w_1))(w_2 - w_1) \leq -C|w_2 - w_1|^2 \tag{2.3}$$
for all $s \in \mathbb{R}$ and $w_1, w_2 \in \mathbb{R}$. Further, for some $C, \delta > 0$, the growth condition
$$|R(s, w)| \leq C(|s|^{2^\# - \delta - 1} + |w|^{2^\# - \delta - 1} + 1) \tag{2.4}$$
holds for all $s \in \mathbb{R}$ and $w \in \mathbb{R}$, where $2^\# := \frac{2(N-1)}{N-2}$ if $N \geq 3$ and $2^\#$ is any fixed constant greater than 2 if $N \leq 2$. We assume there is a constant $C > 0$ such that the pointwise bound
$$|R(s, \pm 1)| \leq C \tag{2.5}$$
is satisfied for any choice of $s \in \mathbb{R}$.

  Testing (2.3) with $w_2 = 1$ or -1 and $w_1 = 0$, and using (2.5) ,we find that $|R(s, 0)| \leq C$ for some constant $C > 0$. From this bound and (2.3) it follows there is $C > 0$ such that, for all $s \in \mathbb{R}$ and $w \in \mathbb{R}$,
$$-wR(s, w) \geq \frac{1}{C}|w|^2 - C. \tag{2.6}$$
  To derive growth conditions of the function G, we may, without loss of generality, suppose that $\partial_s G(s, 0) = R(s, 0)$. Consequently, the fundamental theorem of calculus, (2.4), and Young's inequality imply
$$|G(s, w)| \leq C(|s|^{2^\# - \delta} + |w|^{2^\# - \delta} + 1) \tag{2.7}$$
for all $s \in \mathbb{R}$ and $w \in \mathbb{R}$.

Remark 2.1. *If we assume that $R \in C^1(\mathbb{R}^2)$, then (2.2) is immediately satisfied with*

$$G(s, w) := \int_0^w R(s, \rho)\, d\rho.$$

We note these assumptions are in accordance with those of Kraus and Roggensack (see [42]). To prove strong existence, we will make use of more powerful assumptions:

- We assume that the chemical energy density is governed by a function $f \in C^3(\mathbb{R})$, such that for some $C > 0$,

$$\|f''\|_\infty + \|f'''\|_\infty \leq C. \tag{2.8}$$

  We also make use of the coercivity assumption

$$f(s) \geq \delta|s| - 1/\delta, \tag{2.9}$$

  which holds for all $s \in \mathbb{R}$ and some $\delta > 0$.

- For the reaction rate, we assume that R is Lipschitz, i.e., there is $C > 0$ such that

$$\|\nabla R\|_\infty \leq C. \tag{2.10}$$

  Furthermore, we introduce the growth condition

$$-wR(s, w) \geq -C \tag{2.11}$$

  for all $s \in \mathbb{R}$ and $w \in \mathbb{R}$.

Remark 2.2. *In comparison with those used to obtain weak existence, it is clear that (2.8) imposes much greater restrictions on the growth of f'. This condition arises because we will need to obtain sufficient regularity of the boundary term $R(c, \mu) = R(c, -\Delta c + f'(c))$. Furthermore, the regularity assumptions on R are more stringent, but we are relatively lenient on the structure of R (i.e., we do not need monotonicity). Note that (2.11) is a relaxation of the condition (2.6).*

Remark 2.3. *In view of (2.10),*

$$|R(s, w)| \leq |R(0, 0)| + C(|s| + |w|).$$

In particular, by (2.8),

$$\begin{aligned} |R(s, -w + f'(s))| \leq& |R(0, 0)| + C(|s| + |w| + |f'(s)|) \\ \leq& |R(0, 0)| + C(|s| + |w| + |f'(0)|). \end{aligned}$$

Furthermore, by the chain rule, (2.8), and (2.10), the map $(s, w) \mapsto R(s, -w + f'(s))$ is Lipschitz.

2.3 Estimates

The following result on elliptic regularity may be found in [39] (see also [45], [48]).

Theorem 2.4. *Let $\Omega \subset \mathbb{R}^N$ be an open, bounded set with C^{k+2} boundary for $k \in \mathbb{N}_0$. Let $g \in H^k(\Omega)$ with $\int_\Omega g\, dx = 0$. Let $v \in H^1(\Omega)$ be a weak solution of*

$$\begin{cases} -\Delta v = g & \text{in } \Omega, \\ \partial_\nu v = 0 & \text{on } \Gamma, \end{cases} \tag{2.12}$$

with $\int_\Omega v\, dx = 0$. Then there is a constant $C > 0$, depending only on Ω and N, such that

$$\|v\|_{H^{k+2}(\Omega)} \leq C\|g\|_{H^k(\Omega)}. \tag{2.13}$$

We will also make use of the Gagliardo-Nirenberg inequality [53] (sometimes referred to as just the Nirenberg inequality), which improves upon more the standard Sobolev-Gagliardo-Nirenberg embedding theorem (see, e.g., [44]).

Theorem 2.5 (Gagliardo-Nirenberg inequality)**.** *Suppose that $\Omega \subset \mathbb{R}^N$ is an open, bounded set with Lipschitz boundary. Then the following inequality is satisfied for measurable functions v :*

$$\|\nabla^j v\|_{L^p(\Omega)} \leq C_1 \|\nabla^m v\|^a_{L^r(\Omega)} \|v\|^{1-a}_{L^q(\Omega)} + C_2 \|v\|_{L^q(\Omega)},$$

with $a \geq 0$ satisfying

$$\frac{j}{m} \leq a \leq 1, \quad \frac{1}{p} = \frac{j}{n} + a\left(\frac{1}{r} - \frac{m}{n}\right) + (1-a)\frac{1}{q}.$$

Emulating an argument used in Fonseca et al. [32], we use the previous two results to obtain an interpolation result in the dual of $H^1(\Omega)$. An inequality of this type also follows from interpolation theory, but rather than invoking it, we prove this inequality directly in an effort to stay self-contained where possible.

Corollary 2.6. *Let $\Omega \subset \mathbb{R}^N$ be an open, bounded set with C^3 boundary. Then $g \in H^1(\Omega)$ satisfies the bound*

$$\|g\|_{L^2(\Omega)} \leq C\left(\|g\|^{1/2}_{H^1(\Omega)} \|g\|^{1/2}_{H^1(\Omega)^*} + \|g\|_{H^1(\Omega)^*}\right). \tag{2.14}$$

Proof. First, suppose $\int_\Omega g\, dx = 0$. Let $v \in H^1(\Omega)$ be the weak solution of (2.12). Consequently for all $\xi \in H^1(\Omega)$ we have

$$\int_\Omega \nabla v \cdot \nabla \xi\, dx = \int_\Omega g\xi\, dx.$$

Taking $\xi = v$ gives

$$\|\nabla v\|^2_{L^2(\Omega)} = \int_\Omega gv\, dx \leq \|g\|_{H^1(\Omega)^*} \|v\|_{H^1(\Omega)} \leq C\|g\|_{H^1(\Omega)^*} \|\nabla v\|_{L^2(\Omega)},$$

where the last inequality follows from the Poincaré inequality since $\int_\Omega v\, dx = 0$. It follows that

$$\|\nabla v\|_{L^2(\Omega)} \leq C\|g\|_{H^1(\Omega)^*}. \tag{2.15}$$

We apply the Gagliardo-Nirenberg inequality (Theorem 2.5), (2.13), and (2.15) to bound

$$\|\nabla^2 v\|_{L^2(\Omega)} \leq C\left(\|\nabla^3 v\|^{1/2}_{H^3(\Omega)} \|\nabla v\|^{1/2}_{H^1(\Omega)} + \|\nabla v\|_{H^1(\Omega)}\right) \leq C\left(\|g\|^{1/2}_{H^1(\Omega)} \|g\|^{1/2}_{H^1(\Omega)^*} + \|g\|_{H^1(\Omega)^*}\right).$$

As $-\Delta v = g$ by (2.12), the bound (2.14) follows.

If $\int_\Omega g\, dx \neq 0$, we consider inequality (2.14) for $g - \int_\Omega g\, dx$. Noting that

$$\left|\int_\Omega g\, dx\right| \leq \|g\|_{H^1(\Omega)^*} \|1\|_{H^1(\Omega)} = C(\Omega)\|g\|_{H^1(\Omega)^*},$$

the bound (2.14) follows for g by applications of the triangle inequality and subadditivity of the square-root.

□

2.4 Fractional Sobolev Spaces

In this subsection, we develop a sufficient knowledge of fractional Sobolev spaces to introduce a non-standard norm for anisotropic Sobolev spaces in Subsection 2.5 (see (2.24)). As such, our consideration is restricted to one-dimensional fractional Sobolev spaces $H^s(0,T)$ for $s \in (0,1)$. We note, in this paper, for encounters with higher order fractional derivatives, it will suffice to use the interpolation methods introduced in Subsection 2.5.

On the interval $(a,b) \subset \mathbb{R}$, we define a semi-norm via the difference quotient introduced by Gagliardo (see, e.g., [25], [34], [44]) for $s \in (0,1)$,

$$|u|_{H^s(a,b)} := \left(\int_a^b \int_a^b \frac{|u(x)-u(y)|^2}{|x-y|^{1+2s}}\, dy\, dx \right)^{1/2}. \tag{2.16}$$

We then define the norm

$$\|u\|_{H^s(a,b)} := \|u\|_{L^2(a,b)} + |u|_{H^s(a,b)}. \tag{2.17}$$

The space generated by the closure in this norm of $H^1(a,b)$ in $L^2(a,b)$ is the fractional Sobolev space of order s denoted by $H^s(a,b)$. We prove a result for one-dimensional fractional Sobolev spaces.

Proposition 2.7. *Suppose $u \in H^s(0,T)$ and $\psi \in C^\infty[0,T]$. Then $\psi u \in H^s(0,T)$, and for any $\epsilon \in (0,T)$, it satisfies the bound*

$$\|\psi u\|_{H^s(0,T)} \le C((1+\epsilon^{-s})\|\psi\|_\infty + \epsilon^{1-s}\|\nabla\psi\|_\infty)\|u\|_{L^2(0,T)} + \|\psi\|_\infty |u|_{H^s(0,T)}.$$

Proof. As the control of the L^2 norm of ψu is straightforward, we estimate the seminorm $|\psi u|_{H^s(0,T)}$ as follows.

$$\begin{aligned}\int_0^T \int_0^T \frac{|\psi(x)u(x)-\psi(t)u(t)|^2}{|x-t|^{1+2s}}\, dx\, dt \le& 2\int_0^T \int_0^T |\psi(x)|^2 \frac{|u(x)-u(t)|^2}{|x-t|^{1+2s}}\, dx\, dt \\ &+ 2\iint_{\{|x-t|\le\epsilon\}} |u(t)|^2 \frac{|\psi(x)-\psi(t)|^2}{|x-t|^{1+2s}}\, dx\, dt \\ &+ 2\iint_{\{|x-t|>\epsilon\}} |u(t)|^2 \frac{|\psi(x)-\psi(t)|^2}{|x-t|^{1+2s}}\, dx\, dt.\end{aligned}$$

To bound the terms of the right-hand side, we immediately have

$$\int_0^T \int_0^T |\psi(x)|^2 \frac{|u(x)-u(t)|^2}{|x-t|^{1+2s}}\, dx\, dt \le \|\psi\|_\infty^2 |u|_{H^s(0,T)}^2.$$

For the second term, we use the mean value theorem and Fubini's theorem to find

$$\begin{aligned}\iint_{\{|x-t|\le\epsilon\}} |u(t)|^2 \frac{|\psi(x)-\psi(t)|^2}{|x-t|^{1+2s}}\, dx\, dt \le& \|\nabla\psi\|_\infty^2 \int_0^T |u(t)|^2 \left(\int_{\{x\in(0,T):|x-t|\le\epsilon\}} |x-t|^{1-2s}\, dx \right) dt \\ \le& \|\nabla\psi\|_\infty^2 \int_0^T |u(t)|^2\, dt \int_{-\epsilon}^{\epsilon} |\zeta|^{1-2s}\, d\zeta \\ =& C\|\nabla\psi\|_\infty^2 \|u\|_{L^2(0,T)}^2 \epsilon^{2-2s}.\end{aligned}$$

The third term is directly estimated as

$$\iint_{\{|x-t|>\epsilon\}} |u(t)|^2 \frac{|\psi(x)-\psi(t)|^2}{|x-t|^{1+2s}}\, dx\, dt \le C\|\psi\|_\infty^2 \epsilon^{-2s} \|u\|_{L^2(0,T)}^2.$$

Combining the above inequalities, we conclude the lemma. □

As a result of the above lemma and a reflection argument, we obtain the following extension result.

Corollary 2.8. *Let $0 < T \le T_0$. Suppose that $u \in H^s(0,T)$. There exists an extension of u, $\tilde{u} \in H^s(0,T_0)$, such that the following bound is satisfied:*

$$\|\tilde{u}\|_{H^s(0,T_0)} \le C_{T_0}\Big((1+T^{-s})\|u\|_{L^2(0,T)} + |u|_{H^s(0,T)}\Big), \tag{2.18}$$

with constant C_{T_0} independent of T.

We prove an estimate that is helpful for comparing the semi-norm of a fractional Sobolev space to that of the standard Sobolev space.

Proposition 2.9. *Given $u \in H^1(0,T)$ and $s \in (0,1)$, we have the following estimate*

$$|u|_{H^s(0,T)} \le \frac{1}{s\sqrt{2(1-s)}} T^{1-s}\|\partial_t u\|_{L^2(0,T)}.$$

Proof. By definition of the semi-norm and the fundamental theorem of calculus,

$$\begin{aligned}|u|^2_{H^s(0,T)} &= \int_0^T \int_0^T \frac{|u(x)-u(y)|^2}{|x-y|^{1+2s}}\,dy\,dx \\ &= \int_0^T \left(\left(\int_0^x + \int_x^T\right) \frac{|\int_x^y \partial_t u\,d\sigma|^2}{|x-y|^{1+2s}}\,dy\right)dx.\end{aligned} \tag{2.19}$$

Fixing $x \in (0,T)$, we bound the x variable's integrand using the change of variables $\bar{y} = y - x$ and $\bar{\sigma} = \sigma - x$, and we use Hardy's inequality (see [44]). To be precise,

$$\begin{aligned}\int_x^T \frac{|\int_x^y \partial_t u\,d\sigma|^2}{|x-y|^{1+2s}}\,dy &= \int_0^{T-x} \frac{|\int_x^{x+\bar{y}} \partial_t u\,d\sigma|^2}{|\bar{y}|^{1+2s}}\,d\bar{y} \\ &= \int_0^{T-x} \frac{|\int_0^{\bar{y}} \partial_t u(x+\bar{\sigma})\,d\bar{\sigma}|^2}{|\bar{y}|^{1+2s}}\,d\bar{y} \\ &\le (1/s)^2 \int_0^{T-x} \bar{y}^{1-2s}|\partial_t u(x+\bar{y})|^2\,d\bar{y} \\ &= (1/s)^2 \int_x^T |x-y|^{1-2s}|\partial_t u(y)|^2\,dy.\end{aligned}$$

By the same argument for the other integral, we obtain

$$\begin{aligned}|u|^2_{H^s(0,T)} \le& (1/s)^2 \int_0^T \int_0^T |x-y|^{1-2s}|\partial_t u(y)|^2\,dy\,dx \\ =& (1/s)^2 \int_0^T \left(\int_0^T |x-y|^{1-2s}\,dx\right)|\partial_t u(y)|^2\,dy \\ \le& \left(\frac{1}{s\sqrt{2(1-s)}}\right)^2 T^{2-2s}\|\partial_t u\|^2_{L^2(0,T)},\end{aligned}$$

concluding the result. □

We lastly make note of a simple lemma, which shows how the semi-norm changes for a rescaled domain.

Lemma 2.10. *Let $0 < T$, $s \in (0,1)$, and $u \in H^s(0,T)$. Define $u_T(x) := u(Tx)$ for $x \in (0,1)$. Then it holds that*

$$|u|_{H^s(0,T)} = T^{\frac{1-2s}{2}} |u_T|_{H^s(0,1)}.$$

Proof. We compute

$$\begin{aligned}|u|^2_{H^s(0,T)} &= \int_0^T \int_0^T \frac{|u(x)-u(y)|^2}{|x-y|^{1+2s}}\, dx\, dy \\ &= \int_0^1 \int_0^1 \frac{|u(Tx)-u(Ty)|^2}{|Tx-Ty|^{1+2s}}\, Tdx\, Tdy = T^{1-2s}|u_T|^2_{H^s(0,1)}.\end{aligned}$$

□

2.5 Anisotropic Sobolev spaces

Anisotropic Sobolev spaces naturally arise in the study of PDEs. Let $\Omega \subset \mathbb{R}^N$ be a smooth domain. We recall the notation $\Omega_T := \Omega \times (0,T)$, $\Sigma_T = \partial\Omega \times (0,T)$, and for $r, s \geq 0$, we define

$$H^{r,s}(\Omega_T) := L^2(0,T; H^r(\Omega)) \cap H^s(0,T; L^2(\Omega)), \tag{2.20}$$

where $H^r(\Omega)$ and $H^s(0,T; L^2(\Omega))$ are defined via interpolation of Sobolev spaces of integer order (see [46]). Note that $H^0(\Omega) = L^2(\Omega)$. Likewise, we may define the anisotropic Sobolev space with domain Σ_T. It is standard to endow $H^{r,s}(\Omega_T)$ with the norm arising from interpolation (denoted by "$, I$") given by

$$\|u\|_{H^{r,s}(\Omega_T),I} := \left(\int_0^T \|u(\cdot,t)\|^2_{H^r(\Omega),I}\, dt + \|u\|^2_{H^s(0,T;L^2(\Omega)),I} \right)^{1/2}. \tag{2.21}$$

As an aside, we recall how interpolation and the interpolation norm are defined.

Remark 2.11. *Generically, suppose that X and Y are separable Hilbert spaces, with X densely embedded into Y. There is a positive, self adjoint, and unbounded operator Λ on Y such that $X = \mathrm{dom}(\Lambda)$, and the norm on X is equivalent to the graph norm of Λ :*

$$\frac{1}{C}\|x\|_X \leq (\|x\|_Y^2 + \|\Lambda x\|_Y^2)^{1/2} \leq C\|x\|_X.$$

For the construction of such an operator, we refer the reader to [24], [45], [55]. Using spectral theory for unbounded operators (see [24] and references therein), we may consider fractional powers of the operator Λ. Then the interpolation space $[X,Y]_\theta$ for $\theta \in [0,1]$ is defined by

$$[X,Y]_\theta := \mathrm{dom}(\Lambda^{1-\theta}), \tag{2.22}$$

with norm

$$\|\cdot\|_{[X,Y]_\theta} = (\|\cdot\|_Y^2 + \|\Lambda^{1-\theta}\cdot\|_Y^2)^{1/2}. \tag{2.23}$$

In the context of Sobolev spaces (see Proposition 2.13), for example, we have

$$\|\cdot\|_{H^{1/2}(\Omega),I} := \|\cdot\|_{[H^1(\Omega),H^0(\Omega)]_{1/2}}.$$

We note that the interpolation space defined by (2.22) is norm equivalent to that defined by the K-method (see, e.g., [44], [45]); however, the norm (2.23) is more directly related to norms arising from the Fourier transform.

We may also define the anisotropic Sobolev space (2.20) on the larger cylindrical domain $\Omega \times \mathbb{R}$. For u within such a space, we may consider the Fourier transform of u in the variable t given by the Bochner integral

$$\hat{u}(\xi) := \frac{1}{(2\pi)^{1/2}} \int_{\mathbb{R}} e^{-i\xi t} u(t)\, dt,$$

and define the Fourier norm (denoted by ", F") on the space $H^s(\mathbb{R}; L^2(\Omega))$ (see also [45]) by

$$\|u\|_{H^s(\mathbb{R};L^2(\Omega)),\mathrm{F}} := \left(\int_{\mathbb{R}} (1 + |\xi|^2)^s \|\hat{u}(\xi)\|^2_{L^2(\Omega)}\, d\xi \right)^{1/2}.$$

We will need precise results about the extension properties of the function spaces in (2.20). Consequently, norms akin to (2.19) will prove to be more useful. Let $\|\cdot\|_{H^r(\Omega)}$ denote any standard norm choice for $H^r(\Omega)$.

Lemma 2.12. *Let $\Omega \subset \mathbb{R}^N$ be an open, bounded set with smooth boundary and $r \geq 0$. For $s \in (0,1)$, we recall (2.17) to define the norm*

$$\|u\|_{H^{r,s}(\Omega_T)} := \left(\int_0^T \|u(\cdot,t)\|^2_{H^r(\Omega)}\, dt + \int_{\Omega} \|u(x,\cdot)\|^2_{H^s(0,T)}\, dx \right)^{1/2}. \tag{2.24}$$

The norm (2.24) is equivalent to the norm defined by (2.21). The same is true on domains Σ_T.

Proof. By classical results, we have that $\|\cdot\|_{H^r(\Omega),I}$ is equivalent to $\|\cdot\|_{H^r(\Omega)}$ (see [44] for a proof in the case $\Omega = \mathbb{R}^N$; the following argument proves the result for extension domains Ω). Thus, it suffices to take $r = 0$ and prove that the norm $\|\cdot\|_{H^{0,s}(\Omega_T)}$ is equivalent to $\|\cdot\|_{H^{0,s}(\Omega_T),I}$

By Corollary 2.8, there is an extension operator $\mathcal{T}$, defined via reflection and truncation in the variable t (for each fixed point $x \in \Omega$), such that $\mathcal{T} : H^{0,0}(\Omega_T) \to H^{0,0}(\Omega \times \mathbb{R})$ and $\mathcal{T} : H^{0,1}(\Omega_T) \to H^{0,1}(\Omega \times \mathbb{R})$ are linear and bounded. By interpolation (see Theorem 5.1 of Chapter 1 in [45]), it follows that $\mathcal{T} : H^{0,s}(\Omega_T) \to H^{0,s}(\Omega \times \mathbb{R})$ is linear and bounded in the topology of the interpolation norm (2.21). By a direct computation in the spirit of Corollary 2.8, we have that $\mathcal{T}$ is also continuous in the topology defined by the norm (2.24). Consequently, using the equivalence of the Gagliardo norm and Fourier norm on $\mathbb{R}$ (see [44]) and Fubini's theorem, we have

$$\begin{aligned} \|u\|_{H^{0,s}(\Omega_T)} \leq & C\|\mathcal{T}u\|_{H^{0,s}(\Omega\times\mathbb{R})} \\ \leq & C\|\mathcal{T}u\|_{H^s(\mathbb{R};L^2(\Omega)),\mathrm{F}} \leq C\|\mathcal{T}u\|_{H^s(\mathbb{R};L^2(\Omega)),I} \leq C\|u\|_{H^s(0,T;L^2(\Omega)),I} \end{aligned} \tag{2.25}$$

(see subsection 7.1 of Chapter 1 in [45] for equivalence of Fourier and interpolation norm).

