# LOCAL WELL-POSEDNESS OF THE CONTACT LINE PROBLEM IN 2-D STOKES FLOW

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Abstract. We consider the evolution of contact lines for viscous fluids in a two-dimensional open-top vessel. The domain is bounded above by a free moving boundary and otherwise by the solid wall of the vessel. The dynamics of the fluid are governed by the incompressible Stokes equations under the influence of gravity, and the interface between fluid and air is under the effect of capillary forces. Here we develop a local well-posedness theory of the problem in the framework of nonlinear energy methods. We utilize several techniques, including: energy estimates of a geometric formulation of the Stokes equations, a Galerkin method with a time-dependent basis for an $\epsilon$–perturbed linear Stokes problem in moving domains, the contraction mapping principle for the $\epsilon$–perturbed nonlinear full contact line problem, and a continuity argument for uniform energy estimates.

## 1. Introduction

1.1. **Formulation of the problem in Eulerian coordinates.** We consider a 2–D open top vessel as a bounded, connected open set $\mathcal{V} \subseteq \mathbb{R}^2$ which consists of two "almost" disjoint sections, i.e., $\mathcal{V} = \mathcal{V}_{top} \cup \mathcal{V}_{bot}$. The word "almost" means $\mathcal{V}_{top} \cap \mathcal{V}_{bot}$ is a set of measure 0 in $\mathbb{R}^2$. We assume that the "top" part $\mathcal{V}_{top}$ consists of a rectangular channel defined by

$$\mathcal{V}_{top} = \mathcal{V} \cap \mathbb{R}^2_+ = \{y \in \mathbb{R}^2 : -\ell < y_1 < \ell, 0 \leq y_2 < L\}$$

for some $\ell$, $L > 0$, where $\mathbb{R}^2_+$ is the half plane $\mathbb{R}^2_+ = \{y \in \mathbb{R}^2 : y_2 \geq 0\}$. Similarly, we write the "bottom" part as

$$\mathcal{V}_{bot} = \mathcal{V} \cap \mathbb{R}^2_- = \mathcal{V} \cap \{y \in \mathbb{R}^2 : y_2 \leq 0\}.$$

In addition, we also assume that the boundary $\partial\mathcal{V}$ of $\mathcal{V}$ is $C^2$ away from the points $(\pm\ell, L)$.

Now we consider a viscous incompressible fluid filling the $\mathcal{V}_{bot}$ entirely and $\mathcal{V}_{top}$ partially. More precisely, we assume that the fluid occupies the domain $\Omega(t)$ with an upper free surface,

$$\Omega(t) = \mathcal{V}_{bot} \cup \{y \in \mathbb{R}^2 : -\ell < y_1 < \ell,\ 0 < y_2 < \zeta(y_1, t)\},$$

where the free surface $\zeta(y_1, t)$ is assumed to be a graph of the function $\zeta : [-\ell, \ell] \times \mathbb{R}_+ \to \mathbb{R}$ satisfying $0 < \zeta(\pm\ell, t) \leq L$ for all $t \in \mathbb{R}_+$, which means the fluid does not spill out of the top domain. For simplicity, we write the free surface as $\Sigma(t) = \{y_2 = \zeta(y_1, t)\}$ and the interface between fluid and solid as $\Sigma_s(t) = \partial\Omega(t) \setminus \Sigma(t)$.

For each $t \geq 0$, the fluid is described by its velocity and pressure $(u, P) : \Omega(t) \to \mathbb{R}^2 \times \mathbb{R}$, the dynamics of which are governed by the incompressible Stokes equations for $t > 0$ :

$$\begin{cases} \operatorname{div} S(P,u) = \nabla P - \mu\Delta u = 0 & \text{in } \Omega(t), \\ \operatorname{div} u = 0 & \text{in } \Omega(t), \\ S(P,u)\nu = g\zeta\nu - \sigma\mathcal{H}(\zeta)\nu & \text{on } \Sigma(t), \\ (S(P,u)\nu - \beta u)\cdot\tau = 0 & \text{on } \Sigma_s(t), \\ u\cdot\nu = 0 & \text{on } \Sigma_s(t), \\ \partial_t\zeta = u\cdot\nu = u_2 - u_1\partial_1\zeta & \text{on } \Sigma(t), \\ \partial_t\zeta(\pm\ell, t) = \mathscr{V}\left(\llbracket\gamma\rrbracket \mp \sigma\dfrac{\partial_1\zeta}{(1+|\partial_1\zeta|^2)^{1/2}}(\pm\ell, t)\right). \end{cases} \tag{1.1}$$

with the initial data $\zeta(y_1, t = 0) = \zeta(0)$, $\partial_t\zeta(y_1, t = 0) = \partial_t\zeta(0)$ and $\partial_t^2\zeta(y_1, t = 0) = \partial_t^2\zeta(0)$.

In the above system (1.1), $S(p,u)$ is the viscous stress tensor determined by

$$S(P,u) = PI - \mu \mathbb{D}u,$$

where $I$ is the $2\times 2$ identity matrix, $\mu > 0$ is the coefficient of viscosity, $\mathbb{D}u = \nabla u + \nabla^\top u$ is the symmetric gradient of $u$ for $\nabla^\top u$ the transpose of the matrix $\nabla u$, $P$ is the difference between the full pressure and the hydrostatic pressure. $\nu$ is the outward unit normal. $\tau$ is the unit tangent. $\sigma > 0$ is the coefficient of surface tension, and

$$\mathcal{H}(\zeta) = \partial_1 \left( \frac{\partial_1 \zeta}{(1+|\partial_1 \zeta|^2)^{1/2}} \right)$$

is the twice of mean curvature of the free surface. $\beta > 0$ is the Navier slip friction coefficient on the vessel walls. The function $\mathscr{V} : \mathbb{R} \to \mathbb{R}$ is the contact point velocity response function which is a $C^2$ increasing diffeomorphism satisfying $\mathscr{V}(0) = 0$. $[\![\gamma]\!] := \gamma_{sv} - \gamma_{sf}$ for $\gamma_{sv}$, $\gamma_{sf} \in \mathbb{R}$, where $\gamma_{sv}$, $\gamma_{sf}$ are a measure of the free-energy per unit length with respect to the solid-vapor and solid-fluid intersection. In addition, we assume that the Young relation [21] holds

$$\frac{|[\![\gamma]\!]|}{\sigma} < 1, \tag{1.2}$$

which is necessary for the existence of equilibrium state. For convenience, we introduce the inverse function $\mathscr{W} = \mathscr{V}^{-1}$ and rewrite the final equation in (1.1) as

$$\mathscr{W}(\partial_t \zeta(\pm\ell, t)) = [\![\gamma]\!] \mp \sigma \frac{\partial_1 \zeta}{(1+|\partial_1 \zeta|^2)^{1/2}}(\pm\ell, t). \tag{1.3}$$

1.2. **A steady equilibrium state.** A steady-state equilibrium solution to (1.1) corresponds to $u = 0$, $P(y,t) = P_0(y)$, and $\zeta(y_1,t) = \zeta_0(y)$. These satisfy

$$\begin{cases} \nabla P_0 = 0 & \text{in } \Omega(0), \\ P_0 = g\zeta_0 - \sigma \mathcal{H}(\zeta_0), & \text{on } (-\ell,\ell), \\ \sigma \dfrac{\partial_1 \zeta_0}{\sqrt{1+|\partial_1 \zeta_0|^2}}(\pm\ell) = \pm [\![\gamma]\!]. \end{cases} \tag{1.4}$$

It is well-known (see for instance the discussion in the introduction of [10]) that there exists a smooth solution $\zeta_0 : [-\ell, \ell] \to (0, L)$.

1.3. **Geometric reformulation.** Let $\zeta_0 \in C^\infty[-\ell,\ell]$ be the equilibrium surface given by (1.4). We then define the equilibrium domain $\Omega \subset \mathbb{R}^2$ by

$$\Omega := \mathcal{V}_b \cup \{x \in \mathbb{R}^2 | -\ell < x_1 < \ell, 0 < x_2 < \zeta_0(x_1)\}.$$

The boundary $\partial\Omega$ of the equilibrium $\Omega$ is defined by

$$\partial\Omega := \Sigma \sqcup \Sigma_s,$$

where

$$\Sigma := \{x \in \mathbb{R}^2 | -\ell < x_1 < \ell, x_2 = \zeta_0(x_1)\}, \quad \Sigma_s = \partial\Omega \setminus \Sigma.$$

Here $\Sigma$ is the equilibrium free surface. The corner angle $\omega \in (0,\pi)$ of $\Omega$ is the contact angle formed by the fluid and solid. We will view the function $\zeta(y_1,t)$ of the free surface as the perturbation of $\zeta_0(y_1)$:

$$\zeta(y_1,t) = \zeta_0(y_1) + \eta(y_1,t). \tag{1.5}$$

Let $\phi \in C^\infty(\mathbb{R})$ be such that $\phi(z) = 0$ for $z \le \frac{1}{4}\min\zeta_0$ and $\phi(z) = z$ for $z \ge \frac{1}{2}\min\zeta_0$. Now we define the mapping $\Phi : \Omega \mapsto \Omega(t)$, by

$$\Phi(x_1,x_2,t) = \left( x_1, x_2 + \frac{\phi(x_2)}{\zeta_0(x_1)} \bar{\eta}(x_1,x_2,t) \right) = (y_1,y_2) \in \Omega(t), \tag{1.6}$$

with $\bar{\eta}$ is defined by

$$\bar{\eta}(x_1,x_2,t) := \mathcal{P}E\eta(x_1, x_2 - \zeta_0(x_1), t), \tag{1.7}$$

where $E : H^s(-\ell,\ell) \mapsto H^s(\mathbb{R})$ is a bounded extension operator for all $0 \le s \le 3$ and $\mathcal{P}$ is the lower Poisson extension given by

$$\mathcal{P}f(x_1,x_2) = \int_{\mathbb{R}} \hat{f}(\xi) e^{2\pi|\xi|x_2} e^{2\pi i x_1 \xi}\,\mathrm{d}\xi.$$

If $\eta$ is sufficiently small (in appropriate Sobolev spaces), the mapping $\Phi$ is a $C^1$ diffeomorphism of $\Omega$ onto $\Omega(t)$ that maps the components of $\partial\Omega$ to the corresponding components of $\partial\Omega(t)$.

We have the Jacobian matrix $\nabla\Phi$ and the transform matrix $\mathcal{A}$ of $\Phi$

$$\nabla\Phi = \begin{pmatrix} 1 & 0 \\ A & J \end{pmatrix}, \quad \mathcal{A} = (\nabla\Phi)^{-\top} = \begin{pmatrix} 1 & -AK \\ 0 & K \end{pmatrix}, \tag{1.8}$$

for

$$A = \frac{\phi}{\zeta_0}\partial_1\bar{\eta} - \frac{\phi}{\zeta_0^2}\partial_1\zeta_0\bar{\eta}, \quad J = 1 + \frac{\phi'}{\zeta_0}\bar{\eta} + \frac{\phi}{\zeta_0}\partial_2\bar{\eta}, \quad K = \frac{1}{J}. \tag{1.9}$$

We define the transformed differential operators as follows.

$$(\nabla_{\mathcal{A}} f)_i := \mathcal{A}_{ij}\partial_j f, \quad \operatorname{div}_{\mathcal{A}} X := \mathcal{A}_{ij}\partial_j X_i, \quad \Delta_{\mathcal{A}} f := \operatorname{div}_{\mathcal{A}} \nabla_{\mathcal{A}} f,$$

for appropriate $f$ and $X$. We write the stress tensor

$$S_{\mathcal{A}}(P,u) = PI - \mu\mathbb{D}_{\mathcal{A}} u$$

where $I$ the $2\times 2$ identity matrix and $(\mathbb{D}_{\mathcal{A}}u)_{ij} = \mathcal{A}_{ik}\partial_k u_j + \mathcal{A}_{jk}\partial_k u_i$ the symmetric $\mathcal{A}$–gradient. Note that if we extend $\operatorname{div}_{\mathcal{A}}$ to act on symmetric tensors in the natural way, then $\operatorname{div}_{\mathcal{A}} S_{\mathcal{A}}(P,u) = -\mu\Delta_{\mathcal{A}} u + \nabla_{\mathcal{A}} P$ for vectors fields satisfying $\operatorname{div}_{\mathcal{A}} u = 0$.

We assume that $\Phi$ is a diffeomorphism. Then we can transform the problem (1.1) to the equilibrium domain $\Omega$ for $t \ge 0$. In the new coordinates, (1.1) becomes the $\mathcal{A}$–Stokes problem

$$\begin{cases} \operatorname{div}_{\mathcal{A}} S_{\mathcal{A}}(P,u) = -\mu\Delta_{\mathcal{A}} u + \nabla_{\mathcal{A}} P = 0, & \text{in} \quad \Omega, \\ \operatorname{div}_{\mathcal{A}} u = 0, & \text{in} \quad \Omega, \\ S_{\mathcal{A}}(P,u)\mathcal{N} = g\zeta\mathcal{N} - \sigma\mathcal{H}(\zeta)\mathcal{N}, & \text{on} \quad \Sigma, \\ (S_{\mathcal{A}}(P,u)\nu - \beta u)\cdot\tau = 0, & \text{on} \quad \Sigma_s, \\ u\cdot\nu = 0, & \text{on} \quad \Sigma_s, \\ \partial_t\zeta = u\cdot\mathcal{N}, & \text{on} \quad \Sigma, \\ \mathscr{W}(\partial_t\zeta(\pm\ell,t)) = [\![\gamma]\!] \mp \sigma\dfrac{\partial_1\zeta}{\sqrt{1+|\zeta|^2}}(\pm\ell,t), & \\ \zeta(x_1,0) = \zeta_0(x_1) + \eta_0(x_1), \quad \partial_t\zeta(x_1,0) = \partial_t\eta(x_1,0), \quad \partial_t^2\zeta(x_1,0) = \partial_t^2\eta(x_1,0). & \end{cases} \tag{1.10}$$

Here we have still written $\mathcal{N} := -\partial_1\zeta e_1 + e_2$ for the normal to $\Sigma(t)$.

Since all terms in (1.10) are in terms of $\eta$, (1.10) is connected to the geometry of the free surface. This geometric structure is essential to control higher-order derivatives.

1.4. **Perturbation.** We will construct the solution to (1.10) as a perturbation around the equilibrium state $(0, P_0, \zeta_0)$. To this end we define new perturbed unknowns $(u,p,\eta)$ so that $u = 0+u$, $P = P_0 + p$, and $\zeta = \zeta_0 + \eta$. Then we will reformulate (1.10) in terms of the new unknowns.

First, we rewrite the terms of mean curvature on the equilibrium free surface. By the Taylor expansion in $z$,

$$\frac{y+z}{\sqrt{1+|y+z|^2}} = \frac{y}{\sqrt{1+|y|^2}} + \frac{z}{(1+|y|^2)^{3/2}} + \mathcal{R}(y,z). \tag{1.11}$$

Combining with the assumption (1.5), we then know that

$$\frac{\partial_1\zeta}{\sqrt{1+|\partial_1\zeta|^2}} = \frac{\partial_1\zeta_0}{\sqrt{1+|\partial_1\zeta_0|^2}} + \frac{\partial_1\eta}{(1+|\partial_1\zeta_0|^2)^{3/2}} + \mathcal{R}(\partial_1\zeta_0, \partial_1\eta). \tag{1.12}$$

where the remainder term $\mathcal{R} \in C^\infty(\mathbb{R}^2)$ is given by

$$\mathcal{R}(y,z) = \int_0^z 3\frac{(s-z)(s+y)}{(1+(y+s)^2)^{5/2}}\,\mathrm{d}s. \tag{1.13}$$

Thus

$$\begin{aligned} g\zeta - \sigma\mathcal{H}(\zeta) &= (g\zeta_0 - \sigma\mathcal{H}(\zeta_0)) + g\eta - \sigma\partial_1\left(\frac{\partial_1\eta}{(1+|\partial_1\zeta_0|)^{3/2}}\right) - \sigma\partial_1(\mathcal{R}(\partial_1\zeta_0, \partial_1\eta)) \\ &= p_0 + g\eta - \sigma\partial_1\left(\frac{\partial_1\eta}{(1+|\partial_1\zeta_0|)^{3/2}}\right) - \sigma\partial_1(\mathcal{R}(\partial_1\zeta_0, \partial_1\eta)), \end{aligned} \tag{1.14}$$

and

$$\begin{aligned} [\![\gamma]\!] \mp \sigma\frac{\partial_1\zeta}{\sqrt{1+|\partial_1\zeta|^2}}(\pm\ell, t) &= [\![\gamma]\!] \mp \sigma\frac{\partial_1\zeta_0}{\sqrt{1+|\partial_1\zeta_0|^2}}(\pm\ell, t) \mp \sigma\frac{\partial_1\eta}{(1+|\zeta_0|^2)^{3/2}} \\ \mp \mathcal{R}(\partial_1\zeta_0, \partial_1\eta)(\pm\ell, t) &= \mp\sigma\frac{\partial_1\eta}{(1+|\partial_1\zeta_0|^2)^{3/2}}(\pm\ell, t) \mp \mathcal{R}(\partial_1\zeta_0, \partial_1\eta)(\pm\ell, t). \end{aligned} \tag{1.15}$$

Next we rewrite the terms related to stress tensor in (1.10). Clearly,

$$\begin{aligned} \operatorname{div}_{\mathcal{A}} S_{\mathcal{A}}(P, u) &= \operatorname{div}_{\mathcal{A}} S_{\mathcal{A}}(p, u), \text{ in } \Omega, \\ S_{\mathcal{A}}(P, u)\mathcal{N} &= S_{\mathcal{A}}(p, u)\mathcal{N} + P_0\mathcal{N}, \text{ on } \Sigma, \\ S_{\mathcal{A}}(P, u)\nu \cdot \tau &= S_{\mathcal{A}}(p, u)\nu \cdot \tau, \text{ on } \Sigma_s. \end{aligned} \tag{1.16}$$

Finally, we rewrite the inverse $\mathscr{W} \in C^2(\mathbb{R})$ of contact point response function. Since $\mathscr{W}(0) = 0$, we expand $\mathscr{W}$ as

$$\mathscr{W}(z) = \mathscr{W}'(0)z + \tilde{\mathscr{W}}(z). \tag{1.17}$$

Then we write $\kappa = \mathscr{W}'(0) > 0$, since $\mathscr{W}$ is increasing. For convenience, we write

$$\hat{\mathscr{W}}(z) = \frac{1}{\kappa}\tilde{\mathscr{W}}(z) = \frac{1}{\kappa}\mathscr{W}(z) - z. \tag{1.18}$$

Thus, combining (1.4), (1.14)–(1.16), we arrive at the following perturbative form of Stokes equations

$$\begin{cases} \operatorname{div}_{\mathcal{A}} S_{\mathcal{A}}(p, u) = -\mu\Delta_{\mathcal{A}}u + \nabla_{\mathcal{A}}p = 0, & \text{in } \Omega, \\ \operatorname{div}_{\mathcal{A}} u = 0, & \text{in } \Omega, \\ S_{\mathcal{A}}(p, u)\mathcal{N} = g\eta\mathcal{N} - \sigma\partial_1\left(\dfrac{\partial_1\eta}{(1+|\partial_1\zeta_0|)^{3/2}}\right)\mathcal{N} - \sigma\partial_1(\mathcal{R}(\partial_1\zeta_0, \partial_1\eta))\mathcal{N}, & \text{on } \Sigma, \\ (S_{\mathcal{A}}(p, u)\nu - \beta u) \cdot \tau = 0, & \text{on } \Sigma_s, \\ u \cdot \nu = 0, & \text{on } \Sigma_s, \\ \partial_t\eta = u \cdot \mathcal{N}, & \text{on } \Sigma, \\ \kappa\partial_t\eta(\pm\ell, t) + \kappa\hat{\mathscr{W}}(\partial_t\eta(\pm\ell, t)) = \mp\sigma\left(\dfrac{\partial_1\eta}{(1+|\zeta_0|^2)^{3/2}} + \mathcal{R}(\partial_1\zeta_0, \partial_1\eta)\right)(\pm\ell, t). \end{cases} \tag{1.19}$$

with the initial data $\eta(x_1, 0) = \eta_0(x_1)$, $\partial_t\eta(x_1, 0)$ and $\partial_t^2\eta(x_1, 0)$. Here $\mathcal{A}$ and $\mathcal{N}$ are still determined in terms of $\zeta = \zeta_0 + \eta$. In the following, we write $\mathcal{N}_0$ be the non-unit normal for the equilibrium surface $\Sigma$, and $\mathcal{N} = \mathcal{N}_0 - \partial_1\eta e_1$.

1.5. **Main theorem.** In order to state our result, we need to explain our notation for Sobolev spaces and norms. We take $H^k(\Omega)$ and $H^k(\Sigma)$ for $k \geq 0$ to be the usual Sobolev spaces, and take $W^k_\delta(\Omega)$ and $W^k_\delta(\Sigma)$ for $k \geq 0$ and $\delta \in (0, 1)$ to be the weighted Sobolev spaces defined in (2.24). We write norms $\|\partial_t^j u\|_k$ and $\|\partial_t^j p\|_k$ in the space $H^k(\Omega)$, and $\|\partial_t^j \eta\|_k$ in space $H^k(\Sigma)$.

Now, we define the energy and dissipation used in this paper. The energy is

$$\mathcal{E}(t) = \|u\|^2_{W^2_\delta} + \|\partial_t u\|^2_1 + \|p\|^2_{\mathring{W}^1_\delta} + \|\partial_t p\|^2_{\mathring{H}^0} + \|\eta\|^2_{W^{5/2}_\delta} + \|\partial_t\eta\|^2_{3/2} + \sum_{j=0}^{2}\|\partial_t^j\eta\|^2_{\mathring{H}^1}, \tag{1.20}$$

and the dissipation is

$$\begin{aligned}\mathcal{D}(t) = &\sum_{j=0}^{1}\left(\|\partial_t^j u\|_{W^2_\delta}^2 + \|\partial_t^j p\|_{\mathring{W}^1_\delta}^2 + \|\partial_t^j \eta\|_{W^{5/2}_\delta}^2\right) + \sum_{j=0}^{2}\left(\|\partial_t^j u\|_1^2 + \|\partial_t^j u\|_{H^0(\Sigma_s)}^2 + [\partial_t^j u\cdot\mathcal{N}]_\ell^2\right)\\ &+\sum_{j=0}^{2}\left(\|\partial_t^j p\|_0^2 + \|\partial_t^j \eta\|_{3/2}^2\right) + \|\partial_t^3\eta\|_{W^{1/2}_\delta}^2,\end{aligned} \tag{1.21}$$

where $[f]_\ell^2$ is defined by (2.11) and Remark (2.2), $\mathring{H}^s((-\ell,\ell))$ is defined in (2.3) and $\mathring{W}^k_\delta(\Omega)$ is defined in (2.31).

With the notation established we may now state our main result.

**Theorem 1.1.** *Assume the initial data $\eta_0 \in \mathring{W}^{5/2}_\delta((-\ell,\ell))$, $\partial_t\eta(0) \in \mathring{H}^{3/2}((-\ell,\ell))$ and $\partial_t^2\eta(0) \in \mathring{H}^1((-\ell,\ell))$ satisfy the compatibility condition in Section 3. Then there exists $0<\alpha_0$, $T_0<1$, such that if*

$$\mathfrak{E}_0 := \|\eta_0\|^2_{W^{5/2}_\delta} + \|\partial_t\eta(0)\|^2_{3/2} + \sum_{j=0}^{2}\|\partial_t^j\eta(0)\|_1^2 \le \alpha_0$$

*and $0<T<T_0$, then there exists a unique solution $(u,p,\eta)$ to (1.19) on the interval $[0,T]$ that achieves the initial data and satisfies*

$$\sup_{0\le t\le T}\mathcal{E}(t) + \int_0^T \mathcal{D}(t)\,\mathrm{d}t \le C\mathfrak{E}_0 \tag{1.22}$$

*for a universal constant $C>0$. Moreover, $\Phi$ defined by (1.6) is a $C^1$ diffeomorphism for each $t\in[0,T]$.*

**Remark 1.2.** *Since $\Phi$ is a $C^1$ diffeomorphism, we can change coordinates from $\Omega$ to $\Omega(t)$ to gain solutions of (1.1).*

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The techniques used for the proof of Theorem (1.1) are developed throughout the rest of this paper. We will sketch the main ideas of the proof here.

***$\epsilon$-perturbed linear $\mathcal{A}$-Stokes***. Our fixed point procedure is based on a geometric formulation of linear Stokes equations. We suppose that $\eta$ (and hence $\mathcal{A}$, $\mathcal{N}$, etc.) is given and then solve the linear $\mathcal{A}$–Stokes equations for $(u,p,\xi)$:

$$\begin{cases}\operatorname{div}_{\mathcal{A}} S_{\mathcal{A}}(p,u) = F^1, & \text{in } \Omega,\\ \operatorname{div}_{\mathcal{A}} u = 0, & \text{in } \Omega,\\ S_{\mathcal{A}}(p,u)\mathcal{N} = \left(g\xi - \sigma\partial_1\left(\dfrac{\partial_1\xi}{(1+|\partial_1\zeta_0|^2)^{3/2}}\right) - \sigma\partial_1 F^3\right)\mathcal{N} + F^4, & \text{on } \Sigma,\\ (S_{\mathcal{A}}(p,u)\nu - \beta u)\cdot\tau = F^5, & \text{on } \Sigma_s,\\ u\cdot\nu = 0, & \text{on } \Sigma_s,\\ \partial_t\xi = u\cdot\mathcal{N}, & \text{on } (-\ell,\ell),\\ \mp\sigma\dfrac{\partial_1\xi}{(1+|\partial_1\zeta_0|^2)^{3/2}}(\pm\ell) = \kappa(u\cdot\mathcal{N})(\pm\ell) \pm \sigma F^3(\pm\ell) - \kappa\hat{\mathscr{W}}(\partial_t\eta(\pm\ell)),\end{cases} \tag{1.23}$$

where $F^3 = \mathcal{R}(\partial_1\zeta_0, \partial_1\eta)$. However, if we solve straightforward to (1.23), we might confront one difficulty in the energy estimates. To be more specific , we couldn't estimate

$$\int_{-\ell}^{\ell}\partial_z\mathcal{R}\partial_1\partial_t^2\eta\partial_1\partial_t^3\xi, \tag{1.24}$$

which means we couldn't use Theorem 4.8 to derive Theorem 4.13. That's because in our fixed point procedure, we want to prove that an appropriate map $A$ acting on a complete metric space such that $A\eta = \xi$ has a fixed point. This means that $\partial_t^3\xi$ is only in $W^{1/2}_\delta$ due to the prori estimate for $\partial_t^3\eta$. We

refer to Section 6 in [10] for more details. In order to conquer this difficulty, we consider the following $\epsilon$–perturbed linear $\mathcal{A}$-Stokes instead of (1.23):

$$\begin{cases} \operatorname{div}_{\mathcal{A}} S_{\mathcal{A}}(p,u) = F^1, & \text{in } \Omega, \\ \operatorname{div}_{\mathcal{A}} u = 0, & \text{in } \Omega, \\ S_{\mathcal{A}}(p,u)\mathcal{N} = \left( g(\xi+\epsilon\xi_t) - \sigma\partial_1 \left( \dfrac{\partial_1\xi + \epsilon\partial_1\xi_t}{(1+|\partial_1\zeta_0|^2)^{3/2}} \right) - \sigma\partial_1 F^3 \right)\mathcal{N} + F^4, & \text{on } \Sigma, \\ (S_{\mathcal{A}}(p,u)\nu - \beta u)\cdot\tau = F^5, & \text{on } \Sigma_s, \\ u\cdot\nu = 0, & \text{on } \Sigma_s, \\ \partial_t\xi = u\cdot\mathcal{N}, & \text{on } (-\ell,\ell), \\ \mp\sigma \dfrac{\partial_1\xi + \epsilon\partial_1\xi_t}{(1+|\partial_1\zeta_0|^2)^{3/2}}(\pm\ell) = \kappa(u\cdot\mathcal{N})(\pm\ell) \pm \sigma F^3(\pm\ell) - \kappa\hat{\mathscr{W}}(\partial_t\eta(\pm\ell)), \end{cases} \tag{1.25}$$

From the mean curvature term, $\partial_t^3\xi$ has the same regularity with $\partial_t^2\xi$ that is in $H^1$. Then we could estimate the (1.24). Since $\hat{\mathscr{W}}(\cdot)$ is a nonlinear function, we suppose that its variable $\partial_t\eta$ is given at the corner point $x_1 = \pm\ell$. This assumption could not cause any confusion because of the correspondence with $u\cdot\mathcal{N}$ in the fixed point procedure.

***Galerkin method with a time-dependent basis***. We construct solutions to (1.25) by Galerkin method. This requires a countable basis of our space satisfying $\operatorname{div}_{\mathcal{A}}$ free condition. Since $\operatorname{div}_{\mathcal{A}} u = 0$ is time-dependent, any basis of our space must also be time-dependent. Fortunately, following the path of Theorem 4.3 in [9], we could conquer this technical difficulty.

***Contraction mapping***. Since the linear Stokes has been perturbed, we want to use the contraction mapping principle to the nonlinear $\epsilon$–perturbed Stokes equation for a small $T_\epsilon > 0$. Here we will construct a complete metric space which preserves bounds in higher norm but with some weaker norms as metric. This enables us to prove the strict contraction in a weak topology for this metric space.

***Continuity method for uniform energy estimate***. In order to obtain the solution to the original equations (1.19), we need to pass the limit $\epsilon \to 0$. Due to the Section 8 in [10], we could easily use the continuity method to get the uniform bounds with respect to $\epsilon$.

1.6. **Notation and terminology.** Now, we mention some definitions, notation and conventions that we will use throughout this paper.

1. Constants. The symbol $C > 0$ will denote a universal constant that only depends on the parameters of the problem and $\Omega$, but does not depend on the data, etc. They are allowed to change from line to line. We will write $C = C(z)$ to indicate that the constant $C$ depends on $z$. And we will write $a \lesssim b$ to mean that $a \leq Cb$ for a universal constant $C > 0$.

2. Norms. We will write $H^k$ for $H^k(\Omega)$ for $k \geq 0$, and $H^s(\Sigma)$ with $s \in \mathbb{R}$ for usual Sobolev spaces. W will typically write $H^0 = L^2$, though we will also use $L^2([0,T];H^k)$ (or $L^2([0,T];H^s(\Sigma))$) to denote the space of temporal square–integrable functions with values in $H^k$ (or $H^s(\Sigma)$). Sometimes we will write $\|\cdot\|_k$ instead of $\|\cdot\|_{H^k(\Omega)}$ or $\|\cdot\|_{H^k(\Sigma)}$. When we do this it will be clear on which set the norm is evaluated from the context and the argument of the norm.

1.7. **Plan of the paper.** In Section 2, we review the machinery of time-dependent function spaces, $\operatorname{div}_{\mathcal{A}}$–free vector fields, and weighted Sobolev spaces. In Section 3, we construct the initial data and derive estimates. In Section 4 we study the local well-posedness of the $\epsilon$–linear problem. In Section 5 we construct solutions to (1.19) using a contraction mapping argument and a continuity method, and then we complete the proof of the main result.

## 2. Functional setting and basic estimates

2.1. **Function spaces.** First, we define some time-independent spaces:

$$\mathring{H}^0(\Omega) = \{p \in H^0(\Omega) | \int_\Omega p = 0\}, \tag{2.1}$$

$$\mathring{H}^0((-\ell,\ell)) = \{\eta \in H^0((-\ell,\ell))| \int_{-\ell}^{\ell} \eta = 0\}, \tag{2.2}$$

$$\mathring{H}^k(\Omega) = H^k(\Omega) \cap \mathring{H}^0(\Omega), \quad \mathring{H}^s((-\ell,\ell)) = H^s((-\ell,\ell)) \cap \mathring{H}^0((-\ell,\ell)), \tag{2.3}$$

$$_0H^1(\Omega) = \{u \in H^1(\Omega) | u \cdot \nu = 0 \text{ on } \Sigma_s\}, \tag{2.4}$$

endowed with the usual $H^1$ norm. We also set

$$W := \{u \in {}_0H^1(\Omega) | u \cdot \mathcal{N}_0 \in H^1(-\ell,\ell) \cap \mathring{H}^0(-\ell,\ell)\}, \tag{2.5}$$

endowed with norm $\|u\|_W := \|u\|_1 + \|u \cdot \mathcal{N}_0\|_{H^1((-\ell,\ell))}$, and we write

$$V := \{u \in W | \operatorname{div} u = 0\}. \tag{2.6}$$

Throughout the paper we will often utilize the following Korn-type inequality.

**Lemma 2.1.** *For any $u \in {}_0H^1(\Omega)$, it holds that*

$$\|u\|_1^2 \lesssim \|\mathbb{D}u\|_0^2. \tag{2.7}$$

*Proof.* The inequality (2.7) follows easily from the inequality

$$\|u\|_1^2 \lesssim \|\mathbb{D}u\|_0^2 + \|u\|_0^2 \text{ for all } u \in H^1(\Omega), \tag{2.8}$$

and a standard compactness argument. The inequality (2.8) is may be proved in different ways. See [16] for a direct proof. It can also be derived from the Nečas inequality: see for example Lemma IV.7.6 in [3]. □

Suppose that $\eta$ is given and that $\mathcal{A}$, $J$ and $\mathcal{N}$, etc are determined in terms of $\eta$. Let us define

$$((u,v)) := \int_\Omega \frac{\mu}{2} \mathbb{D}_{\mathcal{A}} u : \mathbb{D}_{\mathcal{A}} v J + \int_{\Sigma_s} \beta (u \cdot \tau)(v \cdot \tau) J. \tag{2.9}$$

We also define

$$(\phi,\psi)_{1,\Sigma} := \int_{-\ell}^{\ell} g\phi\psi + \sigma \frac{\partial_1\phi \partial_1\psi}{(1+|\partial_1\zeta_0|^2)^{3/2}}, \tag{2.10}$$

and

$$[a,b]_\ell := \kappa(a(\ell)b(\ell) + a(-\ell)b(-\ell)). \tag{2.11}$$

**Remark 2.2.** *Throughout this paper, we write $\|\xi\|_{1,\Sigma}$ as*

$$\|\xi\|_{1,\Sigma}^2 = \int_{-\ell}^{\ell} g|\xi|^2 + \sigma \frac{|\partial_1\xi|^2}{(1+|\partial_1\zeta_0|^2)^{3/2}},$$

*and write $[\phi]_\ell$ as*

$$[\phi]_\ell^2 = \kappa(\phi(\ell)^2 + \phi(-\ell)^2).$$

For convenience, let us define the spaces

$$\mathcal{H}^0(\Omega) := \{u : \Omega \to \mathbb{R}^2 | \sqrt{J}u \in H^0(\Omega)\}, \tag{2.12}$$

with norm $\|u\|_{\mathcal{H}^0(\Omega)} := (\int_\Omega |u|^2 J)^{1/2}$ and

$$_0\mathcal{H}^1(\Omega) := \{u : \Omega \to \mathbb{R}^2 | ((u,u)) < \infty, u \cdot \nu = 0 \text{ on } \Sigma_s\}, \tag{2.13}$$

endowed with norm $\|u\|_{{}_0\mathcal{H}^1(\Omega)} = ((u,u))^{1/2}$.

Let us define some time-dependent spaces. We define the space

$$\mathcal{W}(t) := \{v(t) \in {}_0\mathcal{H}^1(\Omega) | v \cdot \mathcal{N} \in H^1(-\ell,\ell) \cap \mathring{H}^0(-\ell,\ell)\}, \tag{2.14}$$

which we endow with the inner-product

$$(u,v)_{\mathcal{W}} = ((u,v)) + (u \cdot \mathcal{N}, v \cdot \mathcal{N})_{1,\Sigma}. \tag{2.15}$$

We also define the subspace $\mathcal{V}(t)$ of $\mathcal{W}(t)$ by

$$\mathcal{V}(t) := \{u(t) \in \mathcal{W}(t) | \operatorname{div}_{\mathcal{A}} u = 0\}. \tag{2.16}$$

Finally, we define the inner products on $L^2([0,T];H^k(\Omega))$ for $k=0,1$ as

$$(u,v)_{\mathcal{H}^1_T}=\int_0^T (u(t),v(t))_{\mathcal{H}^1}\,\mathrm{d}t, \tag{2.17}$$

and write $\mathcal{H}^1_T$ as the corresponding spaces with the corresponding norms $\|u\|_{\mathcal{H}^1_T}$. We define the subspaces of $\mathcal{H}^1_T$ as follows:

$${}_0\mathcal{H}^1_T:=\{u\in\mathcal{H}^1_T | u\cdot\nu=0 \text{ on } \Sigma_s\}, \tag{2.18}$$

which we endow with the norm $\mathcal{H}^1_T$,

$$\mathcal{W}_T:=\{v\in {}_0\mathcal{H}^1_T | v\cdot\mathcal{N}\in H^1(-\ell,\ell)\cap \mathring{H}^0(-\ell,\ell)\}, \tag{2.19}$$

which we endow with the norm $\|v\|_{\mathcal{W}_T}:=\|v\|_{\mathcal{H}^1_T}+\left(\int_0^T \|v\cdot\mathcal{N}\|^2_{H^1(-\ell,\ell)}\right)^{1/2}$, and

$$\mathcal{V}_T:=\{u\in\mathcal{W}_T | \operatorname{div}_{\mathcal{A}} u=0\}. \tag{2.20}$$

The following lemma implies that ${}_0\mathcal{H}^1(\Omega)$ is equivalent to ${}_0H^1(\Omega)$.

**Lemma 2.3.** *There exists a universal $\alpha_0>0$ such that if*

$$\sup_{0\le t\le T}\|\eta(t)\|_{W^{5/2}_\delta}<\alpha_0, \tag{2.21}$$

*then*

$$\frac{1}{\sqrt{2}}\|u\|_k\le\|u\|_{\mathcal{H}^k}\le\sqrt{2}\|u\|_k \tag{2.22}$$

*for $k=0,1$ and for all $t\in[0,T]$. As a consequence, for $k=0,1$,*

$$\|u\|_{L^2H^k(\Omega)}\le\|u\|_{\mathcal{H}^k_T(\Omega)}\le\|u\|_{L^2H^k(\Omega)}. \tag{2.23}$$

*Proof.* The case $k=0$ is proved in Lemma 2.1 in [9]. A result similar to that stated above for $k=1$ is also prove in [9] for a norm not involving the boundary terms. However, the argument used there may be readily coupled to a trace estimate to handle the boundary term. □

For our problem, we need weighted Sobolev spaces. Suppose that $\omega\in(0,\pi)$ is the angle formed by $\zeta_0$ at the corner $M$ of $\Omega$, for $M=\{(-\ell,\zeta_0(-\ell)),(\ell,\zeta_0(\ell))\}$ the corner points of $\Omega$. We now introduce the critical weight $\delta_\omega:=\max\{0,2-\pi/\omega\}\in[0,1)$. For $\delta\in(0,\delta_\omega)$, we define

$$W^k_\delta(\Omega)(t)=:\{u(t) | \|u(t)\|_{W^k_\delta(\Omega)}<\infty\}, \tag{2.24}$$

with the norm

$$\|u(t)\|_{W^k_\delta(\Omega)}=\Big(\sum_{|\alpha|\le k}\int_\Omega d^{2\delta}|\partial^\alpha u(x,t)|^2\,\mathrm{d}x\Big)^{1/2}, \tag{2.25}$$

where $d=dist(\cdot,M)$. A consequence of Hardy's inequality (for example Lemma 7.1.1 in [12]) reveals that we have the continuous embeddings

$$W^1_\delta(\Omega)\hookrightarrow H^0(\Omega),\ W^2_\delta(\Omega)\hookrightarrow H^1(\Omega),\ H^1(\Omega)\hookrightarrow W^0_{-\delta}(\Omega), \tag{2.26}$$

when $\delta\in(0,1)$.

The trace spaces $W^{k-1/2}_\delta(\partial\Omega)$ can be defined in the usual way: see for example Section 7.1.3 in [12]. It can be shown that the following useful lemma holds.

FIXME: is this used explicitly anywhere?

**Lemma 2.4.** *Suppose that $0<\delta<1$ and $\tau$ be the unit tangential of $\partial\Omega$. Then*

$$\left|\int_{\partial\Omega} f(v\cdot\tau)\right|\lesssim\|f\|_{W^{1/2}_\delta(\partial\Omega)}\|v\|_{H^1(\Omega)}, \tag{2.27}$$

*for all $f\in W^{1/2}_\delta(\partial\Omega)$ and $v\in H^1(\Omega)$.*

*Proof.* We choose $p$, $q$ such that $1 < p < \frac{2}{1+\delta}$ and $\frac{1}{p} + \frac{1}{q} = 1$. Employing the Hölder inequality, we may derive that

$$\left| \int_{\partial\Omega} f(v \cdot \tau) \right| \leq \|f\|_{L^p(\partial\Omega)} \|v\|_{L^q(\partial\Omega)}. \tag{2.28}$$

The Sobolev embedding implies that

$$\|f\|_{L^p(\partial\Omega)} \lesssim \|f\|_{W^{1/2}_\delta(\partial\Omega)}. \tag{2.29}$$

And The Sobolev embedding together with usual trace theory imply that

$$\|v\|_{L^q(\partial\Omega)} \lesssim \|v\|_{H^{1/2}(\partial\Omega)} \lesssim \|v\|_{H^1(\Omega)}. \tag{2.30}$$

□

Also, we define the spaces

$$\mathring{W}^k_\delta(\Omega) := \left\{ u \in W^k_\delta(\Omega) | \int_\Omega u = 0 \right\}, \tag{2.31}$$

for $k \geq 1$. The spaces $L^2([0,T]; W^k_\delta(\Omega))$ is defined by

$$\|u\|^2_{L^2([0,T];W^k_\delta(\Omega))} := \int_0^T \|u(t)\|^2_{W^k_\delta(\Omega)} < \infty. \tag{2.32}$$

Now, we want to show that the time-independent spaces are related to the time-dependent spaces. We consider the matrix

$$M := M(t) = K\nabla\Phi = (J\mathcal{A}^\top)^{-1}, \tag{2.33}$$

which induces a linear operator $\mathcal{M}_t : u \mapsto M(t)u$.

**Proposition 2.5.** *Assume that $\eta \in H^r((-\ell,\ell))$ for $r > \frac{3}{2}$.*

*(1) For each $t \in [0,T]$, $\mathcal{M}_t$ is a bounded isomorphism from $H^k(\Omega)$ to $H^k(\Omega)$ for $k = 0,1,2$.*
*(2) For each $t \in [0,T]$, $\mathcal{M}_t$ is a bounded isomorphism from ${}_0H^1(\Omega)$ to ${}_0\mathcal{H}^1(\Omega)$. Moreover,*

$$\|Mu\|_{{}_0\mathcal{H}^1} \lesssim (1 + \|\eta\|_r)\|u\|_1 \tag{2.34}$$

*(3) Let $u \in H^1(\Omega)$. Then $\operatorname{div} u = p$ if and only if $\operatorname{div}_{\mathcal{A}}(Mu) = Kp$.*

*Proof.* See [9] and [10]. □

The following proposition is also useful.

**Proposition 2.6.** *If $u \cdot \nu = 0$ on $\Sigma_s$, then $Ru \cdot \nu = 0$ on $\Sigma_s$, where $R := \partial_t M M^{-1}$.*

*Proof.* According to Proposition 4.4 in [10], we have known that $Mu \cdot \nu = 0 \Leftrightarrow u \cdot \nu = 0$ on $\Sigma_s$, which implies that $M^{-1}u \cdot \nu = 0 \Leftrightarrow u \cdot \nu = 0$ on $\Sigma_s$. Then by definition of $R$,

$$Ru \cdot \nu = \partial_t M M^{-1} u \cdot \nu = -M\partial_t(M^{-1}u) \cdot \nu = 0, \tag{2.35}$$

since $\partial_t(M^{-1}u) \cdot \nu = \partial_t(M^{-1}u \cdot \nu) = 0$. □

## 3. Initial data

3.1. **Construction of initial data.** Before we study the well-posedness of (1.19), we first consider the initial data and the initial energy $\mathcal{E}(0)$. Suppose that $\eta_0 \in W^{5/2}_\delta(\Sigma)$, $\partial_t\eta(0) \in H^{3/2}(\Sigma)$, $\partial_t^2\eta(0) \in H^1(\Sigma)$ and that

$$\mathfrak{E}_0(\eta) := \|\eta_0\|^2_{W^{5/2}_\delta(\Sigma)} + \|\partial_t\eta(0)\|^2_{H^{3/2}(\Sigma)} + \sum_{j=0}^{2} \|\partial_t^j\eta(0)\|^2_{H^1(\Sigma)} \leq \alpha$$

where $\alpha > 0$ is small enough to satisfy the conditions in Lemma 2.3 and Theorem 5.8 in [10]. We now construct the initial data $u(t=0) = u_0$ and $p(t=0) = p_0$. When $t = 0$, we consider the elliptic equation

$$\begin{cases} \operatorname{div}_{\mathcal{A}(0)} S_{\mathcal{A}(0)}(p_0, u_0) = 0, & \text{in} \quad \Omega, \\ \operatorname{div}_{\mathcal{A}(0)} u_0 = 0, & \text{in} \quad \Omega, \\ u_0 \cdot \mathcal{N}(0) = \partial_t \eta(0), & \text{on} \quad \Sigma, \\ \mu \mathbb{D}_{\mathcal{A}(0)} u_0 \mathcal{N}(0) \cdot \mathcal{T}(0) = 0, & \text{on} \quad \Sigma, \\ u_0 \cdot \nu = 0, & \text{on} \quad \Sigma_s, \\ \mu \mathbb{D}_{\mathcal{A}(0)} u_0 \nu \cdot \tau - \beta u_0 \cdot \tau = 0, & \text{on} \quad \Sigma_s. \end{cases} \tag{3.1}$$

We employ the Theorem 5.9 in [10] to deduce that there exists a unique $(u_0, p_0) \in W^2_\delta \times \mathring{W}^1_\delta$, and

$$\|u_0\|^2_{W^2_\delta} + \|p_0\|^2_{\mathring{W}^1_\delta} \lesssim \|\partial_t \eta(0)\|^2_{W^{3/2}_\delta} \lesssim \|\partial_t \eta(0)\|^2_{3/2}. \tag{3.2}$$

Clearly, from the embedding $W^2_\delta(\Omega) \hookrightarrow H^1(\Omega)$ and the boundary condition, $u_0 \in \mathcal{V}(0)$.

Then we construct $\partial_t u(0)$ and $\partial_t p(0)$. In order to preserve the divergence free condition, we construct $D_t u(0)$ instead of $\partial_t u(0)$, where $D_t u$ is defined in (4.11). Now we temporally differentiate the equation (1.19), then take $t = 0$,

$$\begin{cases} \operatorname{div}_{\mathcal{A}(0)} S_{\mathcal{A}(0)}(\partial_t p(0), D_t u(0)) = \tilde{F}(0), & \text{in} \quad \Omega, \\ \operatorname{div}_{\mathcal{A}(0)} D_t u(0) = 0, & \text{in} \quad \Omega, \\ S_{\mathcal{A}(0)}(\partial_t p(0), D_t u(0)) \mathcal{N}(0) = g \partial_t \eta(0) \mathcal{N}(0) - \sigma \partial_1 \left( \dfrac{\partial_1 \partial_t \eta(0)}{(1 + |\partial_1 \zeta_0|)^{3/2}} \right) \mathcal{N}(0) & \\ \quad + \partial_t F^3(0) \mathcal{N}(0) + \tilde{F}^4(0), & \text{on} \quad \Sigma, \\ (S_{\mathcal{A}(0)}(\partial_t p(0), D_t u(0)) \nu - \beta D_t u(0)) \cdot \tau = \tilde{F}^5, & \text{on} \quad \Sigma_s, \\ D_t u(0) \cdot \nu = 0, & \text{on} \quad \Sigma_s, \\ D_t u(0) \cdot \mathcal{N}(0) = \partial_t^2 \eta(0), & \text{on} \quad \Sigma, \\ \kappa \partial_t^2 \eta(\pm \ell, 0) + \kappa \partial_t \hat{\mathscr{W}}(\partial_t \eta(\pm \ell))(0) = \mp \sigma \left( \dfrac{\partial_1 \partial_t \eta(0)}{(1 + |\zeta_0|^2)^{3/2}} + \partial_t F^3(0) \right)(\pm \ell), & \end{cases} \tag{3.3}$$

where

$$\begin{aligned} \tilde{F}^1(0) &= -\operatorname{div}_{\partial_t \mathcal{A}(0)} S_{\mathcal{A}(0)}(p_0, u_0) + \mu \operatorname{div}_{\mathcal{A}(0)} \mathbb{D}_{\partial_t \mathcal{A}(0)} u_0 + \mu \operatorname{div}_{\mathcal{A}(0)} \mathbb{D}_{\mathcal{A}(0)}(R(0) u_0), \\ \partial_t F^3(0) &= \partial_z \mathcal{R}(\partial_1 \zeta_0, \partial_1 \eta_0) \partial_1 \partial_t \eta(0), \\ \tilde{F}^4(0) &= \mu \mathbb{D}_{\mathcal{A}(0)}(R(0) u_0) \mathcal{N}(0) + \mu \mathbb{D}_{\partial_t \mathcal{A}(0)} u_0 \mathcal{N}(0) \\ &\quad + \left[ g \eta_0 - \sigma \partial_1 \left( \frac{\partial_1 \eta_0}{(+|\partial_1 \zeta_0|^2)^{3/2}} + \mathcal{R}(\partial_1 \zeta_0, \partial_1 \eta_0) \right) \right] \partial_t \mathcal{N}(0), \\ \tilde{F}^5(0) &= \mu \mathbb{D}_{\mathcal{A}(0)}(R(0) u_0) \nu \cdot \tau + \mu \mathbb{D}_{\partial_t \mathcal{A}(0)} u_0 \nu \cdot \tau + \beta R(0) u_0 \cdot \tau. \end{aligned}$$

Then we have the pressureless weak formulation

$$\begin{aligned} &((D_t u(0), w)) + [D_t u(0) \cdot \mathcal{N}(0), w \cdot \mathcal{N}(0)]_\ell + [\partial_t \hat{\mathscr{W}}(\partial_t \eta)(0), w \cdot \mathcal{N}(0)]_\ell \\ &= -(\partial_t \eta(0), w \cdot \mathcal{N}(0))_{1,\Sigma} - \int_{-\ell}^{\ell} \partial_z \mathcal{R}(\partial_1 \zeta_0, \partial_1 \eta_0) \partial_1 \partial_t \eta(0) \partial_1 (w \cdot \mathcal{N}(0)) \\ &\quad - \int_{-\ell}^{\ell} \left[ g \eta_0 - \sigma \partial_1 \left( \frac{\partial_1 \eta_0}{(+|\partial_1 \zeta_0|^2)^{3/2}} + \mathcal{R}(\partial_1 \zeta_0, \partial_1 \eta_0) \right) \right] \partial_t \mathcal{N}(0) \cdot w - \int_{\Sigma_s} \beta (R(0) u_0 \cdot \tau)(w \cdot \tau) J(0) \\ &\quad - \int_\Omega \left( \operatorname{div}_{\partial_t \mathcal{A}(0)} S_{\mathcal{A}(0)}(p_0, u_0) \cdot w + \frac{\mu}{2} \mathbb{D}_{\partial_t \mathcal{A}(0)} u_0 : \mathbb{D}_{\mathcal{A}(0)} w + \frac{\mu}{2} \mathbb{D}_{\mathcal{A}(0)}(R(0) u_0) : \mathbb{D}_{\mathcal{A}(0)} w \right) J(0), \end{aligned} \tag{3.4}$$

for each $w \in \mathcal{V}(0)$. Then utilizing the last equation of (3.3), we may rewrite the weak formulation as

$$\begin{aligned}
&B(D_t u(0), w) := ((D_t u(0), w)) + (D_t u(0)\cdot \mathcal{N}(0), w\cdot \mathcal{N}(0))_{1,\Sigma} \\
&= (\partial_t^2 \eta(0), w\cdot \mathcal{N}(0))_{1,\Sigma} - (\partial_t \eta(0), w\cdot \mathcal{N}(0))_{1,\Sigma} - \int_{-\ell}^{\ell} \partial_z \mathcal{R}(\partial_1\zeta_0, \partial_1\eta_0)\partial_1\partial_t\eta(0)\partial_1(w\cdot\mathcal{N}(0)) \\
&- \int_{-\ell}^{\ell} \left[ g\eta_0 - \sigma\partial_1\left( \frac{\partial_1\eta_0}{(+|\partial_1\zeta_0|^2)^{3/2}} + \mathcal{R}(\partial_1\zeta_0, \partial_1\eta_0)\right)\right] \partial_t\mathcal{N}(0)\cdot w - \int_{\Sigma_s} \beta(R(0)u_0\cdot\tau)(w\cdot\tau)J(0) \\
&- \int_{\Omega} \left( \mathrm{div}_{\partial_t\mathcal{A}(0)} S_{\mathcal{A}(0)}(p_0,u_0)\cdot w + \frac{\mu}{2}\mathbb{D}_{\partial_t\mathcal{A}(0)}u_0 : \mathbb{D}_{\mathcal{A}(0)}w + \frac{\mu}{2}\mathbb{D}_{\mathcal{A}(0)}(R(0)u_0) : \mathbb{D}_{\mathcal{A}(0)}w\right) J(0) \\
&- [\partial_t^2\eta(0), w\cdot\mathcal{N}(0)]_\ell - [\partial_t \hat{\mathscr{W}}(\partial_t\eta)(0), w\cdot\mathcal{N}(0)]_\ell := L(w).
\end{aligned} \tag{3.5}$$

Since $B(\cdot,\cdot) : \mathcal{V}(0)\times\mathcal{V}(0) \to \mathbb{R}$ is a bilinear mapping satisfying

$$B(v,w) \lesssim \|v\|_{\mathcal{W}}\|w\|_{\mathcal{W}}, \quad B(v,v) = \|v\|_{\mathcal{W}}^2,$$

and $L : \mathcal{V}(0) \to \mathbb{R}$ is a bounded linear functional on $\mathcal{V}(0)$, the Lax-Milgram Theorem guarantees that there exists a unique $D_t u(0) \in \mathcal{V}(0)$ such that (3.4) holds for each $w \in \mathcal{V}(0)$. Moreover,

$$\|D_t u(0)\|_1^2 \lesssim \|\eta_0\|_{W_\delta^{5/2}}^2 + \|\partial_t\eta(0)\|_{3/2}^2 + \|\partial_t^2\eta(0)\|_1^2. \tag{3.6}$$

Now from Theorem 4.6 in [10], we may recover $\partial_t p(0) \in \mathring{H}^0(\Omega)$ such that

$$\|\partial_t p(0)\|_0^2 \lesssim \|\eta_0\|_{W_\delta^{5/2}}^2 + \|\partial_t\eta(0)\|_{3/2}^2 + \|\partial_t^2\eta(0)\|_1^2. \tag{3.7}$$

3.2. **Compatibility.** In the construction of initial data above, $\eta_0$, $\partial_t\eta(0)$ and $\partial_t^2\eta(0)$ need to satisfy some compatibility conditions. At the corner points $x_1 = \pm\ell$,

$$\kappa\partial_t\eta(0) + \kappa\hat{\mathscr{W}}(\partial_t\eta(0)) = \mp\sigma\left( \frac{\partial_1\eta_0}{(1+|\partial_1\zeta_0|^2)^{3/2}} + \mathcal{R}(\partial_1\zeta_0, \partial_1\eta_0)\right), \tag{3.8}$$

and

$$\kappa\partial_t^2\eta(0) + \kappa\hat{\mathscr{W}}'(\partial_t\eta(0))\partial_t^2\eta(0) = \mp\sigma\left( \frac{\partial_1\partial_t\eta(0)}{(1+|\partial_1\zeta_0|^2)^{3/2}} + \partial_z\mathcal{R}(\partial_1\zeta_0, \partial_1\eta_0)\partial_1\partial_t\eta(0)\right). \tag{3.9}$$

## 4. Linear problem

Suppose that $\eta$ is given and that $\mathcal{A}$, $J$, $\mathcal{N}$, etc. are determined in terms of $\eta$. Before turning to an analysis of the linear problem, we define various quantities in terms of $\eta$:

$$\begin{aligned}
\mathfrak{D}(\eta) &:= \sum_{j=0}^{1} \|\partial_t^j\eta\|_{L^2W_\delta^{5/2}}^2 + \sum_{j=0}^{2}\|\partial_t^j\eta\|_{L^2H^{3/2}}^2 + \|\partial_t^3\eta\|_{L^2W_\delta^{1/2}}^2 + \sum_{j=1}^{3}\|[\partial_t^j\eta]_\ell\|_{L^2([0,T])}^2, \\
\mathfrak{E}(\eta) &:= \|\eta\|_{L^\infty W_\delta^{5/2}}^2 + \|\partial_t\eta\|_{L^\infty H^{3/2}}^2 + \sum_{j=0}^{2}\|\partial_t^j\eta\|_{L^\infty H^1}^2, \\
\mathfrak{K}(\eta) &:= \mathfrak{D}(\eta) + \mathfrak{E}(\eta),
\end{aligned} \tag{4.1}$$

and

$$\mathfrak{E}_0 = \mathfrak{E}_0(\eta) := \|\eta_0\|_{W_\delta^{5/2}}^2 + \|\partial_t\eta(0)\|_{L^\infty H^{3/2}}^2 + \sum_{j=0}^{2}\|\partial_t^j\eta(0)\|_{L^\infty H^1}^2. \tag{4.2}$$

Throughout this section, we always assume that $\mathfrak{K}(\eta) \le \alpha$ and $\alpha > 0$ is sufficiently small.

In the rest sections, we write $d = dist(\cdot, M)$, where $M = \{(-\ell, \zeta_0(-\ell)), (\ell, \zeta_0(\ell))\}$ is the set of corner points of $\partial\Omega$. In the subsequent estimates, the following lemma is useful. The proof is trivial, so we omit it.

**Lemma 4.1.** *Suppose that $d = dist(\cdot, M)$ and that $0 < \delta < 1$. Then $d^{-\delta} \in L^r(\Omega)$ for $2 < r < \frac{2}{\delta}$.*

For the purpose of constructing solutions to the nonlinear system, we need to consider the following modified linear problem

$$\begin{cases} \operatorname{div}_{\mathcal{A}} S_{\mathcal{A}}(p,u) = F^1, & \text{in} \quad \Omega, \\ \operatorname{div}_{\mathcal{A}} u = 0, & \text{in} \quad \Omega, \\ S_{\mathcal{A}}(p,u)\mathcal{N} = \left( g(\xi + \epsilon\xi_t) - \sigma\partial_1 \left( \dfrac{\partial_1\xi + \epsilon\partial_1\xi_t}{(1+|\partial_1\zeta_0|^2)^{3/2}} \right) - \sigma\partial_1 F^3 \right)\mathcal{N} + F^4, & \text{on} \quad \Sigma, \\ (S_{\mathcal{A}}(p,u)\nu - \beta u)\cdot\tau = F^5, & \text{on} \quad \Sigma_s, \\ u\cdot\nu = 0, & \text{on} \quad \Sigma_s, \\ \partial_t\xi = u\cdot\mathcal{N}, & \text{on} \quad (-\ell,\ell), \\ \mp\sigma \dfrac{\partial_1\xi + \epsilon\partial_1\xi_t}{(1+|\partial_1\zeta_0|^2)^{3/2}}(\pm\ell) = \kappa(u\cdot\mathcal{N})(\pm\ell) \pm \sigma F^3(\pm\ell) - \kappa\hat{\mathscr{W}}(\partial_t\eta(\pm\ell)), \end{cases} \tag{4.3}$$

where $F^3 = \mathcal{R}(\partial_1\zeta_0, \partial_1\eta)$ is defined in (1.13). Here we consider the $\epsilon-$ perturbation in order to close the energy estimates for twice temporal differentiation of equations. FIXME: refer to discussion in the intro

4.1. **Weak solution.** To analyze (4.3), we need to consider two notations of solution: weak and strong. Now using the following lemma, we define the weak solutions of (4.3).