To obtain the reverse inequality to prove equivalence of the norms, we may essentially reverse the sequence of inequalities in (2.25). In the same way that $\mathcal{T}$ was shown to be continuous, we may show that the restriction operator $\pi : u \mapsto u|_{\Omega_T}$ mapping from $H^{0,s}(\Omega \times \mathbb{R})$ to $H^{0,s}(\Omega_T)$ is continuous in the interpolation norm. Consequently,

$$\begin{aligned} \|u\|_{H^s(0,T;L^2(\Omega)),I} = & \|\pi(\mathcal{T}u)\|_{H^s(0,T;L^2(\Omega)),I} \\ \leq & C\|\mathcal{T}u\|_{H^s(\mathbb{R};L^2(\Omega)),I} \leq C\|\mathcal{T}u\|_{H^s(\mathbb{R};L^2(\Omega)),\mathrm{F}} \leq C\|\mathcal{T}u\|_{H^{0,s}(\Omega\times\mathbb{R})} \leq C\|u\|_{H^{0,s}(\Omega_T)}, \end{aligned}$$

where we have once again used the equivalence of the Gagliardo and Fourier norms. □

We will make use of some interpolation theorems, which provide regularity of certain quantities. The following result is Proposition 2.1 of Chapter 4 in [46].

Proposition 2.13. *Let Ω be an open, bounded set with smooth boundary. For $r, s \geq 0$ and $\theta \in (0,1)$, we have*

$$[H^{r,s}(\Omega_T), H^{0,0}(\Omega_T)]_\theta = H^{(1-\theta)r,(1-\theta)s}(\Omega_T).$$

The same is true on domains Σ_T.

Theorem 2.14. *Let Ω be an open, bounded set with smooth boundary. Let $u \in H^{r,s}(\Omega_T)$ with $r > 1/2$, $s \geq 0$. If j is an integer such that $0 \leq j < r - 1/2$, we may define the jth normal derivative $\partial_\nu^j u \in H^{\mu_j,\lambda_j}(\Sigma_T)$, where*

$$\frac{\mu_j}{r} = \frac{\lambda_j}{s} = \frac{r - j - 1/2}{r}. \tag{2.26}$$

Furthermore, the map $u \mapsto (\partial_\nu^j u)_{\{0 \leq j < r-1/2\}}$ is surjective, continuous, with continuous right inverse.

Proof. This is primarily a restatement of Theorem 2.1 in Chapter 4 of [46]. Existence of the continuous right inverse follows from Theorem 3.2 of Chapter 1 in [45]. □

Proposition 2.15. *Let $\Omega \subset \mathbb{R}^N$ be a bounded smooth domain. Suppose that $u \in H^{k,k/4}(\Omega_T)$, $k \in \mathbb{N}$. Then, $\nabla^2 u \in H^{k-2,(k-2)/4}(\Omega_T)$, with the map $u \mapsto \nabla^2 u$ continuous in the respective topologies.*

Proof. Let $\mathcal{T} : H^k(\Omega) \to H^k(\mathbb{R}^N)$ be a linear extension operator (defined via reflection and a partition of unity as in Theorem 13.4 and Remark 13.5 of [44]), such that there is $r > 0$ with $\operatorname{supp} \mathcal{T}(\xi) \subset B(0,r)$ for all $\xi \in H^k(\Omega)$. We extend u as $\tilde{u}(x,t) := \mathcal{T}(u(\cdot,x))(x)$. Since $\mathcal{T}(u(\cdot,t))$ is defined by reflecting $u(\cdot,t)$ near $\partial\Omega$ and using a partition of unity (see, e.g., Theorem 13.17 in [44] or Theorem 5.4.1 [29]), the regularity of $\tilde{u}$ in time is preserved, and so $\tilde{u} \in H^{k,k/4}(\mathbb{R}^N \times (0,T))$ with

$$\|\tilde{u}\|_{H^{k,k/4}(\mathbb{R}^N \times (0,T))} \leq C\|u\|_{H^{k,k/4}(\Omega_T)}. \tag{2.27}$$

Write $x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R}$. By identifying $\xi \in H^{k,k/4}(\mathbb{R}^N \times (0,T))$ with the function $x_N \mapsto \xi((\cdot, x_N), \cdot)$, as noted in [46], we may decompose the anisotropic Sobolev space as

$$\begin{aligned} &H^{k,k/4}(\mathbb{R}^N \times (0,T)) = \\ &\qquad H^k\left(\mathbb{R}; L^2(\mathbb{R}^{N-1} \times (0,T))\right) \cap L^2\left(\mathbb{R}; H^{k,k/4}(\mathbb{R}^{N-1} \times (0,T))\right). \end{aligned}$$

Consequently, we apply an intermediate derivative theorem (Theorem 2.3 of Chapter 1 in [45]) and Proposition 2.13 to conclude that $\tilde{u}$ maps continuously to

$$\nabla^2 \tilde{u} \in H^{k-2}(\mathbb{R}; L^2(\mathbb{R}^{N-1} \times (0,T))) \cap L^2(\mathbb{R}; H^{k-2,(k-2)/4}(\mathbb{R}^{N-1} \times (0,T))).$$

Using continuity of the restriction operator, bound (2.27) concludes the lemma. □

3 Weak existence

In this section, we prove existence of weak solutions to the CHR model with and without elasticity. In an effort to illuminate the essential techniques, we first focus on the CHR model without elasticity. We use a generalized gradient structure first introduced by Kraus and Roggensack to prove existence of solutions to the viscous CHR model in [42]. The gradient structure provides a generalization of the H^1 dual gradient flow proposed by Fife [31]. Following De Giorgi's minimizing movements method, we define an implicit scheme to construct approximate solutions of the CHR model. As will be seen, letting $R = 0$, we can recover the standard implicit scheme used to prove existence of solutions for the Cahn-Hilliard equation (1.9). From here, we derive a variety of energy estimates allowing us to pass the approximate solutions to the limit, thereby recovering a weak solution of the CHR model. In Subsection 3.5, elasticity is incorporated by making use of regularity theorems for elliptic systems, thereby proving Theorem 1.4.

We define our notion of weak solution for the CHR model (1.12) without elasticity.

Definition. *We say that c is a* weak solution of the CHR model *(1.12) on Ω_T if for some $\delta > 0$*

$$\begin{aligned}
&c \in L^{(2^\#-\delta)'}(0,T;H^3(\Omega)) \cap C([0,T);L^2(\Omega)),\\
&\partial_t c \in L^{(2^\#-\delta)'}(0,T;H^1(\Omega)^*),\\
&c(0) = c_0 \in H^1(\Omega),
\end{aligned}$$

and for t-a.e. in $(0,T)$ the following equation is satisfied for all $\xi \in H^1(\Omega)$:

$$-\langle \partial_t c(t), \xi \rangle_{H^1(\Omega)^*,H^1(\Omega)} = \int_\Omega \nabla\mu(t) \cdot \nabla\xi \; dx - \int_\Gamma R(c(t),\mu(t))\xi \, d\mathcal{H}^{N-1}, \tag{3.1}$$

where for t-a.e. $\mu(t) \in H^1(\Omega) \subset L^2(\Omega)$ is defined via duality for all $\xi \in H^1(\Omega)$ by

$$\int_\Omega \mu(t)\xi \, dx := \int_\Omega (\nabla c(t) \cdot \nabla \xi + f'(c(t))\xi) \, dx.$$

Subsections 3.1 to 3.4 are devoted to the proof of the following theorem.

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded, open domain with C^3 boundary and $T > 0$. Suppose f and R satisfy assumptions (2.1), (2.3), (2.4), and (2.5). Then a weak solution of the CHR model (1.12) exists in Ω_T.*

3.1 Generalized gradient structure

Much of the exposition in this subsection follows the work of Kraus and Roggensack. For more details we refer the reader to [42]. We do however highlight a new H^1 dual bound in Lemma 3.3, which will be essential to apply a compactness argument in the spirit of Aubin-Lions-Simon [57] in Subsection 3.4.

We introduce functionals to define the gradient structure. Let

$$\begin{aligned}
&\mathcal{A} : L^{2^\#-\delta}(\Gamma) \times L^2(\Omega) \to \mathbb{R} \cup \{\infty\},\\
&\mathcal{A}(c,v) := \begin{cases} \frac{1}{2}\|\nabla v\|^2_{L^2(\Omega)} - \int_\Gamma G(c,v) \; d\mathcal{H}^{N-1} & \text{if } v \in H^1(\Omega),\\ \infty & \text{otherwise.} \end{cases}
\end{aligned} \tag{3.2}$$

This functional is proper, lower semi-continuous, and convex in the second input by assumptions (2.3), (2.2), and (2.7). Furthermore, define

$$\begin{aligned}
&\mathcal{B} : L^{2^\#-\delta}(\Gamma) \times H^1(\Omega) \to H^1(\Omega)^*,\\
&\langle \mathcal{B}(c,v), \xi \rangle_{H^1(\Omega)^*,H^1(\Omega)} := \int_\Omega \nabla v \cdot \nabla \xi \; dx - \int_\Gamma R(c,v)\xi \; d\mathcal{H}^{N-1}.
\end{aligned} \tag{3.3}$$

Under the monotonicity assumption (2.3) on R, we can show for fixed $c \in L^{2^\#-\delta}(\Gamma)$ that $\mathcal{B}_c(\cdot) := \mathcal{B}(c,\cdot)$ is strictly monotone, bounded, and coercive. Consequently, we may define (see Lemma 1 in [42]) the bounded and continuous operator

$$\begin{aligned}
&\overline{\mathcal{B}} : L^{2^\#-\delta}(\Gamma) \times H^1(\Omega)^* \to H^1(\Omega),\\
&\overline{\mathcal{B}}(c,v^*) := \mathcal{B}_c^{-1}(v^*).
\end{aligned} \tag{3.4}$$

Setting $\mathcal{A}_c(\cdot) := \mathcal{A}(c, \cdot)$, using (2.2) and (2.4), one can show that

$$v^* \in \partial \mathcal{A}_c(v) \iff (v^*, \xi)_{L^2(\Omega)} = \langle \mathcal{B}(c, v), \xi \rangle_{H^1(\Omega)^*, H^1(\Omega)} \text{ for all } \xi \in H^1(\Omega).$$

Hence, the gradient structure is given by the inclusion

$$-\partial_t c \in \partial \mathcal{A}_c(\mu),$$

which encapsulates equations $\partial_t c = \Delta\mu$ and $\partial_\nu \mu = R(c, \mu)$ of (1.12). As this is a differential inclusion, there is a natural dual formulation. Let $\mathcal{A}^*$ be the Legendre-Fenchel transform of $\mathcal{A}$ with respect to the second input, that is,

$$\begin{aligned} \mathcal{A}^* &: L^{2^\#-\delta}(\Gamma) \times L^2(\Omega) \to \mathbb{R} \cup \{\infty\} \\ \mathcal{A}^*(c, v^*) &:= (\mathcal{A}_c)^*(v^*) = \sup_{v \in L^2(\Omega)} \{(v^*, v)_{L^2(\Omega)} - \mathcal{A}_c(v)\} \end{aligned} \tag{3.5}$$

As Kraus and Roggensack detail,

$$\partial \mathcal{A}_c^*(v^*) = \{\overline{\mathcal{B}}(c, v^*)\}, \tag{3.6}$$

which provides a way to express μ in terms of c and $\partial_t c$ via convex duality. Explicitly,

$$\mu = \overline{\mathcal{B}}(c, -\partial_t c). \tag{3.7}$$

We note a technical result, which will be helpful in obtaining energetic bounds.

Lemma 3.2 ([42], Lemma 2)**.** *Under the hypotheses of Theorem 3.1, for any $c \in L^{2^\#-\delta}(\Gamma)$ and $v^* \in L^2(\Omega)$, we have the following bound:*

$$\frac{1}{N}\Big|\int_\Omega v^* \, dx\Big| \le \mathcal{A}_c^*(v^*) + \mathcal{A}_c(\overline{\mathcal{B}}(c, 0)) + C$$

for some constant $C > 0$ depending only on G, R, and Ω.

Finally, we conclude this section with the H^1 dual bound.

Lemma 3.3. *Let $\Omega \subset \mathbb{R}^N$ be a bounded, open set with Lipschitz boundary. Assume hypotheses (2.1) to (2.5) hold with $\mathcal{A}^*$ defined as in (3.5). Suppose that $\|c\|_{L^{2^\#-\delta}(\Gamma)} \le \alpha$. Then there exists $C_\alpha > 0$ such that*

$$\mathcal{A}_c^*(v^*) \ge \frac{1}{C_\alpha}\|v^*\|_{H^1(\Omega)^*}^{(2^\#-\delta)'} - C_\alpha. \tag{3.8}$$

Proof. Under the growth assumption (2.7) on G and the definition $\mathcal{A}$ in (3.2), for $\xi \in H^1(\Omega)$, we have

$$\begin{aligned} \mathcal{A}_c(\xi) &\le C(\|\xi\|_{H^1(\Omega)}^2 + \|\xi\|_{L^{2^\#-\delta}(\Gamma)}^{2^\#-\delta} + \|c\|_{L^{2^\#-\delta}(\Gamma)}^{2^\#-\delta} + 1) \\ &\le C_\alpha(\|\xi\|_{H^1(\Omega)}^{2^\#-\delta} + 1) \end{aligned}$$

where in the second inequality we have used the trace inequality $\|\xi\|_{L^{2^\#-\delta}(\Gamma)} \le C\|\xi\|_{H^1(\Omega)}$. As $\mathcal{A}_c = \infty$ on $L^2(\Omega) \setminus H^1(\Omega)$, we compute by definition of the conjugate

$$\begin{aligned} \mathcal{A}_c^*(v^*) &= \sup_{\xi \in H^1(\Omega)} \Big\{(v^*, \xi)_{L^2(\Omega)} - \mathcal{A}_c(\xi)\Big\} \\ &\ge \sup_{\xi \in H^1(\Omega)} \Big\{(v^*, \xi)_{L^2(\Omega)} - C_\alpha\|\xi\|_{H^1(\Omega)}^{2^\#-\delta}\Big\} - C_\alpha. \end{aligned} \tag{3.9}$$

Since $2^\# - \delta > 1$, there is a unique maximizer ξ_0 to the latter supremum, and computing the Gateaux derivative, it must satisfy the following relation for all $\xi \in H^1(\Omega)$:

$$(v^*, \xi)_{L^2(\Omega)} = (2^\# - \delta)C_\alpha \|\xi_0\|_{H^1(\Omega)}^{2^\# - \delta - 2}(\xi_0, \xi)_{H^1(\Omega)}.$$

Furthermore as the right hand side is maximized over the unit ball at $\bar{\xi} = \xi_0/\|\xi_0\|_{H^1(\Omega)}$, this implies

$$\frac{1}{\|\xi_0\|_{H^1(\Omega)}}(v^*, \xi_0)_{L^2(\Omega)} = \|v^*\|_{H^1(\Omega)^*} = (2^\# - \delta)C_\alpha \|\xi_0\|_{H^1(\Omega)}^{2^\# - \delta - 1}.$$

Consequently, evaluating the supremum in (3.9) at its maximizer, we find

$$\begin{aligned}\mathcal{A}_c^*(v^*) \geq& (v^*, \xi_0)_{L^2(\Omega)} - C_\alpha \|\xi_0\|_{H^1(\Omega)}^{2^\# - \delta} - C_\alpha \\ \geq& (2^\# - \delta - 1)C_\alpha \|\xi_0\|_{H^1(\Omega)}^{2^\# - \delta} - C_\alpha \\ \geq& \frac{(2^\# - \delta - 1)C_\alpha}{((2^\# - \delta)C_\alpha)^{(2^\# - \delta)'}} \|v^*\|_{H^1(\Omega)^*}^{(2^\# - \delta)'} - C_\alpha.\end{aligned}$$

Up to redefinition of C_α, this completes the lemma.

□

3.2 Minimizing movements and Euler-Lagrange equations

We fix $T > 0$. For $\tau = \tau_n := T/n > 0$, with $n \in \mathbb{N}$, we define the following iterative scheme. Let $c_\tau^0 = c_0$. Define

$$c_\tau^i = \mathrm{argmin}_{c \in L^2(\Omega)} \left\{ I[c] + \tau \mathcal{A}^*_{c_\tau^{i-1}} \left(-\frac{c - c_\tau^{i-1}}{\tau} \right) \right\}, \tag{3.10}$$

where I and $\mathcal{A}^*$ are defined in (1.7) and (3.5), respectively. Motivated by (3.7), we define the discretized chemical potential by

$$\mu_\tau^i := \overline{\mathcal{B}} \left(c_\tau^{i-1}, -\frac{c_\tau^i - c_\tau^{i-1}}{\tau} \right). \tag{3.11}$$

To gain an intuition for the minimization scheme, we heuristically consider the case of $R = 0$ (note however this choice of R does not satisfy the strict monotonicity assumption (2.3)). A direct computation (see the proof of Lemma 3.3) verifies that $\mathcal{A}^*(v^*) = \frac{1}{2}\|v^*\|^2_{H^1(\Omega)^*}$. Consequently, the above minimization scheme reduces to

$$c_\tau^i = \mathrm{argmin}_{c \in L^2(\Omega)} \Big\{ I[c] + \frac{1}{2\tau} \|c - c_\tau^{i-1}\|^2_{H^1(\Omega)^*} \Big\}.$$

Furthermore, by definition $\overline{\mathcal{B}}(-\partial_t c) = (\Delta)^{-1}(-\partial_t c)$, the inverse Laplacian associated with homogeneous Neumann boundary conditions. Consequently, we obtain the standard implicit scheme for the Cahn-Hilliard equation (1.9) minus fixing the total mass of c.

Lemma 3.4. *Assume $\Omega \subset \mathbb{R}^N$ is a bounded, open set with Lipschitz boundary and hypotheses (2.1) to (2.5) hold. A minimizer, $c_\tau^i \in H^1(\Omega)$, of the iterative scheme (3.10) exists.*

Proof. The functional minimized is lower semi-continuous under weak convergence of c in $H^1(\Omega)$; thus it remains to prove coercivity. This follows directly from the bound $f \geq -C$ (see (2.1)) and applying Lemma 3.2:

$$\begin{aligned}\frac{1}{2}\|\nabla c\|^2 + \left| \int_\Omega c \, dx \right| \leq& I[c] + C + \tau \left| \int_\Omega \frac{c - c_\tau^{i-1}}{\tau} \, dx \right| + \left| \int_\Omega c_\tau^{i-1} \, dx \right| \\ \leq& I[c] + C + \tau \left(\mathcal{A}^*_{c_\tau^{i-1}} \left(-\frac{c - c_\tau^{i-1}}{\tau} \right) + \mathcal{A}_{c_\tau^{i-1}}(\overline{\mathcal{B}}(c_\tau^{i-1}, 0)) \right) + \left| \int_\Omega c_\tau^{i-1} \, dx \right| \\ =& I[c] + \tau \mathcal{A}^*_{c_\tau^{i-1}} \left(-\frac{c - c_\tau^{i-1}}{\tau} \right) + C(c_\tau^{i-1}).\end{aligned}$$

Poincaré's inequality followed by the Direct Method completes the argument. □

We now use the constructed sequence $\{c_\tau^i\}_i$ to define interpolated and piecewise continuous functions as follows. We define c_τ and c_τ^- to be the left and right continuous step functions, respectively:

$$c_\tau(t) := \begin{cases} c_\tau^0 & \text{if } t = 0, \\ c_\tau^{i+1} & \text{if } t \in (i\tau, (i+1)\tau],\ i = 0, \ldots, n-1, \end{cases} \tag{3.12}$$

$$c_\tau^-(t) := c_\tau^i \quad \text{if } t \in [i\tau, (i+1)\tau),\ i = 0, \ldots, n-1. \tag{3.13}$$

Likewise we define μ_τ to be the left continuous step function. Define $\hat{c}_\tau$ to be the piecewise linear interpolation of the sequence:

$$\hat{c}_\tau(t) := \frac{(i+1)\tau - t}{\tau} c_\tau^i + \frac{t - i\tau}{\tau} c_\tau^{i+1} \quad \text{if } t \in [i\tau, (i+1)\tau),\ i = 0, \ldots, n-1. \tag{3.14}$$

Lemma 3.5. *Assume $\Omega \subset \mathbb{R}^N$ is a bounded, open set with Lipschitz boundary and hypotheses (2.1) to (2.5) hold. The functions c_τ, c_τ^-, and $\hat{c}_\tau$ satisfy the "discrete" Euler-Lagrange equations*

$$-(\partial_t \hat{c}_\tau(t), \xi)_{L^2(\Omega)} = \int_\Omega \nabla\mu_\tau(t) \cdot \nabla\xi \, dx - \int_\Gamma R(c_\tau^-(t), \mu_\tau(t))\xi d\mathcal{H}^{N-1}, \tag{3.15}$$

$$(\mu_\tau(t), \xi)_{L^2(\Omega)} = \int_\Omega (\nabla c_\tau(t) \cdot \nabla\xi + f'(c_\tau(t))\xi) \, dx \tag{3.16}$$

for all $\xi \in H^1(\Omega)$ and t a.e. in $[0, T)$, and the energy estimate

$$I[c_\tau(t)] + \int_0^{t_\tau(t)} \Big(\mathcal{A}^*_{c_\tau^-}(-\partial_t \hat{c}_\tau) + \mathcal{A}_{c_\tau^-}(\overline{\mathcal{B}}(c_\tau^-, 0)) \Big) \, ds \le I[c_0], \tag{3.17}$$

where $t_\tau(t) := \min\{k\tau : t \le k\tau\}$.

Proof. Note

$$\begin{aligned} I[c_\tau^i] + \tau\mathcal{A}^*_{c_\tau^{i-1}}\left(-\frac{c_\tau^i - c_\tau^{i-1}}{\tau}\right) &+ \tau\mathcal{A}_{c_\tau^{i-1}}(\overline{\mathcal{B}}(c_\tau^{i-1}, 0)) \\ &\le I[c_\tau^{i-1}] + \tau\mathcal{A}^*_{c_\tau^{i-1}}\left(-\frac{c_\tau^{i-1} - c_\tau^{i-1}}{\tau}\right) + \tau\mathcal{A}_{c_\tau^{i-1}}(\overline{\mathcal{B}}(c_\tau^{i-1}, 0)) \\ &= I[c_\tau^{i-1}] + \tau\Big(\mathcal{A}^*_{c_\tau^{i-1}}(0) + \mathcal{A}_{c_\tau^{i-1}}(\overline{\mathcal{B}}(c_\tau^{i-1}, 0))\Big) \\ &= I[c_\tau^{i-1}] + \tau(0, \overline{\mathcal{B}}(c_\tau^{i-1}, 0))_{L^2} = I[c_\tau^{i-1}], \end{aligned}$$

where we have used the fact c_τ^i is a minimizer (see (3.10)) and Fenchel's (in)equality [56]. Moving $I[c_\tau^{i-1}]$ to the lefthand side, summing up over the inequalities for $i = 1, \ldots, n$, and using telescoping sums, we conclude the energetic bound (3.17). To obtain the Euler-Lagrange equations (3.15) and (3.16), we compute the subdifferential of the minimized equation in $H^1(\Omega)^*$ via (3.6):

$$\begin{aligned} 0 \in &\partial|_{c_\tau^i}\left(I[c] + \tau\mathcal{A}^*_{c_\tau^{i-1}}\left(-\frac{c - c_\tau^{i-1}}{\tau}\right)\right) \iff \\ 0 \in &\partial I[c_\tau^i] - \partial\mathcal{A}^*_{c_\tau^{i-1}}\left(-\frac{c_\tau^i - c_\tau^{i-1}}{\tau}\right) \iff \\ 0 \in &\partial I[c_\tau^i] - \left\{\overline{\mathcal{B}}\left(c_\tau^{i-1}, -\frac{c_\tau^i - c_\tau^{i-1}}{\tau}\right)\right\} \iff \\ 0 \in &\partial I[c_\tau^i] - \{\mu_\tau^i\} \end{aligned}$$

Computing $\partial I[c_\tau^i]$ via differentiation, we obtain (3.16). Equation (3.15) follows by definition of μ_τ in (3.11). □

3.3 Energy estimates

We now obtain some energy estimates, which will be useful in passing to the limit in the "discrete" Euler-Lagrange equations.