**Lemma 4.2.** *Suppose that $(u,p,\xi)$ are smooth enough and satisfy (4.3) and that $v \in \mathcal{W}(t)$. Then*

$$\begin{aligned} &((u,v)) - (p, \operatorname{div}_{\mathcal{A}} v)_{\mathcal{H}^0} + (\xi + \epsilon\xi_t, v\cdot\mathcal{N})_{1,\Sigma} + [u\cdot\mathcal{N}, v\cdot\mathcal{N}]_\ell \\ &= \int_\Omega F^1\cdot vJ - \int_{-\ell}^{\ell} \sigma F^3 \partial_1(v\cdot\mathcal{N}) + F^4\cdot v - \int_{\Sigma_s} F^5(v\cdot\tau)J - [v\cdot\mathcal{N}, \hat{\mathscr{W}}(\partial_t\eta)]_\ell, \end{aligned} \tag{4.4}$$

*Proof.* This can be shown in the usual way by taking the inner product of the first equation in (4.3) with $u$, and integrating by parts over $\Omega$, then employing all of the other equations in (4.3). We omit the details here. □

**Definition 4.3.** *Suppose that $\mathcal{F} \in (\mathcal{H}^1_T)^*$. A weak solution to (1.19) is a triple $(u,p,\xi)$, where*

$$\begin{aligned} u \in L^2([0,T]; {}_0H^1(\Omega)) \quad \text{with} \quad u(\cdot,t) \in \mathcal{V}(t) \text{ for a.e. } t, \\ p \in L^2([0,T], L^2(\Omega)), \quad \xi \in L^2([0,T]; H^1(-\ell,\ell)), \end{aligned} \tag{4.5}$$

*that satisfies*

$$\begin{aligned} &\int_0^T ((u,v)) - \int_0^T (p, \operatorname{div}_{\mathcal{A}} v)_{\mathcal{H}^0} + \int_0^T (\xi + \epsilon\xi_t, v\cdot\mathcal{N})_{1,\Sigma} + \int_0^T [u\cdot\mathcal{N}, v\cdot\mathcal{N}]_\ell \\ &= \int_0^T \int_\Omega F^1\cdot vJ - \int_0^T \int_{-\ell}^{\ell} \sigma F^3\partial_1(v\cdot\mathcal{N}) + F^4\cdot v - \int_0^T \int_{\Sigma_s} F^5(v\cdot\tau)J - \int_0^T [v\cdot\mathcal{N}, \hat{\mathscr{W}}(\partial_t\eta)]_\ell, \end{aligned} \tag{4.6}$$

*for a.e. $t$ and each $v \in \mathcal{W}(t)$. If we take the test function $v \in \mathcal{V}(t)$, we have the pressureless weak solution $(u,\xi)$ satisfies*

$$\begin{aligned} &\int_0^T ((u,v)) + \int_0^T (\xi + \epsilon\xi_t, v\cdot\mathcal{N})_{1,\Sigma} + \int_0^T [u\cdot\mathcal{N}, v\cdot\mathcal{N}]_\ell \\ &= \int_0^T \int_\Omega F^1\cdot vJ - \int_0^T \int_{-\ell}^{\ell} \sigma F^3\partial_1(v\cdot\mathcal{N}) + F^4\cdot v - \int_0^T \int_{\Sigma_s} F^5(v\cdot\tau)J - \int_0^T [v\cdot\mathcal{N}, \hat{\mathscr{W}}(\partial_t\eta)]_\ell. \end{aligned} \tag{4.7}$$

**Remark 4.4.** *For convenience, we write*

$$\langle \mathcal{F}, v\rangle_{(\mathcal{H}^1)^*} = \int_\Omega F^1\cdot vJ - \int_{-\ell}^{\ell} F^4\cdot v - \int_{\Sigma_s} F^5(v\cdot\tau)J, \tag{4.8}$$

*and*

$$\langle \mathcal{F}, v\rangle_{(\mathcal{H}^1_T)^*} = \int_0^T \int_\Omega F^1\cdot vJ - \int_0^T \int_{-\ell}^{\ell} F^4\cdot v - \int_0^T \int_{\Sigma_s} F^5(v\cdot\tau)J, \tag{4.9}$$

*for each $v \in \mathcal{V}$.*

In the following, we will see that weak solutions to (4.7) will arise as a byproduct of the construction of strong solutions to (4.7). Hence, we now ignore the existence of weak solutions and record a uniqueness result based on some integral equalities and bounds satisfied by weak solutions.

**Proposition 4.5.** *Weak solutions to* (4.7) *are unique.*

*Proof.* If $(u^1, \xi^1)$ and $(u^2, \xi^2)$ are both weak solutions to (4.7), then $(w = u^1 - u^2, \theta = \xi^1 - \xi^2)$ is a weak solution with $F^1 = F^3 = F^4 = F^5 = 0$ and the initial data $w(0) = \theta(0) = 0$. Using the test function $w\chi_{[0,t]} \in \mathcal{V}_T$, where $\chi_{[0,t]}$ is a temporal indicator function, we have that

$$\frac{1}{2}\|\theta(t)\|_{1,\Sigma}^2 + \epsilon \int_0^t \|w \cdot \mathcal{N}\|_{1,\Sigma}^2 + \int_0^t ((w(s), w(s)))\,\mathrm{d}s + \int_0^t [w \cdot \mathcal{N}(t)]_\ell^2\,\mathrm{d}s = 0, \tag{4.10}$$

which implies that $w = 0$, $\theta = 0$. Hence, weak solutions to (4.7) are unique. □

4.2. **Strong solution.** Before we define strong solutions, we need to define an operator $D_t$ via

$$D_t u := \partial_t u - Ru \quad \text{for} \quad R := \partial_t M M^{-1}, \tag{4.11}$$

with $M = K\nabla\Phi$, where $K$ and $\Phi$ are defined as in (1.9) and (1.6), respectively. It is easy to see that $D_t$ preserves the $\operatorname{div}_{\mathcal{A}}$–free condition since

$$J \operatorname{div}_{\mathcal{A}}(D_t v) = J \operatorname{div}_{\mathcal{A}}(M\partial_t(M^{-1}v)) = \operatorname{div}(\partial_t(M^{-1}v)) = \partial_t \operatorname{div}(M^{-1}v) = \partial_t(J \operatorname{div}_{\mathcal{A}} v), \tag{4.12}$$

where in the second and last equality, we used the equality $J \operatorname{div}_{\mathcal{A}} v = \operatorname{div}(M^{-1}v)$, which is proved, according to Lemma (A.1) and the definition (2.33) of $M$, as

$$J \operatorname{div}_{\mathcal{A}} v = J\mathcal{A}_{ij}\partial_j v_i = \partial_j(J\mathcal{A}_{ij}v_i) = \operatorname{div}(J\mathcal{A}^\top v) = \operatorname{div}(M^{-1}v). \tag{4.13}$$

We now give our definition of strong solutions.

**Definition 4.6.** *Suppose that the forcing functions satisfy*

$$\begin{gathered} F^1 \in L^2([0,T]; W_\delta^0(\Omega)),\ F^3 \in L^2([0,T]; W_\delta^{3/2}(\Sigma)), \\ F^4 \in L^2([0,T]; W_\delta^{1/2}(\Sigma)),\ F^5 \in L^2([0,T]; W_\delta^{1/2}(\Sigma_s)), \\ \mathcal{F} \in C^0([0,T]; (\mathcal{H}^1)^*),\ \partial_t \mathcal{F} \in L^2([0,T]; (\mathcal{H}^1)^*). \end{gathered} \tag{4.14}$$

*We also assume that the initial data are the same as Section 3. If there exists a pair* $(u, p, \xi)$ *achieving the initial data and satisfying the* (4.3) *in the strong sense of*

$$u \in L^2([0,T]; W_\delta^2(\Omega)) \cap \mathcal{V}_T,\ p \in L^2([0,T]; \mathring{W}_\delta^1(\Omega)),\ \xi \in L^2([0,T]; W_\delta^{5/2}(\Sigma)), \tag{4.15}$$

*and*

$$\begin{gathered} \partial_t^j u \in L^2([0,T]; H^1(\Omega)),\ \partial_t^j u \in L^2([0,T]; H^0(\Sigma_s)),\ [\partial_t^j u \cdot \mathcal{N}]_\ell \in L^2([0,T]), \\ \partial_t^j p \in L^2([0,T]; H^0(\Omega)),\ \partial_t^j \xi \in L^2([0,T]; H^{3/2}(\Sigma)), \end{gathered} \tag{4.16}$$

*for* $j = 0, 1$*, we call it a strong solution.*

**Lemma 4.7.** *Suppose that the right-hand side of the following is finite. Then* $u \in C^0([0,T]; H^1(\Omega))$*, and*

$$\|u\|_{L^\infty H^1}^2 \lesssim \|u_0\|_{W_\delta^2}^2 + \|u\|_{L^2 H^1}^2 + \|\partial_t u\|_{L^2 H^1}^2.$$

*Proof.* We first estimate

$$\begin{aligned} \frac{d}{dt}\|u\|_{H^1}^2 &= \int_\Omega 2u \cdot \partial_t u + 2\nabla u : \nabla \partial_t u \\ &\le \|u\|_{H^1}^2 + \|\partial_t u\|_{H^1}^2. \end{aligned}$$

This may be integrated over $[0,T]$ to see that

$$\begin{aligned} \|u\|_{L^\infty H^1}^2 &\le \|u_0\|_{H^1}^2 + \|u\|_{L^2 H^1}^2 + \|\partial_t u\|_{L^2 H^1}^2 \\ &\lesssim \|u_0\|_{W_\delta^2}^2 + \|u\|_{L^2 H^1}^2 + \|\partial_t u\|_{L^2 H^1}^2, \end{aligned}$$

where the last inequality is obtained by the embedding $W_\delta^2(\Omega) \hookrightarrow H^1(\Omega)$. □

Now we state our main theorem for the strong solutions.

**Theorem 4.8.** *Suppose that the forcing terms $F^1$, $F^4$, and $F^5$ satisfy the condition* (4.14)*, that the initial data are the same as Section 3. Suppose that $\mathfrak{K}(\eta) \leq \alpha$ is smaller than $\alpha_0$ in Lemma 2.3 and Theorem 5.9 in* [10]*. Then there exists a unique strong solution $(u, p, \xi)$ solving* (4.3) *such that $(u, p, \xi)$ satisfies* (4.15) *and* (4.16)*. The solution obeys the estimates*

$$\begin{aligned}
&\|u\|_{L^2H^1}^2 + \|u\|_{L^2H^0(\Sigma_s)}^2 + \|[u\cdot\mathcal{N}]_\ell\|_{L^2([0,T])}^2 + \|u\|_{L^2W^2_\delta}^2 + \|\partial_t u\|_{L^2H^1}^2 + \|\partial_t u\|_{L^2H^0(\Sigma_s)}^2 \\
&\quad + \|[\partial_t u\cdot\mathcal{N}]_\ell\|_{L^2([0,T])}^2 + \|p\|_{L^2H^0}^2 + \|p\|_{L^2\mathring{W}^1_\delta}^2 + \|\partial_t p\|_{L^2H^0}^2 + \|\xi\|_{L^\infty H^1}^2 + \|\xi\|_{L^2H^{3/2}}^2 \\
&\quad + \|\xi\|_{L^2W^{5/2}_\delta}^2 + \|\partial_t\xi\|_{L^\infty H^1}^2 + \|\partial_t\xi\|_{L^2H^{3/2}}^2 \\
&\lesssim C(\epsilon)T\mathfrak{E}(\eta) + \mathfrak{E}_0 + \|\mathcal{F}(0)\|_{(\mathcal{H}^1)^*}^2 + \mathfrak{K}(\eta) + \mathfrak{E}(\eta)(\|F^1\|_{L^2W^0_\delta}^2 + \|F^4\|_{L^2W^{1/2}_\delta}^2 + \|F^5\|_{L^2W^{1/2}_\delta}^2) \\
&\quad + (1+\mathfrak{E}(\eta))\|\partial_t(F^1 - F^4 - F^5)\|_{(\mathcal{H}^1_T)^*}^2.
\end{aligned} \tag{4.17}$$

*Moreover, $(D_t u, \partial_t p, \partial_t\xi)$ satisfies*

$$\begin{cases}
-\mu\Delta_{\mathcal{A}} D_t u + \nabla_{\mathcal{A}}\partial_t p = D_t F^1 + G^1, & \text{in } \Omega, \\
\operatorname{div}_{\mathcal{A}}(D_t u) = 0, & \text{in } \Omega, \\
S_{\mathcal{A}}(\partial_t p, D_t u)\mathcal{N} = \mathcal{L}(\partial_t\xi + \epsilon\partial_t^2\xi)\mathcal{N} - \sigma\partial_1\partial_t F^3\mathcal{N} + \partial_t F^4 + G^4, & \text{on } \Sigma, \\
(S_{\mathcal{A}}(\partial_t p, D_t u)\nu - \beta D_t u)\cdot\tau = \partial_t F^5 + G^5, & \text{on } \Sigma_s, \\
D_t u\cdot\nu = 0, & \text{on } \Sigma_s, \\
\partial_t^2\xi = D_t u\cdot\mathcal{N}, & \text{on } \Sigma, \\
\mp\sigma\dfrac{\partial_1\partial_t\xi + \epsilon\partial_1\partial_t^2\xi}{(1+|\partial_1\zeta_0|^2)^{3/2}}(\pm\ell) = \kappa(D_t u\cdot\mathcal{N})(\pm\ell) \pm \sigma\partial_t F^3(\pm\ell) - \kappa\partial_t\hat{\mathscr{W}}((\partial_t\eta)(\pm\ell)).
\end{cases} \tag{4.18}$$

*in the weak sense of* (4.7)*, where $G^1$ is defined by*

$$G^1 = R^\top\nabla_{\mathcal{A}} p + \operatorname{div}_{\mathcal{A}}\left(\mathbb{D}_{\mathcal{A}}(Ru) + \mathbb{D}_{\partial_t\mathcal{A}}u - R\mathbb{D}_{\mathcal{A}}u\right), \tag{4.19}$$

*and $G^4$ by*

$$G^4 = \mu\mathbb{D}_{\mathcal{A}}(Ru)\mathcal{N} - (pI - \mu\mathbb{D}_{\mathcal{A}}u)\partial_t\mathcal{N} + \mu\mathbb{D}_{\partial_t\mathcal{A}}u\mathcal{N} + \mathcal{L}(\xi + \epsilon\partial_t\xi)\partial_t\mathcal{N} - \sigma\partial_1 F^3\partial_t\mathcal{N}, \tag{4.20}$$

*$G^5$ by*

$$G^5 = (\mu\mathbb{D}_{\mathcal{A}}(Ru)\nu + \mu\mathbb{D}_{\partial_t\mathcal{A}}u\nu + \beta Ru)\cdot\tau \tag{4.21}$$

*More precisely,* (4.18) *holds in the weak sense of*

$$\begin{aligned}
&((\partial_t u, v)) + (\partial_t\xi + \epsilon\partial_t^2\xi, v\cdot\mathcal{N})_{1,\Sigma} + [\partial_t u\cdot\mathcal{N}, v\cdot\mathcal{N}]_\ell + [\hat{\mathscr{W}}'\partial_t^2\eta, v\cdot\mathcal{N}]_\ell \\
&= (\xi + \epsilon\partial_t\xi, Rv\cdot\mathcal{N})_{1,\Sigma} - (p, \operatorname{div}_{\mathcal{A}}(Rv))_{\mathcal{H}^0} + \int_\Omega \left[\partial_t F^1\cdot v + \partial_t JKF^1\cdot v\right]J \\
&\quad - \int_{-\ell}^{\ell}[\partial_t F^3\partial_1(v\cdot\mathcal{N}) + F^3\partial_1(v\cdot\partial_t\mathcal{N}) + \partial_t F^4\cdot v] - \int_{\Sigma_s}\left[\partial_t F^5 v + \partial_t JKF^5 v\right]\cdot\tau J \\
&\quad - \int_\Omega\frac{\mu}{2}(\mathbb{D}_{\partial_t\mathcal{A}}u : \mathbb{D}_{\mathcal{A}}v + \mathbb{D}_{\mathcal{A}}u : \mathbb{D}_{\partial_t\mathcal{A}}v + \partial_t JK\mathbb{D}_{\mathcal{A}}u : \mathbb{D}_{\mathcal{A}}v)J - \int_{\Sigma_s}\beta(u\cdot\tau)(v\cdot\tau)\partial_t J.
\end{aligned} \tag{4.22}$$

*Proof.* Our proof is inspired by a result in [9]. We divide the proof into several steps.

Step 1 – The Galerkin setup.

In order to utilize the Galerkin method, we must first construct a countable basis of $H^2(\Omega)\cap\mathcal{V}(t)$ for each $t\in[0,T]$. Since the requirement $\operatorname{div}_{\mathcal{A}} v = 0$ is time–dependent, any basis of this space must also be time–dependent. For each $t\in[0,T]$, the space $H^2(\Omega)\cap\mathcal{V}(t)$ is separable, so the existence of a countable basis is not an issue. The technical difficulty is that, in order for the basis to be useful in Galerkin method, we must be able to express these time derivatives in terms of finitely many basis elements. Fortunately, it is possible to overcome this difficulty by employing the matrix $M(t)$, defined by (2.33).

Since $H^2(\Omega)\cap V$ is separable, it possess a countable basis $\{w^j\}_{j=1}^\infty$. Note that this basis is not time–dependent. Define $v^j = v^j(t) := M(t)w^j$. According to Proposition 2.5, $v^j(t)\in H^2(\Omega)\cap\mathcal{V}(t)$, and $\{v^j(t)\}_{j=1}^\infty$ is a basis of $H^2(\Omega)\cap\mathcal{V}(t)$ for each $t\in\mathbb{R}^+$. Moreover, we can express $\partial_t v^j(t)$ in terms of $v^j(t)$ as

$$\partial_t v^j(t) = \partial_t M(t)w^j = \partial_t M(t)M^{-1}(t)M(t)w^j = R(t)v^j(t), \tag{4.23}$$

where $R(t)$ is defined by

$$R(t) := \partial_t M(t)M^{-1}(t). \tag{4.24}$$

For any integer $m\geq 1$, we define the finite dimensional space

$$\mathcal{V}_m(t) := \operatorname{span}\{v^1(t),\cdots,v^m(t)\}\subseteq H^2(\Omega)\cap\mathcal{V}(t),$$

and we write

$$\mathcal{P}_t^m : H^2(\Omega)\to\mathcal{V}_m(t) \tag{4.25}$$

for the $H^2(\Omega)$ orthogonal projection onto $\mathcal{V}_m(t)$. Clearly, for each $v\in H^2(\Omega)\cap\mathcal{V}(t)$, we have that $\mathcal{P}_t^m v\to v$ as $m\to\infty$.

Step 2 – Solving the approximate problem.

For our Galerkin problem, we construct a solution to the pressureless problem as follows. For each $m\geq 1$, we define an approximate solution

$$u^m(t) := d_j^m(t)v^j(t),\ \text{with } d_j^m : [0,T]\to\mathbb{R} \text{ for } j=1,\dots,m, \tag{4.26}$$

where as usual we use the Einstein convention of summation of the repeated index $j$. We similarly define

$$\xi^m(t) = \eta_0 + \int_0^t u^m(s)\cdot\mathcal{N}(s)\,\mathrm{d}s, \tag{4.27}$$

where we understand here that $u^m(\cdot)$ denotes the trace onto $\Sigma$.

We want to choose the coefficients $d_j^m(t)\in C^1([0,T])$ so that

$$\begin{aligned}&((u^m,v)) + (\xi^m+\epsilon\partial_t\xi^m, v\cdot\mathcal{N})_{1,\Sigma} + [u^m\cdot\mathcal{N}, v\cdot\mathcal{N}]_\ell\\ &= \int_\Omega F^1\cdot vJ - \int_{-\ell}^{\ell} F^3\partial_1(v\cdot\mathcal{N}) + F^4\cdot v - \int_{\Sigma_s} F^5(v\cdot\tau)J - [v\cdot\mathcal{N}, \hat{\mathscr{W}}(\partial_t\eta)]_\ell,\end{aligned} \tag{4.28}$$

for each $v\in\mathcal{V}_m(t)$. We supplement this with the initial data

$$u^m(0) = \mathcal{P}_0^m u_0\in\mathcal{V}_m(0). \tag{4.29}$$

We may compute

$$\begin{aligned}(\xi^m(t)+\epsilon\partial_t\xi^m, v\cdot\mathcal{N}(t))_{1,\Sigma} &= \left(\eta_0 + \int_0^t u^m(s)\cdot\mathcal{N}(s)\,\mathrm{d}s + \epsilon u^m(t)\cdot\mathcal{N}(t), v\cdot\mathcal{N}(t)\right)_{1,\Sigma}\\ &= (\eta_0, v\cdot\mathcal{N}(t))_{1,\Sigma} + \epsilon d_i^m(t)(v^i\cdot\mathcal{N}(t), v\cdot\mathcal{N}(t))_{1,\Sigma} + \int_0^t d_i^m(s)(v^i\cdot\mathcal{N}(s), v\cdot\mathcal{N}(t))_{1,\Sigma}\,\mathrm{d}s.\end{aligned} \tag{4.30}$$

Then we see that (4.28) is equivalent to the equation of $d_j^m$ given by

$$\begin{aligned}&d_i^m((v^i,v^j)) + \epsilon d_i^m(v^i\cdot\mathcal{N}(t), v^j\cdot\mathcal{N}(t))_{1,\Sigma} + \int_0^t d_i^m(s)(v^i\cdot\mathcal{N}(s), v^j\cdot\mathcal{N}(t))_{1,\Sigma}\,\mathrm{d}s\\ &\quad + d_i^m[v^i\cdot\mathcal{N}, v^j\cdot\mathcal{N}]_\ell\\ &= \int_\Omega F^1\cdot v^jJ - \int_{-\ell}^{\ell} F^3\partial_1(v^j\cdot\mathcal{N}) + F^4\cdot v^j - \int_{\Sigma_s} F^5(v^j\cdot\tau)J - (\eta_0, v^j\cdot\mathcal{N}(t))_{1,\Sigma}\\ &\quad - [v^j\cdot\mathcal{N}, \hat{\mathscr{W}}(\partial_t\eta)]_\ell,\end{aligned} \tag{4.31}$$

for $i,j=1,\dots,m$.

Since $\{v^j(t)\}_{j=1}^\infty$ is a basis of $H^2(\Omega)\cap\mathcal{V}(t)$, the $m\times m$ matrix $A=(A_{jk})$with $j,k$ entry $A_{jk}=((v^j,v^k))+\epsilon(v^j\cdot\mathcal{N},v^k\cdot\mathcal{N})_{1,\Sigma}+[v^j\cdot\mathcal{N},v^k\cdot\mathcal{N}]_\ell$ is positive definite. For any vector $\lambda=(\lambda_1,\dots,\lambda_m)^\top\neq 0$, a straightforward computation shows that

$$\lambda^\top A\lambda=\frac{\mu}{2}\int_\Omega|\lambda_i\mathbb{D}_A v^i|J+\beta\int_{\Sigma_s}|\lambda_i v^i\cdot\tau|^2J+\epsilon\|\lambda_i v^i\cdot\mathcal{N}\|_{1,\Sigma}^2+[\lambda_i v^i\cdot\mathcal{N}]_\ell^2>0$$

where we have used the Einstein convention and the last inequality due to the facts that $\{v^j\}_{j=1}^m$ is a basis of $\mathcal{V}_m$ and $\lambda\neq 0$. Thus $A$ is invertible. Then we view (4.31) as an integral system of the form

$$d^m(t)+\int_0^t\mathfrak{C}(t,s)d^m(s)\,\mathrm{d}s=\mathfrak{F}(t),\tag{4.32}$$

where the $m\times m$ matrix $\mathfrak{C}$ belongs to $C^1(D)$ with $D=\{(t,s)|0\le s\le t,0\le t\le T\}$, and the forcing term $\mathfrak{F}\in C^1([0,T])$ since $\partial_t\eta(\pm\ell,\cdot)\in H^2((0,T))\hookrightarrow C^{1,1/2}([0,T])$.

From the usual theory of integral equations (for instance, see [20]), there exists a unique $d^m\in C^1([0,T])$ satisfying $d^m=Ad^m=\mathfrak{F}(t)-\int_0^t\mathfrak{C}(t,s)d^m(s)\,\mathrm{d}s$.

Step 3 – Estimates for initial data.

For $u^m(0)$, since $\mathcal{P}_0^m$ is the orthogonal projection, we may use Lemma 2.3, Sobolev embedding theory and initial data in Section 3 to obtain the bounds

$$\|u^m(0)\|_{{}_0\mathcal{H}^1}\lesssim\|u^m(0)\|_1\lesssim\|u^m(0)\|_{W^2_\delta}\lesssim\|u_0\|_{W^2_\delta}\lesssim\|\partial_t\eta(0)\|_{3/2},\tag{4.33}$$

and

$$\|\partial_t\xi^m(0)\|_1=\|u^m(0)\cdot\mathcal{N}(0)\|_1\lesssim\|u_0\cdot\mathcal{N}(0)\|_1\lesssim\|\partial_t\eta(0)\|_1.\tag{4.34}$$

Step 4 – Energy estimates for $u^m$.

By construction, $u^m(t)\in\mathcal{V}_m(t)$, so we may choose $v=u^m$ as a test function (4.28). Since $\partial_t\xi^m=u^m\cdot\mathcal{N}$, we have that

$$\begin{aligned}&\frac{d}{dt}\frac{1}{2}\|\xi^m\|_{1,\Sigma}^2+\epsilon\|\partial_t\xi^m\|_{1,\Sigma}^2+\|u^m\|_{{}_0\mathcal{H}^1(\Omega)}^2+[u^m\cdot\mathcal{N}]_\ell^2\\&=\int_\Omega F^1\cdot u^mJ-\int_{-\ell}^\ell F^3\partial_t\partial_1\xi^m+F^4\cdot u^m-\int_{\Sigma_s}F^5(u^m\cdot\tau)J-[\hat{\mathscr{W}}(\partial_t\eta),u^m\cdot\mathcal{N}]_\ell,\end{aligned}\tag{4.35}$$

using the Hölder inequality for $1<q<\frac{2}{1+\delta}$ with $0<\delta<1$ and $\frac{1}{p}+\frac{1}{q}=1$, Lemma 4.1 with $2<r<\frac{2}{\delta}$ and $r'=\frac{2r}{r-2}$, the Cauchy inequality, Sobolev inequality and the usual trace theory, we have that

$$\begin{aligned}&\frac{d}{dt}\frac{1}{2}\|\xi^m\|_{1,\Sigma}^2+\epsilon\|\partial_t\xi^m\|_{1,\Sigma}^2+\|u^m\|_{{}_0\mathcal{H}^1(\Omega)}^2+[u^m\cdot\mathcal{N}]_\ell^2\\&\lesssim\|J\|_{L^\infty(\Sigma)}\|F^1\|_{W^0_\delta}\|d^{-\delta}\|_{L^r}\|u^m\|_{L^{r'}}+\|F^4\|_{L^q}\|u^m\|_{L^p(\Sigma)}+\|J\|_{L^\infty(\Sigma_s)}\|F^5\|_{L^q}\|u^m\|_{L^p(\Sigma_s)}\\&\quad+C(\epsilon)\|\eta\|_1^2\|\eta\|_{W^{5/2}_\delta}^2+[\hat{\mathscr{W}}(\partial_t\eta)]_\ell^2\\&\lesssim(1+\|\eta\|_{W^{5/2}_\delta})(\|F^1\|_{W^0_\delta}+\|F^4\|_{W^{1/2}_\delta}+\|F^5\|_{W^{1/2}_\delta})\|u^m\|_1+C(\epsilon)\|\eta\|_1^2\|\eta\|_{W^{5/2}_\delta}^2+\|\partial_t\eta\|_1^2[\partial_t\eta]_\ell^2\\&\lesssim(1+\|\eta\|_{W^{5/2}_\delta}^2)(\|F^1\|_{W^0_\delta}^2+\|F^4\|_{W^{1/2}_\delta}^2+\|F^5\|_{W^{1/2}_\delta}^2)+C(\epsilon)\|\eta\|_1^2\|\eta\|_{W^{5/2}_\delta}^2+\|\partial_t\eta\|_1^2[\partial_t\eta]_\ell^2.\end{aligned}\tag{4.36}$$

Then we employ the Gronwall's inequality to derive that

$$\begin{aligned}&\sup_{0\le t\le T}\|\xi^m\|_1^2+\epsilon\|\partial_t\xi^m\|_{L^2H^1}^2+\|u^m\|_{L^2H^1(\Omega)}+\|u^m\|_{L^2H^0(\Sigma_s)}+\int_0^T[u^m\cdot\mathcal{N}]_\ell^2\\&\lesssim(1+\|\eta\|_{L^\infty W^{5/2}_\delta}^2)(\|F^1\|_{L^2W^0_\delta}^2+\|F^4\|_{L^2W^{1/2}_\delta}^2+\|F^5\|_{L^2W^{1/2}_\delta}^2)+C(\epsilon)T\|\eta\|_{L^\infty H^1}^2\|\eta\|_{L^\infty W^{5/2}_\delta}^2\\&\quad+\|\partial_t\eta\|_{L^\infty H^1}^2\|[\partial_t\eta]_\ell\|_{L^2}^2+\|\eta_0\|_{W^{5/2}_\delta}^2.\end{aligned}\tag{4.37}$$

Step 5 – Energy estimate for $\partial_t u^m$.

Suppose that $v=b_i^mv^i$ for $b_i^m\in C^{0,1}([0,T])$. It is easily verified that $\partial_tv(t)-R(t)v(t)\in\mathcal{V}_m(t)$ as well. We now use this $v$ in (4.28), temporally differentiate the resulting equation, and then subtract this from

the equation (4.28) with test function $\partial_t v - Rv$. This eliminates the terms of $\partial_t v$ and leaves us with the equality

$$\begin{aligned}
&((\partial_t u^m, v)) + (u^m \cdot \mathcal{N}, v \cdot \mathcal{N})_{1,\Sigma} + \epsilon(\partial_t(u^m \cdot \mathcal{N}), v \cdot \mathcal{N})_{1,\Sigma} + (\xi^m + \epsilon\xi_t^m, Rv \cdot \mathcal{N})_{1,\Sigma} \\
&+ (\xi^m + \epsilon\xi_t^m, v \cdot \partial_t \mathcal{N})_{1,\Sigma} + [\partial_t u^m \cdot \mathcal{N}, v \cdot \mathcal{N}]_\ell + [u^m \cdot \partial_t \mathcal{N}, v \cdot \mathcal{N}]_\ell + [u^m \cdot \mathcal{N}, Rv \cdot \mathcal{N}]_\ell \\
&+ [u^m \cdot \mathcal{N}, v \cdot \partial_t \mathcal{N}]_\ell + \int_{\Sigma_s} \beta(u^m \cdot \tau)(v \cdot \tau)\partial_t J \\
&= \partial_t \mathcal{F}^m(v) - \mathcal{F}^m(\partial_t v) + \mathcal{F}^m(Rv) - ((u^m, Rv)) \\
&\quad - \int_\Omega \frac{\mu}{2}(\mathbb{D}_{\partial_t \mathcal{A}} u^m : \mathbb{D}_{\mathcal{A}} v + \mathbb{D}_{\mathcal{A}} u^m : \mathbb{D}_{\partial_t \mathcal{A}} v + \partial_t J K \mathbb{D}_{\mathcal{A}} u^m : \mathbb{D}_{\mathcal{A}} v) J,
\end{aligned} \tag{4.38}$$

where for brevity we have written

$$\mathcal{F}^m(v) = \int_\Omega F^1 \cdot v J - \int_{-\ell}^{\ell} F^3 \partial_1 (v \cdot \mathcal{N}) + F^4 \cdot v - \int_{\Sigma_s} F^5 (v \cdot \tau) J - [v \cdot \mathcal{N}, \hat{\mathscr{W}}(\partial_t \eta)]_\ell,$$

According to the Lemma (A.1), then

$$\begin{aligned}
&(\xi + \epsilon\xi_t^m, Rv \cdot \mathcal{N})_{1,\Sigma} + (\xi + \epsilon\xi_t^m, v \cdot \partial_t \mathcal{N})_{1,\Sigma} + [u \cdot \mathcal{N}, Rv \cdot \mathcal{N}]_\ell + [u^m \cdot \mathcal{N}, v \cdot \partial_t \mathcal{N}]_\ell \\
&= -(\xi + \epsilon\xi_t^m, v \cdot \partial_t \mathcal{N})_{1,\Sigma} + (\xi + \epsilon\xi_t^m, v \cdot \partial_t \mathcal{N})_{1,\Sigma} - [u \cdot \mathcal{N}, v \cdot \partial_t \mathcal{N}]_\ell + [u^m \cdot \mathcal{N}, v \cdot \partial_t \mathcal{N}]_\ell = 0.
\end{aligned} \tag{4.39}$$

We choose the test function $v = \partial_t u^m - Ru^m$. Then we have that

$$[\partial_t u^m \cdot \mathcal{N}, (\partial_t u^m - Ru^m) \cdot \mathcal{N}]_\ell = [\partial_t u^m \cdot \mathcal{N}]_\ell^2, \tag{4.40}$$

because of the fact that $\mathcal{N} = \mathcal{N}_0 - \partial_1 \eta e_1$ and $Ru^m \cdot \mathcal{N} = u^m \cdot \partial_t \mathcal{N} = u^m \cdot \partial_1 \partial_t \eta e_1 = 0$. Similarly,

$$\partial_t(u^m \cdot \mathcal{N}) = \partial_t u^m \cdot \mathcal{N} - u^m \cdot \partial_t \mathcal{N} = \partial_t u^m \cdot \mathcal{N} - u^m \cdot R^\top \mathcal{N} = (\partial_t u^m - Ru^m) \cdot \mathcal{N}, \tag{4.41}$$

hence

$$\begin{aligned}
&(u^m \cdot \mathcal{N}, (\partial_t u^m - Ru^m) \cdot \mathcal{N})_{1,\Sigma} + \epsilon(\partial_t(u^m \cdot \mathcal{N}), (\partial_t u^m - Ru^m) \cdot \mathcal{N})_{1,\Sigma} \\
&\qquad = (u^m \cdot \mathcal{N}, \partial_t(u^m \cdot \mathcal{N}))_{1,\Sigma} + \epsilon(\partial_t(u^m \cdot \mathcal{N}), \partial_t(u^m \cdot \mathcal{N}))_{1,\Sigma} \\
&\qquad = \frac{d}{dt}\frac{1}{2}\|u^m \cdot \mathcal{N}\|_{1,\Sigma}^2 + \epsilon\|\partial_t(u^m \cdot \mathcal{N})\|_{1,\Sigma}^2.
\end{aligned} \tag{4.42}$$

Plugging the test function $v = \partial_t u^m - Ru^m$ into (4.38), (4.38) becomes that

$$\frac{d}{dt}\frac{1}{2}\|u^m \cdot \mathcal{N}\|_{1,\Sigma}^2 + \epsilon\|\partial_t(u^m \cdot \mathcal{N})\|_{1,\Sigma}^2 + \|\partial_t u^m\|_{{}_0\mathcal{H}^1}^2 + [\partial_t u^m \cdot \mathcal{N}]_\ell^2 = I + II + III, \tag{4.43}$$

where

$$I = -((u^m, R(\partial_t u^m - Ru^m))) + ((\partial_t u^m, Ru^m)), \tag{4.44}$$

$$\begin{aligned}
II = &- \int_\Omega \frac{\mu}{2}(\mathbb{D}_{\partial_t \mathcal{A}} u^m : \mathbb{D}_{\mathcal{A}}(\partial_t u^m - Ru^m)) J \\
&- \int_\Omega \frac{\mu}{2}(\mathbb{D}_{\mathcal{A}} u^m : \mathbb{D}_{\partial_t \mathcal{A}}(\partial_t u^m - Ru^m) + \partial_t J K \mathbb{D}_{\mathcal{A}} u^m : \mathbb{D}_{\mathcal{A}}(\partial_t u^m - Ru^m)) J \\
&- \int_{\Sigma_s} \beta(u^m \cdot \tau)((\partial_t u^m - Ru^m) \cdot \tau)\partial_t J,
\end{aligned} \tag{4.45}$$

and

$$\begin{aligned}
III &= \partial_t \mathcal{F}^m(\partial_t u^m - Ru^m) - \mathcal{F}^m(\partial_t(\partial_t u^m - Ru^m)) + \mathcal{F}^m(R(\partial_t u^m - Ru^m)) \\
&= \int_\Omega \left[\partial_t F^1 \cdot (\partial_t u^m - Ru^m) + \partial_t J K F^1 \cdot (\partial_t u^m - Ru^m) + F^1 \cdot R(\partial_t u^m - Ru^m)\right] J \\
&\quad - \int_{-\ell}^{\ell} [\partial_t F^3 \partial_1((\partial_t u^m - Ru^m) \cdot \mathcal{N}) + \partial_t F^4 \cdot (\partial_t u^m - Ru^m) + F^4 \cdot R(\partial_t u^m - Ru^m)] \\
&\quad - \int_{\Sigma_s} \left[\partial_t F^5(\partial_t u^m - Ru^m) + \partial_t J K F^5(\partial_t u^m - Ru^m) + F^5 R(\partial_t u^m - Ru^m)\right] \cdot \tau J \\
&\quad - [\hat{\mathscr{W}}'(\partial_t \eta)\partial_t^2 \eta, (\partial_t u^m - Ru^m) \cdot \mathcal{N}]_\ell,
\end{aligned} \tag{4.46}$$

where we have used the fact $\partial_t \mathcal{N} = -R^\top \mathcal{N}$ on $\Sigma$.

We now estimate each term of $I$, $II$, $III$. For any fixed small number $\theta > 0$ and $1 < s < \min\{\frac{\pi}{\omega}, 2\}$, we choose $p$ and $q$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$ and $2 < p < \frac{2}{\delta}$, such that, according to the Sobolev inequality, Cauchy inequality and the trace theory of the weighted Sobolev spaces,

$$\begin{aligned}
|I| &\le \|R\|_{L^\infty}\|u^m\|_{{}_0\mathcal{H}^1(\Omega)}\|\partial_t u^m\|_{{}_0\mathcal{H}^1(\Omega)} + \|u^m\|_{{}_0\mathcal{H}^1(\Omega)}\|\mathcal{A}\|_{L^\infty}\|\nabla R\|_{L^p}\|\partial_t u^m\|_{L^q} \\
&\quad + \|R\|_{L^\infty}^2\|u^m\|_{{}_0\mathcal{H}^1(\Omega)}^2 + \|u^m\|_{{}_0\mathcal{H}^1(\Omega)}\|\mathcal{A}\|_{L^\infty}\|R\|_{L^\infty}\|\nabla R\|_{L^p}\|u^m\|_{L^q} \\
&\quad + \|R\|_{L^\infty}\|u^m\|_{{}_0\mathcal{H}^1(\Omega)}\|\partial_t u^m\|_{{}_0\mathcal{H}^1(\Omega)} + \|\partial_t u^m\|_{{}_0\mathcal{H}^1(\Omega)}\|\mathcal{A}\|_{L^\infty}\|\nabla R\|_{L^p}\|u^m\|_{L^q} \\
&\le \|R\|_s\|u^m\|_{{}_0\mathcal{H}^1(\Omega)}\|\partial_t u^m\|_{{}_0\mathcal{H}^1(\Omega)} + \|u^m\|_{{}_0\mathcal{H}^1(\Omega)}\|\mathcal{A}\|_s\|\nabla R\|_{W^1_\delta}\|\partial_t u^m\|_1 \\
&\quad + \|R\|_s^2\|u^m\|_{{}_0\mathcal{H}^1(\Omega)}^2 + \|u^m\|_{{}_0\mathcal{H}^1(\Omega)}\|\mathcal{A}\|_s\|R\|_s\|\nabla R\|_{W^1_\delta}\|u^m\|_1 \\
&\quad + \|R\|_s\|u^m\|_{{}_0\mathcal{H}^1(\Omega)}\|\partial_t u^m\|_{{}_0\mathcal{H}^1(\Omega)} + \|\partial_t u^m\|_{{}_0\mathcal{H}^1(\Omega)}\|\mathcal{A}\|_s\|\nabla R\|_{W^1_\delta}\|u^m\|_1 \\
&\lesssim \theta\|\partial_t u^m\|_{{}_0\mathcal{H}^1(\Omega)}^2 + \left(1 + \frac{1}{\theta}\right) C_2(\eta)\|u^m\|_{{}_0\mathcal{H}^1(\Omega)}^2,
\end{aligned} \tag{4.47}$$

where

$$C_2(\eta) = \|R\|_s^2 + \|\mathcal{A}\|_s^2\|\nabla R\|_{W^1_\delta}^2 + \|\mathcal{A}\|_s\|R\|_s\|\nabla R\|_{W^1_\delta} \lesssim (\|\eta\|_{W^{5/2}_\delta}^2 + \|\partial_t\eta\|_{W^{5/2}_\delta}^2)(1 + \|\eta\|_{W^{5/2}_\delta}^2). \tag{4.48}$$

Similarly,

$$\begin{aligned}
|II| &\lesssim \|\partial_t\mathcal{A}\|_{L^\infty}\|J\|_{L^\infty}\|u^m\|_1\big(\|\partial_t u^m\|_{{}_0\mathcal{H}^1(\Omega)} + \|R\|_{L^\infty}\|u^m\|_{{}_0\mathcal{H}^1(\Omega)} \\
&\quad + \|\mathcal{A}\|_{L^\infty}\|\nabla R\|_{L^p}\|u^m\|_{L^q}\big) + \|\partial_t\mathcal{A}\|_{L^\infty}\|J\|_{L^\infty}\|\partial_t u^m\|_1\|u^m\|_{{}_0\mathcal{H}^1(\Omega)} \\
&\quad + \|\mathcal{A}\|_{L^\infty}\|\nabla R\|_{L^p}\|u^m\|_{L^q}\|u^m\|_{{}_0\mathcal{H}^1(\Omega)} + \|\partial_t JK\|_{L^\infty}\|\partial_t u^m\|_{{}_0\mathcal{H}^1(\Omega)}\|u^m\|_{{}_0\mathcal{H}^1(\Omega)} \\
&\quad + \|\partial_t JK\|_{L^\infty}\|u^m\|_{{}_0\mathcal{H}^1(\Omega)}(\|R\|_{L^\infty}\|u^m\|_{{}_0\mathcal{H}^1(\Omega)} + \|J\mathcal{A}\|_{L^\infty}\|\nabla R\|_{L^p}\|u^m\|_{L^q}) \\
&\quad + \|\partial_t J\|_{L^\infty(\Sigma_s)}\|u^m\|_1(\|\partial_t u^m\|_1 + \|R\|_{L^\infty(\Sigma_s)}\|u^m\|_1) \\
&\lesssim \theta\|\partial_t u^m\|_{{}_0\mathcal{H}^1(\Omega)}^2 + \left(1 + \frac{1}{\theta}\right) C_3(\eta)\|u^m\|_{{}_0\mathcal{H}^1(\Omega)}^2,
\end{aligned} \tag{4.49}$$

where

$$\begin{aligned}
C_3(\eta) &= \big[\|\partial_t\mathcal{A}\|_{L^\infty}\|J\|_{L^\infty}(\|\partial_t\mathcal{A}\|_{L^\infty}\|J\|_{L^\infty} + \|R\|_{L^\infty}) \\
&\quad + \|\mathcal{A}\|_{L^\infty}\|\nabla R\|_{L^p}(1 + \|\partial_t J\|_{L^\infty}) + \|\partial_t JK\|_{L^\infty}(\|\partial_t JK\|_{L^\infty} + \|R\|_{L^\infty}) \\
&\quad + \|\partial_t J\|_{L^\infty(\Sigma_s)}(\|\partial_t J\|_{L^\infty(\Sigma_s)} + \|R\|_{L^\infty(\Sigma_s)})\big] \lesssim (\|\eta\|_{W^{5/2}_\delta}^2 + \|\partial_t\eta\|_{W^{5/2}_\delta}^2)\|\eta\|_{W^{5/2}_\delta}^2.
\end{aligned} \tag{4.50}$$

For $III$, we need more refined estimates. We will separate the estimates for $III$ into several estimates. First, by Cauchy-Schwarz inequality, we have that

$$\int_{-\ell}^{\ell}[\partial_t F^3\partial_1((\partial_t u^m - Ru^m)\cdot\mathcal{N}) = \int_{-\ell}^{\ell}\partial_z\mathcal{R}\partial_1\partial_t\eta\partial_1\partial_t^2\xi^m \le C(\epsilon)\|\partial_t\eta\|_1^2 + \frac{\epsilon}{2}\|\partial_t^2\xi^m\|_1^2, \tag{4.51}$$

here we have used the boundedness for $\partial_z\mathcal{R}$, which can be easily proved by the definition of $\mathcal{R}$.

Then we use Lemma (4.1), Weighted Sobolev embedding theory in Appendix C and D developed by Guo and Tice in [10], usual Sobolev embedding Theorem and Hölder inequality to derive that

$$\begin{aligned}
&\int_\Omega \left[\partial_t F^1\cdot(\partial_t u^m - Ru^m) + \partial_t JKF^1\cdot(\partial_t u^m - Ru^m) + F^1\cdot R(\partial_t u^m - Ru^m)\right] J\\
&\quad - \int_{-\ell}^{\ell}[\partial_t F^4\cdot(\partial_t u^m - Ru^m) + F^4\cdot R(\partial_t u^m - Ru^m)]\\
&\quad - \int_{\Sigma_s}\left[\partial_t F^5(\partial_t u^m - Ru^m) + \partial_t JKF^5(\partial_t u^m - Ru^m) + F^5R(\partial_t u^m - Ru^m)\right]\cdot\tau J\\
&\quad - [\hat{\mathscr{W}}'(\partial_t\eta)\partial_t^2\eta, (\partial_t u^m - Ru^m)\cdot\mathcal{N}]_\ell\\
&\lesssim \theta(\|\partial_t u^m\|_{{}_0\mathcal{H}^1}^2 + [\partial_t u^m\cdot\mathcal{N}]_\ell^2) + \|u^m\|_{{}_0\mathcal{H}^1}^2 + (1+\|R\|_1^2)\|\partial_t(F^1-F^4-F^5)\|_{(\mathcal{H}^1)^*}^2\\
&\quad + (\|\partial_t J\|_1^2 + \|R\|_1^2)(1+\|R\|_1^2)(\|F^1\|_{W^0_\delta}^2 + \|F^4\|_{W^{1/2}_\delta}^2 + \|F^5\|_{W^{1/2}_\delta}^2) + \|\partial_t\eta\|_1^2[\partial_t^2\eta]_\ell^2.
\end{aligned}\tag{4.52}$$

Thus, combining (4.47)–(4.52), we have the energy structure

$$\begin{aligned}
&\frac{d}{dt}\frac{1}{2}\|\partial_t\xi^m\|_{1,\Sigma}^2 + \epsilon\|\partial_t^2\xi^m\|_{1,\Sigma}^2 + \|\partial_t u^m\|_{{}_0\mathcal{H}^1}^2 + [\partial_t u^m\cdot\mathcal{N}]_\ell^2\\
&\lesssim (1+C_2(\eta)+C_3(\eta))\|u^m\|_{{}_0\mathcal{H}^1}^2 + C(\epsilon)\|\partial_t\eta\|_1^2 + \|\partial_t\eta\|_1^2[\partial_t^2\eta]_\ell^2\\
&\quad + C_5(\eta)\|\partial_t(F^1-F^4-F^5)\|_{(\mathcal{H}^1)^*}^2 + C_6(\eta)(\|F^1\|_{W^0_\delta}^2 + \|F^4\|_{W^{1/2}_\delta}^2 + \|F^5\|_{W^{1/2}_\delta}^2),
\end{aligned}\tag{4.53}$$

where

$$C_5(\eta) = (1+\|R\|_1^2) \lesssim (1+\|\partial_t\eta\|_{3/2}^2 + \|\eta\|_{W^{5/2}_\delta}^2),$$

and

$$C_6(\eta) = (\|\partial_t J\|_1^2 + \|R\|_1^2)(1+\|R\|_1^2) \lesssim (\|\partial_t\eta\|_{3/2}^2 + \|\eta\|_{W^{5/2}_\delta}^2)(1+\|\partial_t\eta\|_{3/2}^2 + \|\eta\|_{W^{5/2}_\delta}^2).$$

We then employ the Gronwall's inequality to derive that

$$\begin{aligned}
&\sup_{0\le t\le T}\|\partial_t\xi^m\|_{1,\Sigma}^2 + \epsilon\|\partial_t^2\xi^m\|_{L^2H^1}^2 + \|\partial_t u^m\|_{L^2H^1}^2 + \|\partial_t u^m\|_{L^2H^0(\Sigma_s)}^2 + \int_0^T[\partial_t u^m\cdot\mathcal{N}]_\ell^2\\
&\lesssim C(\epsilon)T\|\partial_t\eta\|_{L^\infty H^1}^2 + \mathfrak{E}_0 + \mathfrak{D}(\eta)\|u^m\|_{L^\infty{}_0\mathcal{H}^1}^2 + \|u^m\|_{L^2H^1}^2 + \mathfrak{E}(\eta)\mathfrak{K}(\eta)\\
&\quad + (1+\mathfrak{E}(\eta))(\|F^1\|_{L^2W^0_\delta}^2 + \|F^4\|_{L^2W^{1/2}_\delta}^2 + \|F^5\|_{L^2W^{1/2}_\delta}^2)\\
&\quad + (1+\mathfrak{E}(\eta))\|\partial_t(F^1-F^4-F^5)\|_{(\mathcal{H}^1_T)^*}^2.
\end{aligned}\tag{4.54}$$

Then applying the smallness of $\mathfrak{K}(\eta)\le\alpha\ll 1$ and Lemma 4.7, we have that

$$\begin{aligned}
&\sup_{0\le t\le T}\|\partial_t\xi^m\|_{1,\Sigma}^2 + \epsilon\|\partial_t^2\xi^m\|_{L^2H^1}^2 + \|\partial_t u^m\|_{L^2H^1}^2 + \|\partial_t u^m\|_{L^2H^0(\Sigma_s)}^2 + \int_0^T[\partial_t u^m\cdot\mathcal{N}]_\ell^2\\
&\lesssim C(\epsilon)T\mathfrak{E}(\eta) + \mathfrak{E}_0 + \mathfrak{E}(\eta)\mathfrak{K}(\eta) + \mathfrak{E}(\eta)(\|F^1\|_{L^2W^0_\delta}^2 + \|F^4\|_{L^2W^{1/2}_\delta}^2 + \|F^5\|_{L^2W^{1/2}_\delta}^2)\\
&\quad + (1+\mathfrak{E}(\eta))\|\partial_t(F^1-F^4-F^5)\|_{(\mathcal{H}^1_T)^*}^2.
\end{aligned}\tag{4.55}$$

Step 6 – Passing to the limit.

We now utilize the energy estimates (4.37) and (4.55) to pass to the limit $m\to\infty$. According to Proposition 2.5 and energy estimates, we have that the sequence $\{u^m\}$ and $\{\partial_t u^m\}$ are uniformly bounded both in $L^2H^1$ and $L^2H^0(\Sigma_s)$, $\{\xi^m\}$ and $\{\partial_t\xi^m\}$ are uniformly bounded in $L^\infty\mathring{H}^1$, $\{[u^m\cdot\mathcal{N}]_\ell\}$ and $\{[\partial_t u^m\cdot\mathcal{N}]_\ell\}$ are uniformly bounded in $L^2([0,T])$. Up to the extraction of a subsequence, we then know that

$$u^m \rightharpoonup u \text{ weakly- in } L^2H^1\cap L^2H^0(\Sigma_s),\quad \partial_t u^m\rightharpoonup\partial_t u \text{ weakly in } L^2H^1\cap L^2H^0(\Sigma_s),$$

$$\xi^m \overset{*}{\rightharpoonup}\xi \text{ weakly-}* \text{ in } L^\infty\mathring{H}^1,\quad \partial_t\xi^m\overset{*}{\rightharpoonup}\partial_t\xi \text{ weakly-}* \text{ in } L^\infty\mathring{H}^1,$$

and

$$[u^m\cdot\mathcal{N}]_\ell\rightharpoonup[u\cdot\mathcal{N}]_\ell \text{ weakly- in } L^2,\ [\partial_t u^m\cdot\mathcal{N}]_\ell\rightharpoonup[\partial_t u\cdot\mathcal{N}]_\ell \text{ weakly in } L^2.$$

By lower semicontinuity, the energy estimates imply that

$$\|u\|_{L^2H^1}^2+\|u\|_{L^2H^0(\Sigma_s)}^2+\|\partial_t u\|_{L^2H^1}^2+\|\partial_t u\|_{L^2H^0(\Sigma_s)}^2+\|[u\cdot\mathcal{N}]_\ell\|_{L^2}^2+\|[\partial_t u\cdot\mathcal{N}]_\ell\|_{L^2}^2 \\ +\|\xi\|_{L^\infty\mathring{H}^1}^2+\|\partial_t\xi\|_{L^\infty\mathring{H}^1}^2$$

is bounded.

Step 7 – Improved bounds for $\xi$ and $\partial_t\xi$.

Form the above step, we know that $\xi^m(t)\in\mathring{H}^1((-\ell,\ell))$, $\partial_t\xi^m(t)\in\mathring{H}^1((-\ell,\ell))$ and $\partial_t^2\xi^m(t)\in\mathring{H}^1((-\ell,\ell))$. In (4.28), for the test function $v\in\mathcal{V}(t)$, according to Theorem 4.11 in [10], we know that

$$\begin{aligned}\|\xi^m+\epsilon\partial_t\xi^m\|_{\mathring{H}^{3/2}}^2&\lesssim\|u^m\|_1^2+[u^m\cdot\mathcal{N}]_\ell^2+\|\eta\|_{W^{5/2}_\delta}^4+\|\mathcal{F}\|_{(\mathcal{H}^1)^*}^2+\|\partial_t\eta\|_1^2[\partial_t\eta]_\ell^2\\&\lesssim\|u^m\|_1^2+[u^m\cdot\mathcal{N}]_\ell^2+\|\eta\|_{W^{5/2}_\delta}^4+\|F^1\|_{W^0_\delta}^2+\|F^4\|_{W^{1/2}_\delta}^2+\|F^5\|_{W^{1/2}_\delta}^2+\|\partial_t\eta\|_1^2[\partial_t\eta]_\ell^2.\end{aligned}\tag{4.56}$$

Then we may employ (4.33), (4.34) and Sobolev theory to obtain the bound for initial data

$$\begin{aligned}\|(\xi^m+\epsilon\partial_t\xi^m)(0)\|_{\mathring{H}^{3/2}}^2&\lesssim\|u^m(0)\|_1^2+[u^m(0)\cdot\mathcal{N}(0)]_\ell^2+\|\eta_0\|_{W^{5/2}_\delta}^4\\&\quad+\|\mathcal{F}(0)\|_{(\mathcal{H}^1)^*}^2+\|\partial_t\eta(0)\|_1^2[\partial_t\eta(0)]_\ell^2\\&\lesssim\mathfrak{E}_0+\|\mathcal{F}(0)\|_{(\mathcal{H}^1)^*}^2.\end{aligned}\tag{4.57}$$

If we let $\vartheta^m=\xi^m+\epsilon\partial_t\xi^m$, we can solve this ODE as

$$\xi^m=\eta_0e^{-\frac{t}{\epsilon}}+\frac{1}{\epsilon}\int_0^t e^{-\frac{t-s}{\epsilon}}\vartheta^m(s)\,\mathrm{d}s.\tag{4.58}$$

Now we estimate $\xi^m$ from (4.58). Applying Cauchy inequality,

$$\int_0^t\frac{1}{\epsilon}e^{-\frac{t-s}{\epsilon}}\|\vartheta^m(s)\|_{3/2}\,\mathrm{d}s\le\left(\int_0^t\frac{1}{\epsilon}e^{-\frac{t-s}{\epsilon}}\|\vartheta^m(s)\|_{3/2}^2\,\mathrm{d}s\right)^{1/2}\left(\int_0^t\frac{1}{\epsilon}e^{-\frac{t-s}{\epsilon}}\,\mathrm{d}s\right)^{1/2}.\tag{4.59}$$

Since

$$\left(\int_0^t\frac{1}{\epsilon}e^{-\frac{t-s}{\epsilon}}\,\mathrm{d}s\right)^{1/2}=(1-e^{-\frac{t}{\epsilon}})^{1/2}\le1,\tag{4.60}$$

then taking $L^2([0,T])$ for (4.59) and employing Fubini Theorem imply that

$$\begin{aligned}\|\int_0^t\frac{1}{\epsilon}e^{-\frac{t-s}{\epsilon}}\|\vartheta^m(s)\|_{3/2}\|_{L^2}^2&\le\int_0^T\int_0^t\frac{1}{\epsilon}e^{-\frac{t-s}{\epsilon}}\|\vartheta(s)\|_{3/2}^2\,\mathrm{d}s\,\mathrm{d}t\\&=\int_0^T\frac{1}{\epsilon}e^{-\frac{s}{\epsilon}}\left(\int_s^T\|\vartheta(t-s)\|_{3/2}^2\,\mathrm{d}t\right)\mathrm{d}s\\&\le\int_0^T\frac{1}{\epsilon}e^{-\frac{s}{\epsilon}}\int_0^T\|\vartheta(t)\|_{3/2}^2\\&\le\|\vartheta\|_{L^2H^{3/2}}^2.\end{aligned}\tag{4.61}$$

Thus, (4.37) reveals that

$$\begin{aligned}&\|\xi^m\|_{L^2\mathring{H}^{3/2}}^2\le C\|\eta_0\|_{W^{5/2}_\delta}^2+\|\int_0^t\frac{1}{\epsilon}e^{-\frac{t-s}{\epsilon}}\|\vartheta^m(s)\|_{3/2}\|_{L^2}^2\\&\lesssim(1+\|\eta\|_{L^\infty W^{5/2}_\delta}^2)(\|F^1\|_{L^2W^0_\delta}^2+\|F^4\|_{L^2W^{1/2}_\delta}^2+\|F^5\|_{L^2W^{1/2}_\delta}^2)+C(\epsilon)T\|\eta\|_{L^\infty H^1}^2\|\eta\|_{L^\infty W^{5/2}_\delta}^2\\&\quad+\|\eta_0\|_{W^{5/2}_\delta}^2+\|\eta\|_{L^\infty W^{5/2}_\delta}^2\|\eta\|_{L^2W^{5/2}_\delta}^2+\|\partial_t\eta\|_{L^\infty H^1}^2\|[\partial_t\eta]_\ell\|_{L^2(0,T)}^2.\end{aligned}\tag{4.62}$$

Similarly, according to (4.38), we know that

$$\begin{aligned}
&\|\partial_t\xi^m+\epsilon\partial_t^2\xi^m\|^2_{L^2\mathring{H}^{3/2}}\\
&\lesssim \|\partial_t u^m\|^2_{L^2H^1}+\int_0^T[\partial_t u^m\cdot\mathcal{N}]^2_\ell+\mathfrak{E}_0+\mathfrak{E}(\eta)\mathfrak{K}(\eta)+(1+\mathfrak{E}(\eta))(\|F^1\|^2_{L^2W^0_\delta}+\|F^4\|^2_{L^2W^{1/2}_\delta}+\|F^5\|^2_{L^2W^{1/2}_\delta})\\
&\quad+(1+\mathfrak{E}(\eta))\|\partial_t(F^1-F^4-F^5)\|^2_{(\mathcal{H}^1_T)^*}\\
&\lesssim C(\epsilon)T\mathfrak{E}(\eta)+\mathfrak{E}_0+\mathfrak{E}(\eta)\mathfrak{K}(\eta)+(1+\mathfrak{E}(\eta))(\|F^1\|^2_{L^2W^0_\delta}+\|F^4\|^2_{L^2W^{1/2}_\delta}+\|F^5\|^2_{L^2W^{1/2}_\delta})\\
&\quad+(1+\mathfrak{E}(\eta))\|\partial_t(F^1-F^4-F^5)\|^2_{(\mathcal{H}^1_T)^*}.
\end{aligned}\tag{4.63}$$

If we denote $\partial_t\vartheta^m=\partial_t\xi^m+\epsilon\partial_t^2\xi^m$, and the extension $\partial_t\bar{\vartheta}^m=\partial_t\bar{\xi}^m+\epsilon\partial_t^2\bar{\xi}^m$, then by a standard computation for energy formulation, we may get

$$\epsilon\frac{d}{dt}\|\partial_t\bar{\xi}^m\|_{H^2}+\|\partial_t\bar{\xi}^m\|_{H^2}\le\|\partial_t\bar{\vartheta}^m\|_{H^2},\tag{4.64}$$

Then by the trace theory and (4.57), we may derive that

$$\begin{aligned}
&\|\partial_t\xi^m\|^2_{L^2\mathring{H}^{3/2}}\lesssim\epsilon^2\|\partial_t\xi^m(0)\|^2_{3/2}+\|\partial_t\vartheta^m\|^2_{L^2\mathring{H}^{3/2}}\\
&\lesssim\|\xi^m(0)\|^2_{3/2}+\|\xi^m(0)+\epsilon\partial_t\xi^m(0)\|^2_{3/2}+\|\partial_t\vartheta^m\|^2_{L^2\mathring{H}^{3/2}}\\
&\lesssim C(\epsilon)T\mathfrak{E}(\eta)+\mathfrak{E}_0+\|\mathcal{F}(0)\|^2_{(\mathcal{H}^1)^*}+\mathfrak{E}(\eta)\mathfrak{K}(\eta)\\
&\quad+(1+\mathfrak{E}(\eta))(\|F^1\|^2_{L^2W^0_\delta}+\|F^4\|^2_{L^2W^{1/2}_\delta}+\|F^5\|^2_{L^2W^{1/2}_\delta})+(1+\mathfrak{E}(\eta))\|\partial_t(F^1-F^4-F^5)\|^2_{(\mathcal{H}^1_T)^*}.
\end{aligned}\tag{4.65}$$

Then, up to an extraction of subsequence, we know that

$$\xi^m\rightharpoonup\xi\text{ weakly in }L^2H^{3/2},\ \partial_t\xi^m\rightharpoonup\partial_t\xi\text{ weakly in }L^2H^{3/2}.\tag{4.66}$$

By lower semicontinuity, we have

$$\|\xi\|_{L^2H^{3/2}}+\|\partial_t\xi\|_{L^2H^{3/2}}$$

is bounded.