Lemma 3.6. *Assume $\Omega \subset \mathbb{R}^N$ is a bounded, open set with C^3 boundary and hypotheses (2.1) to (2.5) hold. The functions c_τ, c_τ^-, and $\hat{c}_\tau$ defined in (3.12), (3.13), and (3.14) satisfy the following estimates uniformly in τ :*

$$\|c_\tau\|_{L^\infty(0,T;H^1(\Omega))} \leq C, \tag{3.18}$$

$$\|\partial_t \hat{c}_\tau\|_{L^{(2^\# - \delta)'}(0,T;H^1(\Omega)^*)} \leq C, \tag{3.19}$$

$$\|\mu_\tau\|_{L^{(2^\# - \delta)'}(0,T;H^1(\Omega))} \leq C, \tag{3.20}$$

$$\|c_\tau\|_{L^{(2^\# - \delta)'}(0,T;H^3(\Omega))} \leq C. \tag{3.21}$$

Proof. To see that (3.18) holds, note that $\nabla c_\tau \in L^\infty(0,T;L^2(\Omega))$ by (3.17), and thus by the Poincaré inequality, we simply need to bound $\int_\Omega c_\tau \, dx$ to conclude. Making use of Lemma 3.2 and (3.17) once again, we find

$$\begin{aligned}
\Big|\int_\Omega c_\tau^i \, dx\Big| \leq & \tau \sum_{j=1}^{i} \Big|\int_\Omega \frac{c_\tau^j - c_\tau^{j-1}}{\tau} \, dx\Big| + \int_\Omega |c_0| \, dx \\
\leq & N\tau \sum_{j=1}^{i} \Big(\mathcal{A}^*_{c_\tau^{j-1}}\big(-\frac{c_\tau^j - c_\tau^{j-1}}{\tau}\big) + \mathcal{A}_{c_\tau^{j-1}}(\overline{\mathcal{B}}(c_\tau^{j-1}, 0))\Big) + TC + \int_\Omega |c_0| \, dx \\
\leq & N \int_0^T \Big(\mathcal{A}^*_{c_\tau^-}(-\partial_t \hat{c}_\tau) + \mathcal{A}_{c_\tau^-}(\overline{\mathcal{B}}(c_\tau^-, 0))\Big) \, ds + C \leq C(c_0)
\end{aligned}$$

Furthermore, (3.18) implies that c_τ is in $L^\infty(0,T;L^{2^\# - \delta}(\Gamma))$ by continuity of the trace. The coercivity of $\mathcal{A}_c^*$ given by Lemma 3.3 along with the energy estimate (3.17) then conclude (3.19). Making use of the bound

$$\|\overline{\mathcal{B}}(c, v^*)\|_{H^1(\Omega)} \leq C(\|v^*\|_{H^1(\Omega)^*} + 1),$$

which holds for $C > 0$ independent of c (see pg. 6 in [42]), in conjunction with (3.19), and the definition (3.11) of μ_τ provides (3.20).

Note for a.e. $t \in (0,T)$, $c_\tau(t)$ satisfies the Neumann problem

$$\begin{cases} \mu_\tau = -\Delta c_\tau + f'(c_\tau) & \text{in } \Omega, \\ \partial_\nu c_\tau = 0 & \text{on } \Gamma. \end{cases} \tag{3.22}$$

We use the growth condition

$$|f'(s)| \leq C(|s|^{2^*/2} + 1) \qquad \text{for all } s \in \mathbb{R}$$

obtained by integrating (2.1), the Sobolev-Gagliardo-Nirenberg embedding theorem, (3.18), and Theorem 2.4 applied to (3.22) to conclude that

$$\begin{aligned}
\|c_\tau\|^2_{H^2(\Omega)} \leq & C\Big(\|\mu_\tau - f'(c_\tau)\|^2_{L^2(\Omega)} + \|c_\tau\|^2_{H^1(\Omega)}\Big) \\
\leq & C\Big(\|\mu_\tau\|^2_{L^2(\Omega)} + \|c_\tau\|^{2^*}_{L^{2^*}(\Omega)}\Big) + C \\
\leq & C\Big(\|\mu_\tau\|^2_{L^2(\Omega)} + \|c_\tau\|^{2^*}_{H^1(\Omega)}\Big) + C \\
\leq & C\|\mu_\tau\|^2_{L^2(\Omega)} + C.
\end{aligned}$$

Consequently, bound (3.20) implies

$$\|c_\tau\|_{L^{(2^\# - \delta)'}(0,T;H^2(\Omega))} \leq C. \tag{3.23}$$

We show that $f'(c_\tau) \in L^{(2^\# - \delta)'}(0,T;H^1(\Omega))$ by the assumptions on f given in (2.1). Assume $N \geq 3$; $N = 2$ follows similarly. By the chain rule in Sobolev spaces [44] and Hölder's inequality, we compute

$$\begin{aligned}
\|\nabla(f' \circ c_\tau)\|_{L^2(\Omega)}^2 =& \int_\Omega |f''(c_\tau)|^2 \|\nabla c_\tau\|^2 \, dx \\
\leq& C \int_\Omega |c_\tau|^{2^*-2} \|\nabla c_\tau\|^2 \, dx + C \int_\Omega \|\nabla c_\tau\|^2 \, dx \\
\leq& C \| |c_\tau|^{2^*-2} \|_{L^{(N/(N-2))'}(\Omega)} \| (\|\nabla c_\tau\|^2) \|_{L^{N/(N-2)}(\Omega)} + C \\
\leq& C \|c_\tau\|_{H^2(\Omega)}^2 + C,
\end{aligned} \tag{3.24}$$

where in the last inequality, we have used the following two bounds:

$$\| |\nabla c_\tau|^2 \|_{L^{N/(N-2)}(\Omega)}^{N/(N-2)} = \int_\Omega \|\nabla c_\tau\|^{2^*} \, dx \leq \|c_\tau\|_{H^2(\Omega)}^{2^*},$$

by the Sobolev-Gagliardo-Nirenberg embedding; likewise, $(N/(N-2))' = N/2$, leading to

$$\| |c_\tau|^{2^*-2} \|_{L^{N/2}(\Omega)}^{N/2} = \int_\Omega |c_\tau|^{2^*} \, dx \leq \|c_\tau\|_{H^1(\Omega)}^{2^*} \leq C.$$

As desired, (3.23) and (3.24) then imply

$$\|f'(c_\tau)\|_{L^{(2^\# - \delta)'}(0,T;H^1(\Omega))} \leq C.$$

We once again make use of Theorem 2.4 for the problem (3.22) along with the previous bound and (3.20) to conclude (3.21). □

3.4 Passing to the limit

We wish to pass to the limit with respect to τ in the "discrete" Euler-Lagrange equations (3.15) and (3.16). To do this, we will look directly at the underlying compactness result used to obtain the Aubin-Lions-Simon compactness theorem [57]. For $h \in \mathbb{R}$, we introduce the translation operator $\mathcal{T}_h$ defined by action on a function g with domain $(0,T)$:

$$\mathcal{T}_h(g)(x) := g(x+h), \quad x \in (-h, T-h). \tag{3.25}$$

We further defined the auxillary set $\mathcal{O}_h := (0,T) \cap (-h, T-h)$, the interval of common definition for the functions $\mathcal{T}_h(g)$ and g. We recall a result.

Theorem ([57], Theorem 5)**.** *Suppose $(\mathcal{B}_i, \|\cdot\|_{\mathcal{B}_i})$, $i \in \{0,1,2\}$, are Banach spaces such that $\mathcal{B}_0 \hookrightarrow\hookrightarrow \mathcal{B}_1 \hookrightarrow \mathcal{B}_2$. Let $p \in [1,\infty)$ and $\mathcal{U} \subset L^p(0,T;\mathcal{B}_0)$ be a bounded set such that*

$$\|\mathcal{T}_h(g) - g\|_{L^p(\mathcal{O}_h;\mathcal{B}_2)} \to 0 \quad \text{as} \quad h \to 0 \quad \text{uniformly for} \quad g \in \mathcal{U}.$$

Then $\mathcal{U}$ is relatively compact in $L^p(0,T;\mathcal{B}_1)$.

We will not make direct use of this result, but the proof of the above result may be applied to prove the following lemma.

Lemma 3.7. *Let $\Omega \subset \mathbb{R}^N$ be a bounded, open set with Lipschitz boundary. Consider the triple*

$$c \in H^3(\Omega) \mapsto \Psi(c) := (c, \mathrm{Tr}(\nabla^2 c)) \in H^2(\Omega) \times [L^{2^\# - \delta}(\Gamma)]^{N\times N}$$
$$\mapsto \pi(c, \mathrm{Tr}(\nabla^2 c)) := c \in L^2(\Omega) \subset H^1(\Omega)^*.$$

Let $p \in [1, \infty)$ and $\mathcal{U} \subset L^p(0,T; H^3(\Omega))$ be a bounded set such that

$$\|\mathcal{T}_h(c) - c\|_{L^p(\mathcal{O}_h; H^1(\Omega)^*)} \to 0 \quad \textit{as} \quad h \to 0 \quad \textit{uniformly for} \quad c \in \mathcal{U}.$$

Then $\Psi(\mathcal{U})$ is relatively compact in $L^p(0,T; H^2(\Omega) \times [L^{2^\# - \delta}(\Gamma)]^{N\times N})$.

Proof. The proof of this lemma primarily follows as in the proof of Theorem 5 in [57]. We do not repeat the entire proof, but show why it still holds, despite the mappings no longer being embeddings. We **claim** that for every $\epsilon > 0$ there is $C_\epsilon > 0$ such that given any $c \in H^3(\Omega)$ the bound

$$\|\Psi(c)\|_{H^2(\Omega)\times[L^{2^\#-\delta}(\Gamma)]^{N\times N}} \le \epsilon \|c\|_{H^3(\Omega)} + C_\epsilon \|c\|_{H^1(\Omega)^*}$$

holds (see also Lemma 8 of [57]). We prove this by contradiction. Suppose not; thus for each $n \in \mathbb{N}$ there is $\xi_n \in H^3(\Omega)$ such that

$$\|\Psi(\xi_n)\|_{H^2(\Omega)\times[L^{2^\#-\delta}(\Gamma)]^{N\times N}} > \epsilon \|\xi_n\|_{H^3(\Omega)} + n\|\xi_n\|_{H^1(\Omega)^*}. \tag{3.26}$$

Normalizing in $H^3(\Omega)$, we may assume that $\|\xi_n\|_{H^3(\Omega)} = 1$. Thus by compactness of the map Ψ, there is $\xi_0 \in H^3(\Omega)$ such that, up to a subsequence (not relabeled), $\Psi(\xi_n) \to \Psi(\xi_0)$ in $H^2(\Omega) \times [L^{2^\#-\delta}(\Gamma)]^{N\times N}$. Additionally, as $\Psi(\xi_n)$ is bounded in $H^2(\Omega) \times [L^{2^\#-\delta}(\Gamma)]^{N\times N}$, (3.26) implies $\xi_n \to 0$ in $H^1(\Omega)^*$. As $\xi_n \to \pi \circ \Psi(\xi_0)$ in $H^1(\Omega)^*$, we have that $\xi_0 = 0$ necessarily (this is a key feature we needed satisfied by the mappings). However, (3.26) shows

$$0 = \|\Psi(\xi_0)\|_{H^2(\Omega)\times[L^{2^\#-\delta}(\Gamma)]^{N\times N}} = \lim_{n\to\infty} \|\Psi(\xi_n)\|_{H^2(\Omega)\times[L^{2^\#-\delta}(\Gamma)]^{N\times N}} \ge \epsilon,$$

a contradiction. The rest of the proof follows as in [57].

□

We now prove the desired convergences. In this proof, we make use of the composition symbol $\circ$ to differentiate the behavior of a map like $\Psi(c)$ from the composition $f'(c)$.

Lemma 3.8. *Assume $\Omega \subset \mathbb{R}^N$ is a bounded, open set with C^3 boundary and hypotheses (2.1) to (2.5) hold. There is $c \in L^{(2^\#-\delta)'}(0,T; H^3(\Omega)) \cap W^{1,(2^\#-\delta)'}(0,T; H^1(\Omega)^*)$ such that the functions c_τ, c_τ^-, and $\hat{c}_\tau$ defined in (3.12), (3.13), and (3.14) satisfy the following (up to a subsequence of τ approaching 0 not relabeled):*

$$\hat{c}_\tau \rightharpoonup c \qquad \textit{in} \quad L^{(2^\#-\delta)'}(0,T; H^3(\Omega)) \cap W^{1,(2^\#-\delta)'}(0,T; H^1(\Omega)^*), \tag{3.27}$$
$$\Psi(c_\tau),\ \Psi(c_\tau^-),\ \Psi(\hat{c}_\tau) \to \Psi(c) \quad \textit{in} \quad L^{(2^\#-\delta)'}(0,T; H^2(\Omega) \times [L^{2^\#-\delta}(\Gamma)]^{N\times N}), \tag{3.28}$$
$$f' \circ c_\tau \to f' \circ c \quad \textit{in} \quad L^{(2^\#-\delta)'}(0,T; H^1(\Omega)), \tag{3.29}$$
$$\mu_\tau \rightharpoonup \mu \qquad \textit{in} \quad L^{(2^\#-\delta)'}(0,T; H^1(\Omega)), \tag{3.30}$$

where $\Psi(c) := (c, \mathrm{Tr}(\nabla^2 c)) \in H^2(\Omega) \times [L^{2^\#-\delta}(\Gamma)]^{N\times N}$ for $c \in H^3(\Omega)$.

Proof. We first show that

$$\|\mathcal{T}_h(c_\tau) - c_\tau\|_{L^{(2^\#-\delta)'}(\mathcal{O}_h; H^1(\Omega)^*)} \to 0 \quad \text{as} \quad h \to 0 \quad \text{uniformly in } \tau \tag{3.31}$$

where $\mathcal{T}_h$ is defined in (3.25) and $\mathcal{O}_h := (0,T) \cap (-h, T-h)$. We estimate the $L^{(2^\#-\delta)'}$ norm directly in the case that $|h| < \tau$ and $|h| \ge \tau$. Without loss of generality, we perform the

computation for $h > 0$ so that $\mathcal{O}_h = (0, T-h)$. Partition $(0,T)$ into n intervals $J_1, \ldots, J_n$ of length $\tau = T/n$.
Case $h < \tau$: In this case $\mathcal{O}_h = (0, T-h)$. Making use of (3.12), the fundamental theorem of calculus, properties of the Bochner integral (see Chapter 8 of [44]), and Hölder's inequality, we have

$$\begin{aligned}\|\mathcal{T}_h(c_\tau) - c_\tau\|^{(2^\#-\delta)'}_{L^{(2^\#-\delta)'}(\mathcal{O}_h, H^1(\Omega)^*)} =& \sum_{i=1}^{n-1} \int_{i\tau - h}^{i\tau} \|c_\tau^{i+1} - c_\tau^i\|^{(2^\#-\delta)'}_{H^1(\Omega)^*} \, dt \\ =& h \sum_{i=1}^{n-1} \left\| \int_{J_{i+1}} \partial_t \hat{c}_\tau \, ds \right\|^{(2^\#-\delta)'}_{H^1(\Omega)^*} \\ \leq& h \sum_{i=1}^{n-1} |J_{i+1}|^{(2^\#-\delta)'-1} \int_{J_{i+1}} \|\partial_t \hat{c}_\tau\|^{(2^\#-\delta)'}_{H^1(\Omega)^*} \, ds \\ \leq& h \tau^{(2^\#-\delta)'-1} \int_0^T \|\partial_t \hat{c}_\tau\|^{(2^\#-\delta)'}_{H^1(\Omega)^*} \, dt.\end{aligned} \tag{3.32}$$

Case $h \geq \tau$: We define some auxillary variables to help with the computation. Let $[h]_\tau$ be h modulo τ, and let $k \in \mathbb{N}$ satisfy $k\tau = h - [h]_\tau$. Again by (3.12), the fundamental theorem of calculus, properties of the Bochner integral [44], and Hölder's inequality, we find

$$\begin{aligned}&\|\mathcal{T}_h(c_\tau) - c_\tau\|^{(2^\#-\delta)'}_{L^{(2^\#-\delta)'}(\mathcal{O}_h, H^1(\Omega)^*)} \\ &= \sum_{i=1}^{n-k} \left(\int_{(i-1)\tau}^{i\tau - [h]_\tau} \|c_\tau^{i+k} - c_\tau^i\|^{(2^\#-\delta)'}_{H^1(\Omega)^*} \, dt \right) + \sum_{i=1}^{n-k-1} \left(\int_{i\tau - [h]_\tau}^{i\tau} \|c_\tau^{i+k+1} - c_\tau^i\|^{(2^\#-\delta)'}_{H^1(\Omega)^*} \, dt \right) \\ &\leq \sum_{i=1}^{n-k} \left((\tau - [h]_\tau) |\cup_{j=i+1}^{i+k} J_{j+1}|^{(2^\#-\delta)'-1} \int_{\cup_{j=i+1}^{i+k} J_{j+1}} \|\partial_t \hat{c}_\tau\|^{(2^\#-\delta)'}_{H^1(\Omega)^*} \, ds \right) \\ &\quad + \sum_{i=1}^{n-k-1} \left([h]_\tau |\cup_{j=i}^{i+k} J_{j+1}|^{(2^\#-\delta)'-1} \int_{\cup_{j=i}^{i+k} J_{j+1}} \|\partial_t \hat{c}_\tau\|^{(2^\#-\delta)'}_{H^1(\Omega)^*} \, ds \right) \\ &\leq (2h)^{(2^\#-\delta)'-1} \left(\int_0^T \|\partial_t \hat{c}_\tau\|^{(2^\#-\delta)'}_{H^1(\Omega)^*} \, dt \right) (n-k)\tau.\end{aligned} \tag{3.33}$$

Making use of (3.19), (3.32), and (3.33), we conclude the proof of (3.31). Similarly (3.31) holds for c_τ^- and $\hat{c}_\tau$ too.

Consequently making use of (3.21), we apply Lemma 3.7 to c_τ, c_τ^-, and $\hat{c}_\tau$. Thus up to a subsequence of τ (not relabeled), there is $c \in L^{(2^\#-\delta)'}(0,T; H^3(\Omega)) \cap W^{1,(2^\#-\delta)'}(0,T; H^1(\Omega)^*)$ such that $\Psi(c_\tau)$, $\Psi(c_\tau^-)$, $\Psi(\hat{c}_\tau) \to \Psi(c)$ in $L^{(2^\#-\delta)'}(0,T; H^2(\Omega) \times [L^{2^\#-\delta}(\Gamma)]^{N\times N})$.

We **remark** a priori, it is not clear that c_τ, c_τ^- and $\hat{c}_\tau$ converge to the same c. Let us show that c_τ and $\hat{c}_\tau$ converge to the same limit. Suppose $c_\tau \to c_1$ in $L^{(2^\#-\delta)'}(0,T; L^2(\Omega))$ and $\hat{c}_\tau \to c_2$ in $L^{(2^\#-\delta)'}(0,T; L^2(\Omega))$. Letting $\Phi : L^2(\Omega) \to H^1(\Omega)^*$ be the natural, continuous inclusion given by $\Phi(\xi)(\phi) := \int_\Omega \xi \phi \, dx$ for all $\phi \in H^1(\Omega)$. On the interval J_i defined above, we have $c_\tau|_{J_i} = \hat{c}_\tau(i\tau)$. By the fundamental theorem of calculus, (3.19), and Hölder's inequality, we compute for any $t \in J_i$,

$$\begin{aligned}&\|\Phi(c_\tau)(t) - \Phi(\hat{c}_\tau)(t)\|_{H^1(\Omega)^*} \\ &\quad = \|\Phi(\hat{c}_\tau)(i\tau) - \Phi(\hat{c}_\tau)(t)\|_{H^1(\Omega)^*} \leq \int_t^{i\tau} \|\partial_t \hat{c}_\tau(t)\|_{H^1(\Omega)^*} \, dt \leq C(i\tau - t)^{\frac{1}{2^\#-\delta}} \leq C\tau^{\frac{1}{2^\#-\delta}}.\end{aligned}$$

The above estimate holds for any $t \in (0,T)$, and we conclude

$$\Phi(c_\tau) - \Phi(\hat{c}_\tau) \to 0 \quad \text{in} \quad L^\infty(0,T; H^1(\Omega)^*) \quad \text{as} \quad \tau \to 0.$$

But we also have that

$$\Phi(c_\tau) - \Phi(\hat{c}_\tau) \to \Phi(c_1) - \Phi(c_2) = \Phi(c_1 - c_2) \quad \text{in} \quad L^{(2^\# - \delta)'}(0,T; H^1(\Omega)^*) \quad \text{as} \quad \tau \to 0.$$

Consequently, $\Phi(c_1 - c_2) = 0$, which by the injectivity of Φ, implies $c_1 = c_2$ as desired.

Furthermore, c_τ's strong convergence allows us to show (3.29). We show convergence of the gradient $\nabla(f' \circ c_\tau)$, with convergence of $f' \circ c_\tau$ in $L^{(2^\# - \delta)'}(0,T; L^2(\Omega))$ being easier to conclude. To see this, we apply Lebesgue dominated convergence theorem in an iterative fashion. Consider a subsequence of τ such that both $c_\tau \to c$ and $\nabla c_\tau \to \nabla c$ pointwise a.e. in $\Omega \times (0,T)$ and for t-a.e., $c_\tau(t) \to c(t)$ in $H^2(\Omega)$. By (2.1) and Young's inequality

$$|\nabla(f' \circ c_\tau)|^2 = |f''(c_\tau)|^2 |\nabla c_\tau|^2 \leq C(|c_\tau|^{2^*} + |\nabla c_\tau|^{2^*} + |\nabla c_\tau|^2),$$

and so $\nabla(f' \circ c_\tau)(t) \to \nabla(f' \circ c)(t)$ in $L^2(\Omega)$ for t-a.e. in $(0,T)$ by the generalized Lebesgue dominated convergence theorem and the Sobolev-Gagliardo-Nirenberg embedding theorem. We look at the previously derived bound (3.24), which shows us that for t-a.e., we have

$$\|\nabla(f' \circ c_\tau - f' \circ c)\|_{L^2(\Omega)} \leq C(\|c_\tau\|_{H^2(\Omega)} + \|c\|_{H^2(\Omega)} + 1).$$

As the right-hand side of the above bound converges in $L^{(2^\# - \delta)'}(0,T; \mathbb{R})$ and the left-hand side converges to 0 for t-a.e., another application of the generalized Lebesgue dominated convergence theorem concludes.

Convergence given in (3.30) follows from $\mu_\tau = -\Delta c_\tau + f' \circ c_\tau$ (see (3.16)) in conjunction with the convergences (3.27) and (3.29) □

We consider a definition, which will prove useful in the next proof.