Step 8 – The strong solution

Due to the convergence, we may pass to the limit in (4.28) for almost every $t\in[0,T]$.

$$\begin{aligned}
&((u,v))+(\xi+\epsilon\partial_t\xi,v\cdot\mathcal{N})_{1,\Sigma}+[u\cdot\mathcal{N},v\cdot\mathcal{N}]_\ell\\
&=\int_\Omega F^1\cdot vJ-\int_{-\ell}^{\ell}F^3\partial_1(v\cdot\mathcal{N})+F^4\cdot v-\int_{\Sigma_s}F^5(v\cdot\tau)J-+[\hat{\mathscr{W}}(\partial_t\eta),v\cdot\mathcal{N}]_\ell.
\end{aligned}\tag{4.67}$$

We now introduce the pressure. Define the functional $\Lambda_t\in(\mathcal{W}(t))^*$ so that $\Lambda_t(v)$ equals the difference between the left and right sides of (4.67) with $v\in\mathcal{W}(t)$. Then $\Lambda_t(v)=0$ for all $v\in\mathcal{V}(t)$. So, by the Theorem 4.6 in [10], there exists a unique $p(t)\in\mathring{H}^0(t)$ such that $(p(t),\operatorname{div}_{\mathcal{A}}v)_{\mathcal{H}^0}=\Lambda_t(v)$ for all $v\in\mathcal{W}(t)$. This is equivalent to

$$\begin{aligned}
&((u,v))+(\xi+\epsilon\partial_t\xi,v\cdot\mathcal{N})_{1,\Sigma}-(p,\operatorname{div}_{\mathcal{A}}v)_{\mathcal{H}^0}+[u\cdot\mathcal{N},v\cdot\mathcal{N}]_\ell\\
&=\int_\Omega F^1\cdot vJ-\int_{-\ell}^{\ell}F^3\partial_1(v\cdot\mathcal{N})+F^4\cdot v-\int_{\Sigma_s}F^5(v\cdot\tau)J-[\hat{\mathscr{W}}(\partial_t\eta),v\cdot\mathcal{N}]_\ell.
\end{aligned}\tag{4.68}$$

Moreover,

$$\|p\|^2_0\lesssim\|u\|^2_1+\|F^1\|^2_{W^0_\delta}+\|F^5\|^2_{W^{1/2}_\delta}.\tag{4.69}$$

On the other hand, we pass the limit in (4.38) For a.e. $t\in[0,T]$, $(u(t),p(t),\xi(t))$ is the unique weak solution to the elliptic problem (**??**) with $F^3$ replaced by $F^3(t)$. Since $\partial_1F^3(t)\in W^{1/2}_\delta$, and also according to the elliptic theory of [10], this elliptic problem admits a unique strong solution with

$$\begin{aligned}
\|u(t)\|^2_{W^2_\delta}+\|p(t)\|^2_{\mathring{W}^1_\delta}+\|\xi(t)+\epsilon\partial_t\xi(t)\|^2_{W^{5/2}_\delta}&\lesssim\|F^1\|^2_{L^2W^0_\delta}+\|F^4\|^2_{L^2W^{1/2}_\delta}+\|F^5\|^2_{L^2W^{1/2}_\delta}\\
&+\|\partial_t\xi(t)\|^2_{W^{3/2}_\delta}+\|\partial_1F^3(t)\|^2_{W^{1/2}_\delta}+[u(t)\cdot\mathcal{N}]^2_\ell+[\hat{\mathscr{W}}(\partial_t\eta)]^2_\ell+[\sigma F^3(t)]^2_\ell.
\end{aligned}\tag{4.70}$$

Then using the extension and restriction of weighted Sobolev spaces theory, similar to (4.64), we may derive that

$$\|\xi\|^2_{L^2W^{5/2}_\delta} \lesssim \|\eta_0\|^2_{L^2W^{5/2}_\delta} + \|\xi(t)+\epsilon\partial_t\xi(t)\|^2_{L^2W^{5/2}_\delta}. \tag{4.71}$$

Integrating temporally from 0 to $T$ for (4.70), we employ (4.71) to derive that

$$\begin{aligned}
&\|u\|^2_{L^2W^2_\delta} + \|p\|^2_{L^2\mathring{W}^1_\delta} + \|\xi\|^2_{L^2W^{5/2}_\delta}\\
&\lesssim \|\eta_0\|^2_{W^{5/2}_\delta} + \|\partial_t\xi\|^2_{L^2W^{3/2}_\delta} + \|\partial_1F^3\|^2_{L^2W^{1/2}_\delta} + \int_0^T \left([\kappa u\cdot\mathcal{N}]^2_\ell + [\hat{\mathscr{W}}(\partial_t\eta)]^2_\ell + [\sigma F^3]^2_\ell\right)\\
&\quad + \|F^1\|^2_{L^2W^0_\delta} + \|F^4\|^2_{L^2W^{1/2}_\delta} + \|F^5\|^2_{L^2W^{1/2}_\delta}\\
&\lesssim C(\epsilon)T\mathfrak{E}(\eta) + \mathfrak{E}_0 + \|\mathcal{F}(0)\|^2_{(\mathcal{H}^1)^*} + \mathfrak{E}(\eta)\mathfrak{K}(\eta) + (1+\mathfrak{E}(\eta))(\|F^1\|^2_{L^2W^0_\delta} + \|F^4\|^2_{L^2W^{1/2}_\delta} + \|F^5\|^2_{L^2W^{1/2}_\delta})\\
&\quad + (1+\mathfrak{E}(\eta))\|\partial_t(F^1-F^4-F^5)\|^2_{(\mathcal{H}^1_T)^*}.
\end{aligned} \tag{4.72}$$

Step 9 – The weak solution for $D_tu$ and $\partial_tp$. Now we seek to use (4.38) to determine the PDE satisfied by $D_tu$ and $\partial_tp$. We may pass to the limit $m\to\infty$, and use (4.68) with the test function $v$ replaced by $Rv$ to derive that

$$\begin{aligned}
&((\partial_tu,v)) + (\partial_t\xi+\epsilon\partial_t^2\xi, v\cdot\mathcal{N})_{1,\Sigma} + [\partial_tu\cdot\mathcal{N}, v\cdot\mathcal{N}]_\ell\\
&= (\xi+\epsilon\partial_t\xi, Rv\cdot\mathcal{N})_{1,\Sigma} - (p, \mathrm{div}_{\mathcal{A}}(Rv))_{\mathcal{H}^0} + \int_\Omega \left[\partial_tF^1\cdot v + \partial_tJKF^1\cdot v\right]J\\
&\quad - \int_{-\ell}^{\ell}[\partial_tF^3\partial_1(v\cdot\mathcal{N}) + F^3\partial_1(v\cdot\partial_t\mathcal{N}) + \partial_tF^4\cdot v] - \int_{\Sigma_s}\left[\partial_tF^5v + \partial_tJKF^5v\right]\cdot\tau J\\
&\quad - \int_\Omega\frac{\mu}{2}(\mathbb{D}_{\partial_t\mathcal{A}}u:\mathbb{D}_{\mathcal{A}}v + \mathbb{D}_{\mathcal{A}}u:\mathbb{D}_{\partial_t\mathcal{A}}v + \partial_tJK\mathbb{D}_{\mathcal{A}}u:\mathbb{D}_{\mathcal{A}}v)J - \int_{\Sigma_s}\beta(u\cdot\tau)(v\cdot\tau)\partial_tJ\\
&\quad - [\hat{\mathscr{W}}'(\partial_t\eta)\partial_t^2\eta, v\cdot\mathcal{N}]_\ell.
\end{aligned} \tag{4.73}$$

According to the Lemma A.1, we know that $-R^\top\mathcal{N} = \partial_t\mathcal{N}$ on $\Sigma$. Then integrating by parts, we have that

$$-(p,\mathrm{div}_{\mathcal{A}}(Rv))_{\mathcal{H}^0} = (R^\top\nabla_{\mathcal{A}}p, v)_{\mathcal{H}^0} + \langle p\partial_t\mathcal{N}, v\rangle_{-1/2}, \tag{4.74}$$

where we have used the Proposition 2.6 to cancel the term on boundary of solid wall. Then the definition of $R$ and integration by parts yields that

$$\begin{aligned}
&-\int_\Omega\frac{\mu}{2}(\mathbb{D}_{\partial_t\mathcal{A}}u:\mathbb{D}_{\mathcal{A}}v + \mathbb{D}_{\mathcal{A}}u:\mathbb{D}_{\partial_t\mathcal{A}}v + \partial_tJK\mathbb{D}_{\mathcal{A}}u:\mathbb{D}_{\mathcal{A}}v)J\\
&= -\int_\Omega\mu(\mathbb{D}_{\partial_t\mathcal{A}}u - R\mathbb{D}_{\mathcal{A}}u):\nabla_{\mathcal{A}}vJ\\
&= (\mathrm{div}_{\mathcal{A}}(\mathbb{D}_{\partial_t\mathcal{A}}u - R\mathbb{D}_{\mathcal{A}}u), v)_{\mathcal{H}^0} - \langle\mathbb{D}_{\partial_t\mathcal{A}}u\mathcal{N} + \mathbb{D}_{\mathcal{A}}u\partial_t\mathcal{N}, v\rangle_{-1/2}.
\end{aligned} \tag{4.75}$$

Similarly, we have that

$$\begin{aligned}
-\int_{\Sigma_s}\beta(u\cdot\tau)(v\cdot\tau)\partial_tJ &= \int_{\Sigma_s}\mu\mathbb{D}_{\mathcal{A}}u\nu\cdot v\partial_tJ\\
&= \int_{\Sigma_s}\mu\mathbb{D}_{\mathcal{A}}u\nu\cdot v\partial_t\tilde{J}KJ + \mu R\mathbb{D}_{\mathcal{A}}u\nu\cdot vJ - \mu R\mathbb{D}_{\mathcal{A}}u\nu\cdot vJ\\
&= \int_{\Sigma_s}\mu\mathbb{D}_{\partial_t\mathcal{A}}u\nu\cdot vJ - \mu R\mathbb{D}_{\mathcal{A}}u\nu\cdot vJ\\
&= \int_{\Sigma_s}\mu\mathbb{D}_{\partial_t\mathcal{A}}u\nu\cdot vJ + \beta Ru\cdot vJ.
\end{aligned} \tag{4.76}$$

Combining the above equalities (4.73)–(4.76),

$$\begin{aligned}
&((\partial_t u, v)) + (\partial_t \xi + \epsilon \partial_t^2 \xi, v \cdot \mathcal{N})_{1,\Sigma} + [\partial_t u \cdot \mathcal{N}, v \cdot \mathcal{N}]_\ell \\
&= \int_\Omega \left[ \mathrm{div}_{\mathcal{A}} (\mathbb{D}_{\partial_t \mathcal{A}} u - R \mathbb{D}_{\mathcal{A}} u) + R^\top \nabla_{\mathcal{A}} p \right] \cdot v J + \int_\Omega (\partial_t F^1 + \partial_t J K F^1) \cdot v J \\
&\quad + \int_{\Sigma_s} (\mu \mathbb{D}_{\partial_t \mathcal{A}} u \nu + \beta R u) \cdot \tau (\tau \cdot v) J + \int_{\Sigma_s} (\partial_t F^5 + \partial_t J K F^5)(v \cdot \tau) J \\
&\quad - \int_{-\ell}^{\ell} \partial_t F^3 \partial_1 (v \cdot \mathcal{N}) + F^3 \partial_1 (v \cdot \partial_t \mathcal{N}) + F^4 \cdot v + (\mathbb{D}_{\partial_t \mathcal{A}} u \mathcal{N} + \mathbb{D}_{\mathcal{A}} u \partial_t \mathcal{N}) \cdot v \\
&\quad + \int_{-\ell}^{\ell} \left( -p + g(\xi + \epsilon \partial_t \xi) + \partial_1 \left( F^3 + \frac{\xi + \epsilon \partial_t \xi}{(1 + |\partial_1 \zeta_0|^2)^{3/2}} \right) \right) \partial_t \mathcal{N} \cdot v \\
&\quad - [\mathscr{W}'(\partial_t \eta) \partial_t^2 \eta, v \cdot \mathcal{N}]_\ell,
\end{aligned} \tag{4.77}$$

where we have used the integration by parts for the term $(\xi + \epsilon \partial_t \xi, Rv \cdot \mathcal{N})_{1,\Sigma}$ and the fact that $v \cdot \partial_t \mathcal{N} = 0$ at $x_1 = \pm \ell$. Then there exists a unique $\partial_t q \in \mathring{H}^0$, such that

$$\begin{aligned}
&((\partial_t u, v)) - (\partial_t p, \mathrm{div}_{\mathcal{A}} v)_0 + (\partial_t \xi + \epsilon \partial_t^2 \xi, v \cdot \mathcal{N})_{1,\Sigma} \\
&= \int_\Omega \left[ \mathrm{div}_{\mathcal{A}} (\mathbb{D}_{\partial_t \mathcal{A}} u - R \mathbb{D}_{\mathcal{A}} u) + R^\top \nabla_{\mathcal{A}} p \right] \cdot v J + \int_\Omega (\partial_t F^1 + \partial_t J K F^1) \cdot v J \\
&\quad + \int_{\Sigma_s} (\mu \mathbb{D}_{\partial_t \mathcal{A}} u \nu + \beta R u) \cdot \tau (\tau \cdot v) J + \int_{\Sigma_s} (\partial_t F^5 + \partial_t J K F^5)(v \cdot \tau) J \\
&\quad - \int_{-\ell}^{\ell} \partial_t F^3 \partial_1 (v \cdot \mathcal{N}) + F^3 \partial_1 (v \cdot \partial_t \mathcal{N}) + F^4 \cdot v + (\mathbb{D}_{\partial_t \mathcal{A}} u \mathcal{N} + \mathbb{D}_{\mathcal{A}} u \partial_t \mathcal{N}) \cdot v \\
&\quad + \int_{-\ell}^{\ell} \left( -p + g(\xi + \epsilon \partial_t \xi) + \partial_1 \left( F^3 + \frac{\xi + \epsilon \partial_t \xi}{(1 + |\partial_1 \zeta_0|^2)^{3/2}} \right) \right) \partial_t \mathcal{N} \cdot v \\
&\quad - [\hat{\mathscr{W}}'(\partial_t \eta) \partial_t^2 \eta, v \cdot \mathcal{N}]_\ell,
\end{aligned} \tag{4.78}$$

and

$$\begin{aligned}
\|\partial_t p\|_0^2 \lesssim\ & \|\partial_t u\|_1^2 + [\partial_t u \cdot \mathcal{N}]_\ell^2 + \|\eta\|_{3/2}^2 \|\partial_t \eta\|_{3/2}^2 (\|u\|_{W_\delta^2}^2 + \|p\|_{\mathring{W}_\delta^1}^2 + \|\xi + \epsilon \partial_t \xi\|_{W_\delta^{5/2}}^2 \\
&+ \|\eta\|_{W_\delta^{5/2}}^2 + \|\eta\|_{3/2}^2 + 1 + \|F^1\|_{W_\delta^0}^2 + \|F^4\|_{W_\delta^{1/2}}^2 + \|F^5\|_{W_\delta^{1/2}}^2 \\
&+ \|\partial_t (F^1 - F^4 - F^5)\|_{(\mathcal{H}^1)^*}^2) + [\hat{\mathscr{W}}'(\partial_t \eta) \partial_t^2 \eta]_\ell^2.
\end{aligned} \tag{4.79}$$

Thus integrating temporally from 0 to $T$ reveals that

$$\begin{aligned}
\|\partial_t p\|_{L^2 H^0}^2 \lesssim\ & C(\epsilon) T \mathfrak{E}(\eta) + \mathfrak{E}_0 + \|\mathcal{F}(0)\|_{(\mathcal{H}^1)^*}^2 + \mathfrak{E}(\eta) \mathfrak{K}(\eta) + (1 + \mathfrak{E}(\eta))(\|F^1\|_{L^2 W_\delta^0}^2 + \|F^4\|_{L^2 W_\delta^{1/2}}^2 \\
&+ \|F^5\|_{L^2 W_\delta^{1/2}}^2) + (1 + \mathfrak{E}(\eta)) \|\partial_t (F^1 - F^4 - F^5)\|_{(\mathcal{H}_T^1)^*}^2.
\end{aligned} \tag{4.80}$$

□

4.3. **Higher regularity.** In order to state our higher regularity results for the problem (4.3), we must be able to define the forcing terms and initial data the problem that results from temporally differentiating (4.3) one time. First, we define some mappings. Given $F^3$, $v$, $q$, $\tilde{\xi}$, we define the vector fields $\mathfrak{G}^1$ in $\Omega$, $\mathfrak{G}^3$ on $\Sigma$ and $\mathfrak{G}^4$ on $\Sigma_s$ by

$$\begin{aligned}
\mathfrak{G}^1(v, q) &= R^\top \nabla_{\mathcal{A}} q + \mathrm{div}_{\mathcal{A}} \left( \mathbb{D}_{\mathcal{A}} (Rv) + \mathbb{D}_{\partial_t \mathcal{A}} v - R \mathbb{D}_{\mathcal{A}} v \right), \\
\mathfrak{G}^4(v, q, \tilde{\xi}) &= \mu \mathbb{D}_{\mathcal{A}} (Rv) \mathcal{N} - (qI - \mu \mathbb{D}_{\mathcal{A}} v) \partial_t \mathcal{N} + \mu \mathbb{D}_{\partial_t \mathcal{A}} v) \mathcal{N} + \mathcal{L}(\tilde{\xi}) \partial_t \mathcal{N} - \sigma \partial_1 F^3 \partial_t \mathcal{N}, \\
\mathfrak{G}^5(v) &= (\mu \mathbb{D}_{\mathcal{A}} (Rv) \nu + \mu \mathbb{D}_{\partial_t \mathcal{A}} v \nu + \beta R v) \cdot \tau,
\end{aligned} \tag{4.81}$$

These mappings allow us to define the forcing terms as follows. We write $F^{1,0} = F^1$, $F^{4,0} = F^4$ and $F^{5,0} = F^5$. Then we write

$$F^{1,1} := D_t F^1 + G^1, \quad F^{4,1} := \partial_t F^4 + G^4, \quad F^{5,1} := \partial_t F^5 + G^5. \tag{4.82}$$

When $F^3$, $u$, $p$ and $\xi$ are sufficiently regular for the following to make sense, we define the vectors

$$F^{1,2} := \mathfrak{G}^1(D_t u, \partial_t p) + D_t G^1, F^{4,2} := \mathfrak{G}^4(D_t u, \partial_t p, \partial_t \xi) + \partial_t G^4, F^{5,2} := \mathfrak{G}^5(D_t u) + \partial_t G^5. \tag{4.83}$$

In order to deduce the higher regularity, we need to control the forcing terms $F^{i,j}$. But for the purpose of solving the nonlinear problem (5.1), it's necessary to assume that $F^{i,0} = 0$, $j = 1, 4, 5$. Before that, we need the following useful lemma.

**Lemma 4.9.** *Suppose that the right-hand side of the following estimates are finite. Then $u \in C^0([0,T]; W^2_\delta(\Omega))$, $p \in C^0([0,T]; \mathring{W}^1_\delta(\Omega))$, $\xi \in C^0([0,T]; W^{5/2}_\delta((-\ell,\ell)))$, and*

$$\|u\|^2_{L^\infty W^2_\delta} \lesssim \|\partial_t \eta(0)\|^2_{3/2} + \|u\|^2_{L^2 W^2_\delta} + \|\partial_t u\|^2_{L^2 W^2_\delta}, \tag{4.84}$$

$$\|p\|^2_{L^\infty \mathring{W}^1_\delta} \lesssim \|\partial_t \eta(0)\|^2_{3/2} + \|p\|^2_{L^2 \mathring{W}^1_\delta} + \|\partial_t p\|^2_{L^2 \mathring{W}^1_\delta}, \tag{4.85}$$

$$\|\xi\|^2_{L^\infty W^{5/2}_\delta} \lesssim \|\eta_0\|^2_{W^{5/2}_\delta} + \|\xi\|^2_{L^2 W^{5/2}_\delta} + \|\partial_t \xi\|^2_{L^2 W^{5/2}_\delta}. \tag{4.86}$$

$$\|\xi + \epsilon \partial_t \xi\|^2_{L^\infty W^{5/2}_\delta} \lesssim \|\eta_0\|^2_{W^{5/2}_\delta} + \|\xi + \epsilon \partial_t \xi\|^2_{L^2 W^{5/2}_\delta} + \|\partial_t \xi + \epsilon \partial_t^2 \xi\|^2_{L^2 W^{5/2}_\delta}. \tag{4.87}$$

*Proof.* First, (4.84) and (4.85) are obtained by the similar computation of Lemma 4.7 and then utilizing the initial data for $u_0$, $p_0$ in Section 3. By Theorem 4.6 in [10] and Stokes equation, we have that (4.86) could easily obtained by the similar computation as (4.84) after employing the extension theory on weighted Sobolev spaces, and then using the restriction theory on Sobolev spaces. From the third equation of (4.3), we know that

$$\|(\xi + \epsilon \partial_t \xi)(0)\|^2_{W^{5/2}_\delta} \lesssim \|\eta_0\|^2_{W^{5/2}_\delta} + (1 + \|\eta_0\|^2_{W^{5/2}_\delta}) \|u_0\|^2_{W^2_\delta} \lesssim \|\eta_0\|^2_{W^{5/2}_\delta} + \|\partial_t \eta(0)\|^2_{3/2},$$

which together with (4.86) imply (4.87). □

Now, we need to estimate the forcing terms of $F^{i,j}$.

**Lemma 4.10.** *The following estimates hold whenever the right hand side are finite.*

$$\|F^{1,1}\|^2_{L^2 W^0_\delta} \lesssim \mathfrak{K}(\eta)(\|u\|^2_{L^2 H^1} + \|\partial_t u\|^2_{L^2 H^1} + \|u\|^2_{L^2 W^2_\delta} + \|\partial_t u\|^2_{L^2 W^2_\delta} + \|p\|^2_{L^2 \mathring{W}^1_\delta} + \|\partial_t p\|^2_{L^2 \mathring{W}^1_\delta}), \tag{4.88}$$

$$\begin{aligned} \|F^{4,1}\|^2_{L^2 W^{1/2}_\delta} \lesssim \mathfrak{K}(\eta)\Big((1 + \|\eta_0\|^2_{W^{5/2}_\delta})(1 + \|u_0\|^2_{W^2_\delta}) + \|p\|^2_{L^2 W^1_\delta} + \|u\|^2_{L^2 W^2_\delta} + \|\xi + \epsilon \partial_t \xi\|^2_{L^2 W^{5/2}_\delta} \\ + \|\partial_t p\|^2_{L^2 W^1_\delta} + \|\partial_t u\|^2_{L^2 W^2_\delta} + \|\partial_t \xi + \epsilon \partial_t^2 \xi\|^2_{L^2 W^{5/2}_\delta}\Big), \end{aligned} \tag{4.89}$$

$$\|F^{5,1}\|^2_{L^2 W^{1/2}_\delta} \lesssim \mathfrak{K}(\eta) \left(\|u\|^2_{L^2 W^2_\delta} + \|\partial_t u\|^2_{L^2 W^2_\delta} + \|u\|^2_{L^2 H^1}\right). \tag{4.90}$$

*Proof.* The estimates follow from simple but lengthy computations, invoking the arguments of Appendix C and Appendix D in [10]. For this reason, we only give a sketch of proving these estimates.

According to the definition of $F^{1,1}$, $F^{4,1}$ and $F^{5,1}$ in (4.82), we use Leibniz rule to rewrite $F^{i,1}$ as a sum of products for two terms. One term is a product of various derivatives of $\bar{\eta}$, and the other is linear for derivatives of $u$, $p$ and $\xi$. Then for a.e. $t \in [0,T]$, we estimate these resulting products using the weighted Sobolev theory in Appendix C and Appendix D in [10], the usual Sobolev embedding theorems and Lemma 4.7. Then the resulting inequalities after integrating over $[0,T]$ reveals

$$\begin{aligned} \|F^{1,1}\|^2_{L^2 W^0_\delta} \lesssim P(\mathfrak{E}(\eta))\mathfrak{D}(\eta)(\|u\|^2_{L^2 H^1} + \|\partial_t u\|^2_{L^2 H^1} + \|u\|^2_{L^2 W^2_\delta} + \|\partial_t u\|^2_{L^2 W^2_\delta} \\ + \|p\|^2_{L^2 \mathring{W}^1_\delta} + \|\partial_t p\|^2_{L^2 \mathring{W}^1_\delta}), \end{aligned} \tag{4.91}$$

where $P(\cdot)$ is a polynomial. Since $\mathfrak{K}(\eta) \leq 1$, we know that $P(\mathfrak{E}(\eta))\mathfrak{D}(\eta) \lesssim \mathfrak{K}(\eta)$. Thus we have the bounds for (4.88). Similarly, we have the bounds for (4.89) and (4.90), and (4.89) also needs (4.87). □

**Lemma 4.11.** *It holds that*

$$\|F^{1,1}-F^{4,1}-F^{5,1}\|^2_{L^2(\mathcal{H}^1)^*}\lesssim \mathfrak{E}(\eta)(\|p\|^2_{L^2\mathring{W}^1_\delta}+\|u\|^2_{L^2W^2_\delta}+\|\xi+\epsilon\partial_t\xi\|^2_{L^2W^{5/2}_\delta}), \tag{4.92}$$

*and*

$$\begin{aligned}\|\partial_t(F^{1,1}-F^{4,1}-F^{5,1})\|^2_{(\mathcal{H}^1_T)^*}\lesssim \mathfrak{K}(\eta)(1+\|u\|^2_{L^2H^1}+\|\partial_t u\|^2_{L^2H^1}+\|u\|^2_{L^2W^2_\delta}+\|\partial_t u\|^2_{L^2W^2_\delta}\\ +\|p\|^2_{L^2\mathring{W}^1_\delta}+\|\partial_t p\|^2_{L^2\mathring{W}^1_\delta}+\|\xi+\epsilon\partial_t\xi\|^2_{L^2W^{5/2}_\delta}+\|\partial_t\xi+\epsilon\partial_t^2\xi\|^2_{L^2W^{5/2}_\delta}).\end{aligned} \tag{4.93}$$

*Then* $F^{1,1}-F^{4,1}-F^{5,1}\in C([0,T];(\mathcal{H}^1)^*)$. *Moreover,*

$$\|(F^{1,1}-F^{4,1}-F^{5,1})(0)\|^2_{(\mathcal{H}^1)^*}\lesssim \mathfrak{E}_0. \tag{4.94}$$

*Proof.* Since the proof of the first two inequalities are similar, we only give the proof second inequality. From the notation in Remark 4.4, we have that

$$\big\langle\partial_t(F^{1,1}-F^{4,1}-F^{5,1}),v\big\rangle_{(\mathcal{H}^1_T)^*}=\int_0^T\int_\Omega\partial_tF^{1,1}\cdot vJ-\int_0^T\int_{-\ell}^{\ell}\partial_tF^{4,1}\cdot v-\int_0^T\int_{\Sigma_s}\partial_tF^{5,1}(v\cdot\tau)J, \tag{4.95}$$

for each $v\in\mathcal{V}$. Since we assume that $F^i=0$, (4.95) reduces to

$$\big\langle\partial_t(F^{1,1}-F^{4,1}-F^{5,1}),v\big\rangle_{(\mathcal{H}^1_T)^*}=\int_0^T\int_\Omega\partial_tG^{1}\cdot vJ-\int_0^T\int_{-\ell}^{\ell}\partial_tG^{4}\cdot v-\int_0^T\int_{\Sigma_s}\partial_tG^{5}(v\cdot\tau)J, \tag{4.96}$$

for each $v\in\mathcal{V}$. Then we use an integration by parts

$$\int_\Omega\operatorname{div}_{\mathcal{A}}(\mathbb{D}_{\mathcal{A}}(Ru))vJ=-\frac{1}{2}\int_\Omega\mathbb{D}_{\mathcal{A}}(Ru):\mathbb{D}_{\mathcal{A}}vJ+\int_{-\ell}^{\ell}\mathbb{D}_{\mathcal{A}}(Ru)\mathcal{N}\cdot v+\int_{\Sigma_s}\mathbb{D}_{\mathcal{A}}(Ru)\nu\cdot\tau(v\cdot\tau)J \tag{4.97}$$

to reduce (4.96) to the following equality

$$\begin{aligned}\big\langle\partial_t(F^{1,1}-F^{4,1}-F^{5,1}),v\big\rangle_{(\mathcal{H}^1_T)^*}&=\frac{1}{2}\int_0^T\int_\Omega\partial_t(\mathbb{D}_{\mathcal{A}}(Ru)):\mathbb{D}_{\mathcal{A}}vJ\\ &+\int_0^T\int_\Omega\big[\partial_t(G^1-\mu\operatorname{div}_{\mathcal{A}}(\mathbb{D}_{\mathcal{A}}(Ru)))+\mu\operatorname{div}_{\partial_t\mathcal{A}}(\mathbb{D}_{\mathcal{A}}(Ru))\big]\cdot vJ\\ &-\int_0^T\int_{-\ell}^{\ell}\partial_t(G^4-\mu\mathbb{D}_{\mathcal{A}}(Ru)\mathcal{N})\cdot v-\int_0^T\int_{\Sigma_s}\partial_t(G^5-\mu\mathbb{D}_{\mathcal{A}}(Ru)\nu\cdot\tau)(v\cdot\tau)J.\end{aligned} \tag{4.98}$$

Then we use Hölder inequality and the same computation in Lemma 4.10 to derive the resulting bounds. □

Now, we give some estimates for the difference between $\partial_t u$ and $D_t u$. The proof is similar as that of Lemma 4.10, so we omit it here.

**Lemma 4.12.**

$$\|\partial_t u-D_tu\|^2_{L^2W^2_\delta}\lesssim\mathfrak{D}(\eta)(\|u_0\|^2_{W^2_\delta}+\|u\|^2_{L^2W^2_\delta}+\|\partial_tu\|^2_{L^2W^2_\delta}), \tag{4.99}$$

$$\|\partial_t u-D_tu\|^2_{L^2H^1}+\|\partial_t u-D_tu\|^2_{L^2H^0(\Sigma_s)}\lesssim\mathfrak{E}(\eta)(\|u\|^2_{L^2H^1}+\|u\|^2_{L^2W^2_\delta}), \tag{4.100}$$

*and*

$$\|\partial_t^2 u-\partial_tD_tu\|^2_{L^2H^1}+\|\partial_t^2 u-\partial_tD_tu\|^2_{L^2H^0(\Sigma_s)}\lesssim\mathfrak{K}(\eta)(\|u\|^2_{L^2W^2_\delta}+\|\partial_tu\|^2_{L^2W^2_\delta}). \tag{4.101}$$

Now we turn to construct the initial data $D_t^ju(0)$, $\partial_t^jp(0)$ for $j=0,1$ and $\partial_t^j\xi(0)$ for $j=0,1,2$. For $j=0$, we write $D_t^0u(0)=u_0\in W^2_\delta(\Omega)$, $\partial_t^0p(0)=p(0)\in\mathring{W}^1_\delta(\Omega)$ and $\partial_t^0\xi(0)=\eta_0\in W^{5/2}_\delta(\Sigma)$. For $j=1$, we define that $\partial_t\xi(0)=u_0\cdot\mathcal{N}(0)=\partial_t\eta(0)\in H^{3/2}(\Sigma)$ and $D_tu(0)$, $\partial_tp(0)$ are the same as Section 3. $\partial_t^2\xi(0)=D_tu(0)\cdot\mathcal{N}(0)=\partial_t^2\eta(0)\in H^1(\Sigma)$.

Now, we define the quantities we need to estimate as follows.

$$\begin{aligned}\mathfrak{D}(u,p,\xi) :=& \sum_{j=0}^{2}\left(\|\partial_t^j u\|_{L^2H^1}^2+\|\partial_t^j u\|_{L^2H^0(\Sigma_s)}^2+\int_0^T\left[\partial_t^j u\cdot\mathcal{N}\right]_\ell^2\right)\\&+\sum_{j=0}^{2}\left(\|\partial_t^j p\|_{L^2H^0}^2+\|\partial_t^j\xi\|_{L^2H^{3/2}}^2\right)+\|\partial_t^3\xi\|_{L^2W_\delta^{1/2}}^2\\&+\sum_{j=0}^{1}\left(\|\partial_t^j u\|_{L^2W_\delta^2}^2+\|\partial_t^j p\|_{L^2\mathring{W}_\delta^1}^2+\|\partial_t^j\xi\|_{L^2W_\delta^{5/2}}^2\right),\end{aligned}\tag{4.102}$$

$$\begin{aligned}\mathfrak{E}(u,p,\xi) :=& \|u\|_{L^\infty W_\delta^2}^2+\|\partial_t u\|_{L^\infty H^1}^2+\|p\|_{L^\infty\mathring{W}_\delta^1}^2+\|\partial_t p\|_{L^\infty H^0}^2\\&+\|\xi\|_{L^\infty W_\delta^{5/2}}^2+\|\partial_t\xi\|_{L^\infty H^{3/2}}^2+\sum_{j=0}^{2}\|\partial_t^j\xi\|_{L^\infty H^1}^2,\end{aligned}\tag{4.103}$$

and

$$\mathfrak{K}(u,p,\xi) := \mathfrak{E}(u,p,\xi)+\mathfrak{D}(u,p,\xi).\tag{4.104}$$

Now for convenience, we introduce two new spaces

$$\begin{aligned}\mathcal{X} =& \Big\{(u,p,\eta)|u\in C^0([0,T];W_\delta^2(\Omega)),\partial_t u\in C^0([0,T];H^1(\Omega)),p\in C^0([0,T];\mathring{W}_\delta^1(\Omega)),\\&\partial_t p\in C^0([0,T];\mathring{H}^0(\Omega)),\eta\in C^0([0,T];W_\delta^{5/2}(-\ell,\ell)),\partial_t\eta\in C^0([0,T];H^{3/2}(-\ell,\ell)),\\&\partial_t\eta\in C^0([0,T];\mathring{H}^1(-\ell,\ell)),\partial_t^2\eta\in C^0([0,T];\mathring{H}^1(-\ell,\ell))\Big\},\end{aligned}\tag{4.105}$$

endowed with norm $\|(u,p,\eta)\|_{\mathcal{X}}=\sqrt{\mathfrak{E}(u,p,\eta)}$, and

$$\begin{aligned}\mathcal{Y} =& \Big\{(u,p,\eta)|u\in L^2([0,T];W_\delta^2(\Omega))\cap L^2([0,T];H^1(\Omega))\cap L^2([0,T];H^0(\Sigma_s)),\\&\partial_t u\in L^2([0,T];W_\delta^2(\Omega))\cap L^2([0,T];H^1(\Omega))\cap L^2([0,T];H^0(\Sigma_s)),j=0,1,2,\\&\partial_t^2 u\in L^2([0,T];H^1(\Omega))\cap L^2([0,T];H^0(\Sigma_s)),[\partial_t^j u\cdot\mathcal{N}]_\ell\in L^2([0,T]),\\&p\in L^2([0,T];\mathring{W}_\delta^1(\Omega))\cap L^2([0,T];H^0(\Omega)),\partial_t p\in L^2([0,T];\mathring{W}_\delta^1(\Omega))\cap L^2([0,T];H^0(\Omega)),\\&\partial_t^2 p\in L^2([0,T];\mathring{H}^0(\Omega)),\eta\in L^2([0,T];W_\delta^{5/2}((-\ell,\ell)))\cap L^2([0,T];\mathring{H}^{3/2}((-\ell,\ell))),\\&\partial_t\eta\in L^2([0,T];W_\delta^{5/2}((-\ell,\ell)))\cap L^2([0,T];\mathring{H}^{3/2}((-\ell,\ell))),\partial_t^2\eta\in L^2([0,T];\mathring{H}^{3/2}((-\ell,\ell))),\\&\partial_t^3\eta\in L^2([0,T];\mathring{W}_\delta^{1/2}((-\ell,\ell)))\Big\}\end{aligned}\tag{4.106}$$

endowed with the norm $\|(u,p,\eta)\|_{\mathcal{Y}}=\sqrt{\mathfrak{D}(u,p,\eta)}$.

In the following theorem, we set the forcing terms $F^i=0$, $i=1,4,5$ for the brevity. But it also holds by adding the forcing terms and assuming the regularity of them respectively.

**Theorem 4.13.** *Suppose that $\eta_0\in W_\delta^{5/2}(-\ell,\ell)$, $\partial_t\eta(0)\in\mathring{H}^{3/2}((-\ell,\ell))$ and $\partial_t^2\eta(0)\in\mathring{H}^1((-\ell,\ell))$ satisfy the compatibility in Section 3, that $\mathfrak{K}(\eta)\le\alpha$ is sufficiently small satisfying the assumption in Lemma 2.3 and Theorem 5.9 in* [10]*, and that $F^i=0$, $i=1,4,5$. Let $u_0\in W_\delta^2(\Omega)$, $D_t u(0)\in H^1(\Omega)$, $p_0\in\mathring{W}_\delta^1(\Omega)$, $\partial_t p(0)\in\mathring{H}^0(\Omega)$, $\partial_t\xi(0)\in H^{3/2}((-\ell,\ell))$ and $\partial_t^2\xi(0)\in H^1((-\ell,\ell))$, all be determined in terms of $\eta_0$, $\partial_t\eta(0)$ and $\partial_t^2\eta(0)$ as above. Suppose that the initial data satisfy the compatibility conditions in Section 3. Then for each $0<\epsilon\le 1$ there exists $T_\epsilon>0$ such that for $0<T\le T_\epsilon$, then there exists a unique strong solution $(u,p,\xi)$ to* (4.3) *on $[0,T]$ such that*

$$(u,p,\xi)\in\mathcal{X}\cap\mathcal{Y}.\tag{4.107}$$

*The pair* $(D_t^j u, \partial_t^j p, \partial_t^j \xi)$ *satisfies*

$$\begin{cases} -\mu\Delta_{\mathcal{A}}(D_t^j u) + \nabla_{\mathcal{A}}\partial_t^j p = F^{1,j}, & \text{in} \quad \Omega, \\ \operatorname{div}_{\mathcal{A}}(D_t^j u) = 0, & \text{in} \quad \Omega, \\ S_{\mathcal{A}}(\partial_t^j p, D_t^j u)\mathcal{N} = \mathcal{L}(\partial_t^j \xi + \epsilon\partial_t^{j+1}\xi)\mathcal{N} - \sigma\partial_1\partial_t^j F^3\mathcal{N} + F^{4,j}, & \text{on} \quad \Sigma, \\ (S_{\mathcal{A}}(\partial_t^j p, D_t^j u)\nu - \beta(D_t^j u))\cdot\tau = F^{5,j}, & \text{on} \quad \Sigma_s, \\ D_t^j u\cdot\nu = 0, & \text{on} \quad \Sigma_s, \\ \partial_t^{j+1}\xi = D_t^j u\cdot\mathcal{N}, & \text{on} \quad \Sigma, \\ \mp\sigma\dfrac{\partial_1\partial_t^j\xi + \epsilon\partial_1\partial_t^{j+1}\xi}{(1+|\partial_1\zeta_0|^2)^{3/2}}(\pm\ell) = \kappa(D_t^j u\cdot\mathcal{N})(\pm\ell) \pm \sigma\partial_t^j F^3(\pm\ell) - \kappa\partial_t^j\hat{\mathscr{W}}(\partial_t\eta), \end{cases} \tag{4.108}$$

*in the strong sense with initial data* $(D_t^j u(0), \partial_t^j p(0), \partial_t^j\xi(0))$ *for* $j = 0, 1$ *and in the weak sense for* $j = 2$. *Moreover, the solution satisfies the estimate*

$$\mathfrak{K}(u,p,\xi) \le C(\epsilon)T(\mathfrak{K}(\eta) + \mathfrak{E}_0) + C_0(\mathfrak{E}_0 + \mathfrak{E}(\eta)\mathfrak{K}(\eta)), \tag{4.109}$$

*where* $C_0$ *is a positive constant independent of* $\epsilon$.

*Proof.* Step 1 – FIXME

Following Theorem 4.8. When $j = 0$, since the compatibility condition in Section 3 is satisfied and $\mathfrak{K}(\eta)$ is small enough, Theorem 4.8 guarantees the existence of $(u, p, \xi)$ satisfying (4.15) and (4.16). $(D_t^j u, \partial_t^j p, \partial_t^j\xi)$ is a unique solution of (4.108) in the strong sense when $j = 0$ and in the weak sense when $j = 1$. For $j = 1$, the assumption of Theorem 4.8 are satisfied by Lemma 4.10, Lemma 4.11 and compatibility condition in section 3. Then according to Theorem 4.8 and the elliptic estimate for $\xi + \epsilon\partial_t\xi$, we have that $(D_t u, \partial_t p, \partial_t\xi)$ is a unique strong solution of (4.108), and $(D_t^2 u, \partial_t^2 p, \partial_t^2\xi)$ is a unique weak solution of (4.108). Moreover,

$$\begin{aligned} &\|D_t u\|^2_{L^2H^1} + \|D_t u\|^2_{L^2H^0(\Sigma_s)} + \|[D_t u\cdot\mathcal{N}]_\ell\|^2_{L^2([0,T])} + \|D_t u\|^2_{L^2W^2_\delta} + \|\partial_t D_t u\|^2_{L^2H^1} \\ &\quad + \|\partial_t D_t u\|^2_{L^2H^0(\Sigma_s)} + \|[\partial_t D_t u\cdot\mathcal{N}]_\ell\|^2_{L^2([0,T])} + \|\partial_t p\|^2_{L^2H^0} + \|\partial_t p\|^2_{L^2\mathring{W}^1_\delta} + \|\partial_t^2 p\|^2_{L^2H^0} \\ &\quad + \|\partial_t\xi\|^2_{L^\infty H^1} + \|\partial_t^2\xi\|^2_{L^2H^{3/2}} + \|\partial_t\xi + \epsilon\partial_t^2\xi\|^2_{L^2W^{5/2}_\delta} + \|\partial_t^2\xi\|^2_{L^\infty H^1} + \|\partial_t\xi\|^2_{L^2H^{3/2}} \\ &\lesssim C(\epsilon)T\mathfrak{E}(\eta) + \mathfrak{E}_0 + \|(F^{1,1} - F^{4,1} - F^{5,1})(0)\|^2_{(\mathcal{H}^1)^*} + \mathfrak{E}(\eta)\mathfrak{K}(\eta) \\ &\quad + (1+\mathfrak{E}(\eta))(\|F^{1,1}\|^2_{L^2W^0_\delta} + \|F^{4,1}\|^2_{L^2W^{1/2}_\delta} + \|F^{5,1}\|^2_{L^2W^{1/2}_\delta}) \\ &\quad + (1+\mathfrak{E}(\eta))\|\partial_t(F^{1,1} - F^{4,1} - F^{5,1})\|^2_{(\mathcal{H}^1_T)^*}. \end{aligned} \tag{4.110}$$

Since $\mathfrak{K}(\eta)$ is sufficiently small, then Lemma 4.12 and the fact that $D_t u\cdot\mathcal{N}(\pm\ell) = \partial_t u\cdot\mathcal{N}(\pm\ell)$ can reduce the above estimate to

$$\begin{aligned} &\|\partial_t u\|^2_{L^2H^1} + \|\partial_t u\|^2_{L^2H^0(\Sigma_s)} + \|[\partial_t u\cdot\mathcal{N}]_\ell\|^2_{L^2([0,T])} + \|\partial_t u\|^2_{L^2W^2_\delta} + \|\partial_t^2 u\|^2_{L^2H^1} \\ &\quad + \|\partial_t^2 u\|^2_{L^2H^0(\Sigma_s)} + \|[\partial_t^2 u\cdot\mathcal{N}]_\ell\|^2_{L^2([0,T])} + \|\partial_t p\|^2_{L^2H^0} + \|\partial_t p\|^2_{L^2\mathring{W}^1_\delta} + \|\partial_t^2 p\|^2_{L^2H^0} \\ &\quad + \|\partial_t\xi\|^2_{L^\infty H^1} + \|\partial_t^2\xi\|^2_{L^2H^{3/2}} + \|\partial_t\xi + \epsilon\partial_t^2\xi\|^2_{L^2W^{5/2}_\delta} + \|\partial_t^2\xi\|^2_{L^\infty H^1} + \|\partial_t\xi\|^2_{L^2H^{3/2}} \\ &\lesssim C(\epsilon)T\mathfrak{E}(\eta) + \mathfrak{E}_0 + \mathfrak{E}(\eta)\mathfrak{K}(\eta) + \mathfrak{K}(\eta)(\|u\|^2_{L^2H^1} + \|u\|^2_{L^2W^2_\delta} + \|p\|^2_{L^2\mathring{W}^1_\delta} + \|\xi + \epsilon\partial_t\xi\|^2_{L^2W^{5/2}_\delta}) \\ &\lesssim C(\epsilon)T(\mathfrak{E}_0 + \mathfrak{K}(\eta)) + \mathfrak{E}_0 + \mathfrak{E}(\eta)\mathfrak{K}(\eta). \end{aligned} \tag{4.111}$$

Then from the extension and restriction theory of weighted Sobolev spaces, we may derive that

$$\begin{aligned} \|\partial_t\xi\|^2_{L^2W^{5/2}_\delta} &\lesssim \epsilon^2\|\partial_t\xi(0)\|^2_{L^2W^{5/2}_\delta} + \|\partial_t\xi + \epsilon\partial_t^2\xi\|^2_{L^2W^{5/2}_\delta} \\ &\lesssim \|\xi(0)\|^2_{L^2W^{5/2}_\delta} + \|\xi(0) + \epsilon\partial_t\xi(0)\|^2_{L^2W^{5/2}_\delta} + \|\partial_t\xi + \epsilon\partial_t^2\xi\|^2_{L^2W^{5/2}_\delta} \\ &\lesssim \mathfrak{E}_0 + \|\partial_t\xi + \epsilon\partial_t^2\xi\|^2_{L^2W^{5/2}_\delta}. \end{aligned} \tag{4.112}$$

We can directly estimate $\partial_t^3\xi$ by

$$\begin{aligned}\|\partial_t^3\xi\|_{W^{1/2}_\delta}^2 &= \|\partial_t D_t u\cdot\mathcal{N}+D_t u\cdot\partial_t\mathcal{N}\|_{W^{1/2}_\delta}^2\\ &\lesssim \|\partial_t D_t u\|_1^2\|\eta\|_{W^{5/2}_\delta}^2+\|D_t u\|_1^2\|\partial_t\eta\|_{W^{5/2}_\delta}^2.\end{aligned}\tag{4.113}$$

Then from (4.12), we might derive that

$$\|\partial_t^3\xi\|_{L^2W^{1/2}_\delta}^2\lesssim\mathfrak{K}(\eta)(\|u\|_{L^2H^1}^2+\|u\|_{L^2W^2_\delta}^2+\|\partial_t u\|_{L^2W^2_\delta}^2)\tag{4.114}$$

Thus(4.111) – (4.113) imply

$$\mathfrak{D}(u,p,\xi)\le C(\epsilon)T(\mathfrak{K}(\eta)+\mathfrak{E}_0)+C_0(\mathfrak{E}_0+\mathfrak{E}(\eta)\mathfrak{K}(\eta)).\tag{4.115}$$

Step 2 – Other terms in $\mathfrak{E}$.

Similar to Lemma 4.7,

$$\begin{aligned}\|\partial_t u\|_{L^\infty H^1}^2&\lesssim\|\partial_t u(0)\|_{H^1}^2+\|\partial_t u\|_{L^2H^1}^2+\|\partial_t^2 u\|_{L^2H^1}^2,\\ \|\partial_t p\|_{L^\infty H^0}^2&\lesssim\|\partial_t p(0)\|_{H^1}^2+\|\partial_t p\|_{L^2H^1}^2+\|\partial_t^2 p\|_{L^2H^1}^2,\\ \|\partial_t\xi\|_{L^\infty H^{3/2}}^2&\lesssim\|\partial_t\xi(0)\|_{H^{3/2}}^2+\|\partial_t\xi\|_{L^2H^{3/2}}^2+\|\partial_t^2\xi\|_{L^2H^{3/2}}^2\end{aligned}$$

which together with Lemma 4.7, Lemma 4.9 and initial data imply that

$$\mathfrak{E}(u,p,\xi)\le C(\epsilon)T(\mathfrak{K}(\eta)+\mathfrak{E}_0)+C_0(\mathfrak{E}_0+\mathfrak{E}(\eta)\mathfrak{K}(\eta)).\tag{4.116}$$

Then (4.115) and (4.116) imply the conclusion (4.109). □

## 5. Local well-posedness for the full nonlinear equation

We now consider the local well-posedness of the full problem (1.19). We first construct an approximate solution $(u^\epsilon,p^\epsilon,\eta^\epsilon)$ for (1.19) and for each $\epsilon>0$. Then our plan is to let $\epsilon\to0$ to obatin the solution of (1.19).

### 5.1. Existence of approximate solutions.

We now construct a sequence of approximate solution $(u^\epsilon,p^\epsilon,\eta^\epsilon)$ for each $0<\epsilon\le1$. For simplicity, we still denote the unknown as $(u,p,\eta)$ instead of $(u^\epsilon,p^\epsilon,\eta^\epsilon)$. But it should be understood as $(u^\epsilon,p^\epsilon,\eta^\epsilon)$.

Now we consider the $\epsilon$–perturbation problem of the original system (1.19) as

$$\begin{cases}\operatorname{div}_{\mathcal{A}} S_{\mathcal{A}}(p,u)=-\mu\Delta_{\mathcal{A}}u+\nabla_{\mathcal{A}}p=0, & \text{in}\quad\Omega,\\ \operatorname{div}_{\mathcal{A}} u=0, & \text{in}\quad\Omega,\\ S_{\mathcal{A}}(p,u)\mathcal{N}=g(\eta+\epsilon\eta_t)\mathcal{N}-\sigma\partial_1\left(\dfrac{\partial_1\eta+\epsilon\partial_1\eta_t}{(1+|\partial_1\zeta_0|)^{3/2}}\right)\mathcal{N}-\sigma\partial_1(\mathcal{R}(\partial_1\zeta_0,\partial_1\eta))\mathcal{N}, & \text{on}\quad\Sigma,\\ (S_{\mathcal{A}}(p,u)\nu-\beta u)\cdot\tau=0, & \text{on}\quad\Sigma_s,\\ u\cdot\nu=0, & \text{on}\quad\Sigma_s,\\ \partial_t\eta=u\cdot\mathcal{N}, & \text{on}\quad\Sigma,\\ \kappa\partial_t\eta(\pm\ell,t)=\mp\sigma\dfrac{\partial_1\eta+\epsilon\partial_1\eta_t}{(1+|\zeta_0|^2)^{3/2}}(\pm\ell,t)\mp\mathcal{R}(\partial_1\zeta_0,\partial_1\eta)(\pm\ell,t)-\kappa\hat{\mathscr{W}}(\partial_t\eta(\pm\ell,t)).\end{cases}\tag{5.1}$$

where $\mathcal{A},\mathcal{N}$ are in terms of $\eta^\epsilon$ and the initial data are $\eta(x_1,0)=\eta_0(x_1)$, $\partial_t\eta(x_1,0)$ and $\partial_t^2\eta(x_1,0)$.

Our strategy is to work in a metric space that requires high regularity estimates to hold but that is endowed with a low-regularity metric. First we will find a complete metric space, endowed with a weak choice of a metric, compatible with the linear estimates in Theorem 4.13. Then we will prove that the fixed point on this metric space gives a solution to (5.1).

Now, we give the definition of a metric space.

**Definition 5.1.** *Suppose that $T>0$, the initial data are the same as Section 3. For $\sigma\in(0,\infty)$, we define*

$$S(T,\sigma)=\Big\{(u,p,\eta)\in L^2H^1\times L^2\mathring{H}^0\times(L^\infty W^{5/2}_\delta\cap\dot{H}^1([0,T];\pm\ell),\Big|(u,p,\eta))\in\mathcal{X}\cap\mathcal{Y},\ \text{with}\\ \mathfrak{K}(u,p,\eta)^{1/2}\le\sigma\ \text{and}\ (u,p,\eta)\ \text{achieve the initial data as Section 3}\Big\}. \tag{5.2}$$

*We endow this space with the metric*

$$d((u,p,\eta),(v,q,\xi))=\|u-v\|_{L^2H^1}+\|p-q\|_{L^2\mathring{H}^0}+\|\eta-\xi\|_{L^\infty W^{5/2}_\delta}+\|[\partial_t\eta-\partial_t\xi]_\ell\|_{L^2([0,T])}, \tag{5.3}$$

*where the the temporal norm is through $[0,T]$.*

In order to use the contraction mapping principle, we need to show that this metric space is complete.

**Theorem 5.2.** *$S(T,\sigma)$ is a complete metric space.*

*Proof.* Suppose that $\{(u^m,p^m,\eta^m)\}_{m=0}^\infty\subseteq S(T,\sigma)$ is a Cauchy sequence. Since $L^2H^1\times L^2\mathring{H}^0\times(L^\infty W^{5/2}_\delta\cap\dot{H}^1([0,T];\pm\ell))$ is a Banach space, there exists $(u,p,\eta)\in L^2H^1\times L^2\mathring{H}^0\times(L^\infty W^{5/2}_\delta\cap\dot{H}^1([0,T];\pm\ell))$ such that

$$u^m\to u\text{ in }L^2H^1,\quad p^m\to p\text{ in }L^2\mathring{H}^0,\quad \eta^m\to\eta\text{ in }L^\infty W^{5/2}_\delta,\quad \partial_t\eta^m\to\partial_t\eta\text{ in }L^2([0,T];\pm\ell)$$

as $m\to\infty$.

For each $m$, we have that $\mathfrak{K}(u^m,p^m,\eta^m)\le\sigma^2$. Then up to an extraction of a subsequence we have that

$$(u^m,p^m,\eta^m)\rightharpoonup(u,p,\eta)\text{ weakly}-*\text{ in }\mathcal{X},\quad (u^m,p^m,\eta^m)\rightharpoonup(u,p,\eta)\text{ weakly in }\mathcal{Y}, \tag{5.4}$$

which imply that $(u,p,\eta)\in\mathcal{X}\cap\mathcal{Y}$. Then according to lower semicontinuity,

$$\mathfrak{K}(u,p,\eta)^{1/2}\le\inf_m\mathfrak{K}(u^m,p^m,\eta^m)^{1/2}\le\sigma. \tag{5.5}$$

This completes the proof. □

**Theorem 5.3.** *There exists a constant $C$ such that for each $0<\epsilon\le\min\{1,1/(8C)\}$, then there exists a unique solution $(u^\epsilon,p^\epsilon,\eta^\epsilon)$ in the metric space $S(T_\epsilon,\sigma)$ with $T_\epsilon>0$ and $\sigma>0$ sufficiently small, and so in particular $(u^\epsilon,p^\epsilon,\eta^\epsilon)\in\mathcal{X}\cap\mathcal{Y}$ solving the equation* (5.1)*, where $\mathcal{X}$ and $\mathcal{Y}$ are defined in* (4.105) *and* (4.106)*.*

*Proof.* Throughout the proof, $P(\cdot)$ is a polynomial, which is allowed to be changed from line to line and $P(0)=0$.

Step 1 – The metric space.

Suppose that $\mathfrak{K}(\eta)\le\alpha$ is sufficiently small. Then $C_0\mathfrak{E}(\eta)\mathfrak{K}(\eta)\le\alpha/4$. Now, we take $T_\epsilon>0$ small enough such that $C(\epsilon)T_\epsilon\alpha\le\alpha/4$. Then we take the initial data small enough such that $C(\epsilon)T_\epsilon\mathfrak{E}_0\le\alpha/4$ and $C_0\mathfrak{E}_0\le\alpha/4$. Here $C(\epsilon)$ and $C_0$ are the same as in (4.109). Then we take $\sigma\le\alpha^{1/2}$. For every $(u,p,\eta)\in S(T_\epsilon,\sigma)$, let $(\tilde{u},\tilde{p},\tilde{\eta})$ be the unique solution of the linear problem of

$$\begin{cases}
\operatorname{div}_{\mathcal{A}}S_{\mathcal{A}}(\tilde{p},\tilde{u})=-\mu\Delta_{\mathcal{A}}\tilde{u}+\nabla_{\mathcal{A}}\tilde{p}=0, & \text{in}\quad\Omega,\\
\operatorname{div}_{\mathcal{A}}\tilde{u}=0, & \text{in}\quad\Omega,\\
S_{\mathcal{A}}(\tilde{p},\tilde{u})\mathcal{N}=g(\tilde{\eta}+\epsilon\partial_t\tilde{\eta})\mathcal{N}-\sigma\partial_1\left(\dfrac{\partial_1\tilde{\eta}+\epsilon\partial_1\partial_t\tilde{\eta}}{(1+|\partial_1\zeta_0|)^{3/2}}\right)\mathcal{N}-\sigma\partial_1(\mathcal{R}(\partial_1\zeta_0,\partial_1\eta))\mathcal{N}, & \text{on}\quad\Sigma,\\
(S_{\mathcal{A}}(\tilde{p},\tilde{u})\nu-\beta\tilde{u})\cdot\tau=0, & \text{on}\quad\Sigma_s,\\
\tilde{u}\cdot\nu=0, & \text{on}\quad\Sigma_s,\\
\partial_t\tilde{\eta}=\tilde{u}\cdot\mathcal{N}, & \text{on}\quad\Sigma,\\
\kappa\partial_t\tilde{\eta}(\pm\ell,t)=\mp\sigma\dfrac{\partial_1\tilde{\eta}+\epsilon\partial_1\partial_t\tilde{\eta}}{(1+|\zeta_0|^2)^{3/2}}(\pm\ell,t)\mp\mathcal{R}(\partial_1\zeta_0,\partial_1\eta)(\pm\ell,t)-\kappa\hat{\mathscr{W}}(\partial_t\eta(\pm\ell,t)),
\end{cases} \tag{5.6}$$

where $\mathcal{A}$ and $\mathcal{N}$ are in terms of $\eta$, and the initial data $\tilde{\eta}(0)=\eta_0$, $\partial_t\tilde{\eta}(0)=\partial_t\eta(0)$ and $\partial_t^2\tilde{\eta}(0)=\partial_t^2\eta(0)$. By the Theorem 4.13 of linear case, we have the estimate

$$\mathfrak{K}(\tilde{u},\tilde{p},\tilde{\eta})\le\sigma^2, \tag{5.7}$$

which imply

$$(\tilde{u},\tilde{p},\tilde{\eta})\in S(T_\epsilon,\sigma). \tag{5.8}$$

Step 2 – Contraction.