Definition 3.9. *For $\Omega \subset \mathbb{R}^N$, an open, bounded with Lipschitz boundary, we define the inclusion operator $\mathcal{I} : L^{(2^\# - \delta)'}(\Gamma) \to H^{1/2}(\Gamma)^*$ by*

$$\langle \mathcal{I}(g), \xi \rangle_{H^{1/2}(\Gamma)^*, H^{1/2}(\Gamma)} := \int_\Gamma g\xi \, d\mathcal{H}^{N-1}.$$

To see that the inclusion makes sense, for any $\xi \in H^{1/2}(\Gamma)$ there exists $\overline{\xi} \in H^1(\Omega)$ such that $\mathrm{Tr}(\overline{\xi}) = \xi$ *and* $\frac{1}{C}\|\overline{\xi}\|_{H^1(\Omega)} \leq \|\xi\|_{H^{1/2}(\Gamma)} \leq C\|\overline{\xi}\|_{H^1(\Omega)}$, *for constant $C > 0$ independent of ξ (see Theorem 18.40 in [44]). Thus for $g \in L^{(2^\# - \delta)'}(\Gamma)$, we have*

$$\begin{aligned}\left|\langle \mathcal{I}(g), \xi \rangle_{H^{1/2}(\Gamma)^*, H^{1/2}(\Gamma)}\right| \leq \|g\|_{L^{(2^\# - \delta)'}(\Gamma)} \|\xi\|_{L^{2^\# - \delta}(\Gamma)} &\leq C\|g\|_{L^{(2^\# - \delta)'}(\Gamma)} \|\overline{\xi}\|_{H^1(\Omega)} \\ &\leq C\|g\|_{L^{(2^\# - \delta)'}(\Gamma)} \|\xi\|_{H^{1/2}(\Gamma)}.\end{aligned}$$

We now have enough machinery in place to prove that there is a weak solution of the CHR model (1.12).

Proof of Theorem 3.1. We prove that the function c from Lemma 3.8 is a weak solution of the CHR model in two steps. First, we prove that the Euler-Lagrange equation (3.1) is satisfied. Second, we prove the initial condition is satisfied.

Step 1: Integrating equation (3.15) in time for $\xi \in L^{2^\# - \delta}(0,T; H^1(\Omega))$, we have

$$-\int_0^T (\partial_t \hat{c}_\tau(t), \xi)_{L^2} \, dt = \int_0^T \int_\Omega \nabla \mu_\tau(t) \cdot \nabla \xi \, dx \, dt - \int_0^T \int_\Gamma R(c_\tau^-(t), \mu_\tau(t))\xi \, d\mathcal{H}^{N-1} \, dt, \quad (3.34)$$

which makes sense by Lemma 3.6 and (2.4). We wish to pass $\tau \to 0$. Making use of Lemma 3.8, the only term for which this is difficult is the boundary term. Denote $R_\tau := R(c_\tau^-, \mu_\tau)$ for notational simplicity. Reconfiguring equation (3.34) and applying Hölder's inequality, we obtain

$$\begin{aligned}&\left|\int_0^T \int_\Gamma R_\tau \xi \; d\mathcal{H}^{N-1} \; dt\right| \\ &\leq \|\nabla \mu_\tau\|_{L^{(2^\# -\delta)'}(0,T;L^2(\Omega))} \|\nabla \xi\|_{L^{2^\# -\delta}(0,T;L^2(\Omega))} + \|\partial_t c_\tau\|_{L^{(2^\# -\delta)'}(0,T;H^1(\Omega)^*)} \|\xi\|_{L^{2^\# -\delta}((0,T),H^1(\Omega))}.\end{aligned}$$

Making use of Lemma 3.6, we conclude

$$\sup_{\substack{\xi \in L^{2^\# -\delta}(0,T;H^1(\Omega)) \\ \|\xi\|_{L^{2^\# -\delta}(0,T;H^1(\Omega))} \leq 1}} \left|\int_0^T \int_\Gamma R_\tau \xi \; d\mathcal{H}^{N-1} \; dt\right| \leq C. \tag{3.35}$$

We now consider the naturally defined inclusion operator $\mathcal{I} : L^{(2^\# -\delta)'}(\Gamma) \to H^{1/2}(\Gamma)^*$ (see Definition 3.9). By the supremum (3.35), we have that $\mathcal{I}(R_\tau)$ is bounded in $L^{2^\# -\delta}(0,T;H^{1/2}(\Gamma))^* = L^{(2^\# -\delta)'}(0,T;H^{1/2}(\Gamma)^*)$ uniformly with respect to τ. Furthermore, as $H^{1/2}(\Gamma)$ is reflexive, so is $L^{(2^\# -\delta)'}(0,T;H^{1/2}(\Gamma)^*)$ [33], and by weak compactness, up to a subsequence, $\mathcal{I}(R_\tau) \rightharpoonup \zeta$ for some $\zeta \in L^{(2^\# -\delta)'}(0,T;H^{1/2}(\Gamma)^*)$.

We **claim** that for t-a.e. $\mathcal{I}(R_\tau) \to \mathcal{I}(R(c,\mu))$ in $H^{1/2}(\Gamma)^*$. Looking to the convergence given by (3.28), we see that $\Delta c_\tau \to \Delta c$ in the space of real-valued functions $L^{(2^\# -\delta)'}(\Gamma \times (0,T))$, and up to a subsequence of τ (not relabeled), we may apply classical results for L^p spaces to conclude that $\Delta c_\tau \to \Delta c$ pointwise $\mathcal{H}^N$-a.e. in $\Gamma \times (0,T)$. Furthermore, by definition of the convergence in $L^{(2^\# -\delta)'}(0,T;L^{2^\# -\delta}(\Gamma))$, we have $\|\Delta c_\tau - \Delta c\|_{L^{2^\# -\delta}(\Gamma)} \to 0$ in $L^{(2^\# -\delta)'}(0,T)$. It follows by classical results for L^p spaces that up to a subsequence of τ (not relabeled) $\Delta c_\tau(t) \to \Delta c(t)$ in $L^{2^\# -\delta}(\Gamma)$ for t-a.e. in $(0,T)$. Repeating this argument using (3.28), (3.29), and continuity of the trace, we may assume, up to another subsequence of τ, for t-a.e. $c_\tau^-(t) \to c(t)$ and $f'(c_\tau)(t) \to f'(c)(t)$ in $L^{2^\# -\delta}(\Gamma)$ and pointwise a.e. in the domain Γ. Recalling the growth estimate on R given by (2.4) and (3.22), we have

$$\begin{aligned}|R_\tau(t)|^{(2^\# -\delta)'} &\leq C(|c_\tau^-(t)|^{2^\# -\delta -1} + |\mu_\tau(t)|^{2^\# -\delta -1} + 1)^{(2^\# -\delta)'} \\ &\leq C(|c_\tau^-(t)|^{2^\# -\delta} + |\Delta c_\tau(t)|^{2^\# -\delta} + |f'(c_\tau)(t)|^{2^\# -\delta} + 1).\end{aligned}$$

As R is a continuous function, we utilize the generalized Lebesgue Dominated Convergence theorem to conclude $R_\tau(t) \to R(c,\mu)(t)$ in $L^{(2^\# -\delta)'}(\Gamma)$ for t-a.e. Continuity of $\mathcal{I}$ implies for t-a.e. $\mathcal{I}(R_\tau(t)) \to \mathcal{I}(R(c,\mu)(t))$ in $H^{1/2}(\Gamma)^*$, proving the claim. Applying Mazur's Lemma [13], this further implies $\zeta = \mathcal{I}(R(c,\mu))$.

Passing τ to the limit in (3.34) and using the variety of convergences derived herein and in Lemma 3.8, we obtain

$$-\int_0^T \langle \partial_t c, \xi \rangle_{H^1(\Omega)^*, H^1(\Omega)} \; dt = \int_0^T \int_\Omega \nabla \mu \cdot \nabla \xi \; dx \; dt - \int_0^T \langle \mathcal{I}(R(c,\mu)), \xi \rangle_{H^{1/2}(\Gamma)^*, H^{1/2}(\Gamma)} \; dt.$$

By definition of $\mathcal{I}$, this is rewritten as

$$-\int_0^T \langle \partial_t c, \xi \rangle_{H^1(\Omega)^*, H^1(\Omega)} \; dt = \int_0^T \int_\Omega \nabla \mu \cdot \nabla \xi \; dx \; dt - \int_0^T \int_\Gamma R(c,\mu) \xi \; d\mathcal{H}^{N-1} \; dt.$$

Considering a dense collection of $\{\xi_k\}_{k \in \mathbb{N}} \subset H^1(\Omega)$, we let $\xi = \xi_k$ (constant in time) in the above equation. By a standard analysis using Lebesgue points (see also Theorem 6.2), we find the Euler-Lagrange equation (3.1) is satisfied by c for t-a.e.

Step 2: The initial condition and continuity of c follows directly from bound (3.18), (3.19), and the Aubin-Simon-Lions Compactness theorem with $p = \infty$ [57]. Or we may avoid the use of high-level compactness theorems as follows.

Note that

$$\|\hat{c}_\tau(t_2) - \hat{c}_\tau(t_1)\|_{H^1(\Omega)^*} \leq \int_{t_1}^{t_2} \|\partial_t \hat{c}_\tau(t)\|_{H^1(\Omega)^*}\ dt \leq C(t_2 - t_1)^{1/(2^\# - \delta)}$$

by an application of Hölder's inequality and (3.19). Applying Corollary 2.6, we find

$$\|\hat{c}_\tau(t_2) - \hat{c}_\tau(t_1)\|_{L^2(\Omega)} \leq \|\hat{c}_\tau(t_2) - \hat{c}_\tau(t_1)\|_{H^1(\Omega)}^{1/2} \|\hat{c}_\tau(t_2) - \hat{c}_\tau(t_1)\|_{H^1(\Omega)^*}^{1/2} + \|\hat{c}_\tau(t_2) - \hat{c}_\tau(t_1)\|_{H^1(\Omega)^*}$$

By (3.18),

$$\|\hat{c}_\tau(t_2) - \hat{c}_\tau(t_1)\|_{H^1(\Omega)} \leq C.$$

Synthesizing these three inequalities, we have

$$\|\hat{c}_\tau(t_2) - \hat{c}_\tau(t_1)\|_{L^2(\Omega)} \leq C \max\{(t_2 - t_1)^{1/(2(2^\# - \delta))}, (t_2 - t_1)^{1/(2^\# - \delta)}\}. \tag{3.36}$$

Passing $\tau \to 0$, we see that c satisfies the relation (3.36) for t-a.e. Furthermore, letting $t_1 = 0$ in (3.36), we have

$$\|\hat{c}_\tau(t_2) - c_0\|_{L^2(\Omega)} \leq C \max\{(t_2)^{1/(3(2^\# - \delta))}, (t_2)^{1/(2^\# - \delta)}\}.$$

Letting $\tau \to 0$, we find that $c(0) = c_0$ as desired.

□

3.5 Including elasticity

We complete the proof of Theorem 1.4. We highlight where the argument differs from the proof of Theorem 3.1.

Proof of Theorem 1.4. We construct an iterative scheme via the minimization

$$(c_\tau^i, u_\tau^i) = \underset{(c,u) \in L^2(\Omega)^2}{\operatorname{argmin}} \left\{ I_{el}[c, u] + \tau \mathcal{A}^*_{c_\tau^{i-1}} \left(-\frac{c - c_\tau^{i-1}}{\tau} \right) \right\}. \tag{3.37}$$

As before we are able to obtain the "discrete" Euler-Lagrange equations for all $\xi \in H^1(\Omega)$ and $\psi \in H^1(\Omega; \mathbb{R}^N)$:

$$\begin{aligned} -(\partial_t \hat{c}_\tau(t), \xi)_{L^2(\Omega)} &= \int_\Omega \nabla \mu_\tau(t) \cdot \nabla \xi\ dx - \int_\Gamma R(c_\tau^-(t), \mu_\tau(t)) \xi\ d\mathcal{H}^{N-1} \\ (\mu_\tau(t), \xi)_{L^2(\Omega)} &= \int_\Omega (\nabla c_\tau(t) \cdot \nabla \xi + f'(c_\tau(t))\xi + \mathbb{C}(c_\tau(t) e_0 - e(u_\tau(t))) : e_0 \xi)\ dx, \\ 0 &= \int_\Omega \mathbb{C}(e(u_\tau(t)) - c_\tau(t) e_0) : e(\psi)\ dx. \end{aligned} \tag{3.38}$$

The estimate (3.17) continues to hold, with I replaced by I_{el}. In turn, one has the bounds (3.18), (3.19), and (3.20) of Lemma 3.6. To conclude (3.21), we **claim** that u_τ is bounded in $L^\infty(0, T; \dot{H}^2(\Omega; \mathbb{R}^N))$ uniformly with respect to τ, where $\dot{H}^2(\Omega)$ is $H^2(\Omega)$ quotiented by skew affine functions. Note by (3.17), which holds with I_{el}, and Korn's inequality [54], we have u_τ bounded in $L^\infty(0, T; \dot{H}^1(\Omega; \mathbb{R}^N))$. By the last equation of (3.38), we have

$$\int_\Omega \mathbb{C}(e(u_\tau)) : e(\xi)\ dx = \int_\Omega \mathbb{C}(c_\tau e_0) : e(\xi)\ dx = \int_\Omega c_\tau L_1(\nabla \xi)\ dx,$$

where $L_1 : \mathbb{R}^{N\times N} \to \mathbb{R}$ is a linear function. Applying integration by parts, we find

$$\int_\Omega \mathbb{C}(e(u_\tau)) : e(\xi)\, dx = \int_\Omega L_2(\nabla c_\tau)\cdot \xi\, dx + \int_\Gamma L_3(c_\tau \nu)\cdot \xi\, d\mathcal{H}^{N-1},$$

where $L_2 : \mathbb{R}^N \to \mathbb{R}^N$ and $L_3 : \mathbb{R}^N \to \mathbb{R}^N$ are linear functions. It follows that u_τ is a weak solution of the PDE

$$\begin{cases} \operatorname{div}[\mathbb{C}(e(u_\tau))] = L_2(\nabla c_\tau) & \text{in } \Omega, \\ \mathbb{C}(e(u_\tau))\cdot \nu = L_3(c_\tau \nu) & \text{on } \Gamma. \end{cases}$$

By regularity results for linearized elastostatic problems on C^2 domains [4] (see also [20]), we have

$$\|u_\tau\|_{\dot{H}^2(\Omega;\mathbb{R}^N)} \le C\left(\|L_2(\nabla c_\tau)\|_{L^2(\Omega;\mathbb{R}^N)} + \|L_3(c_\tau\nu)\|_{H^{1/2}(\Gamma;\mathbb{R}^N)}\right) \le C\|c_\tau\|_{H^1(\Omega)}, \tag{3.39}$$

which proves the claim by (3.18). With this, we may proceed as in Lemma 3.6 to conclude (3.21). With these estimates, we once again obtain the convergences provided by Lemma 3.8. Estimating $u_{\tau_k} - u_{\tau_m}$, for $k, m \in \mathbb{N}$, by a bound analogous to (3.39) and using (3.28), we see that u_τ is Cauchy in $L^{(2^\#-\delta)'}(0,T;\dot{H}^2(\Omega))$, and hence converges strongly in $L^{(2^\#-\delta)'}(0,T;\dot{H}^2(\Omega))$ to $u \in L^\infty(0,T;\dot{H}^2(\Omega))$. Thus $\operatorname{Tr}(e(u_\tau)) \to \operatorname{Tr}(e(u))$ in $L^{(2^\#-\delta)'}(0,T;L^{2^\#-\delta}(\Gamma))$.

From this, we may proceed as in Theorem 3.1 to pass to the limit in the "discrete" Euler-Lagrange Equations (3.38).

□

4 Strong solution

In this section, we prove strong existence of solutions to the CHR model (1.12) in dimensions $N = 2$ and 3. In this section we will depart from the variational perspective developed in Section 3 to prove existence of a weak solution. Although it it possible that a bootstrapping argument applied to the weak solution already recovered could lead to a strong solution, restrictions on the possible choices of f and R would still be governed by properties of composition; hence it is not clear we would obtain existence of strong solutions under more general hypotheses than in Theorem 1.5. Furthermore, we will see that the limitations highlighted by Remark 5.3 require us to apply more sophisticated methods than directly bootstrapping to obtain higher regularity. Consequently, we find it instrumental to develop these methods in the simpler context of strong solutions. Lastly, our proof of regularity in Section 5 will require smallness estimates which we will derive with the aid of function spaces developed in Section 2 (e.g. in Theorem 4.1).

We will make extensive use of Schaefer's fixed point theorem and interpolation theory. As such it will be useful to look at the CHR model (1.12) in the equivalent formulation:

$$\begin{cases} \partial_t c + \Delta^2 c = \Delta f'(c) & \text{in } \Omega_T, \\ \partial_\nu c = 0 & \text{on } \Sigma_T, \\ \partial_\nu(\Delta c) = \mathcal{R}(c, \Delta c) & \text{on } \Sigma_T, \\ c(0) = c_0 & \text{in } \Omega, \end{cases} \tag{4.1}$$

where we define

$$\begin{aligned} \mathcal{R}(s,w) &:= -R(s, -w + f'(s)), \\ \mu &:= -\Delta c + f'(c). \end{aligned} \tag{4.2}$$

In the language of anisotropic Sobolev spaces, we seek a solution of (4.1) belonging to $H^{4,1}(\Omega_T)$, so we need $\Delta f'(c) \in H^{0,0}(\Omega_T)$. As is well known, for Lipschitz f the composition map defined by $u \mapsto f(u)$ is linearly bounded from $H^1(\Omega)$ to $H^1(\Omega)$, i.e.

$$\|f(u)\|_{H^1(\Omega)} \le C_f(\|u\|_{H^1(\Omega)} + 1).$$

It is then natural to hope that the same would hold of composition operators from $H^2(\Omega)$ to $H^2(\Omega)$, but this is too much to ask. To illuminate this problem, consider

$$\Delta f'(c) = f'''(c)\|\nabla c\|^2 + f''(c)\Delta c.$$

Clearly the second term is linearly bounded by the $H^2(\Omega)$ norm of c, but the first term is quadratic, and it will be impossible to avoid this nonlinearity without modification. Hence for $\alpha > 0$, we can introduce a truncation function $\psi_\alpha \in C^\infty(\mathbb{R})$ such that $\psi_\alpha(x) = x$ for all $x \in (-\alpha, \alpha)$, $\|\psi'_\alpha\|_\infty \leq 2$, and $\psi'_\alpha = 0$ on $(-\alpha-1, \alpha+1)^C$. We define $\Psi_\alpha(x) := (\psi_\alpha(x_1), \ldots, \psi_\alpha(x_N))$. We then consider the *truncated Laplacian*:

$$(\Delta f')_\alpha(c) := f'''(c)\|\Psi_\alpha(\nabla c)\|^2 + f''(c)\psi_\alpha(\Delta c). \tag{4.3}$$

Although the preceeding discussion shows we don't need to truncate the second term, we do so as it will be useful in Section 5. In addition to analysis of (4.1), we will consider strong solutions of the *truncated CHR model*:

$$\begin{cases} \partial_t c + \Delta^2 c = (\Delta f')_\alpha(c) & \text{in } \Omega_T, \\ \partial_\nu c = 0 & \text{on } \Sigma_T, \\ \partial_\nu(\Delta c) = \mathcal{R}(c, \Delta c) & \text{on } \Sigma_T, \\ c(0) = c_0 & \text{in } \Omega, \end{cases} \tag{4.4}$$

where $\mathcal{R}$ is still defined as in (4.2), i.e., the "chemical potential" is unmodified on the boundary. The truncated CHR model (4.4) is relevant for two primary reasons.

- It allows us to circumnavigate analytical complications arising from composition, which will be especially helpful in proving higher regularity.
- If ∇c and $\nabla^2 c$ are continuous, for T sufficiently small and α well chosen, we will recover a solution to (4.1).

After proving the claims of existence given in the introduction (e.g. Theorem 1.5), we introduce an a priori estimate, which holds for any solution of the the aforementioned PDEs. These estimates will be essential to prove existence of a regular solution in Section 5. These estimates show that given initial data sufficiently small (in a specific sense), the solution maintains quantifiably small energy for short times.

Theorem 4.1. *Suppose $\Omega \subset \mathbb{R}^N$, where $N = 2$ or 3, is a bounded, open set with smooth boundary. Further suppose f and $\mathcal{R}$, defined by (4.2), satisfy assumptions (2.8) and (2.10). Let $c_0 \in H^2(\Omega)$ with $\partial_\nu c_0 = 0$ on Γ. Then any strong solution c of the truncated CHR model (4.4) satisfies the estimate*

$$\|c\|_{H^{4,1}(\Omega_T)} + \|\nabla^2 c\|_{L^\infty(0,T;L^2(\Omega))} \leq C\|\Delta c_0\|_{L^2(\Omega)} + \eta(c_0, T), \tag{4.5}$$

for some $C > 0$ with $\eta(c_0, T)$ tending to 0 as $T \to 0$. If additionally f and $\mathcal{R}$ satisfy (2.9) and (2.11), then any strong solution c of the CHR model (4.1) satisfies estimate (4.5).

4.1 Proofs of Theorem 1.5 and Corollary 1.6

We prove existence with the use of Schaefer's fixed point theorem. To control the bulk data as necessary we look to the analysis of Elliott and Songmu [28], wherein the Gagliardo-Nirenberg inequality (see Theorem 2.5) makes an appearance. The Gagliardo-Nirenberg inequality will provide us with the means to decompose nonlinear terms that arise from repeated differentiation into two pieces, typically one controlled in L^∞ and the other in L^2. Many of the methods applied herein will be applied once again in the slightly more technical proof of Theorem 5.2.

Proof of Theorem 1.5. We give the full proof in the more delicate case that $N = 3$ and indicate the changes in the case $N = 2$.
Step 1: Assume N = 3. We define the Banach space

$$\mathcal{B} := H^{3,3/4}(\Omega_T).$$

Define the operator

$$\begin{aligned} A :&\mathcal{B} \to \mathcal{B}, \\ &v \mapsto c \end{aligned}$$

by the PDE

$$\begin{cases} \partial_t c + \Delta^2 c = \Delta f'(v) & \text{in } \Omega_T, \\ \partial_\nu c = 0 & \text{on } \Sigma_T, \\ \partial_\nu(\Delta c) = \mathcal{R}(v, \Delta v) & \text{on } \Sigma_T, \\ c(0) = c_0 & \text{in } \Omega. \end{cases} \tag{4.6}$$

To prove existence of a strong solution to CHR model (4.1), we **claim** that $A : \mathcal{B} \to \mathcal{B}$ satisfies the hypotheses of Schaefer's fixed point theorem [29]. These hypotheses are characterized as

- Compactness: The functional $A : \mathcal{B} \to \mathcal{B}$ is compact.
- Continuity: The functional $A : \mathcal{B} \to \mathcal{B}$ is continuous.
- Bounded: The set $\{c \in \mathcal{B} : \lambda A[c] = c, \lambda \in (0, 1])\}$ is bounded in the norm of $\mathcal{B}$.

Supposing the claim, by Schaefer's fixed point theorem, we conclude there is $c \in \mathcal{B}$ such that $A[c] = c$, and by the argument for compactness, $c \in H^{4,1}(\Omega_T)$. Therefore c is a strong solution of the CHR model 4.1. Thus, it only remains to verify the claim.
Compactness: Given Theorem 2.14 and Remark 2.3, it follows

$$\begin{aligned} \|\mathcal{R}(v, \Delta v)\|_{H^{1/2,1/8}(\Sigma_T)} \leq& C(\Omega, T)\|\mathcal{R}(v, \Delta v)\|_{H^{1,1/4}(\Omega_T)} \\ \leq& C(\mathcal{R}, \Omega, T)(\|v\|_{H^{1,1/4}(\Omega_T)} + \|\nabla^2 v\|_{H^{1,1/4}(\Omega_T)} + 1) \\ \leq& C(\mathcal{R}, \Omega, T)(\|v\|_{H^{3,3/4}(\Omega_T)} + 1), \end{aligned} \tag{4.7}$$

where in the third inequality we have made use of Proposition 2.15 with $k = 3$. We have

$$\Delta f'(v) = f'''(v)\|\nabla v\|^2 + f''(v)\Delta v. \tag{4.8}$$

Using the Gagliardo-Nirenberg inequality (see Theorem 2.5) to control the quadratic term of (4.8), we have

$$\begin{aligned} \|\nabla v\|_{L^4(\Omega)} \leq& C_1\|\nabla^3 v\|^a_{L^2(\Omega)}\|\nabla v\|^{1-a}_{L^2(\Omega)} + C_2\|\nabla v\|_{L^2(\Omega)}, \\ \frac{1}{4} =& a\left(\frac{1}{2} - \frac{2}{3}\right) + (1-a)\frac{1}{2} \implies a = 3/8. \end{aligned} \tag{4.9}$$

As $a < 1/2$, it follows from (2.8) and (4.8) that

$$\begin{aligned} \|\Delta f'(v)\|_{L^2(\Omega)} \leq& C(f)\left(\|\nabla^3 v\|^{2a}_{L^r(\Omega)}\|\nabla v\|^{2(1-a)}_{L^2(\Omega)} + \|\nabla v\|^2_{L^2(\Omega)} + \|\nabla^2 v\|_{L^2(\Omega)}\right) \\ \leq& C\left(\|\nabla^3 v\|_{L^r(\Omega)}\|v\|^{2(1-a)}_{L^\infty(0,T;H^1(\Omega))} + \|v\|^{2(1-a)+1}_{L^\infty(0,T;H^1(\Omega))} + \|\nabla^2 v\|_{L^2(\Omega)} + 1\right). \end{aligned}$$

We integrate in time and use that $\mathcal{B} = H^{3,3/4}(\Omega_T) \hookrightarrow L^\infty(0, T; H^1(\Omega))$ (see Theorem 3.1 of Chapter 1 in [45]) to find

$$\begin{aligned} &\|\Delta f'(v)\|_{H^{0,0}(\Omega_T)} \\ &\leq C(T, f)\left(\|v\|^{2(1-a)}_{L^\infty(0,T;H^1(\Omega))}\|\nabla^3 v\|_{L^2(0,T;L^2(\Omega))} + \|v\|^{2(1-a)+1}_{L^\infty(0,T;H^1(\Omega))} + \|\nabla^2 v\|_{H^{0,0}(\Omega_T)} + 1\right) \\ &\leq C\left(\|v\|^{2(1-a)+1}_{\mathcal{B}} + 1\right), \end{aligned} \tag{4.10}$$

where in the last inequality we used the fact that $2(1-a)+1 = 11/4 > 2$.

By (4.6), (4.7), (4.10), and Theorem 6.7 (with $k = 0$), we have

$$\|c\|_{H^{4,1}(\Omega_T)} \le C(c_0)\left(\|v\|_{\mathcal{B}}^{2(1-a)+1} + 1\right). \tag{4.11}$$

As $H^{4,1}(\Omega_T) \hookrightarrow\hookrightarrow \mathcal{B}$ by standard interpolation results (see [44]), bound (4.11) proves compactness of the operator A.

Continuity: This follows from the generalized Lebesgue dominated convergence theorem and the estimates derived in showing compactness (see the proof of Theorem 5.2 for details in an analogous case).

Boundedness: Suppose that $c \in \mathcal{B}$ satisfies $\lambda A[c] = c$; without loss of generality, we assume that $\lambda = 1$. As c is a fixed point of A, $c \in H^{4,1}(\Omega_T)$ by (4.11) and satisfies (4.1). Arguing via mollification, it is straightforward to show that $\|\nabla c\|^2_{L^2(\Omega)}$ is absolutely continuous in time (for a related perspective see also pg. 330 of [12]). Consequently, the integral $I[c(t)]$ (see (1.7)) is absolutely continuous as a function of time and may therefore be differentiated. Making use of integration by parts and the embedding $H^{4,1}(\Omega_T) \hookrightarrow BUC(0,T;H^2(\Omega))$ [45], for t-a.e. we differentiate the gradient term of the integrand as follows:

$$\begin{aligned}
\partial_t\left(\frac{1}{2}\int_\Omega \|\nabla c\|^2\,dx\right) &= \lim_{\delta\to 0}\int_\Omega \frac{\|\nabla c(t+\delta)\|^2 - \|\nabla c(t)\|^2}{2\delta}\,dx \\
&= \lim_{\delta\to 0}\int_\Omega \frac{(\nabla c(t+\delta),\nabla c(t+\delta)) - (\nabla c(t),\nabla c(t))}{2\delta}\,dx \\
&= \lim_{\delta\to 0}\int_\Omega \frac{(\nabla c(t+\delta)-\nabla c(t),\nabla c(t+\delta)) + (\nabla c(t),\nabla c(t+\delta)-\nabla c(t))}{2\delta}\,dx \\
&= -\lim_{\delta\to 0}\int_\Omega \left(\frac{\Delta c(t+\delta)+\Delta c(t)}{2}\right)\left(\frac{c(t+\delta)-c(t)}{\delta}\right)dx \\
&= -\int_\Omega \Delta c\,\partial_t c\,dx
\end{aligned}$$

Then, we compute the derivative

$$\begin{aligned}
\partial_t I[c](t) &= \int_\Omega \left(f'(c)\partial_t c - \Delta c\,\partial_t c\right)\,dx \\
&= \int_\Omega \left(f'(c)(-\Delta^2 c + \Delta f'(c)) - \Delta c(-\Delta^2 c + \Delta f'(c))\right)\,dx \\
&= -\int_\Omega \left(\|\nabla f'(c)\|^2 - 2\nabla f'(c)\cdot\nabla(\Delta c) + \|\nabla(\Delta c)\|^2\right)dx - \int_{\partial\Omega}\partial_\nu(\Delta c)\mu\,d\mathcal{H}^2 \\
&\le -\int_\Omega \|\nabla\mu\|^2\,dx + C(\mathcal{R})\mathcal{H}^2(\partial\Omega),
\end{aligned}$$

where we have used (2.11) and (4.6). It follows that

$$I[c(t)] \le I[c_0] + TC(\mathcal{R})\mathcal{H}^2(\partial\Omega) \tag{4.12}$$

for all $t \in (0,T)$. By the coercivity of f in (2.9), the Poincaré inequality implies

$$\|c\|_{L^\infty(0,T;H^1(\Omega))} \le C(T,R,\delta)(I[c_0]+1) = C(T,R,\delta,c_0).$$

We use the above estimate and the first inequality of (4.10) to conclude

$$\|\Delta f'(c)\|_{H^{0,0}(\Omega_T)} \le C(T,f,R,\delta,c_0)\|c\|_{H^{3,0}(\Omega_T)} + C(T,f,R,\delta,c_0). \tag{4.13}$$

Applying Theorem 6.7 (with $k = 0$) to (4.6) as before, we control c in terms of its data:

$$\|c\|_{H^{4,1}(\Omega_T)} \leq C(\|\mathcal{R}(c, \Delta c)\|_{H^{1/2,1/8}(\Sigma_T)} + \|\Delta f'(c)\|_{H^{0,0}(\Omega_T)}).$$

Using (4.7) and (4.13), we have

$$\|c\|_{H^{4,1}(\Omega_T)} \leq C(R, f)(\|c\|_{H^{3,3/4}(\Omega_T)} + 1) + C(T, f, R, \delta, c_0).$$

The interpolation inequality $\|c\|_{H^{3,3/4}(\Omega_T)} \leq C\|c\|^{1/4}_{H^{0,0}(\Omega_T)}\|c\|^{3/4}_{H^{4,1}(\Omega_T)}$ (see [45]) and the particular Young's inequality $a^{1/4}b^{3/4} \leq \epsilon b + C(\epsilon)a$ for $\epsilon > 0$ then show that for ϵ sufficiently small

$$\|c\|_{H^{4,1}(\Omega_T)} \leq C(T, f, R, \delta, c_0, \epsilon, \Omega).$$

Step 2: Assume $\mathbf{N = 2}$. The only dimension dependent inequality arising in the above argument was the Gagliardo-Nirenberg inequality. In inequality (4.9), a is now $a = 1/4$. This is of course sufficient to repeat the above proof.

□

Proof of Corollary 1.6. Given $c_0 \in H^2(\Omega)$, we have $c_0 \in C^{0,\alpha}(\Omega)$ by the Morrey embedding theorem [44]. Choose $\tilde{f}$ which satisfies hypotheses (2.8) and (2.9) and $\tilde{f}|_{\mathcal{O}} = f$, where $\mathcal{O} = (-\|c_0\|_{C^{0,\alpha}} - 1, \|c_0\|_{C^{0,\alpha}} + 1)$. Applying Theorem 1.5, there is $c \in H^{4,1}(\Omega_1)$ a solution of the CHR model 4.1 with $\tilde{f}$. As $H^{4,1}(\Omega_1) \hookrightarrow BUC(0, 1; H^2(\Omega)) \hookrightarrow BUC(0, 1; C^{0,\alpha}(\Omega))$ [45], for sufficiently small $T > 0$, $c|_{\Omega_T}$ is a solution of the CHR model (1.12) with f. □

4.2 Proof of Theorem 4.1

We emulate the above proof of *boundedness* keeping estimates of smallness to show that a solution with small data and short time stays small in energy. Two challenges occur which make the proof of the following more involved:

- We know that $\|v\|_{L^\infty(0,T;H^2(\Omega))} \leq C(?)\|v\|_{H^{4,1}(\Omega_T)}$ from the trace theory detailed by Lions and Magenes (see [45], [46]). Necessarily though, $C(?)$ depends on T, blowing up as $T \to 0$ (consider a function constant in time).
- Returning to the notation of subsection 2.5, there is insufficient literature detailing the constants by which $\|\cdot\|_{H^s(0,T),I}$ is equivalent to $\|\cdot\|_{H^s(0,T)}$, where the later norm is given by the integral of the derivative or difference quotients. Existing results of which the authors are aware address this relation with the use of extensions (see, e.g., [18]).

As we will send $T \to 0$, i.e., shrink the size of our domain, these constants are critical. To navigate this problem, we use a variety of estimates developed in Subsection 2.4, which work directly with the Gagliardo semi-norm for fractional Sobolev spaces.

Proof of Theorem 4.1. Assuming f and $\mathcal{R}$ (see (4.2)) satisfy hypotheses (2.9) and (2.11), we prove the theorem for a solution of the CHR model in dimension $N = 3$. The proof in dimension $N = 2$ follows as in Theorem 1.5. The result for the truncated CHR model follows from a simplified version of the following argument.

Let c be a strong solution of the CHR model (4.1) on Ω_T for $0 < T \leq 1$. For convenience, we will define $\mathcal{R}_c := \mathcal{R}(c, \Delta c)$. Note that $\mathcal{R}_c \in H^{1/2,1/8}(\Sigma_T)$ (see Theorem 2.14) and by Corollary 2.8 may be extended to $\tilde{\mathcal{R}}_c \in H^{1/2,1/8}(\Sigma_1)$ satisfying the bound

$$\|\tilde{\mathcal{R}}_c\|_{H^{1/2,1/8}(\Sigma_1)} \leq C\Big((1 + T^{-1/8})\|\mathcal{R}_c\|_{H^{0,0}(\Sigma_T)} + \|\mathcal{R}_c\|_{H^{1/2,1/8}(\Sigma_T)}\Big).$$

Let F be the extension by 0 of $\Delta f'(c) \in H^{0,0}(\Omega_T)$ to $H^{0,0}(\Omega_1)$.

We consider the PDE for $\bar{c}$ on the extended domain Ω_1 :

$$\begin{cases} \partial_t \bar{c} + \Delta^2 \bar{c} = F & \text{in } \Omega_1, \\ \partial_\nu \bar{c} = 0 & \text{on } \Sigma_1, \\ \partial_\nu (\Delta \bar{c}) = \tilde{\mathcal{R}}_c & \text{on } \Sigma_1, \\ \bar{c}(0) = c_0 & \text{in } \Omega. \end{cases} \tag{4.14}$$

If problem (4.14) admits a solution $\bar{c} \in H^{4,1}(\Omega)$, then as (4.14) coincides with the CHR model (4.1) on Ω_T, by uniqueness (see Theorem 6.7), $\bar{c}|_{\Omega_T} = c$. We may apply Theorem 6.7 (with $k = 0$) to conclude that (4.14) admits a unique solution $\bar{c}$ satisfying the slightly modified bound:

$$\begin{aligned} &\|\bar{c}\|_{H^{4,1}(\Omega_1)} + \|\nabla^2 \bar{c}\|_{L^\infty(0,1;L^2(\Omega))} \\ &\quad \leq C\Big(\|\Delta c_0\|_{L^2(\Omega)} + \|F\|_{H^{0,0}(\Omega_1)} + \|\tilde{\mathcal{R}}_c\|_{H^{1/2,1/8}(\Sigma_1)} \Big) \\ &\quad \leq C\Big(\|\Delta c_0\|_{L^2(\Omega)} + \|\Delta f'(c)\|_{H^{0,0}(\Omega_T)} + (1 + T^{-1/8}) \|\mathcal{R}_c\|_{H^{0,0}(\Sigma_T)} + \|\mathcal{R}_c\|_{H^{1/2,1/8}(\Sigma_T)} \Big) \\ &\quad =: C(\|\Delta c_0\|_{L^2(\Omega)} + A_1 + A_2 + A_3). \end{aligned} \tag{4.15}$$

We note that C in the above estimate is independent of T, and the extension Ω_T to Ω_1 was specifically done to control dependence of constants on T in the above expression. We now estimate each term in the above expression.

Term A_1: Recalling the proof of Theorem 1.5 (see (4.12)), we have

$$I[c(t)] \leq I[c_0] + TC\mathcal{H}^2(\partial\Omega) =: \bar{\eta}^2(c_0, T)$$

for all $t \in (0, T)$, implying

$$\|\nabla c\|_{L^\infty(0,T;L^2(\Omega))} \leq \bar{\eta}(c_0, T). \tag{4.16}$$

We have

$$\Delta f'(c) = f'''(c)\|\nabla c\|^2 + f''(c)\Delta c. \tag{4.17}$$

Making use of the Gagliardo-Nirenberg inequality (Theorem 2.5), we control the quadratic term of the previous function:

$$\|\nabla c\|_{L^4(\Omega)} \leq C(\Omega) \|\nabla c\|_{H^3(\Omega)}^{1/4} \|\nabla c\|_{L^2(\Omega)}^{3/4}.$$

Consequently for $\epsilon > 0$, (4.17) is bounded in L^2 as

$$\begin{aligned} \|\Delta f'(c)\|_{L^2(\Omega)} &\leq C(f)(\|\nabla c\|_{L^4(\Omega)}^2 + \|\Delta c\|_{L^2(\Omega)}) \\ &\leq C(f, \Omega)(\bar{\eta}^{3/2} \|\nabla c\|_{H^3(\Omega)}^{1/2} + \|\Delta c\|_{L^2(\Omega)}) \\ &\leq C(f, \Omega) \left(\epsilon \|\nabla c\|_{H^3(\Omega)} + \frac{1}{\epsilon} \bar{\eta}^3 + \|\Delta c\|_{L^2(\Omega)} \right), \end{aligned}$$

where we used (2.8), (4.16), and the previous inequality. Taking the L^2 norm over $(0, T)$, we find

$$\|\Delta f'(c)\|_{H^{0,0}(\Omega_T)} \leq C(f, \Omega) \left(\epsilon \|\nabla c\|_{H^{3,0}(\Omega_T)} + \frac{1}{\epsilon} \sqrt{T} \bar{\eta}^3 + \sqrt{T} \|\Delta c\|_{L^\infty(0,T;L^2(\Omega))} \right). \tag{4.18}$$

Term A_2: We note that $H^3(\Omega) = [H^2(\Omega), H^4(\Omega)]_{1/2}$, so $\|\cdot\|_{H^3(\Omega)} \leq C(\Omega) \|\cdot\|_{H^2(\Omega)}^{1/2} \|\cdot\|_{H^4(\Omega)}^{1/2}$ for some constant $C(\Omega) > 0$ (see [45]). It follows by Hölder's inequality that

$$\begin{aligned} \|c\|_{H^{3,0}(\Omega_T)}^2 &\leq C(\Omega) \int_0^T \|c(t)\|_{H^2(\Omega)} \|c(t)\|_{H^4(\Omega)} \, dt \\ &\leq C(\Omega) \|c\|_{H^{2,0}(\Omega_T)} \|c\|_{H^{4,0}(\Omega_T)} \\ &\leq C(\Omega) \sqrt{T} \|c\|_{L^\infty(0,T;H^2(\Omega))} \|c\|_{H^{4,0}(\Omega_T)} \\ &\leq C(\Omega) \sqrt{T} \Big(\|c\|_{L^\infty(0,T;H^2(\Omega))}^2 + \|c\|_{H^{4,0}(\Omega_T)}^2 \Big). \end{aligned} \tag{4.19}$$

Using this computation, Remark 2.3, and the continuity of the trace in $H^1(\Omega)$, we estimate Term A_2 as

$$\begin{aligned}(1+T^{-1/8})\|\mathcal{R}_c\|_{H^{0,0}(\Sigma_T)} \leq& C(\Omega)(1+T^{-1/8})\|\mathcal{R}_c\|_{H^{1,0}(\Omega_T)}\\ \leq& C(\mathcal{R},\Omega)(1+T^{-1/8})\Big(\|c\|_{H^{3,0}(\Omega_T)}+|\mathcal{R}(0,0)|\sqrt{T}\Big)\\ \leq& C(\mathcal{R},\Omega)T^{1/8}\Big(\|c\|_{L^\infty(0,T;H^2(\Omega))}+\|c\|_{H^{4,0}(\Omega_T)}+T^{1/4}\Big).\end{aligned} \tag{4.20}$$

Term A_3: Since $\|\cdot\|_{H^{1/2}(\Gamma)} \leq C\|\cdot\|_{H^1(\Omega)}$, it follows $\|\mathcal{R}_c\|_{H^{1/2,0}(\Sigma_T)}$ can be estimated by the same method as term A_2, so we restrict our attention to the semi-norm $|\mathcal{R}_c|_{H^{0,1/8}(\Sigma_T)}$.

Setting $(\mathcal{R}_c)_T(x,t) := \mathcal{R}_c(x,Tt)$ and $c_T(x,t) := c(x,Tt)$, using Lemma 2.10, Remark 2.3, Theorem 2.14, and Proposition 2.15, it follows that

$$\begin{aligned}|\mathcal{R}_c|_{H^{0,1/8}(\Sigma_T)} =& T^{3/8}|(\mathcal{R}_c)_T|_{H^{0,1/8}(\Sigma_1)}\\ =& T^{3/8}|\mathcal{R}_{(c_T)}|_{H^{0,1/8}(\Sigma_1)}\\ \leq& C(\mathcal{R})T^{3/8}\left(|c_T|_{H^{0,1/8}(\Sigma_1)}+|\Delta c_T|_{H^{0,1/8}(\Sigma_1)}\right)\\ \leq& CT^{3/8}\|c_T\|_{H^{3,3/4}(\Omega_1)}\\ \leq& CT^{3/8}(\|c_T\|_{H^{3,0}(\Omega_1)}+|c_T|_{H^{0,3/4}(\Omega_1)}).\end{aligned}$$

Using a direct change of variables, we have that $\|c_T\|_{H^{3,0}(\Omega_1)} = T^{-1/2}\|c\|_{H^{3,0}(\Omega_T)}$. By Proposition 2.9 and a change of variables, we have

$$|c_T|_{H^{0,3/4}(\Omega_1)} \leq C\|\partial_t(c_T)\|_{H^{0,0}(\Omega_1)} = CT\|(\partial_t c)_T\|_{H^{0,0}(\Omega_1)} = CT^{1/2}\|\partial_t c\|_{H^{0,0}(\Omega_T)}.$$

Consolidating these estimates along with (4.19), we find

$$|\mathcal{R}_c|_{H^{0,1/8}(\Sigma_T)} \leq C(\mathcal{R},\Omega)T^{1/8}\left(\|c\|_{H^{4,1}(\Omega_T)}+\|c\|_{L^\infty(0,T;H^2(\Omega))}\right). \tag{4.21}$$

Returning to (4.15) and combining the bounds (4.18), (4.20), and (4.21), we find

$$\begin{aligned}&\|c\|_{H^{4,1}(\Omega_T)}+\|\nabla^2 c\|_{L^\infty(0,T;L^2(\Omega))}\\ &\leq \|\bar{c}\|_{H^{4,1}(\Omega_1)}+\|\nabla^2\bar{c}\|_{L^\infty(0,1;L^2(\Omega))}\\ &\leq C\|\Delta c_0\|_{L^2(\Omega)}+C(f,\Omega)\left(\epsilon\|\nabla c\|_{H^{3,0}(\Omega_T)}+\frac{1}{\epsilon}\sqrt{T}\bar{\eta}^3+\sqrt{T}\|\Delta c\|_{L^\infty(0,T;L^2(\Omega))}\right)\\ &\quad + C(\mathcal{R},\Omega)T^{1/8}\left(\|c\|_{H^{4,1}(\Omega_T)}+\|c\|_{L^\infty(0,T;H^2(\Omega))}+T^{1/4}\right).\end{aligned}$$

By the coercivity of f (2.9), the definition of $\bar{\eta}$ (see above (4.16)), and the Poincaré inequality, we have

$$\|c\|_{H^2(\Omega)} \leq C(\Omega)\left(\bar{\eta}(c_0,T)+\frac{1}{\delta}\right)+\|\nabla^2 c\|_{L^2(\Omega)},$$

and returning to the above estimate, we have

$$\begin{aligned}&\|c\|_{H^{4,1}(\Omega_T)}+\|\nabla^2 c\|_{L^\infty(0,T;L^2(\Omega))}\\ &\leq C\|\Delta c_0\|_{L^2(\Omega)}+C(f,\Omega)\left(\epsilon\|\nabla c\|_{H^{3,0}(\Omega_T)}+\frac{1}{\epsilon}\sqrt{T}\bar{\eta}^3+\sqrt{T}\|\Delta c\|_{L^\infty(0,T;L^2(\Omega))}\right)\\ &\quad + C(\mathcal{R},\Omega)T^{1/8}\left(\bar{\eta}+\frac{1}{\delta}+\|\nabla^2 c\|_{L^\infty(0,T;L^2(\Omega))}+\|c\|_{H^{4,1}(\Omega_T)}+T^{1/4}\right)\\ &\leq C\|\nabla^2 c_0\|_{L^2(\Omega)}+C(f,\mathcal{R},\Omega)\left(\epsilon+T^{1/8}\right)\left(\|c\|_{H^{4,1}(\Omega_T)}+\|\nabla^2 c\|_{L^\infty(0,T;L^2(\Omega))}\right)\\ &\quad + C(\mathcal{R},\Omega)T^{1/8}\left(\bar{\eta}+\frac{1}{\delta}+T^{1/4}\right)+\frac{C(f,\Omega)}{\epsilon}\sqrt{T}\bar{\eta}^3.\end{aligned}$$

Choosing $0 < \epsilon < 1/(4C(f,\mathcal{R},\Omega))$, we have for all $0 < T < (1/(4C(f,\mathcal{R},\Omega)))^8$, that

$$\begin{aligned}
&\|c\|_{H^{4,1}(\Omega_T)} + \|\nabla^2 c\|_{L^\infty(0,T;L^2(\Omega))} \\
\leq &C\|\Delta c_0\|_{L^2(\Omega)} + C(\mathcal{R},\Omega)T^{1/8}\left(\bar{\eta} + \frac{1}{\delta} + T^{1/4}\right) + \frac{C(f,\Omega)}{\epsilon}\sqrt{T}\bar{\eta}^3 \\
=:&C\|\Delta c_0\|_{L^2(\Omega)} + \eta(c_0,T).
\end{aligned}$$

□

5 Regularity for $N = 2$

This section is devoted to proving sufficient regularity of a solution to the truncated CHR model (4.4) such that we recover a solution of the CHR model (1.12) (see also (4.1)) with R given by (1.6) and f given by (1.3). We will prove inclusion of a solution of the truncated CHR model in a higher order anisotropic Sobolev space, at which point we will make use of embedding theorems to recover continuity of the second derivatives in space and time.