Define $A:(u,p,\eta)=(\tilde{u},\tilde{p},\tilde{\eta})$. Now we prove that

$$A: S(T_\epsilon,\sigma)\to S(T_\epsilon,\sigma)$$

is a strict contraction mapping with the metric in the definition 5.1. Choose $(u^i,p^i,\eta^i)\in S(T_\epsilon,\sigma)$, and define $A(u^i,p^i,\eta^i)=(\tilde{u}^i,\tilde{p}^i,\tilde{\eta}^i)$ as above, $i=1,2$. For simplicity, we may abuse the notation, and denote $u=u^1-u^2$, $p=p^1-p^2$, $\eta=\eta^1-\eta^2$ and the same as $\tilde{u},\tilde{p},\tilde{\eta}$. From the difference of equation for $(\tilde{u}^i,\tilde{p}^i,\tilde{\eta}^i)$, $i=1,2$, we know that

$$\begin{cases}
\operatorname{div}_{\mathcal{A}^1} S_{\mathcal{A}^1}(\tilde{p},\tilde{u})=\mu\operatorname{div}_{\mathcal{A}^1}(\mathbb{D}_{\mathcal{A}^1-\mathcal{A}^2}\tilde{u}^2)+R^1, & \text{in}\quad \Omega,\\
\operatorname{div}_{\mathcal{A}^1}\tilde{u}=R^2, & \text{in}\quad \Omega,\\
S_{\mathcal{A}^1}(\tilde{p},\tilde{u})\mathcal{N}^1=\mu\mathbb{D}_{\mathcal{A}^1-\mathcal{A}^2}\tilde{u}^2\mathcal{N}^1+g(\tilde{\eta}+\epsilon\partial_t\tilde{\eta})\mathcal{N}^1-\sigma\partial_1\left(\dfrac{\partial_1\tilde{\eta}+\epsilon\partial_1\partial_t\tilde{\eta}}{(1+|\partial_1\zeta_0|)^{3/2}}\right)\mathcal{N}^1 & \\
\quad -\sigma\partial_1 F^3\mathcal{N}^1+R^3, & \text{on}\quad \Sigma,\\
(S_{\mathcal{A}^1}(\tilde{p},\tilde{u})\nu-\beta\tilde{u})\cdot\tau=\mu\mathbb{D}_{\mathcal{A}^1-\mathcal{A}^2}\tilde{u}^2\nu\cdot\tau, & \text{on}\quad \Sigma_s,\\
\tilde{u}\cdot\nu=0, & \text{on}\quad \Sigma_s,\\
\partial_t\tilde{\eta}=\tilde{u}\cdot\mathcal{N}^1+R^5, & \text{on}\quad \Sigma,\\
\kappa\partial_t\tilde{\eta}(\pm\ell,t)=\mp\sigma\dfrac{\partial_1\tilde{\eta}+\epsilon\partial_1\partial_t\tilde{\eta}}{(1+|\zeta_0|^2)^{3/2}}(\pm\ell,t)\mp F^3(\pm\ell,t))-R^6, & \\
\tilde{u}(x,0)=0,\quad \tilde{\eta}(x_1,0)=0, &
\end{cases} \tag{5.9}$$

where $R^1$, $R^2$, $R^3$, $R^4$, $R^5$, $R^6$ are defined by

$$\begin{aligned}
R^1&=\mu\operatorname{div}_{(\mathcal{A}^1-\mathcal{A}^2)}(\mathbb{D}_{\mathcal{A}^2}\tilde{u}^2)-\nabla_{(\mathcal{A}^1-\mathcal{A}^2)}\tilde{p}^2,\\
R^2&=-\operatorname{div}_{(\mathcal{A}^1-\mathcal{A}^2)}\tilde{u}^2,\\
R^3&=-\tilde{p}^2(\mathcal{N}^1-\mathcal{N}^2)+\mathbb{D}_{\mathcal{A}^2}\tilde{u}^2(\mathcal{N}^1-\mathcal{N}^2)+g(\tilde{\eta}^2+\epsilon\partial_1\partial_t\tilde{\eta}^2)(\mathcal{N}^1-\mathcal{N}^2)\\
&\quad-\sigma\partial_1\left(\frac{\partial_1\tilde{\eta}^2+\epsilon\partial_1\partial_t\tilde{\eta}^2}{(1+|\zeta_0|^2)^{3/2}}\right)(\mathcal{N}^1-\mathcal{N}^2)-\sigma\partial_1 F^{3,2}(\mathcal{N}^1-\mathcal{N}^2),\\
R^5&=\tilde{u}^2\cdot(\mathcal{N}^1-\mathcal{N}^2),\\
R^6&=\kappa(\hat{\mathscr{W}}(\partial_t\eta^1(\pm\ell,t))-\hat{\mathscr{W}}(\partial_t\eta^2(\pm\ell,t))),
\end{aligned}$$

and $\mathcal{A}^i$, $\mathcal{N}^i$, $F^{3,i}=\mathcal{R}(\partial_1\zeta_0,\partial_1\eta^i)$ are in terms of $\eta^i$, $i=1,2$. Here $F^3=F^{3,1}-F^{3,2}$.

We now have the pressureless weak formulation of (5.9) as

$$\begin{aligned}
&\frac{\mu}{2}\int_\Omega\mathbb{D}_{\mathcal{A}^1}\tilde{u}:\mathbb{D}_{\mathcal{A}^1}wJ^1+\beta\int_{\Sigma_s}J^1(\tilde{u}\cdot\tau)(w\cdot\tau)+(\tilde{\eta}+\epsilon\partial_t\tilde{\eta},w\cdot\mathcal{N}^1)_{1,\Sigma}+[\tilde{u}\cdot\mathcal{N}^1,w\cdot\mathcal{N}^1]_\ell\\
&=\mu\int_\Omega\mathbb{D}_{\mathcal{A}^1-\mathcal{A}^2}\tilde{u}^2:\mathbb{D}_{\mathcal{A}^1}wJ^1+\int_\Omega R^1\cdot wJ^1-\int_{-\ell}^{\ell}\sigma F^3\partial_1(w\cdot\mathcal{N}^1)+R^3\cdot w-[w\cdot\mathcal{N}^1,R^5+R^6]_\ell,
\end{aligned} \tag{5.10}$$

for each $w\in\mathcal{V}(t)$. Then according to Theorem 4.6 in [10], there exists a unique $\tilde{p}\in\mathring{H}^0(\Omega)$ such that

$$\begin{aligned}
&\frac{\mu}{2}\int_\Omega\mathbb{D}_{\mathcal{A}^1}\tilde{u}:\mathbb{D}_{\mathcal{A}^1}wJ^1+\beta\int_{\Sigma_s}J^1(\tilde{u}\cdot\tau)(w\cdot\tau)-(\tilde{p},\operatorname{div}_{\mathcal{A}^1}w)_0+(\tilde{\eta}+\epsilon\partial_t\tilde{\eta},w\cdot\mathcal{N}^1)_{1,\Sigma}\\
&\quad+[\tilde{u}\cdot\mathcal{N}^1,w\cdot\mathcal{N}^1]_\ell\\
&=\mu\int_\Omega\mathbb{D}_{\mathcal{A}^1-\mathcal{A}^2}\tilde{u}^2:\mathbb{D}_{\mathcal{A}^1}wJ^1+\int_\Omega R^1\cdot wJ^1-\int_{-\ell}^{\ell}\sigma F^3\partial_1(w\cdot\mathcal{N}^1)+R^3\cdot w\\
&\quad-[\tilde{u}\cdot\mathcal{N}^1,R^5+R^6]_\ell\quad,
\end{aligned} \tag{5.11}$$

for each $w \in \mathcal{W}(t)$. Moreover,

$$\|\tilde{p}\|_0 \lesssim \|\tilde{u}\|_1 + \|\eta\|_{3/2}\left((\|\eta^1\|_{3/2}+\|\eta^2\|_{3/2})\|\tilde{u}^2\|_{W^2_\delta} + \|\tilde{p}^2\|_{\mathring{W}^1_\delta}\right). \tag{5.12}$$

Multiplying the first equation of (5.9) by $\tilde{u}J^1$, then integrating by parts,

$$\begin{aligned}
&\partial_t\left(\int_{-\ell}^{\ell}\frac{g}{2}|\tilde{\eta}|^2 + \frac{\sigma}{2}\frac{|\partial_1\tilde{\eta}|^2}{(1+|\partial_1\zeta_0|^2)^{3/2}}\right) + \epsilon\int_{-\ell}^{\ell} g|\partial_t\tilde{\eta}|^2 + \sigma\frac{|\partial_1\partial_t\tilde{\eta}|^2}{(1+|\partial_1\zeta_0|^2)^{3/2}}\\
&\quad + \frac{\mu}{2}\int_\Omega |\mathbb{D}_{\mathcal{A}^1}\tilde{u}|^2 J^1 + [\tilde{u}\cdot\mathcal{N}^1]^2_\ell\\
&= \frac{\mu}{2}\int_\Omega \mathbb{D}_{\mathcal{A}^1-\mathcal{A}^2}\tilde{u}^2 : \mathbb{D}_{\mathcal{A}^1}\tilde{u}J^1 + \int_\Omega R^1\cdot\tilde{u}J^1 + \tilde{p}R^2J^1 - \int_{\Sigma_s} J^1(\tilde{u}\cdot\tau)R^4\\
&\quad - \int_{-\ell}^{\ell}\sigma F^3\partial_1(\tilde{u}\cdot\mathcal{N}^1) + R^3\cdot\tilde{u} - g(\tilde{\eta}+\epsilon\partial_t\tilde{\eta})R^5 - \sigma\frac{\partial_1(\tilde{\eta}+\epsilon\partial_t\tilde{\eta})\partial_1 R^5}{(1+|\partial_1\zeta_0|^2)^{3/2}}\\
&\quad - [\tilde{u}\cdot\mathcal{N}^1, R^5+R^6]_\ell,
\end{aligned} \tag{5.13}$$

Then we estimate the terms in right-hand side of (5.13).

$$\begin{aligned}
&\int_\Omega R^1\cdot\tilde{u}J^1 + \tilde{p}R^2J^1\\
&= \int_\Omega \left(\mathrm{div}_{(\mathcal{A}^1-\mathcal{A}^2)}(\mathbb{D}_{\mathcal{A}^2}\tilde{u}^2) - \nabla_{(\mathcal{A}^1-\mathcal{A}^2)}\tilde{p}^2\right)\cdot\tilde{u}J^1 + \tilde{p}\,\mathrm{div}_{(\mathcal{A}^1-\mathcal{A}^2)}\,\tilde{u}^2J^1\\
&\lesssim \int_\Omega |\nabla\bar{\eta}|\left(|\nabla^2\bar{\eta}^2||\nabla\tilde{u}^2| + |\nabla\bar{\eta}^2||\nabla^2\tilde{u}^2| + |\nabla\tilde{p}^2|\right)|\tilde{u}| + |\tilde{p}||\nabla\bar{\eta}||\nabla\tilde{u}^2|\\
&\lesssim \left(\|\tilde{u}\|_1\|\eta^2\|_{W^{5/2}_\delta} + \|\tilde{p}\|_0\right)\|\eta\|_{3/2}\|\eta^1\|_{W^{5/2}_\delta}\|\tilde{u}^2\|_{W^2_\delta},
\end{aligned}$$

Now we consider the integration on $(-\ell,\ell)$. First, we know that $\mathcal{N}^1-\mathcal{N}^2 = (-\partial_1\eta, 0)$ and

$$\partial_1 F^{3,2} = \partial_1\mathcal{R}(\partial_1\zeta_0, \partial_1\eta^2) = \partial_y\mathcal{R}\partial_1^2\zeta_0 + \partial_z\mathcal{R}\partial_1^2\eta^2$$

where $|\partial_y\mathcal{R}| \lesssim |\partial_1\eta^2|^2$, $|\partial_z\mathcal{R}| \lesssim |\partial_1\eta^2|$. Then we take $\frac{1}{p}+\frac{3}{q}=1$, $\frac{1}{p}+\frac{2}{r}=1$, with $1<p<\frac{2}{1+\delta}$ and use Hölder inequality, Sobolev inequality and trace theory to derive that

$$\begin{aligned}
&\int_{-\ell}^{\ell} R^3\cdot\tilde{u} = \int_{-\ell}^{\ell}\Bigg[-\tilde{p}^2(\mathcal{N}^1-\mathcal{N}^2) + \mathbb{D}_{\mathcal{A}^2}\tilde{u}^2(\mathcal{N}^1-\mathcal{N}^2) + g(\tilde{\eta}^2+\epsilon\partial_1\partial_t\tilde{\eta}^2)(\mathcal{N}^1-\mathcal{N}^2)\\
&\quad - \sigma\partial_1\left(\frac{\partial_1\tilde{\eta}^2+\epsilon\partial_1\partial_t\tilde{\eta}^2}{(1+|\zeta_0|^2)^{3/2}}\right)(\mathcal{N}^1-\mathcal{N}^2) - \sigma\partial_1 F^{3,2}(\mathcal{N}^1-\mathcal{N}^2)\Bigg]\cdot\tilde{u}\\
&\lesssim \|\tilde{p}^2\|_0\|\partial_1\eta\|_{L^4}\|\tilde{u}\|_{L^4(\Sigma)} + \|\partial_1\eta^2\|_{L^q(\Sigma)}\|\nabla\tilde{u}^2\|_{L^p(\Sigma)}\|\partial_1\eta\|_{L^q(\Sigma)}\|\tilde{u}\|_{L^q(\Sigma)}\\
&\quad + \left\|g(\tilde{\eta}^2+\epsilon\partial_1\partial_t\tilde{\eta}^2) - \sigma\partial_1\left(\frac{\partial_1\tilde{\eta}^2+\epsilon\partial_1\partial_t\tilde{\eta}^2}{(1+|\zeta_0|^2)^{3/2}}\right)\right\|_{L^p(\Sigma)}\|\partial_1\eta\|_{L^r(\Sigma)}\|\tilde{u}\|_{L^r(\Sigma)}\\
&\quad + \|\partial_y\mathcal{R}\|_{L^2}\|\partial_1\eta\|_{L^4(\Sigma)}\|\tilde{u}\|_{L^4(\Sigma)} + \|\partial_z\mathcal{R}\|_{L^q}\|\partial_1\eta^2\|_{L^p(\Sigma)}\|\partial_1\eta\|_{L^q(\Sigma)}\|\tilde{u}\|_{L^q(\Sigma)}\\
&\lesssim \Bigg(\|\tilde{p}^2\|_0\|\partial_1\eta\|_{1/2} + \|\partial_1\eta^2\|_{1/2}\|\nabla\tilde{u}^2\|_{W^{1/2}_\delta(\Sigma)}\|\partial_1\eta\|_{1/2} + \|\tilde{\eta}^2+\epsilon\partial_t\tilde{\eta}^2\|_{W^{5/2}_\delta}\|\partial_1\eta\|_{1/2}\\
&\quad + \|\partial_1\eta^2\|^2_{1/2}\|\partial_1\eta\|_{1/2} + \|\partial_1\eta^2\|_{1/2}\|\eta^2\|_{W^{5/2}_\delta}\|\partial_1\eta\|_{1/2}\Bigg)\|\tilde{u}\|_{1/2(\Sigma)}\\
&\lesssim \Bigg(\|\tilde{p}^2\|_0\|\eta\|_{3/2} + \|\eta^2\|_{3/2}\|\tilde{u}^2\|_{W^2_\delta}\|\eta\|_{3/2} + \|\tilde{\eta}^2+\epsilon\partial_t\tilde{\eta}^2\|_{W^{5/2}_\delta}\|\eta\|_{3/2}\\
&\quad + \|\eta^2\|^2_{3/2}\|\eta\|_{3/2} + \|\eta^2\|_{3/2}\|\eta^2\|_{W^{5/2}_\delta}\|\eta\|_{3/2}\Bigg)\|\tilde{u}\|_1
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\int_{-\ell}^{\ell} \sigma F^3 \partial_1(\tilde{u}\cdot\mathcal{N}^1) - g(\tilde{\eta}+\epsilon\partial_t\tilde{\eta})R^5 - \sigma\frac{\partial_1(\tilde{\eta}+\epsilon\partial_t\tilde{\eta})\partial_1 R^5}{(1+|\partial_1\zeta_0|^2)^{3/2}}\\
&\lesssim \|\eta\|_{3/2}\left(\|\eta\|_{3/2}+\|\eta^2\|_{3/2}\right)\|\partial_t\tilde{\eta}\|_1 + \|\tilde{u}\|_1\|\tilde{u}^2\|_1\|\eta\|_{3/2}\|\eta^1\|_{3/2}\\
&\quad + \|\tilde{\eta}^2+\epsilon\partial_t\tilde{\eta}^2\|_{W^{5/2}_\delta}\|\tilde{u}\|_1\|\eta\|_{3/2} + \|\eta^2\|_{3/2}\|\eta\|_{3/2} + \|\eta^2\|_{3/2}^2\|\eta\|_{3/2}\\
&\quad + \|\tilde{\eta}+\epsilon\partial_t\tilde{\eta}\|_1\left(\|\eta\|_{3/2}\|\tilde{u}^2\|_{W^2_\delta} + \|\tilde{u}^2\|_1\|\eta\|_{W^{5/2}_\delta}\right),
\end{aligned}$$

$$[\tilde{u}\cdot\mathcal{N}^1, R^5+R^6]_\ell \lesssim [\tilde{u}\cdot\mathcal{N}^1]_\ell([\partial_t\eta^1]_\ell + [\partial_t\eta^2]_\ell)[\partial_t\eta]_\ell,$$

here we have used the fact that $R^5=0$ at the end points $x_1=\pm\ell$ since $u_1^i$ vanishes there, for $i=1,2$ and we denote that $u^i=(u_1^i,u_2^i)$. Then, the Cauchy-Schwarz inequality, weighted Sobolev embedding theorem and Gronwall's inequality imply that

$$\begin{aligned}
&\sup_{0\le t\le T}\|\tilde{\eta}\|_1^2 + \epsilon\int_0^T\|\partial_t\tilde{\eta}\|_1^2 + \int_0^T\|\tilde{u}\|_1^2 + [\tilde{u}\cdot\mathcal{N}^1]_\ell^2\\
&\lesssim \int_0^T \|\eta\|_{W^{5/2}_\delta}^2\left((\|\eta^1\|_{W^{5/2}_\delta}^2+\|\eta^2\|_{W^{5/2}_\delta}^2)\|\tilde{u}^2\|_{W^2_\delta}^2 + \|\eta^1\|_{3/2}^2 + \|\eta^2\|_{3/2}^2 + \|\tilde{p}^2\|_0^2\right)\\
&\quad + \int_0^T \|\tilde{\eta}^2+\epsilon\partial_t\tilde{\eta}^2\|_{W^{5/2}_\delta}^2\|\eta\|_{3/2}^2 + \|\eta\|_{W^{5/2}_\delta}^2\|\tilde{u}^2\|_{W^2_\delta}^2 + \int_0^T(\|\partial_t\eta^1\|_1^2+\|\partial_t\eta^2\|_1^2)[\partial_t\eta]_\ell^2.
\end{aligned}\tag{5.14}$$

From the weak formulation (5.10) and the Theorem 4.11 in [10],

$$\begin{aligned}
\|\tilde{\eta}+\epsilon\partial_t\tilde{\eta}\|_{3/2}^2 &\lesssim \|\tilde{u}\|_1^2 + [\tilde{u}\cdot\mathcal{N}^1]_\ell^2 + \|\eta\|_{3/2}^2(\|\eta^1\|_{W^{5/2}_\delta}^2\|\tilde{u}^2\|_{W^2_\delta}^2 + \|\tilde{p}^2\|_{\mathring{W}^1_\delta}^2 + \|\tilde{\eta}^2+\epsilon\partial_t\tilde{\eta}^2\|_{W^{5/2}_\delta}^2)\\
&\quad + \|F^3\|_{1/2}^2 + [R^5+R^6]_\ell^2\\
&\lesssim \|\tilde{u}\|_1^2 + [\tilde{u}\cdot\mathcal{N}^1]_\ell^2 + \|\eta\|_{3/2}^2(\|\eta^1\|_{W^{5/2}_\delta}^2\|\tilde{u}^2\|_{W^2_\delta}^2 + \|\tilde{p}^2\|_{\mathring{W}^1_\delta}^2 + \|\tilde{\eta}^2+\epsilon\partial_t\tilde{\eta}^2\|_{W^{5/2}_\delta}^2)\\
&\quad + \|\eta\|_{3/2}^2(\|\eta\|_{3/2}^2+\|\tilde{u}^2\|_{W^2_\delta}^2) + (\|\partial_t\eta^1\|_1^2+\|\partial_t\eta^2\|_1^2)[\partial_t\eta]_\ell^2.
\end{aligned}\tag{5.15}$$

Since

$$\tilde{\eta} = \frac{1}{\epsilon}\int_0^t e^{-\frac{t-s}{\epsilon}}(\tilde{\eta}+\epsilon\partial_t\tilde{\eta}),\tag{5.16}$$

thus

$$\partial_t\tilde{\eta} = \frac{1}{\epsilon}(\tilde{\eta}+\epsilon\partial_t\tilde{\eta}) - \frac{1}{\epsilon^2}\int_0^t e^{-\frac{t-s}{\epsilon}}(\tilde{\eta}+\epsilon\partial_t\tilde{\eta}),$$

then we have

$$\begin{aligned}
\|\partial_t\tilde{\eta}\|_{L^2H^{3/2}}^2 &\le \frac{1}{\epsilon^2}\|\tilde{\eta}+\epsilon\partial_t\tilde{\eta}\|_{L^2H^{3/2}}^2 + \int_0^T\left(\frac{1}{\epsilon^2}\int_0^t e^{-\frac{t-s}{\epsilon}}\|\tilde{\eta}+\epsilon\partial_t\tilde{\eta}\|_{3/2}\right)^2\\
&\lesssim C(\epsilon)\|\tilde{\eta}+\epsilon\partial_t\tilde{\eta}\|_{L^2H^{3/2}}^2 \lesssim C(\epsilon)(\|\eta\|_{L^\infty W^{5/2}_\delta}^2 + \|[\partial_t\eta]_\ell\|_{L^2([0,T])}^2)P(\sigma).
\end{aligned}\tag{5.17}$$

From the Theorem 5.9 in [10], we have that

$$\begin{aligned}
&\|\tilde{u}\|_{W^2_\delta}^2 + \|\tilde{p}\|_{\mathring{W}^1_\delta}^2\\
&\lesssim \|-\mu\,\mathrm{div}_{\mathcal{A}^1}(\mathbb{D}_{\mathcal{A}^1-\mathcal{A}^2}\tilde{u}^2) + R^1\|_{W^0_\delta}^2 + \|R^2\|_{W^1_\delta}^2 + \|\partial_t\tilde{\eta}-R^5\|_{W^{3/2}_\delta}^2\\
&\quad + \|\mu\mathbb{D}_{\mathcal{A}^1-\mathcal{A}^2}\tilde{u}^2\mathcal{N}^1 + R^3\|_{W^{1/2}_\delta}^2 + \|\mu\mathbb{D}_{\mathcal{A}^1-\mathcal{A}^2}\tilde{u}^2\nu\cdot\tau\|_{W^{1/2}_\delta}^2\\
&\lesssim \|\eta\|_{W^{5/2}_\delta}^2\left(\left(1+\|\eta^1\|_{W^{5/2}_\delta}^2+\|\eta^2\|_{W^{5/2}_\delta}^2\right)\|\tilde{u}^2\|_{W^2_\delta}^2 + \|\tilde{p}^2\|_{\mathring{W}^1_\delta}^2\right) + \|\partial_t\tilde{\eta}\|_{W^{3/2}_\delta}^2\\
&\quad + \|\eta\|_{W^{5/2}_\delta}^2\left(\|\tilde{\eta}^2+\epsilon\partial_t\tilde{\eta}^2\|_{W^{5/2}_\delta}^2 + \|\eta^1\|_{W^{5/2}_\delta}^2 + \|\eta^2\|_{W^{5/2}_\delta}^2\right),
\end{aligned}\tag{5.18}$$

then the Theorem 5.10 in [10] implies

$$\begin{aligned}
&\|\tilde{\eta}+\epsilon\partial_t\tilde{\eta}\|^2_{W^{5/2}_\delta}\\
&\lesssim \|\tilde{u}\|^2_{W^2_\delta}+\|\tilde{p}\|^2_{\mathring{W}^1_\delta}+\|\partial_1 F^3\|^2_{W^{1/2}_\delta}+\|\mu\mathbb{D}_{\mathcal{A}^1-\mathcal{A}^2}\tilde{u}^2\mathcal{N}^1+R^3\|^2_{W^{1/2}_\delta}+[F^3]^2_\ell+[R^6]^2_\ell\\
&\lesssim \|\eta\|^2_{W^{5/2}_\delta}\left(\left(1+\|\eta^1\|^2_{W^{5/2}_\delta}+\|\eta^2\|^2_{W^{5/2}_\delta}\right)\|\tilde{u}^2\|^2_{W^2_\delta}+\|\tilde{p}^2\|^2_{\mathring{W}^1_\delta}\right)+\|\partial_t\tilde{\eta}\|^2_{W^{3/2}_\delta}\\
&\quad+\|\eta\|^2_{W^{5/2}_\delta}\Big(\|\tilde{\eta}^2+\epsilon\partial_t\tilde{\eta}^2\|^2_{W^{5/2}_\delta}+\|\eta^1\|^2_{W^{5/2}_\delta}+\|\eta^2\|^2_{W^{5/2}_\delta}\Big).
\end{aligned}\tag{5.19}$$

Combining (5.14)–(5.17), then Cauchy-Schwarz inequality, weighted Sobolev embedding theorem and linear estimates in Theorem 4.8 and 4.13 imply that,

$$\begin{aligned}
\|\tilde{\eta}\|^2_{L^\infty W^{5/2}_\delta}&\le C(\epsilon)T\|\tilde{\eta}+\epsilon\partial_t\tilde{\eta}\|^2_{L^2W^{5/2}_\delta}\\
&\lesssim C(\epsilon)TP(\|\eta^1\|^2_{L^\infty W^{5/2}_\delta},\|\eta^2\|^2_{L^\infty W^{5/2}_\delta},\|\tilde{u}^2\|^2_{L^2W^2_\delta},\|\tilde{p}^2\|^2_{L^2\mathring{W}^1_\delta})\\
&\quad\times\left(\|\eta\|^2_{L^\infty W^{5/2}_\delta}+\|[\partial_t\eta]_\ell\|^2_{L^2([0,T])}\right)\\
&\lesssim C(\epsilon)TP(\sigma)\left(\|\eta\|^2_{L^\infty W^{5/2}_\delta}+\|[\partial_t\eta]_\ell\|^2_{L^2([0,T])}\right),
\end{aligned}\tag{5.20}$$

where the first inequality is obtained by (5.16) and the second inequality used the fact that $\|\|\tilde{\eta}^2+\epsilon\partial_t\tilde{\eta}^2\|^2_{L^2W^{5/2}_\delta}\le\sigma_2$ which is included in the proof of Theorem 4.8.

Then (5.12) and (5.14) imply that

$$\begin{aligned}
&\|\tilde{u}\|^2_{L^2H^1}+\|\tilde{p}\|^2_{L^2\mathring{H}^0}+\|[\tilde{u}\cdot N^1]_\ell\|^2_{L^2([0,T])}\\
&\lesssim P(\sigma)\left(\|\eta\|^2_{L^\infty W^{5/2}_\delta}+\|[\partial_t\eta]_\ell\|^2_{L^2([0,T])}\right).
\end{aligned}\tag{5.21}$$

Since at the corner points, $[\partial_t\tilde{\eta}]_\ell=[\tilde{u}\cdot N^1]_\ell$, (5.20) and (5.21) reveals that

$$\begin{aligned}
&\|\tilde{u}\|^2_{L^2H^1}+\|\tilde{p}\|^2_{L^2\mathring{H}^0}+\|[\partial_t\tilde{\eta}]_\ell\|^2_{L^2([0,T])}+\|\tilde{\eta}\|^2_{L^\infty W^{5/2}_\delta}\\
&\le(C(\epsilon)T+C)P(\sigma)\left(\|\eta\|^2_{L^\infty W^{5/2}_\delta}+\|[\partial_t\eta]_\ell\|^2_{L^2([0,T])}\right),
\end{aligned}\tag{5.22}$$

where $C$ is a universal constant independent of $\epsilon$.