We refine the **assumptions** previously used to prove strong existence:

- We assume that the chemical energy density is governed by
$$f \in C^{4,1}(\mathbb{R}). \tag{5.1}$$
- For the reaction rate, we assume
$$R \in C^{2,1}(\mathbb{R}^2). \tag{5.2}$$

Remark 5.1. *For R and f satisfying (5.2) and (5.1), respectively, and recalling (4.2), the chain rule provides the bound*

$$\|\nabla\mathcal{R}\|_{C^{1,1}(\mathbb{R}^2;\mathbb{R}^2)} \leq C.$$

The proof of the existence as claimed in Theorem 1.8 proceeds in two primary steps. First, a fixed point argument analogous to Theorem 1.5 is applied to obtain existence of a solution to the truncated CHR model (4.4) belonging to $H^{5,1+1/4}(\Omega_T)$ for sufficiently small $T > 0$. Looking to Remark 5.3, we see that the use of another fixed point argument is driven by necessity–versus being able to directly bootstrap from a strong solution to higher regularity. The argument will make sharp use of the growth given by the exponents of the Gagliardo-Nirenberg inequality (Theorem 2.5), and hence critically relies on the a priori "smallness" estimate provided by Theorem 4.1. Second, we directly bootstrap to show that given appropriate initial conditions, a solution of the truncated CHR model (4.4) belongs to $H^{6,1+1/2}(\Omega_T)$, at which point we may directly apply embedding theorems to conclude existence of the desired solution to the CHR model (4.1). Given that our argument places restrictions on the class of admissible initial conditions, we last show that this class of functions is non-empty.

Theorem 5.2. *Suppose $\Omega \subset \mathbb{R}^2$ is an open bounded set with smooth boundary, and f and $\mathcal{R}$ (see (4.2)) satisfy assumptions (5.1) and (5.2). There is $\lambda(\mathcal{R},\Omega) > 0$ such that if $c_0 \in H^4(\Omega)$ such that $\partial_\nu c_0 = 0$ and $\|\nabla^2 c_0\|_{L^2(\Omega)} \leq \lambda$, then there is $T > 0$ such that a solution of the truncated CHR model (4.4) exists on the interval Ω_T satisfying the estimate*

$$\|c\|_{H^{5,1+1/4}(\Omega)} \leq C(f,c_0,\mathcal{R},\Omega,T).$$

Proof. We apply Schaefer's fixed point theorem [29] to obtain existence. Define the operator $A : v \mapsto c$ by the PDE

$$\begin{cases} \partial_t c + \Delta^2 c = (\Delta f')_\alpha(v) & \text{in } \Omega_T, \\ \partial_\nu c = 0 & \text{on } \Sigma_T, \\ \partial_\nu(\Delta c) = \mathcal{R}(v, \Delta v) & \text{on } \Sigma_T, \\ c(0) = c_0 & \text{in } \Omega. \end{cases} \tag{5.3}$$

We choose the domain of A such that A is both compact and $\mathrm{range(A)} \subset \mathrm{H}^{5,1+1/4}(\Omega_\mathrm{T})$. Define the Banach space

$$\mathcal{B} := H^{4,1}(\Omega_T) \cap L^2(0,T;W^{4,r}(\Omega)) \cap L^\infty(0,T;W^{2,r}(\Omega))$$

equipped with the sum of norms, for r yet to be determined. We **claim** $H^{5,1+1/4}(\Omega_T) \hookrightarrow\hookrightarrow \mathcal{B}$. As

$$H^{5,1+1/4}(\Omega_T) \hookrightarrow\hookrightarrow [H^{0,0}(\Omega_T), H^{5,1+1/4}(\Omega_T)]_\theta$$

for $\theta \in (0,1)$ (see, e.g., Exercise 16.26 of [44] and [46]), it suffices to show that

$$[H^{0,0}(\Omega_T), H^{5,1+1/4}(\Omega_T)]_\theta = H^{\theta 5, \theta(5/4)}(\Omega_T) \hookrightarrow \mathcal{B} \tag{5.4}$$

for some $\theta \in (0,1)$. Note, $H^{\theta 5,\theta(5/4)}(\Omega_T) \hookrightarrow H^{4,1}(\Omega_T)$ for θ sufficiently close to 1. Using a Besov embedding theorem (see Theorem 17.51 of [44]), we have $H^{\theta 5}(\Omega) \hookrightarrow W^{4,r}(\Omega)$ for all $r < \frac{2}{(1-\theta)5}$. It immediately follows that $H^{\theta 5,\theta(5/4)}(\Omega_T) \hookrightarrow L^2(0,T;W^{4,r}(\Omega))$. To conclude the last embedding necessary to prove (5.4) we make use of a trace theorem (see Theorem 3.1 of [45]), which shows that $c \in H^{\theta 5,\theta(5/4)}(\Omega_T)$ is continuously embedded in the space of $BUC(0,T;[H^{\theta 5}(\Omega), L^2(\Omega)]_{2/(\theta 5)})$. As $[H^{\theta 5}(\Omega), L^2(\Omega)]_{2/(\theta 5)} = H^{\theta 5-2}(\Omega)$ by Proposition 2.13, we may once again make use of a Besov embedding theorem [44] to conclude for any $r \geq 2$, there is θ sufficiently close to 1 such that $H^{\theta 5,\theta(5/4)}(\Omega_T) \hookrightarrow L^\infty(0,T;W^{2,r}(\Omega))$. This concludes the claim.

We now prove that the hypotheses of Schaefer's fixed point theorem are satisfied by the operator $A : \mathcal{B} \to \mathcal{B}$ for appropriate initial data c_0; these are summarily referred to as Compactness, Continuity, and Boundedness (see the proof of Theorem 1.5). This will complete the proof.

Compactness: We estimate $\mathcal{R}(v, \Delta v)$ and $(\Delta f')_\alpha(v)$ in the norms for $H^{3/2,3/8}(\Omega_T)$ and $H^{1,1/4}(\Omega_T)$ respectively. Hence we define $\beta_v := \mathcal{R}(v, \Delta v)$. By continuity of the trace map (see Theorem 2.14) and Remarks 2.3 and 5.1, we have

$$\begin{aligned} \|\beta_v\|_{H^{3/2,3/8}(\Sigma_T)} &\leq C(\Omega,T)\|\beta_v\|_{H^{2,1/2}(\Omega_T)} \\ &\leq C(\mathcal{R},\Omega,T)(\|\nabla^2\beta_v\|_{H^{0,0}(\Omega_T)} + \|v\|_{H^{4,1}(\Omega_T)} + 1), \end{aligned} \tag{5.5}$$

For simplicity, we show how to control the higher order spatial derivatives of β_v by looking at two derivatives in the same direction (mixed derivatives are similar):

$$\begin{aligned} \partial_i^2 \beta_v =& (\partial_s^2 \mathcal{R}(v,\Delta v)(\partial_i v)^2 + 2(\partial_s\partial_w \mathcal{R})(v,\Delta v)\partial_i v \partial_i \Delta v + (\partial_s \mathcal{R})(v,\Delta v)\partial_i^2 v \\ &+ (\partial_w^2 \mathcal{R})(v,\Delta v)(\partial_i \Delta v)^2 + (\partial_w \mathcal{R})(v,\Delta v)\partial_i^2 \Delta v. \end{aligned} \tag{5.6}$$

Given Remark 5.1, the terms in (5.6) are controlled in $H^{0,0}(\Omega_T)$ by the norm of v in $H^{4,1}(\Omega_T)$ plus $C(\mathcal{R})\|(\partial_i \Delta v)^2\|_{H^{0,0}(\Omega_T)}$. To control this term, we make use of the Gagliardo-Nirenberg inequality (Theorem 2.5):

$$\|\partial_i \Delta v\|_{L^4(\Omega)} \leq C(\|\nabla^2 \Delta v\|^a_{L^r(\Omega)} \|\Delta v\|^{1-a}_{L^r(\Omega)} + \|\Delta v\|_{L^r(\Omega)}), \tag{5.7}$$

where $a = 1/4 + 1/r$. Consequently, choosing $r \geq 4$, we have

$$\begin{aligned}
\|(\partial_i \Delta v)^2\|_{H^{0,0}(\Omega_T)} =& \left(\int_0^T \|\partial_i \Delta v\|_{L^4(\Omega)}^4 \, dt\right)^{1/2} \\
\leq& C\|\Delta v\|_{L^\infty(0,T;L^r(\Omega))}^{2(1-a)} \left(\int_0^T \|\nabla^2 \Delta v\|_{L^r(\Omega)}^{4a} \, dt\right)^{1/2} + C\|\Delta v\|_{L^\infty(0,T;L^r(\Omega))}^2 \\
\leq& C\|\Delta v\|_{L^\infty(0,T;L^r(\Omega))}^{2(1-a)} \left(\int_0^T \|\nabla^2 \Delta v\|_{L^r(\Omega)}^{2} \, dt\right)^{1/2} + C\|\Delta v\|_{L^\infty(0,T;L^r(\Omega))}^{2(1-a)+1} + C \\
\leq& C\|v\|_{\mathcal{B}}^{2(1-a)+1} + C,
\end{aligned} \tag{5.8}$$

where we used the inequality $s^2 \leq C(s^{2(1-a)+1} + 1)$ since $a \leq 1/2$. Thus, using the definition of the space $\mathcal{B}$, (5.5), and (5.8), we have

$$\|\beta_v\|_{H^{3/2,3/8}(\Sigma_T)} \leq C\|v\|_{\mathcal{B}}^{2(1-a)+1} + C.$$

Due to the truncated Laplacian (4.3), estimation of the bulk term $(\Delta f')_\alpha(v)$ in $H^{0,0}(\Omega_T)$ is straightforward. By Theorem 6.7 (with $k = 1$), we conclude

$$\|c\|_{H^{5,1+1/4}(\Omega_T)} \leq C\|v\|_{\mathcal{B}}^{2(1-a)+1} + C,$$

which implies $A : \mathcal{B} \to \mathcal{B}$ is compact by the claim regarding (5.4).

Continuity: Suppose $v_n \to v$ in $\mathcal{B}$. To show that $A[v_n] \to A[v]$, by Theorem 6.7 (with $k = 1$) and the claim preceding (5.4), it is sufficient to show that the data converges as follows:

$$\begin{aligned}
(\Delta f')_\alpha(v_n) &\to (\Delta f')_\alpha(v) && \text{in } H^{1,1/4}(\Omega_T), \\
\beta_{v_n} := \mathcal{R}(v_n, \Delta v_n) &\to \mathcal{R}(v, \Delta v) = \beta_v && \text{in } H^{2,1/2}(\Omega_T),
\end{aligned}$$

where we have used Theorem 2.14 to reduce our consideration to convergence on Ω_T versus Σ_T. We focus our attention on the second convergence, the first being similar.

Up to a subsequence, we may assume $\nabla^k v_n \to \nabla^k v$ a.e. in Ω_T for $k \in \{0, \ldots, 4\}$. To see that $\beta_{v_n} \to \beta_v$ in $H^{0,0}(\Omega_T)$, recall Remark 2.3 to find

$$\begin{aligned}
\|\beta_{v_n} - \beta_v\|_{H^{0,0}(\Omega_T)}^2 =& \int_0^T \int_\Omega |\mathcal{R}(v_n, \Delta v_n) - \mathcal{R}(v, \Delta v)|^2 \, dx \, dt \\
\leq& C(\mathcal{R}) \int_0^T \int_\Omega \left(|v_n - v|^2 + |\Delta v_n - \Delta v|^2\right) \, dx \, dt.
\end{aligned}$$

Thus by the convergence of v_n in $\mathcal{B}$, we directly have convergence in $H^{0,0}(\Omega_T)$. To prove convergence in $H^{0,1/2}(\Omega_T)$ we argue using Remark 2.3 and the Galiardo type semi-norm (see (2.24)):

$$\begin{aligned}
&|\beta_{v_n} - \beta_v|_{H^{0,1/2}(\Omega_T)}^2 \\
=& \int_\Omega \int_0^T \int_0^T \frac{|\mathcal{R}(v_n, \Delta v_n)(t) - \mathcal{R}(v, \Delta v)(s)|^2}{|t-s|^2} \, dt \, ds \, dx \\
\leq& C(\mathcal{R}) \int_\Omega \int_0^T \int_0^T \frac{|v_n(t) - v(s)|^2}{|t-s|^2} + \frac{|\Delta v_n(t) - \Delta v(s)|^2}{|t-s|^2} \, dt \, ds \, dx \\
=& C(\mathcal{R}) \left(|v_n - v|_{H^{0,1/2}(\Omega_T)}^2 + |\Delta v_n - \Delta v|_{H^{0,1/2}(\Omega_T)}^2\right).
\end{aligned}$$

As $(v_n, \Delta v_i) \to (v, \Delta v)$ in $[H^{2,1/2}(\Omega_T)]^2$ (see Proposition 2.15), we are done. Convergence of first order derivatives in space is done similarly.

To show that the second order derivatives converge is more involved. We show convergence for repeated derivatives as in (5.6), with mixed derivatives being similar. We explicitly show convergence of the term $(\partial_w^2\mathcal{R})(v_n,\Delta v_n)(\partial_i\Delta v_n)^2$ with the remaining terms being simpler. Decomposing the difference of products, for $t-$a.e. we compute

$$\begin{aligned}
&\|(\partial_w^2\mathcal{R})(v_n,\Delta v_n)(\partial_i\Delta v_n)^2-(\partial_w^2\mathcal{R})(v,\Delta v)(\partial_i\Delta v)^2\|_{L^2(\Omega)}\\
&\quad\le\|(\partial_w^2\mathcal{R})(v_n,\Delta v_n)\left[(\partial_i\Delta v_n)^2-(\partial_i\Delta v)^2\right]\|_{L^2(\Omega)}\\
&\qquad+\|\left[(\partial_w^2\mathcal{R})(v_n,\Delta v_n)-(\partial_w^2\mathcal{R})(v,\Delta v)\right](\partial_i\Delta v)^2\|_{L^2(\Omega)}\\
&\quad\le C(\mathcal{R})\|(\partial_i\Delta v_n)^2-(\partial_i\Delta v)^2\|_{L^2(\Omega)}\\
&\qquad+\|\left[(\partial_w^2\mathcal{R})(v_n,\Delta v_n)-(\partial_w^2\mathcal{R})(v,\Delta v)\right](\partial_i\Delta v)^2\|_{L^2(\Omega)},
\end{aligned}\tag{5.9}$$

where we used Remark 5.1. Up to another subsequence of n, for t-a.e., the second term goes to 0 by the Lebesgue dominated convergence theorem. Taking another subsequence if necessary, we apply Hölder's inequality and the Sobolev-Gagliardo-Nirenberg embedding theorem to show that the first term also goes to 0 for t-a.e.:

$$\begin{aligned}
\|(\partial_i\Delta v_n)^2-(\partial_i\Delta v)^2\|_{L^2(\Omega)}&\le\|\partial_i\Delta v_n-\partial_i\Delta v\|_{L^4(\Omega)}\|\partial_i\Delta v_n+\partial_i\Delta v\|_{L^4(\Omega)}\\
&\le C_\Omega\|\nabla\Delta(v_n-v)\|_{H^1(\Omega)}\|\nabla\Delta(v_n+v)\|_{H^1(\Omega)}\\
&\to 0\cdot 2\|\nabla\Delta v\|_{H^1(\Omega)}=0.
\end{aligned}$$

We now apply the generalized Lebesgue dominated convergence theorem to prove $(\partial_i\Delta v_n)^2\to(\partial_i\Delta v)^2$ in $H^{0,0}(\Omega_T)$:

$$\|(\partial_i\Delta v_n)^2-(\partial_i\Delta v)^2\|^2_{H^{0,0}(\Omega_T)}=\int_0^T\|(\partial_i\Delta v_n)^2-(\partial_i\Delta v)^2\|^2_{L^2(\Omega)}\,dt.\tag{5.10}$$

We bound the integrand pointwise for t-a.e. using estimate (5.7) and that $\|v_n\|_{\mathcal{B}}\to\|v\|_{\mathcal{B}}$ in $\mathcal{B}$:

$$\begin{aligned}
&\|(\partial_i\Delta v_n)^2-(\partial_i\Delta v)^2\|^2_{L^2(\Omega)}\\
&\le C\left(\|\partial_i\Delta v_n\|^4_{L^4(\Omega)}+\|\partial_i\Delta v\|^4_{L^4(\Omega)}\right)\\
&\le C\left(\|\Delta v_n\|^{4a}_{W^{2,r}(\Omega)}\|\Delta v_n\|^{4(1-a)}_{L^r(\Omega)}+\|\Delta v\|^{4a}_{W^{2,r}(\Omega)}\|\Delta v\|^{4(1-a)}_{L^r(\Omega)}\right)\\
&\le C\sup_n\left\{\|\Delta v_n\|^{4(1-a)}_{L^\infty(0,T;L^r(\Omega))}\right\}\left(\|\Delta v_n\|^2_{W^{2,r}(\Omega)}+\|\Delta v\|^2_{W^{2,r}(\Omega)}+1\right)\\
&\le C\sup_n\left\{\|v_n\|^{4(1-a)}_{\mathcal{B}}\right\}\left(\|\Delta v_n\|^2_{W^{2,r}(\Omega)}+\|\Delta v\|^2_{W^{2,r}(\Omega)}+1\right)\in L^1(0,T).
\end{aligned}$$

Thus we apply the generalized Lebesgue dominated convergence theorem to conclude (5.10) converges to 0. Likewise, we conclude that the left-hand side of (5.9) goes to 0 in $L^2(0,T)$, from which we conclude the desired convergence of second order terms, and finally continuity of the operator A.

Boundedness: We show that the set

$$\{c\in\mathcal{B}:c=\lambda A[c]\text{ for }\lambda\in(0,1]\}$$

is bounded in $\mathcal{B}$ for $\lambda\in(0,1]$. We assume $\lambda=1$; the argument is the same for other λ. Thus, suppose $c=A[c]\in H^{5,1+1/4}(\Omega_T)$. Making use of the bound (4.5) and assumptions on f, it straightforward to show that $(\Delta f')_\alpha(c)$ is bounded in $H^{1,1/4}(\Omega_T)$ in terms of $\|c\|_{H^{4,1}(\Omega_T)}\le C(c_0,\Omega,T)$. Now, we control $\beta_c:=\mathcal{R}(c,\Delta c)$ in $H^{3/2,3/8}(\Sigma_T)$. Given Proposition 2.15, the norm of β_c in $H^{0,3/8}(\Sigma_T)$ is controlled by $\|c\|_{H^{4,1}(\Omega_T)}\le C(c_0,\Omega,T)$. We summarize these initial bounds as

$$\|\beta_c\|_{H^{0,3/8}(\Sigma_T)}+\|(\Delta f')_\alpha(c)\|_{H^{1,1/4}(\Omega_T)}\le C(c_0,f,\mathcal{R},\Omega,T).\tag{5.11}$$

To bound β_c in $H^{3/2,0}(\Sigma_T)$, we first look at $\nabla^2\beta_c$ in $H^{0,0}(\Omega_T)$. We control the repeated derivative $\partial_i^2\beta_c$, as in (5.6); control of the mixed derivatives is analogous.

First, we impose a restriction on the initial condition such that

$$\|\Delta c_0\|_{L^2(\Omega)} \leq \frac{1}{k},$$

where $k \in \mathbb{N}$ is yet to be chosen. By (4.5), for all $T > 0$ sufficiently small, we have

$$\|c\|_{H^{4,1}(\Omega_T)} + \|\nabla^2 c\|_{L^\infty(0,T;L^2(\Omega))} \leq \frac{C}{k} + \eta, \tag{5.12}$$

where the constant $C > 0$ is independent of T. Looking to terms arising in (5.6), we square the Gagliardo-Nirenberg inequality (Theorem 2.5) in dimension $N = 2$ and use (5.12) to find for t-a.e.

$$\|\partial_i(\Delta c)\|_{L^4(\Omega)}^2 \leq C(\Omega)\|\Delta c\|_{H^3(\Omega)}\|\Delta c\|_{L^2(\Omega)} \leq C(\Omega)\left(\frac{C}{k} + \eta\right)\|\Delta c\|_{H^3(\Omega)}. \tag{5.13}$$

Using Hölder's inequality, Young's inequality, the Sobolev-Gagliardo-Nirenberg embedding theorem, and a trace theorem (Theorem 3.1 of [45]), we have for t-a.e.

$$\begin{aligned}\|\partial_i c\, \partial_i(\Delta c)\|_{L^2(\Omega)} &\leq \frac{1}{2}\left(C(\Omega)\|c\|_{L^\infty(0,T;H^2(\Omega))}^2 + \|\partial_i(\Delta c)\|_{L^4(\Omega)}^2\right) \\ &\leq \frac{1}{2}\left(C(\Omega, T)\|c\|_{H^{4,1}(\Omega_T)}^2 + \|\partial_i(\Delta c)\|_{L^4(\Omega)}^2\right).\end{aligned} \tag{5.14}$$

Recalling (5.2) and noting the terms in (5.6), we see that the bounds (5.13), (5.14), and $\|c\|_{H^{4,1}(\Omega_T)} \leq C$ (from (4.5)) imply

$$\|\partial_i^2\beta_c\|_{H^{0,0}(\Omega_T)} \leq C(c_0, f, \mathcal{R}, \Omega)\left(C(T) + \left(\frac{C}{k} + \eta\right)\|\Delta c\|_{H^{3,0}(\Omega_T)}\right). \tag{5.15}$$

As noted previously, an argument analogous to the above succeeds in controlling the full derivative $\nabla^2\beta_c$. Furthermore, as $\mathcal{R}$ is Lipschitz (see Remark 5.1), it is direct to conclude β_c is controlled in $H^{1,0}(\Omega_T)$ by $\|c\|_{H^{4,1}(\Omega_T)}$. Thus by (5.15) and the trace inequality $\|\cdot\|_{H^{3/2}(\Gamma)} \leq C(\Omega)\|\cdot\|_{H^2(\Omega)}$ [44], we have

$$\|\beta_c\|_{H^{3/2,0}(\Sigma_T)} \leq C(c_0, f, \mathcal{R}, \Omega)\left(C(T) + \left(\frac{C}{k} + \eta\right)\|\Delta c\|_{H^{3,0}(\Omega_T)}\right). \tag{5.16}$$

Lastly, we will use the method established in the proof of Theorem 4.1 to extend the bulk and boundary data to Ω_1 and Σ_1, and then apply Theorem 6.7 (with $k = 1$) to bound $c \in H^{5,1+1/4}(\Omega_T)$ by

$$\|c\|_{H^{5,1+1/4}(\Omega_T)} \leq C(c_0, f, \mathcal{R}, \Omega, T) + C(c_0, f, \mathcal{R}, \Omega)\left(\frac{C}{k} + \eta\right)\|\Delta c\|_{H^{3,0}(\Omega_T)}. \tag{5.17}$$

Supposing we have estimate (5.17), choosing $T > 0$ sufficiently small and $k \in \mathbb{N}$ large enough such that

$$C(c_0, f, \mathcal{R}, \Omega)\left(\frac{C}{k} + \eta\right) < 1,$$

we may directly conclude the proof of boundedness. So it only remains to prove (5.17).

We use Corollary 2.8 to find an extension $\tilde{\beta_c} \in H^{3/2,3/8}(\Omega_1)$ of β_c such that

$$\|\tilde{\beta_c}\|_{H^{3/2,3/8}(\Sigma_1)} \leq C\left((1 + T^{-3/8})\|\beta_c\|_{H^{0,0}(\Sigma_T)} + \|\beta_c\|_{H^{3/2,3/8}(\Sigma_T)}\right).$$

Likewise, we find an extension $\tilde{f} \in H^{1,1/4}$ of $(\Delta f')_\alpha(c)$ such that

$$\|\tilde{f}\|_{H^{1,1/4}(\Omega_1)} \leq C\left((1+T^{-1/4})\|(\Delta f')_\alpha(c)\|_{H^{0,0}(\Omega_T)} + \|(\Delta f')_\alpha(c)\|_{H^{1,1/4}(\Omega_T)}\right).$$

With this in hand, we consider the PDE for $\tilde{c}$

$$\begin{cases} \partial_t \tilde{c} + \Delta^2 \tilde{c} = \tilde{f} & \text{in } \Omega_1, \\ \partial_\nu \tilde{c} = 0 & \text{on } \Sigma_1, \\ \partial_\nu(\Delta \tilde{c}) = \tilde{\beta}_c & \text{on } \Sigma_1, \\ \tilde{c}(0) = c_0 & \text{in } \Omega. \end{cases}$$

Note, by uniqueness (see Theorem 6.7), $\tilde{c}|_{\Omega_T} = c$. Theorem 6.7 (with $k=1$), bound (5.11), and (5.16) then show

$$\begin{aligned} \|c\|_{H^{5,1+1/4}(\Omega_T)} \leq \|\tilde{c}\|_{H^{5,1+1/4}(\Omega_1)} &\leq C(\Omega,1)(\|\tilde{f}\|_{H^{1,1/4}(\Omega_1)} + \|\tilde{\beta}_c\|_{H^{3/2,3/8}(\Sigma_1)}) \\ &\leq C(c_0,f,\mathcal{R},\Omega,T) + C(c_0,f,\mathcal{R},\Omega)\left(\frac{C}{k}+\eta\right)\|\Delta c\|_{H^{3,0}(\Omega_T)}. \end{aligned}$$

Note the first constant may blow up as $T \to 0$, and the extensions in time have been used to guarantee this does not happen to the coefficient of $\frac{1}{k}$.

□

Remark 5.3. *In the previous theorem, we were unable to directly bootstrap to higher regularity and had to re-do the fixed point argument first used to gain a strong solution. This is because $c \in H^{4,1}(\Omega_T)$ is not sufficient to guarantee $\mathcal{R}(c,\Delta c) \in H^{3/2,3/8}(\Sigma_T)$. However, we will see that $c \in H^{5,1+1/4}(\Omega_T)$ is sufficient to guarantee $\mathcal{R}(c,\Delta c) \in H^{5/2,5/8}(\Sigma_T)$.*

We now prove the next step in regularity, inclusion of the solution in $H^{6,1+1/2}(\Omega_T)$.

Theorem 5.4. *Suppose $\Omega \subset \mathbb{R}^2$ is an open bounded smooth domain. Further suppose f and $\mathcal{R}$ satisfy (5.1) and (5.2). There is $\lambda(\mathcal{R},\Omega) > 0$ such that if $c_0 \in H^4(\Omega)$ such that $\partial_\nu c_0 = 0$, $\partial_\nu(\Delta c_0) = \mathcal{R}(c_0,\Delta c_0)$ and $\|\nabla^2 c_0\|_{L^2(\Omega)} \leq \lambda$, then there is $T > 0$ such that a solution of the truncated CHR model (4.4) exists on the domain Ω_T satisfying the estimate*

$$\|c\|_{H^{6,1+1/2}(\Omega)} \leq C(f,c_0,R,\Omega,T).$$

Proof. Let $\lambda > 0$ be as in Theorem 5.2. By the previous theorem, we have a solution of the truncated CHR model (4.4) given by $c \in H^{5,1+1/4}(\Omega_T)$. We show that $\mathcal{R}(c,\Delta c) \in H^{3,3/4}(\Omega_T)$, which will imply that $\mathcal{R}(c,\Delta c) \in H^{5/2,5/8}(\Sigma_T)$ (see Theorem 2.14). As $\mathcal{R}$ is Lipschitz, making use of Proposition 2.15 and Theorem 5.2, it is straightforward to show that

$$\|\mathcal{R}(c,\Delta c)\|_{H^{0,3/4}(\Omega_T)} \leq C(\mathcal{R},\Omega)\left(\|c\|_{H^{0,3/4}(\Omega_T)} + \|\Delta c\|_{H^{0,3/4}(\Omega_T)} + 1\right) \leq C(f,c_0,\mathcal{R},\Omega,T).$$

Thus it remains to bound the third derivative of $\mathcal{R}(c,\Delta c)$. Looking to (5.6), we see that the difficult terms which will need bounded in $H^{0,0}(\Omega_T)$ are $\partial_i^2(\Delta c)\partial_i(\Delta c)$ and $(\partial_i \Delta c)^3$; using Young's inequality, we reduce this to consideration of the terms $(\partial_i^2(\Delta c))^{3/2}$ and $(\partial_i \Delta c)^3$. The Gagliardo-Nirenberg inequality (Theorem 2.5) provides the estimates for t-a.e. in $(0,T)$

$$\begin{aligned} \|\partial_i^2(\Delta c)\|_{L^3(\Omega)} &\leq C(\Omega)\|\partial_i(\Delta c)\|_{H^2(\Omega)}^{2/3}\|\partial_i(\Delta c)\|_{L^2(\Omega)}^{1/3}, \\ \|\partial_i \Delta c\|_{L^6(\Omega)} &\leq C(\Omega)\|\partial_i(\Delta c)\|_{H^2(\Omega)}^{1/3}\|\partial_i(\Delta c)\|_{L^2(\Omega)}^{2/3}, \end{aligned}$$

which in turn by Theorem 5.2 and $H^{5,1+1/4}(\Omega_T) \hookrightarrow BUC(0,T;H^3(\Omega))$ [45] shows

$$\begin{aligned} \|(\partial_i^2(\Delta c))^{3/2}\|_{H^{0,0}(\Omega_T)} + \|(\partial_i(\Delta c))^3\|_{H^{0,0}(\Omega_T)} &\leq C(\Omega)(\|\nabla^3 c\|_{L^\infty(0,T;L^2(\Omega))}^2 + 1)\|c\|_{H^{5,0}(\Omega_T)} \\ &\leq C(f,c_0,\mathcal{R},\Omega,T). \end{aligned}$$

Combining the above calculation with the analogous calculation for the bulk data, we have

$$\|\mathcal{R}(c,\Delta c)\|_{H^{5/2,5/8}(\Sigma_T)} + \|(\Delta f')_\alpha(c)\|_{H^{2,1/2}(\Omega_T)} \le C(f, c_0, \mathcal{R}, \Omega, T).$$

To apply the regularity Theorem 6.7 (with $k = 2$), we must make sure that the compatibility condition is satisfied (i.e., $\beta(0) = \partial_\nu(\Delta c_0)$). To see that this is the case, we again note that $c \in H^{5,1+1/4}(\Omega_T)$ implies $c \in BUC(0,T; H^3(\Omega)) \hookrightarrow (c, \Delta c) \in BUC(0,T; [H^{1/2}(\Gamma)]^2)$. Consequently,

$$\mathcal{R}(c,\Delta c)(\cdot,0) = \mathcal{R}(c(\cdot,0), \Delta c(\cdot,0)) = \mathcal{R}(c_0,\Delta c_0) = \partial_\nu(\Delta c_0), \tag{5.18}$$

verifying the compatibility condition.

□

To understand the utility of a solution in $H^{6,1+1/2}(\Omega_T)$, we have the following lemma.

Lemma 5.5. *$H^{6,1+1/2}(\Omega_T)$ continuously embeds into $BUC(0,T;H^4(\Omega))$.*

Proof. This is a consequence of Theorem 3.1 in [45], which holds for noninteger derivatives. □

We now have sufficient power to prove existence of a solution to the CHR model with exponential boundary conditions for sufficiently small intervals of time.

Proof of Theorem 1.8. This is proof is mainly a matter of choosing truncations. As $c_0 \in H^4(\Omega)$, $\|c_0\|_{C^2(\Omega)} =: \alpha < \infty$. We can construct functions $\tilde{\mathcal{R}}$ and $\tilde{f}$ such that $\tilde{\mathcal{R}} = \mathcal{R}$ on $B(0,\alpha+1)$ and $\tilde{f} = f$ on $[\epsilon/2, 1-\epsilon/2]$ and the hypotheses of Theorem 5.4 are satisfied. Consider the PDE

$$\begin{cases} \partial_t c + \Delta^2 c = (\Delta \tilde{f})_{\alpha+1}(c) & \text{in } \Omega_T, \\ \partial_\nu c = 0 & \text{on } \Sigma_T, \\ \partial_\nu(\Delta c) = \tilde{\mathcal{R}}(c,\Delta c) & \text{on } \Sigma_T, \\ c(0) = c_0 & \text{in } \Omega, \end{cases}$$

for which there is a solution $c \in H^{6,1+1/2}(\Omega_{T_0})$ for some $T_0 > 0$ by Theorem 5.4. By Lemma 5.5 and the Sobolev-Gagliardo-Nirenberg and Morrey embedding theorems in dimension 2, $c \in C^0([0,T]; C^{2,a}(\Omega))$ for some $a > 0$. By continuity, there is some interval $[0,T]$ for which $\tilde{\mathcal{R}}(c,\Delta c) = \mathcal{R}(c,\Delta c)$ and $(\Delta\tilde{f})_{\alpha+1}(c) = \Delta f(c)$, proving the theorem. □

Remark 5.6. *Lastly, we would like to make sure we have proven something which is not trivially true. Explicitly, we claim there are initial conditions which satisfy the hypothesis of Theorem 1.8. Recall, as in Singh et. al. [58], by (1.6) and (4.2), we have that*

$$\mathcal{R}(c,\Delta c) = -(R_{\text{ins}} - R_{\text{ext}}) = k_{\text{ext}} c \exp(\beta(\mu - \mu_e)) - k_{\text{ins}} \exp(\beta(\mu_e - \mu)),$$

where all constants k_{ext}, k_{ins}, β, μ_e are positive. Consider the case of a constant $c_0 \in (0,1)$, then we have

$$\mathcal{R}(c_0,\Delta c_0) = -R(c_0, f'(c_0)) = k_{\text{ext}} c_0 \exp(\beta(f'(c_0) - \mu_e)) - k_{\text{ins}} \exp(\beta(\mu_e - f'(c_0))).$$

Since $\lim_{z\to 0} f'(z) = -\infty$ and $\lim_{z\to 1} f'(z) = \infty$, it follows that

$$\lim_{c_0\to 0} \mathcal{R}(c_0,\Delta c_0) = -\infty, \qquad \lim_{c_0\to 1} \mathcal{R}(c_0,\Delta c_0) = \infty.$$

By the intermediate value theorem, there is $c_0 \in (0,1)$ such that $\mathcal{R}(c_0,\mu_0) = 0$. It then follows that c_0 is an admissible condition for Theorem 1.8 as $\|\nabla^2 c_0\|_{L^2(\Omega)} = 0 \le \lambda$ and $\partial_\nu(\Delta c_0) = 0 = \mathcal{R}(c_0,\mu_0)$. Considering sufficiently small perturbations of c_0, we may find other admissible initial conditions.

Remark 5.7. *In Theorem 1.8, the method of proof can also account for the case in which the initial chemical potential is small, i.e. $\|-\Delta c_0 + f'(c_0)\|_{L^2(\Omega)} \le \lambda$. However, we do not know if this class of admissible functions is non-empty.*

6 Appendix

The purpose of this appendix is to develop a relatively self contained presentation for regularity of the PDE (6.1) below. The first two results show how gradient flows provide regularity in the case of weak data, and then in the case of L^2 data. It was originally our desire to use interpolation theorems to obtain regularity for the case of intermediate data in the space $H^{4+k,1+k/4}(\Omega_T)$. However, this approach demands a little too much of an appendix (see Remark 6.8), and we refer to a classical result of Lions and Magenes [46].

We analyze the PDE

$$\begin{cases} \partial_t c - \operatorname{div}(\Lambda\nabla\mu) = g & \text{in } \Omega_T, \\ -\operatorname{div}(\Lambda\nabla c) = \mu & \text{in } \Omega_T, \\ (\Lambda\nabla c)\cdot\nu = 0 & \text{on } \Sigma_T, \\ (\Lambda\nabla\mu)\cdot\nu = 0 & \text{on } \Sigma_T, \\ c(0) = c_0 & \text{in } \Omega, \end{cases} \tag{6.1}$$

for $\Lambda \in C^\infty(\bar{\Omega}, \mathrm{Pos}(N))$ with data g and c_0. Note that

$$\lambda_1(x)\|y\|^2 \leq (\Lambda(x)y)\cdot y \leq \lambda_N(x)\|y\|^2, \quad x \in \bar{\Omega},\ y \in \mathbb{R}^N,$$

where $\lambda_1(x)$ and $\lambda_N(x)$ are the smallest and largest eigenvalues of $\Lambda(x)$, respectively. We further remark that we only use $\Lambda = \mathbb{I}$ in the previous sections, but including this generality here does not create additional complications.

Let us now make clear by what we mean by a solution. First, define the Hilbert space $V := \{w \in H^1(\Omega) : \fint_\Omega w = 0\}$, with inner product

$$(w, v)_V := \int_\Omega (\nabla w, \nabla v)_\Lambda \, dx, \tag{6.2}$$

where for vectors $x, y \in \mathbb{R}^N$

$$(x, y)_\Lambda := (\Lambda x)\cdot y.$$

Note that $\|w\|_V$ is equivalent to the standard H^1 norm by (6.2) the Poincaré inequality. For any element $L \in V^*$, by the Riesz representation theorem, we have that $\langle L, w\rangle_{V^*,V} = \int_\Omega (\nabla z_L, \nabla w)_\Lambda$ for some $z_L \in V$; i.e., z_L is a weak solution of the Neumann problem

$$\begin{cases} -\operatorname{div}(\Lambda\nabla w) = L & \text{in } \Omega, \\ \Lambda\nabla w\cdot\nu = 0 & \text{on } \Gamma. \end{cases} \tag{6.3}$$

Furthermore,

$$\|z_L\|_V = \|L\|_{V^*}. \tag{6.4}$$

We also have the following result on regularity of z_L. This is a more general case of the estimate in Theorem 2.4.

Lemma 6.1. *[39] Let Ω be an open, bounded set with C^{k+2} boundary. Suppose $L \in H^k(\Omega)$ with $\int_\Omega L\,dx = 0$, $\Lambda \in C^\infty(\bar{\Omega}, Pos(N))$, and z_L is a weak solution of the Neumann problem (6.3). Then*

$$\|z_L\|_{H^{k+2}(\Omega)} \leq C\|L\|_{H^k(\Omega)}.$$

Assuming $g \in L^2(0,T;V^*)$, we may define $z_g \in L^2(0,T;V)$ pointwise in t by the aforementioned isomorphism. Consequently, we may rewrite (6.1) as

$$\begin{cases} \partial_t c - \operatorname{div}(\Lambda\nabla\mu) = 0 & \text{in } \Omega_T, \\ -\operatorname{div}(\Lambda\nabla c) - z_g = \mu & \text{in } \Omega_T, \\ (\Lambda\nabla c)\cdot\nu = 0 & \text{on } \Sigma_T, \\ (\Lambda\nabla\mu)\cdot\nu = 0 & \text{on } \Sigma_T, \\ c(0) = c_0 & \text{on } \Omega. \end{cases}$$

This motivates our definition.

Definition. *We say that c is a* weak solution *of (6.1) in* Ω_T *if*

$$\begin{aligned} &c \in L^2(0,T;H^3(\Omega)\cap V)\cap C([0,T),L^2(\Omega)),\\ &\partial_t c \in L^2(0,T;V^*),\\ &c(0)=c_0\in V, \end{aligned} \tag{6.5}$$

and for t-a.e. and $\xi \in V$*,*

$$\langle \partial_t c(t),\xi\rangle_{V^*,V} + \int_\Omega (\nabla\mu(t),\nabla\xi)_\Lambda \, dx = 0,$$

where for t-a.e. $\mu(t)\in H^1(\Omega)\subset L^2(\Omega)$ *is defined by duality as*

$$(\mu(t),\xi)_{L^2(\Omega)} := \int_\Omega ((\nabla c(t),\nabla\xi)_\Lambda - z_g(t)\xi)\, dx, \tag{6.6}$$

which holds for all $\xi\in H^1(\Omega)$*.*

We proceed in stages of increasing regularity of the data.

Theorem 6.2. *Let* $\Omega\subset\mathbb{R}^N$ *be an open, bounded set with boundary of class* C^3*. Let* $T>0$*,* $\Lambda\in C^\infty(\bar\Omega, Pos(N))$*,* $g\in L^2(0,T;V^*)$*, and* $c_0\in V$*. Then there exists a unique weak solution c to (6.1) satisfying the following bound*

$$\|c\|_{L^2(0,T;H^3(\Omega))} + \|\partial_t c\|_{L^2(0,T;V^*)} \le C\left(\|c_0\|_V + \|g\|_{L^2(0,T;V^*)}\right)$$

for some constant $C=C(\Lambda,\Omega)>0$*.*

Proof. **Step 1: Minimizing movements scheme.** We construct a solution via the method of minimizing movements. For $n\in\mathbb{N}$, we partition our time interval $(0,T)$ into n equal size steps of length $\tau=T/n$ and recursively define the finite sequence $\{c_\tau^i\}_{i=0}^n$ in V as follows:

$$c_\tau^i = \operatorname*{argmin}_{c\in V}\left\{\frac{1}{2\tau}\|c-c_\tau^{i-1}\|_{V^*}^2 + \int_\Omega\left(\frac12\langle\nabla c,\nabla c\rangle_\Lambda - z_{g_\tau^i}c\right)dx\right\}, \tag{6.7}$$

where $c_\tau^0 := c_0$,

$$g_\tau^i := \fint_{(i-1)\tau}^{i\tau} g(t)\,dt, \tag{6.8}$$

and we have implicitly used the embedding of V in $L^2(\Omega)\subset V^*$. Note that a minimizer exists as the functional being minimized is coercive and lower semicontinuous with respect to the weak topology of V. To see the lower semicontinuity, note that $\|\cdot\|_{V^*}$ is lower semi-continuous with respect to the weak topology of V as the inclusion $i: V\to L^2(\Omega)$ is compact and $\|\cdot\|_{V^*}$ is continuous with respect to the strong topology on $L^2(\Omega)$.

Step 2: "Discrete" Euler-Lagrange equations. We now compute the "discrete" Euler-Lagrange Equations associated with the minimzation problem (6.7). Since $\xi\mapsto z_\xi$ is linear, using (6.2), (6.3), and (6.4), we can compute the Frechet derivative of the norm in V^* :

$$\begin{aligned} \lim_{t\to0}\frac{\|v+tw\|_{V^*}^2-\|v\|_{V^*}^2}{2t} &= \lim_{t\to0}\frac{1}{2t}\int_\Omega\Big((\nabla z_{v+tw},\nabla z_{v+tw})_\Lambda - (\nabla z_v,\nabla z_v)_\Lambda\Big)\,dx\\ &= \lim_{t\to0}\int_\Omega\Big((\nabla z_w,\nabla z_v)_\Lambda + \frac{t}{2}(\nabla z_w,\nabla z_w)_\Lambda\Big)\,dx\\ &= \int_\Omega(\nabla z_w,\nabla z_v)_\Lambda\,dx = \langle v, z_w\rangle_{V^*,V}. \end{aligned}$$

Consequently, the "discrete" Euler-Lagrange equation for a minimizer is

$$\frac{1}{\tau}\langle c_\tau^i - c_\tau^{i-1}, z_w\rangle_{V^*,V} + \int_\Omega \Big((\nabla c_\tau^i, \nabla w)_\Lambda - z_{g_\tau^i} w \Big)\, dx = 0, \tag{6.9}$$

which holds for all $w \in V$.

Step 3: Energy estimates and convergence. Letting $w = c_\tau^i - c_\tau^{i-1}$ be the test function in (6.9), we find that

$$\frac{1}{\tau}\|c_\tau^i - c_\tau^{i-1}\|_{V^*}^2 + \int_\Omega \Big((\nabla c_\tau^i, \nabla (c_\tau^i - c_\tau^{i-1}))_\Lambda - z_{g_\tau^i}(c_\tau^i - c_\tau^{i-1}) \Big)\, dx = 0.$$

But using the representation of $c_\tau^i - c_\tau^{i-1}$ in V^*, we rewrite this as

$$\frac{1}{\tau}\|c_\tau^i - c_\tau^{i-1}\|_{V^*}^2 + \int_\Omega \Big((\nabla c_\tau^i, \nabla (c_\tau^i - c_\tau^{i-1}))_\Lambda - (\nabla z_{g_\tau^i}, \nabla z_{c_\tau^i - c_\tau^{i-1}})_\Lambda \Big)\, dx = 0.$$

Applying the Cauchy-Schwarz inequality, we compute

$$\begin{aligned}\int_\Omega (\nabla z_{g_\tau^i}, \nabla z_{c_\tau^i - c_\tau^{i-1}})_\Lambda\, dx &\le \|z_{g_\tau^i}\|_V \|z_{c_\tau^i - c_\tau^{i-1}}\|_V \\ &= \|g_\tau^i\|_{V^*}\|c_\tau^i - c_\tau^{i-1}\|_{V^*} \\ &\le \frac{\tau}{2}\|g_\tau^i\|_{V^*}^2 + \frac{1}{2\tau}\|c_\tau^i - c_\tau^{i-1}\|_{V^*}^2.\end{aligned}$$

Similarly,

$$\int_\Omega (\nabla c_\tau^i, \nabla c_\tau^{i-1})_\Lambda\, dx \le \frac{1}{2}\int_\Omega (\nabla c_\tau^i, \nabla c_\tau^i)_\Lambda\, dx + \frac{1}{2}\int_\Omega (\nabla c_\tau^{i-1}, \nabla c_\tau^{i-1})_\Lambda\, dx.$$

With this, we obtain the following energy estimate:

$$\frac{1}{2\tau}\|c_\tau^i - c_\tau^{i-1}\|_{V^*}^2 + \frac{1}{2}\int_\Omega (\nabla c_\tau^i, \nabla c_\tau^i)_\Lambda\, dx \le \frac{\tau}{2}\|g_\tau^i\|_{V^*}^2 + \frac{1}{2}\int_\Omega (\nabla c_\tau^{i-1}, \nabla c_\tau^{i-1})_\Lambda\, dx. \tag{6.10}$$

Let $\hat{c}_\tau$ is the linear interpolant and c_τ is the left continuous step function associated with the sequence $\{c_\tau^i\}_{i=0}^n$ as in (3.14) and (3.12), respectively. Likewise we define g_τ to be the left continuous step function,

$$g_\tau(t) := g_\tau^i \qquad t \in ((i-1)\tau, i\tau],\ i = 0, \ldots, n-1. \tag{6.11}$$

Let k be a positive integer less than or equal to n. From (6.8) and Jensen's inequality, we have

$$\|g_\tau\|_{L^2(0,k\tau;V^*)}^2 = \sum_{i=1}^k \tau \left\| \fint_{(i-1)\tau}^{i\tau} g(t)\, dt \right\|_{V^*}^2 \le \sum_{i=1}^k \int_{(i-1)\tau}^{i\tau} \|g(t)\|_{V^*}^2\, dt = \|g\|_{L^2(0,k\tau;V^*)}^2. \tag{6.12}$$

Using the above bound, recalling (6.2), and that $\partial_t \hat{c}_\tau(t) = \frac{c_\tau^i - c_\tau^{i-1}}{\tau}$ for $t \in ((i-1)\tau, i\tau)$, we sum inequality (6.10) over $i = 1, \ldots, k$ to find

$$\|\partial_t \hat{c}_\tau\|_{L^2(0,k\tau;V^*)}^2 + \|c_\tau^k\|_V^2 \le \|g\|_{L^2(0,k\tau;V^*)}^2 + \|c_0\|_V^2.$$

This bound implies the control

$$\|\partial_t \hat{c}_\tau\|_{L^2(0,T;V^*)}^2 + \|c_\tau\|_{L^\infty(0,T;V)}^2 \le \|g\|_{L^2(0,T;V^*)}^2 + \|c_0\|_V^2. \tag{6.13}$$

We apply the Aubin-Lions-Simon compactness theorem [57], with the evolution triple (V, L^2, V^*), to find that $\hat{c}_\tau$ converges in $L^2(0,T;L^2(\Omega))$ as $\tau \to 0$ to some $c \in L^2(0,T;V)$. We would also like to consider convergence of the left continuous step functions c_τ and hence compute the difference

$$\|\hat{c}_\tau(t_2) - \hat{c}_\tau(t_1)\|_{V^*} \leq \int_{t_1}^{t_2} \|\partial_t \hat{c}_\tau\|_{V^*}\, dt \leq C(t_2 - t_1)^{1/2}.$$

By Corollary 2.6 (which holds with $H^1(\Omega)^*$ replaced by V^*) and (6.13),

$$\begin{aligned}\|\hat{c}_\tau(t_2) - \hat{c}_\tau(t_1)\|_{L^2(\Omega)} \leq & C\big(\|\nabla(\hat{c}_\tau(t_2) - \hat{c}_\tau(t_1))\|_{L^2(\Omega)}^{1/2} \|\hat{c}_\tau(t_2) - \hat{c}_\tau(t_1)\|_{V^*}^{1/2} \\ & + \|\hat{c}_\tau(t_2) - \hat{c}_\tau(t_1)\|_{V^*}\big) \\ \leq & C \max\{(t_2 - t_1)^{1/4}, (t_2 - t_1)^{1/2}\},\end{aligned} \tag{6.14}$$

which implies convergence of c_τ to c in $L^2(0,T;L^2(\Omega))$ by a direct application of the triangle inequality, and consequently $c_\tau \rightharpoonup c$ in $L^2(0,T;V)$. Lastly, we note it is clear that $\partial_t \hat{c}_\tau \rightharpoonup \partial_t c$ and (6.13) holds with $\hat{c}_\tau$ and c_τ replaced by c.

Step 4: c is the desired solution. Now, let $\{w_k\}_{k\in\mathbb{N}}$ be dense in V. We then integrate the "discrete" Euler-Lagrange equation (6.9) in time to find

$$\int_{t_1}^{t_2} \langle \partial_t c_\tau, z_{w_k}\rangle_{V^*,V}\, dt + \int_{t_1}^{t_2} \left(\int_\Omega \big((\nabla c_\tau^-, \nabla w_k)_\Lambda - z_{g_\tau} w_k\big)\, dx \right) dt = 0. \tag{6.15}$$

We would like to show that $z_{g_\tau} \to z_g$ in $L^2(0,T;L^2(\Omega))$. By the Lebesgue differentiation theorem [33], $g_\tau \to g$ in V^* for t-a.e. in $(0,T)$. Using (6.12), Fatou's lemma, and the uniform convexity of a Hilbert space [13], we conclude $g_\tau \to g$ in $L^2(0,T;V^*)$. Consequently,

$$\|z_{g_\tau} - z_g\|_{L^2(0,T;H^1(\Omega))} \leq C\|g_\tau - g\|_{L^2(0,T;V^*)} \to 0,$$

where we have used (6.4) and linearity of z.

Passing $\tau \to 0$ in (6.15) and using the various modes of convergence, we have

$$\int_{t_1}^{t_2} \langle \partial_t c, z_{w_k}\rangle_{V^*,V}\, dt + \int_{t_1}^{t_2} \left(\int_\Omega \big((\nabla c, \nabla w_k)_\Lambda - z_g w_k\big)\, dx \right) dt = 0$$

for all $t_1, t_2 \in [0,T]$. Using Lebesgue points, we find for all $k \in \mathbb{N}$ and t-a.e.

$$-\langle \partial_t c, z_{w_k}\rangle_{V^*,V} = \int_\Omega \big((\nabla c, \nabla w_k)_\Lambda - z_g w_k\big)\, dx. \tag{6.16}$$

By density the above equation holds for all $w \in V$. Using duality, this implies

$$-\int_\Omega w z_{\partial_t c}\, dx = -\langle w, z_{\partial_t c}\rangle_{V^*,V} = -\langle \partial_t c, z_w\rangle_{V^*,V} = \int_\Omega \big((\nabla c, \nabla w)_\Lambda - z_g w\big)\, dx. \tag{6.17}$$

Since $\partial_t c \in L^2(0,T;V^*)$ by (6.13), the function $z_{\partial_t c}$ belongs to $L^2(0,T;H^1(\Omega))$ by (6.3) and (6.4). Similarly $z_g \in L^2(0,T;H^1(\Omega))$; hence Lemma 6.1 and (6.17) imply $c \in L^2(0,T;H^3(\Omega))$.

Furthermore, if we define $\mu := -z_{\partial_t c}$, by (6.17) we have that

$$\int_\Omega \big((\nabla c, \nabla w)_\Lambda - z_g w\big)\, dx = \int_\Omega \mu w\, dx, \tag{6.18}$$

and by (6.3),

$$\int_\Omega (\nabla \mu, \nabla w)_\Lambda\, dx = -\int_\Omega (\nabla z_{\partial_t c}, \nabla w)_\Lambda\, dx = \langle \partial_t c, w\rangle_{V^*,V}. \tag{6.19}$$

Noting that the initial condition $c(0) = c_0$ and continuity are consequences of (6.14), we have c is a weak solution of (6.1).

Step 5: Uniqueness. This follows from an energy argument as in the proof of uniqueness for a solution of the heat equation. Suppose c_1 and c_2 solve (6.1). Then $c := c_1 - c_2$ solves (6.1) with 0 data. Testing the weak formulation of c with $w = c$, for t-a.e. we have

$$\langle \partial_t c, c\rangle_{V^*,V} = \int_\Omega (\nabla(\mathrm{div}(\Lambda c)), \nabla c)_\Lambda \, dx,$$

which Theorem II.5.12 of [12] (which shows $2\langle \partial_t c, c\rangle_{V^*,V} = \partial_t \|c\|^2_{L^2(\Omega)}$) and integrating by parts imply

$$\partial_t \left(\frac{1}{2} \|c\|^2_{L^2(\Omega)} \right) + \int_\Omega (\mathrm{div}(\Lambda c))^2 \, dx = 0,$$

for t-a.e. Thus $\|c\|^2_{L^2(\Omega)}$ satisfies the differential inequality $\partial_t(\|c\|^2_{L^2(\Omega)}) \leq 0$ with $\|c(0)\|^2_{L^2(\Omega)} = 0$, which implies $c = 0$ as desired. □

We would now like to extend the above analysis to more regular data, and do so in the following theorem.

Theorem 6.3. *Let Ω be an open, bounded set with C^4 boundary. Let $T > 0$, $\Lambda \in C^\infty(\bar{\Omega}, Pos(N))$, $g \in L^2(0, T; L^2(\Omega))$, and $c_0 \in H^2(\Omega)$ with $(\Lambda \nabla c_0) \cdot \nu = 0$ on Γ. Then there exists a unique solution to (6.1) given by $c \in H^{4,1}(\Omega_T)$ satisfying the following bound*

$$\|c\|_{H^{4,1}(\Omega_T)} + \|c\|_{L^\infty(0,T;H^2(\Omega))} \leq C \left(\|c_0\|_{H^2(\Omega)} + \|g\|_{L^2(0,T;L^2(\Omega))} \right) \tag{6.20}$$

for some constant $C = C(\Lambda, \Omega) > 0$.

Proof. Note that up to a shift by a constant function, we can assume that $c_0 \in V$. We highlight the parts of the analysis differing from the argument in the proof of Theorem 6.2. Making use of the Neumann boundary conditions, we integrate by parts the "discrete" Euler-Lagrange equations (see also (6.18) and (6.19)) derived in the previous proof and use a density argument to conclude for all $w \in V$ and t-a.e.

$$\langle \partial_t \hat{c}_\tau, w\rangle_{V^*,V} = \int_\Omega \left(\nabla(\mathrm{div}(\Lambda \nabla c_\tau^-) + z_{g_\tau}), \nabla w\right)_\Lambda dx.$$

We compute for $t \in ((i-1)\tau, i\tau)$. Letting $w = \partial_t \hat{c}_\tau(t) = \frac{c_\tau^i - c_\tau^{i-1}}{\tau}$ and recalling properties of the isomorphism z_L for $L \in L^2(\Omega)$ provided by (6.3), we have

$$\begin{aligned}
\|\partial_t \hat{c}_\tau\|^2_{L^2(\Omega)} &= \int_\Omega \left(\nabla(\mathrm{div}(\Lambda \nabla c_\tau^i) + z_{g_\tau}), \nabla\Big(\frac{c_\tau^i - c_\tau^{i-1}}{\tau}\Big) \right)_\Lambda dx \\
&= \int_\Omega \left(-\mathrm{div}(\Lambda \nabla c_\tau^i)\mathrm{div}\Big(\Lambda\Big(\frac{c_\tau^i - c_\tau^{i-1}}{\tau}\Big)\Big) - \mathrm{div}(\Lambda \nabla z_{g_\tau})(\partial_t \hat{c}_\tau) \right) dx \\
&\leq \int_\Omega -\mathrm{div}(\Lambda \nabla c_\tau^i)\mathrm{div}\Big(\Lambda\Big(\frac{c_\tau^i - c_\tau^{i-1}}{\tau}\Big)\Big) \, dx + \frac{1}{2}\|\mathrm{div}(\Lambda \nabla z_{g_\tau})\|^2_{L^2(\Omega)} + \frac{1}{2}\|\partial_t \hat{c}_\tau\|^2_{L^2(\Omega)} \\
&\leq \int_\Omega -\mathrm{div}(\Lambda \nabla c_\tau^i)\mathrm{div}\Big(\Lambda\Big(\frac{c_\tau^i - c_\tau^{i-1}}{\tau}\Big)\Big) \, dx + \frac{1}{2}\|g_\tau\|^2_{L^2(\Omega)} + \frac{1}{2}\|\partial_t \hat{c}_\tau\|^2_{L^2(\Omega)}.
\end{aligned}$$

Note it was in the second equality, when $i = 1$, that we used the hypothesis $(\Lambda \nabla c_0) \cdot \nu = 0$. Multiplying the previous equation by τ (equivalently integrating in time over $((i-1)\tau, i\tau)$), utilizing Cauchy's inequality, and rearranging we find

$$\frac{1}{2}\|\partial_t \hat{c}_\tau\|^2_{L^2((i-1)\tau, i\tau; L^2(\Omega))} + \frac{1}{2}\int_\Omega (\mathrm{div}(\Lambda \nabla c_\tau^i))^2 \leq \frac{1}{2}\int_\Omega (\mathrm{div}(\Lambda \nabla c_\tau^{i-1}))^2 + \frac{1}{2}\|g_\tau\|^2_{L^2((i-1)\tau, i\tau; L^2(\Omega))}$$

Summing the above inequality over i,

$$\|\partial_t \hat{c}_\tau\|^2_{L^2(0,T;L^2(\Omega))} + \|\mathrm{div}(\Lambda\nabla c_\tau)\|^2_{L^\infty(0,T;L^2(\Omega))} \leq \|\mathrm{div}(\Lambda\nabla c_0)\|^2_{L^2(\Omega)} + \|g_\tau\|^2_{L^2(0,T;L^2(\Omega))}$$
$$\leq \|\mathrm{div}(\Lambda\nabla c_0)\|^2_{L^2(\Omega)} + \|g\|^2_{L^2(0,T;L^2(\Omega))}.$$

Thus $\partial_t \hat{c}_\tau$ converges weakly in $L^2(0,T;L^2(\Omega))$, which allows us to conclude that $\partial_t c \in L^2(0,T;L^2(\Omega))$ by uniqueness of limits. Consequently by Lemma 6.1, $z_{\partial_t c} \in L^2(0,T;H^2(\Omega))$ for t-a.e., and we use elliptic regularity once again conclude the bound (6.20).

□

Remark 6.4. *Following the above analysis, $\partial_t \hat{c}_\tau \in L^2(0,T;L^2(\Omega))$ for the approximate solutions. Looking at (6.17), we have $\hat{c}_\tau \in L^2(0,T;H^4(\Omega))$. The norms associated with the aforementioned inclusions are uniformly bounded. Consequently, we may apply the compactness theorem of Aubin-Lions-Simon [57], with $H^4(\Omega) \hookrightarrow\hookrightarrow H^3(\Omega) \hookrightarrow L^2(\Omega)$, to conclude (up to a subsequence) $\hat{c}_\tau \to c \in L^2(0,T;H^3(\Omega))$.*

The above results are nearly sufficient to tackle the problems of strong solutions in Section 4. It remains to extend to the case of inhomogeneous boundary conditions, but this is achieved with the aid of liftings from Theorem 2.14. For consideration of regular solutions in Section 5, we will also need results for higher regularity data. We specifically consider (6.1) with $\Lambda = \mathbb{I}$ and inhomogeneous boundary conditions:

$$\begin{cases} \partial_t c + \Delta^2 c = g & \text{in } \Omega_T, \\ \partial_\nu c = \alpha & \text{on } \Sigma_T, \\ \partial_\nu(\Delta c) = \beta & \text{on } \Sigma_T, \\ c(0) = c_0 & \text{in } \Omega. \end{cases} \tag{6.21}$$

For regularity and existence, we have the subsequent theorem. However, let us first make an two important remarks.

Remark 6.5. *The following theorem holds for any choice of norms on the anisotropic Sobolev spaces, so long as you are willing to change the constant $C(\Omega,T)$. In applications within this paper, it will be important to control exactly how this constant depends on T, so we will often extend our considerations to a domain with $T = 1$, and control the dependence on T by other means.*

Remark 6.6. *We remark that the compatibility conditions for the initial data necessarily arise due to the embedding $H^{4+k,1+k/4}(\Omega_T) \hookrightarrow BUC(0,T;H^{2+k}(\Omega))$.*

Theorem 6.7. *Let $\Omega \subset \mathbb{R}^N$ be an open, bounded set with smooth boundary and $k \in \{0,1,2\}$. Suppose $g \in H^{k,k/4}(\Omega_T)$, $c_0 \in H^{k+2}(\Omega)$, $\alpha \in H^{\mu_1,\lambda_1}(\Sigma_T)$, and $\beta \in H^{\mu_3,\lambda_3}(\Sigma_T)$, where μ_j and λ_j are defined in (2.26) with $r = 4+k$ and $s = 1+k/4$. We further assume the compatibility condition $\partial_\nu c_0 = \alpha(\cdot,0)$ on Γ. If $k = 2$, we additionally suppose $\partial_\nu(\Delta c_0) = \beta(\cdot,0)$ on Γ. Then there is a unique solution of the PDE (6.21) given by $c \in H^{4+k,1+k/4}(\Omega_T)$ satisfying the bound*

$$\|c\|_{H^{4+k,1+k/4}(\Omega_T)} \leq C(\Omega,T)\left(\|c_0\|_{H^{k+2}(\Omega)} + \|g\|_{H^{k,k/4}(\Omega_T)} + \|\alpha\|_{H^{\mu_1,\lambda_1}(\Sigma_T)} + \|\beta\|_{H^{\mu_3,\lambda_3}(\Sigma_T)}\right). \tag{6.22}$$

Proof. This theorem is a case of Theorem 5.3 of Chapter 4 in [46]. We only prove the special case $k = 0$ and $\alpha = 0$, which is the extent of this theorem's use in Section 4.

We reduce to the case of homogeneous boundary conditions by lifting the boundary condition. Looking to Theorem 2.14, this is a matter of writing $\partial_\nu(\Delta w)$ in terms of $\partial_\nu^3 w$. Consider smooth $\gamma \in C^\infty(\bar{\Omega})$ such that $\gamma = 0$ on Γ. Let $\mathrm{Pr}(x) := x - d(x)\nabla d(x)$ be the (locally well-defined) projection of x onto Γ, where d is the signed distance from Γ (negative on the interior of Ω) [10]. We have that

$$\gamma(\mathrm{Pr}(x)) = 0.$$

We then compute the ith derivative by the chain rule:

$$0 = \partial_i(\gamma(\mathrm{Pr}(x))) = \langle \nabla\gamma(\mathrm{Pr}(x)), e_i - \partial_i d(x)\nabla d(x) - d(x)\nabla(\partial_i d(x))\rangle.$$

Consequently,

$$\partial_i\gamma(\mathrm{Pr}(x)) = \langle \nabla\gamma(\mathrm{Pr}(x)), \partial_i d(x)\nabla d(x) + d(x)\nabla(\partial_i d(x))\rangle.$$

We compute the derivative again with respect to the ith direction:

$$\begin{aligned}&\langle \nabla\partial_i\gamma(\mathrm{Pr}(x)), e_i - \partial_i d(x)\nabla d(x) - d(x)\nabla(\partial_i d(x))\rangle\\ =&\langle \nabla^2\gamma(\mathrm{Pr}(x))(e_i - \partial_i d(x)\nabla d(x) - d(x)\nabla(\partial_i d(x))), \partial_i d(x)\nabla d(x) + d(x)\nabla(\partial_i d(x))\rangle\\ &+ \langle \nabla\gamma(\mathrm{Pr}(x)), \partial_i(\partial_i d(x)\nabla d(x) + d(x)\nabla(\partial_i d(x)))\rangle\end{aligned}$$

Choosing $x \in \Gamma$ and recalling $\nabla d(x) = \nu$ [10] and $d(x) = 0$, we have

$$\begin{aligned}\partial_i^2\gamma(x) =& 2\langle \nabla\partial_i\gamma(x), \partial_i d(x)\nabla d(x)\rangle - (\partial_i d(x))^2\langle \nabla^2\gamma(x)\nabla d(x), \nabla d(x)\rangle\\ &+ \langle \nabla\gamma(x), \partial_i^2 d(x)\nabla d(x) + 2\partial_i d(x)\nabla(\partial_i d(x))\rangle\\ =& 2\,\partial_i d(x)\langle \nabla\partial_i\gamma(x), \nabla d(x)\rangle - (\partial_i d(x))^2\partial_\nu^2\gamma(x)\\ &+ \langle \nabla\gamma(x), \partial_i^2 d(x)\nabla d(x) + 2\partial_i d(x)\nabla(\partial_i d(x))\rangle.\end{aligned}$$

Summing over i, we have

$$\Delta\gamma(x) = \partial_\nu^2\gamma(x) + \partial_\nu\gamma(x) + 2\langle \nabla\gamma(x), \nu\nabla\nu\rangle = \partial_\nu^2\gamma(x) + \partial_\nu\gamma(x),$$

where we have used that $\nabla d(x) \in \ker(\nabla^2 d(x))$ [10]. Consequently, for w satisfying the boundary condition $\partial_\nu w = 0$, for $x \in \Gamma$ we have that

$$\Delta(\partial_\nu w) = \partial_\nu^2(\partial_\nu w(x)) + \partial_\nu(\partial_\nu w(x)) = \partial_\nu^3 w(x) + \partial_\nu^2 w(x).$$

Note we have to be careful as to why $\partial_\nu\partial_\nu = \partial_\nu^2$. Then using product rule, we have

$$\Delta(\partial_\nu w) = \partial_\nu\Delta w(x) + \nabla^2 w(x) : \nabla^2 d(x) + \nabla(\Delta d)(x)\cdot\nabla w(x).$$

Consequently,

$$\partial_\nu\Delta w(x) = \partial_\nu^3 w(x) + \partial_\nu^2 w(x) - \nabla^2 w(x) : \nabla^2 d(x) - \nabla(\Delta d)(x)\cdot\nabla w(x). \tag{6.23}$$

We note this formula holds in the trace sense for any $w \in H^4(\Omega)$ as w can be approximated by smooth functions w_i satisfying $\partial_\nu w_i = 0$. Such w_i can be found by considering the elliptic PDE $\Delta w_i = f_i$, $\partial_\nu w_i = 0$ where f_i belongs to $C^\infty(\bar{\Omega})$ and $f_i \to \Delta w$ in $H^2(\Omega)$. Furthermore, for smooth w satisfying $w = \partial_\nu w = \partial_\nu^2 w = 0$, it follows that $\nabla w = 0$ and $\nabla^2 w = 0$ on Γ, so these equalities also hold in the trace sense for any $w \in H^3(\Omega)$ (to approximate such w by smooth function with smooth w_i satisfying $w_i = \partial_\nu w_i = \partial_\nu^2 w_i = 0$, see the space $H_0^3(\Omega)$ [45]).

We then apply Theorem 2.14 to find $w \in H^{4,1}(\Omega_T)$ satisfying the bound

$$\|w\|_{H^{4,1}(\Omega_T)} \le C(\Omega, T)\|\beta\|_{H^{\mu_3,\lambda_3}(\Sigma_T)} \tag{6.24}$$

such that $w = \partial_\nu w = \partial_\nu^2 w = 0$ and $\partial_\nu^3 w = \beta$ on Γ. By (6.23) and the comment following, we have $\partial_\nu\Delta w = \beta$. Considering the trace of w in time, it follows $w(0) \in H^2(\Omega)$ with $\partial_\nu w(0) = 0$ (see Theorem 3.1 of [45]). Let $\bar{c}$ be a strong solution of

$$\begin{cases}\partial_t\bar{c} + \Delta^2\bar{c} = g - (\partial_t + \Delta^2)w & \text{in } \Omega_T,\\ \partial_\nu\bar{c} = 0 & \text{on } \Sigma_T,\\ \partial_\nu(\Delta\bar{c}) = 0 & \text{on } \Sigma_T,\\ \bar{c}(0) = c_0 - w(0) & \text{in } \Omega,\end{cases} \tag{6.25}$$

as guaranteed by Theorem 6.3. Then $c := \bar{c} + w$ solves (6.21), and satisfies the bound (6.22) by (6.24) and (6.20). □

Remark 6.8. *If $g \in H^{4,1}(\Omega_T)$ and $c_0 \in H^6(\Omega)$, and we wish to conclude $c \in H^{8,2}(\Omega_T)$, we must impose the compatibility condition $\partial_\nu(\Delta^2 c_0) = \partial_\nu g(\cdot, 0)$ on Γ, which is well defined by a trace theorem of [45]. To approach intermediate regularity via interpolation, we can conveniently decouple the initial condition from the bulk data. When $g = 0$, interpolation of the map $c_0 \mapsto c$ can be done with the aid of Grisvard's interpolation results for Sobolev spaces with boundary conditions defined by normal operators [38]. However, for the map $g \mapsto c$, we must have a keen understanding of how $H^{0,0}(\Omega_T)$ and $\{g \in H^{4,1}(\Omega_T) : \partial_\nu g(\cdot, 0) = 0\}$ interpolate. Though we may intuitively speculate as to what will be the result of this interpolation, such an undertaking is outside of the scope of an appendix.*

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