We may restrict $\sigma$ such that $CP(\sigma)\le 1/8$. For each $0<\epsilon\le 1/(8CP(\sigma))$, we choose $T'>0$ such that $C(\epsilon)T'P(\sigma_2)\le 1/8$. This implies

$$d(A(u^1,p^1,\eta^1),A(u^2,p^2,\eta^2))=d((\tilde{u}^1,\tilde{p}^1,\tilde{\eta}^1),(\tilde{u}^2,\tilde{p}^2,\tilde{\eta}^2))\le\frac{1}{2}d((u^1,p^1,\eta^1),(u^2,p^2,\eta^2)).\tag{5.23}$$

If $0<T'<T_\epsilon$, we can repeat the above argument on intervals $[0,T']$, $[T',2T']$,etc. Finally we see that $A$ is a strict contraction on $S(T_\epsilon,\sigma)$. Since the metric space $S(T_\epsilon,\sigma)$ is complete, the contraction mapping principle reveals the existence of a unique $(u,p,\eta)\in S(T_\epsilon,\sigma)$ such that $A(u,p,\eta)=(\tilde{u},\tilde{p},\tilde{\eta})=(u,p,\eta)$. □

5.2. **Energy estimates.** We want to send $\epsilon\to 0$ to get a uniform $T>0$ independent of $\epsilon$, so we need some uniform estimates. For simplicity, we may abuse the same symbol of energy and dissipation in section 2.1 of [10] and still denote the unknown $(u^\epsilon,p^\epsilon,\eta^\epsilon)$ as $(u,p,\eta)$.

**Theorem 5.4.** *There exists a universal constant $C$ and a universal $T>0$ independent of $\epsilon$ such that for each $\epsilon>0$ sufficiently small,*

$$\sup_{0\le t\le T}\mathcal{E}(t)+\int_0^T\mathcal{D}(t)\,\mathrm{d}t\le C.\tag{5.24}$$

*Proof.* We shall use the continuity argument to prove the uniform bounds. First, we define some variants of energy, dissipation and forcing terms.

$$\mathscr{E} := \sum_{j=0}^{2} \int_{-\ell}^{\ell} \frac{g}{2} |\partial_t^j \eta|^2 + \frac{\sigma}{2} \frac{|\partial_1 \partial_t^j \eta|^2}{(1+|\partial_1 \zeta_0|^2)^{3/2}}, \tag{5.25}$$

$$\mathscr{D} := \sum_{j=0}^{2} \left( \frac{\mu}{2} \int_{\Omega} \left| \mathbb{D}_{\mathcal{A}} \partial_t^j u \right|^2 J + \beta \int_{\Sigma_s} \left| \partial_t^j u \cdot \tau \right|^2 J + \left[ \partial_t^j u \cdot \mathcal{N} \right]_{\ell}^2 \right), \tag{5.26}$$

and

$$\mathscr{F} := \int_{-\ell}^{\ell} \left[ \sigma \mathcal{Q}(\partial_1 \zeta_0, \partial_1 \eta) + \sigma \partial_z \mathcal{R}(\partial_1 \zeta_0, \partial_1 \eta) \frac{|\partial_1 \partial_t^2 \eta|^2}{2} + \sigma \partial_z^2 \mathcal{R}(\partial_1 \zeta_0, \partial_1 \eta)(\partial_1 \partial_t \eta)^2 \partial_1 \partial_t^2 \eta \right]. \tag{5.27}$$

Suppose that

$$\sup_{0<s\leq t} \mathcal{E}(s) + \int_0^t \mathcal{D} \leq \alpha \text{ for each } t \in [0,T),$$

where $\alpha > 0$ is sufficiently small and $0 < T < 1$ is to be determined later. Similar to the energy estimate and section 8 of [10], we can derive that

$$\frac{d}{dt}(\mathscr{E} + \mathscr{F}) + \mathscr{D} + \epsilon \left( \|\partial_t \eta\|_{1,\Sigma}^2 + \|\partial_t^2 \eta\|_{1,\Sigma}^2 + \|\partial_t^3 \eta\|_{1,\Sigma}^2 \right) \lesssim \sqrt{\mathcal{E}} \mathcal{D}. \tag{5.28}$$

Then in order to follow the proof of Theorem 8.2 in [10], we need to prove the uniform bounds of $\|\eta\|_{3/2}^2 + \|\partial_t \eta\|_{3/2}^2 + \|\partial_t^2 \eta\|_{3/2}^2$ independent of $\epsilon$ . First, by following the proof of Theorem 8.2 in [10], we have known that

$$\|\eta + \epsilon \partial_t \eta\|_{3/2}^2 + \|\partial_t \eta + \epsilon \partial_t^2 \eta\|_{3/2}^2 + \|\partial_t^2 \eta + \epsilon \partial_t^3 \eta\|_{3/2}^2 \lesssim \bar{\mathcal{D}}_{\shortparallel} + \sqrt{\mathcal{E}} \mathcal{D} \tag{5.29}$$

and

$$\|\eta\|_{3/2}^2 + \|\partial_t \eta\|_{3/2}^2 \lesssim (\|\eta_0\|_{3/2}^2 + \|\partial_t \eta(0)\|_{3/2}^2) e^{-2\frac{t}{\epsilon}} + \left( \frac{1}{\epsilon} \int_0^t e^{-\frac{t-s}{\epsilon}} (\bar{\mathcal{D}}_{\shortparallel} + \sqrt{\mathcal{E}} \mathcal{D})^{1/2} \right)^2, \tag{5.30}$$

Then we denote $\vartheta = \partial_t^2 \eta + \epsilon \partial_t^3 \eta$ and the extension $\bar{\vartheta} = \partial_t^2 \bar{\eta} + \epsilon \partial_t^3 \bar{\eta}$, then the standard calculation and trace theory reveals that,

$$\epsilon \frac{d}{dt} \|\partial_t^2 \bar{\eta}\|_2 + \|\partial_t^2 \bar{\eta}\|_2 \leq \|\bar{\vartheta}\|_2 \lesssim \|\vartheta\|_{3/2}. \tag{5.31}$$

This implies that

$$\begin{aligned} \epsilon^2 \|\partial_t^2 \eta\|_{L^\infty H^{3/2}}^2 &\leq t \int_0^t \|\vartheta\|_{3/2}^2 + \epsilon^2 \|\partial_t^2 \eta(0)\|_{3/2}^2 \\ &\leq t \int_0^t \|\vartheta\|_{3/2}^2 + \|\partial_t \eta(0)\|_{3/2}^2 + \|\partial_t \eta(0) + \epsilon \partial_t^2 \eta(0)\|_{3/2}^2 \lesssim t \int_0^t \|\vartheta\|_{3/2}^2 + \mathfrak{E}_0, \end{aligned} \tag{5.32}$$

which also implies

$$\int_0^t \|\partial_t^2 \eta\|_{3/2}^2 \leq \int_0^t \|\vartheta\|_{3/2}^2 + t^2 \epsilon^2 \|\partial_t^2 \eta\|_{L^\infty H^{3/2}} \lesssim (1+t^3) \int_0^t \|\vartheta\|_{3/2}^2 + t^2 \mathfrak{E}_0. \tag{5.33}$$

Then following the proof of Theorem 8.2 in [10] together wiht (5.30) and (5.33), for $t \leq T < 1$, we may derive that

$$\int_0^t \mathcal{D}_{\shortparallel} \lesssim \int_0^t (\mathscr{D} + \sqrt{\mathcal{E}} \mathcal{D}) + \mathfrak{E}_0, \tag{5.34}$$

which reveals

$$\int_0^t \mathcal{D} \lesssim \int_0^t (\mathscr{D} + \sqrt{\mathcal{E}} \mathcal{D}) + \mathfrak{E}_0 \tag{5.35}$$

after similar estimate for $\|\partial_t\eta\|^2_{W^{5/2}_\delta}$ derived from $\|\partial_t\eta+\epsilon\partial_t^2\eta\|_{W^{5/2}_\delta}$. Then similar to the proof of Theorem 8.4 in [10], combining (5.28) and (5.35), we have

$$\sup_{0<s\le t}\mathcal{E}(s)+\int_0^t\mathcal{D}\le C\mathcal{E}(0)\le C'\mathfrak{E}_0, \tag{5.36}$$

for each $t\in[0,T]$, and the second inequality follows from the initial data in section 3. Restricting the initial data implies that

$$\sup_{0<s\le t}\mathcal{E}(s)+\int_0^t\mathcal{D}\le\frac{\alpha}{2}, \tag{5.37}$$

for each $t\in[0,T]$. □

5.3. **Existence of solutions.** In this section, we consider the solution of original problem (1.19).

**Theorem 5.5.** *There exists a solution $(u,p,\eta)\in\mathcal{X}\cap\mathcal{Y}$ solving the equation* (1.19).

*Proof.* According to the energy estimate in Theorem 5.4, there exists a sequence $\epsilon_k$ tends to zero and a pair $(u,p,\eta)$ such that $(u,p,\eta)\in\mathcal{X}\cap\mathcal{Y}$ with

$$\begin{cases}(u^{\epsilon_k},p^{\epsilon_k},\eta^{\epsilon_k})\overset{*}{\rightharpoonup}(u,p,\eta) & \text{weakly-}*\text{ in }\mathcal{X},\\ (u^{\epsilon_k},p^{\epsilon_k},\eta^{\epsilon_k})\rightharpoonup(u,p,\eta) & \text{weakly in }\mathcal{Y}.\end{cases} \tag{5.38}$$

Choose a function $w\in\mathcal{W}$, then from the weak formulation, we deduce that

$$\begin{aligned}&\int_0^T\int_{-\ell}^{\ell}g\eta^{\epsilon_k}(w\cdot\mathcal{N}^{\epsilon_k})+\sigma\frac{\partial_1\eta_k^\epsilon\partial_1(w\cdot\mathcal{N}^{\epsilon_k})}{(1+|\partial_1\zeta_0|^2)^{3/2}}+\epsilon_k\int_0^T\int_{-\ell}^{\ell}g\partial_t\eta^{\epsilon_k}(w\cdot\mathcal{N}^{\epsilon_k})+\sigma\frac{\partial_1\partial_t\eta_k^\epsilon\partial_1(w\cdot\mathcal{N}^{\epsilon_k})}{(1+|\partial_1\zeta_0|^2)^{3/2}}\\ &\quad+\int_0^T\int_\Omega\frac{\mu}{2}\mathbb{D}_{\mathcal{A}^{\epsilon_k}}u^{\epsilon_k}:\mathbb{D}_{\mathcal{A}^{\epsilon_k}}wJ^{\epsilon_k}+\int_0^T\int_{\Sigma_s}\beta(u^{\epsilon_k}\cdot\tau)(w\cdot\tau)J^{\epsilon_k}-\int_0^T\int_\Omega p^{\epsilon_k}\operatorname{div}_{\mathcal{A}^{\epsilon_k}}wJ^{\epsilon_k}\\ &\quad+[u^{\epsilon_k}\cdot\mathcal{N}^{\epsilon_k}+\hat{\mathscr{W}}(u^{\epsilon_k}\cdot\mathcal{N}^{\epsilon_k}),w\cdot\mathcal{N}^{\epsilon_k}]=-\sigma\int_0^T\int_{-\ell}^{\ell}\mathcal{R}(\partial_1\zeta_0,\partial_1\eta_k^\epsilon)\partial_1(w\cdot\mathcal{N}^{\epsilon_k}).\end{aligned} \tag{5.39}$$

Passing the limit $\epsilon_k\to0$, the convergence (5.38) reveals that

$$\begin{aligned}&\int_0^T\int_{-\ell}^{\ell}g\eta(w\cdot\mathcal{N})+\sigma\frac{\partial_1\eta\partial_1(w\cdot\mathcal{N})}{(1+|\partial_1\zeta_0|^2)^{3/2}}+\int_0^T\int_\Omega\frac{\mu}{2}\mathbb{D}_{\mathcal{A}}u:\mathbb{D}_{\mathcal{A}}wJ+\int_0^T\int_{\Sigma_s}\beta(u\cdot\tau)(w\cdot\tau)J\\ &\quad-\int_0^T\int_\Omega p\operatorname{div}_{\mathcal{A}}wJ+[u\cdot\mathcal{N}+\hat{\mathscr{W}}(u\cdot\mathcal{N}),w\cdot\mathcal{N}]_\ell=-\sigma\int_0^T\int_{-\ell}^{\ell}\mathcal{R}(\partial_1\zeta_0,\partial_1\eta)\partial_1(w\cdot\mathcal{N}).\end{aligned} \tag{5.40}$$

Thus the limit $(u,p,\eta)$ is a weak solution of (1.19). Then integrating by parts,

$$\begin{aligned}&\int_0^T\int_{-\ell}^{\ell}g\eta(w\cdot\mathcal{N})-\sigma\partial_1\left(\frac{\partial_1\eta}{(1+|\partial_1\zeta_0|^2)^{3/2}}+\mathcal{R}(\partial_1\zeta_0,\partial_1\eta)\right)w\cdot\mathcal{N}-\int_0^T\int_\Omega\mu(\Delta_{\mathcal{A}}u)wJ\\ &\quad+\int_0^T\int_{-\ell}^{\ell}\mu\mathbb{D}_{\mathcal{A}}u\mathcal{N}\cdot w+\int_0^T\int_{\Sigma_s}\mu\mathbb{D}_{\mathcal{A}}u\nu\cdot w+\beta(u\cdot\tau)(w\cdot\tau)J+\int_0^T\int_\Omega\nabla_{\mathcal{A}}p\cdot wJ\\ &\quad-\int_0^T\int_{-\ell}^{\ell}p\mathcal{N}\cdot w-\int_0^T\int_{\Sigma_s}p\nu\cdot wJ+\left[\sigma\left(\frac{\partial_1\eta}{(1+|\partial_1\zeta_0|^2)^{3/2}}+\mathcal{R}(\partial_1\zeta_0,\partial_1\eta)\right),w\cdot\mathcal{N}\right]_\ell\\ &\quad+[u\cdot\mathcal{N}+\hat{\mathscr{W}}(u\cdot\mathcal{N}),w\cdot\mathcal{N}]_\ell=0,\end{aligned} \tag{5.41}$$

we know that $(u,p,\eta)$ satisfy the boundary condition of (1.19). Thus $(u,p,\eta)$ is a strong solution of (1.19) because of its regularity. □

5.4. **Uniqueness.** We refer to velocities as $u^j$, pressures as $p^j$, surface functions as $\eta^j$, for $j = 1, 2$.

**Theorem 5.6.** *Let $u^1$, $u^2$, $p^1$, $p^2$ and $\eta^1$, $\eta^2$ satisfy*

$$\sup_{0\le t\le T}\{\mathcal{E}(u^1,p^1,\eta^1),\mathcal{E}(u^2,p^2,\eta^2)\}<\varepsilon,\quad \text{and}\ \int_0^T\{\mathcal{D}(u^1,p^1,\eta^1),\mathcal{D}(u^2,p^2,\eta^2)\}<\varepsilon, \tag{5.42}$$

*with $T > 0$. Suppose that for $j = 1, 2$,*

$$\begin{cases} -\mu\Delta_{\mathcal{A}^j}u^j+\nabla_{\mathcal{A}^j}p^j=0, & \text{in}\quad \Omega,\\ \operatorname{div}_{\mathcal{A}^j}u^j=0, & \text{in}\quad \Omega,\\ S_{\mathcal{A}^j}(p^j,u^j)\mathcal{N}^j=g\eta^j\mathcal{N}^j-\sigma\partial_1\left(\dfrac{\partial_1\eta^j}{1+|\partial_1\zeta_0|^2}+F^{3,j}\right)\mathcal{N}^j, & \text{on}\quad \Sigma,\\ \left(S_{\mathcal{A}^j}(p^j,u^j)\nu-\beta u^j\right)\cdot\tau=0, & \text{on}\quad \Sigma_s,\\ u^j\cdot\nu=0, & \text{on}\quad \Sigma_s,\\ \partial_t\eta^j=u^j\cdot\mathcal{N}^j, & \text{on}\quad \Sigma,\\ \kappa\partial_t\eta^j(\pm\ell,t)+\kappa\hat{\mathscr{W}}(\partial_t\eta^j(\pm\ell,t))=\mp\sigma\left(\dfrac{\partial_1\eta^j}{(1+|\zeta_0|^2)^{3/2}}+F^{3,j}\right)(\pm\ell,t). \end{cases} \tag{5.43}$$

*where $\mathcal{A}^j$, $\mathcal{N}^j$, $F^{3,j}$ are determined by $\eta^j$ as usual. Suppose that $u^1(0) = u^2(0)$, $p^1(0) = p^2(0)$ and $\partial_t^k\eta^1(0) = \partial_t^k\eta^2(0)$ for $k = 0, 1$.*

*Then there exist $\varepsilon_1 > 0$, $T_1 > 0$ such that if $0 < \varepsilon \le \varepsilon_1$ and $0 < T \le T_1$, then*

$$u^1=u^2,\quad p^1=p^2,\quad \eta^1=\eta^2. \tag{5.44}$$

*Proof.* First, we define $v = u^1 - u^2$, $q = p^1 - p^2$, $\theta = \eta^1 - \eta^2$ and derive the PDEs satisfied by $v$, $q$, $\theta$. We still use $F^3$ to denote $F^3 = F^{3,1} - F^{3,2}$.

Step 1 – PDEs and energy for differences.

Subtracting equations in (5.43) with $j = 2$ from the same equations with $j = 1$, we can write the resulting equations in terms of $v$, $q$, $\theta$ as

$$\begin{cases} \operatorname{div}_{\mathcal{A}^1}S_{\mathcal{A}^1}(q,v)=\mu\operatorname{div}_{\mathcal{A}^1}\left(\mathbb{D}_{(\mathcal{A}^1-\mathcal{A}^2)}u^2\right)+H^1, & \text{in}\quad \Omega,\\ \operatorname{div}_{\mathcal{A}^1}v=H^2, & \text{in}\quad \Omega,\\ S_{\mathcal{A}^1}(q,v)\mathcal{N}^1=\mu\mathbb{D}_{(\mathcal{A}^1-\mathcal{A}^2)}u^2\mathcal{N}^1+g\theta\mathcal{N}^1-\sigma\partial_1\left(\dfrac{\partial_1\theta}{(1+|\partial_1\zeta_0|^2)^{3/2}}\right)\mathcal{N}^1 & \\ \qquad -\sigma\partial_1F^3\mathcal{N}^1+H^3, & \text{on}\quad \Sigma,\\ (S_{\mathcal{A}^1}(q,v)\nu-\beta v)\cdot\tau=\mu\mathbb{D}_{(\mathcal{A}^1-\mathcal{A}^2)}u^2\nu\cdot\tau, & \text{on}\quad \Sigma_s,\\ v\cdot\nu=0, & \text{on}\quad \Sigma_s,\\ \partial_t\theta=v\cdot\mathcal{N}^1+H^5, & \text{on}\quad \Sigma,\\ \kappa\partial_t\theta(\pm\ell,t)=\mp\sigma\dfrac{\partial_1\theta}{(1+|\zeta_0|^2)^{3/2}}(\pm\ell,t)\mp F^3-H^6, & \\ v(t=0)=0,\quad \theta(t=0)=0. & \end{cases} \tag{5.45}$$

where $H^1$, $H^2$, $H^3$, $H^4$, $H^5$, $H^6$ are defined by

$$\begin{aligned} H^1&=\mu\operatorname{div}_{(\mathcal{A}^1-\mathcal{A}^2)}(\mathbb{D}_{\mathcal{A}^2}u^2)-\nabla_{(\mathcal{A}^1-\mathcal{A}^2)}p^2,\\ H^2&=-\operatorname{div}_{(\mathcal{A}^1-\mathcal{A}^2)}u^2,\\ H^3&=-p^2(\mathcal{N}^1-\mathcal{N}^2)+\mathbb{D}_{\mathcal{A}^1}u^2(\mathcal{N}^1-\mathcal{N}^2)-\mathbb{D}_{(\mathcal{A}^1-\mathcal{A}^2)}u^2\mathcal{N}^2+g\eta^2(\mathcal{N}^1-\mathcal{N}^2)\\ &\quad-\sigma\partial_1\left(\frac{\partial_1\eta^2}{(1+|\zeta_0|^2)^{3/2}}\right)(\mathcal{N}^1-\mathcal{N}^2)-\sigma\partial_1F^{3,2}(\mathcal{N}^1-\mathcal{N}^2),\\ H^5&=u^2\cdot(\mathcal{N}^1-\mathcal{N}^2), \end{aligned}$$

$$H^6 = \kappa(\hat{\mathscr{W}}(\partial_t\eta^1(\pm\ell,t)) - \hat{\mathscr{W}}(\partial_t\eta^2(\pm\ell,t))).$$

The solutions are sufficiently regular for us to differentiate (5.45) in time, which results in the equations

$$\begin{cases} \operatorname{div}_{\mathcal{A}^1} S_{\mathcal{A}^1}(\partial_t q, \partial_t v) = \mu \operatorname{div}_{\mathcal{A}^1}\left(\mathbb{D}_{(\partial_t\mathcal{A}^1 - \partial_t\mathcal{A}^2)}u^2\right) + \tilde{H}^1, & \text{in } \Omega, \\ \operatorname{div}_{\mathcal{A}^1}\partial_t v = \tilde{H}^2, & \text{in } \Omega, \\ S_{\mathcal{A}^1}(\partial_t q, \partial_t v)\mathcal{N}^1 = \mu\mathbb{D}_{(\partial_t\mathcal{A}^1 - \partial_t\mathcal{A}^2)}u^2\mathcal{N}^1 + g\partial_t\theta\mathcal{N}^1 - \sigma\partial_1\left(\dfrac{\partial_1\partial_t\theta}{(1+|\partial_1\zeta_0|^2)^{3/2}}\right)\mathcal{N}^1 \\ \qquad - \sigma\partial_1\partial_t(F^{3,1} - F^{3,2})\mathcal{N}^1 + \tilde{H}^3, & \text{on } \Sigma, \\ (S_{\mathcal{A}^1}(\partial_t q, \partial_t v)\nu - \beta\partial_t v)\cdot\tau = \mu\mathbb{D}_{(\partial_t\mathcal{A}^1 - \partial_t\mathcal{A}^2)}u^2\nu\cdot\tau + \tilde{H}^4, & \text{on } \Sigma_s, \\ \partial_t v\cdot\nu = 0, & \text{on } \Sigma_s, \\ \partial_t^2\theta = \partial_t v\cdot\mathcal{N}^1 + \tilde{H}^5, & \text{on } \Sigma, \\ \kappa\partial_t^2\theta(\pm\ell,t) = \mp\sigma\dfrac{\partial_1\partial_t\theta}{(1+|\zeta_0|^2)^{3/2}}(\pm\ell,t) \mp \tilde{H}^6, \\ \partial_t v(t=0) = 0, \quad \partial_t\theta(t=0) = 0, \end{cases} \tag{5.46}$$

where

$$\begin{aligned} \tilde{H}^1 &= \partial_t H^1 + \operatorname{div}_{\partial_t\mathcal{A}^1}(\mathbb{D}_{(\mathcal{A}^1-\mathcal{A}^2)}u^2) + \operatorname{div}_{\mathcal{A}^1}(\mathbb{D}_{(\mathcal{A}^1-\mathcal{A}^2)}\partial_t u^2) + \operatorname{div}_{\partial_t\mathcal{A}^1}(\mathbb{D}_{\mathcal{A}^1}v) \\ &\quad + \operatorname{div}_{\mathcal{A}^1}(\mathbb{D}_{\partial_t\mathcal{A}^1}v) - \nabla_{\partial_t\mathcal{A}^1}q, \\ \tilde{H}^2 &= \partial_t H^2 - \operatorname{div}_{\partial_t\mathcal{A}^1}v, \\ \tilde{H}^3 &= \partial_t H^3 + \mathbb{D}_{(\mathcal{A}^1-\mathcal{A}^2)}\partial_t u^2\mathcal{N}^1 + \mathbb{D}_{(\mathcal{A}^1-\mathcal{A}^2)}u^2\partial_t\mathcal{N}^1 - S_{\mathcal{A}^1}(q,v)\partial_t\mathcal{N}^1 + \mathbb{D}_{\partial_t\mathcal{A}^1}v\mathcal{N}^1 \\ &\quad + g\theta\partial_t\mathcal{N}^1 - \sigma\partial_1\left(\frac{\partial_1\theta}{(1+|\partial_1\zeta_0|^2)^{3/2}}\right)\partial_t\mathcal{N}^1, \\ \tilde{H}^4 &= \mu\mathbb{D}_{(\mathcal{A}^1-\mathcal{A}^2)}\partial_t u^2\nu\cdot\tau + \mathbb{D}_{\partial_t\mathcal{A}^1}v\nu\cdot\tau, \\ \tilde{H}^5 &= \partial_t H^5 + v\cdot\partial_t\mathcal{N}^1, \\ \tilde{H}^6 &= \partial_t H^6. \end{aligned}$$

Now we multiply (5.46) by $J^1\partial_t u^1$, integrate over $\Omega$ and integrate by parts to deduce that

$$\begin{aligned} &\partial_t\left(\int_{-\ell}^{\ell}\frac{g}{2}|\partial_t\theta|^2 + \frac{\sigma}{2}\frac{|\partial_1\partial_t\theta|^2}{(1+|\partial_1\zeta_0|^2)^{3/2}}\right) + \frac{\mu}{2}\int_\Omega|\mathbb{D}_{\mathcal{A}^1}\partial_t v|^2J^1 + \beta\int_{\Sigma_s}J^1|\partial_t v\cdot\tau|^2 + [\partial_t v\cdot\mathcal{N}^1]_\ell^2 \\ &= \int_\Omega\mu\operatorname{div}_{\mathcal{A}^1}(\mathbb{D}_{(\partial_t\mathcal{A}^1-\partial_t\mathcal{A}^2)}u^2)\cdot\partial_t vJ^1 + \tilde{H}^1\cdot\partial_t vJ^1 + \partial_t q\tilde{H}^2J^1 - \int_{\Sigma_s}J^1(\partial_t v\cdot\tau)\tilde{H}^4 \\ &\quad - \int_{-\ell}^{\ell}\sigma\partial_t F^3\partial_1(\partial_t v\cdot\mathcal{N}^1) + (\mathbb{D}_{(\partial_t\mathcal{A}^1-\partial_t\mathcal{A}^2)}u^2\mathcal{N}^1 + \tilde{H}^3)\cdot\partial_t v - g\partial_t\theta\tilde{H}^5 - \sigma\frac{\partial_1\partial_t\theta\partial_1\tilde{H}^5}{(1+|\partial_1\zeta_0|^2)^{3/2}} \\ &\quad - \int_{\Sigma_s}J^1(\partial_t v\cdot\tau)\mu\mathbb{D}_{(\partial_t\mathcal{A}^1-\partial_t\mathcal{A}^2)}u^2\nu\cdot\tau - \left[\partial_t v\cdot\mathcal{N}^1, \tilde{H}^5 + \tilde{H}^6\right]_\ell. \end{aligned} \tag{5.47}$$

Here we notice that

$$\sum_{a=\pm1}\kappa(\partial_t v\cdot\mathcal{N}^1)(a\ell)\tilde{H}^5(a\ell) = 0, \tag{5.48}$$

since $v_1^1 = v_1^2 = 0$ at the endpoints $x_1 = \pm\ell$, where we denote that $u^1 = (v_1^1, v_2^1)$ and $u^2 = (v_1^2, v_2^2)$.

Another integration by parts reveals that

$$\begin{aligned} \int_\Omega\mu\operatorname{div}_{\mathcal{A}^1}(\mathbb{D}_{(\partial_t\mathcal{A}^1-\partial_t\mathcal{A}^2)}u^2)\cdot\partial_t vJ^1 &= -\frac{\mu}{2}\int_\Omega J^1\mathbb{D}_{(\partial_t\mathcal{A}^1-\partial_t\mathcal{A}^2)}u^2 : \mathbb{D}_{\mathcal{A}^1}\partial_t v \\ &\quad + \int_{-\ell}^{\ell}\mathbb{D}_{(\partial_t\mathcal{A}^1-\partial_t\mathcal{A}^2)}u^2\mathcal{N}^1\cdot\partial_t v + \int_{\Sigma_s}\mathbb{D}_{(\partial_t\mathcal{A}^1-\partial_t\mathcal{A}^2)}u^2\nu\cdot\partial_t vJ^1. \end{aligned} \tag{5.49}$$

We combine (5.47) and (5.49), and then integrate in time from 0 to $t<T$ to derive that

$$\begin{aligned}&\int_{-\ell}^{\ell}\frac{g}{2}|\partial_t\theta|^2+\frac{\sigma}{2}\frac{|\partial_1\partial_t\theta|^2}{(1+|\partial_1\zeta_0|^2)^{3/2}}+\frac{\mu}{2}\int_0^t\int_\Omega|\mathbb{D}_{\mathcal{A}^1}\partial_t v|^2J^1+\beta\int_0^t\int_{\Sigma_s}J^1|\partial_t v\cdot\tau|^2+\int_0^t[\partial_t v\cdot\mathcal{N}^1]_\ell^2\\&=-\frac{\mu}{2}\int_\Omega J^1\mathbb{D}_{(\partial_t\mathcal{A}^1-\partial_t\mathcal{A}^2)}u^2:\mathbb{D}_{\mathcal{A}^1}\partial_t v+\int_0^t\int_\Omega\tilde{H}^1\cdot\partial_t vJ^1+\partial_t q\tilde{H}^2J^1-\int_0^t\int_{\Sigma_s}J^1(\partial_t v\cdot\tau)\tilde{H}^4\\&\quad-\int_0^t\int_{-\ell}^{\ell}\sigma\partial_tF^3\partial_1(\partial_t v\cdot\mathcal{N}^1)+\tilde{H}^3\cdot\partial_t v-g\partial_t\theta\tilde{H}^5-\sigma\frac{\partial_1\partial_t\theta\partial_1\tilde{H}^5}{(1+|\partial_1\zeta_0|^2)^{3/2}}-[\partial_t v\cdot\mathcal{N}^1,\tilde{H}^6]_\ell.\end{aligned}\tag{5.50}$$

Step 2 – Estimate of pressure.

In order to handle the term related to $\partial_t q$, we multiply (5.46) by $J^1w$, integrate over $\Omega$ and integrate by parts to deduce that

$$\begin{aligned}&\frac{\mu}{2}\int_\Omega\mathbb{D}_{\mathcal{A}^1}\partial_t v:\mathbb{D}_{\mathcal{A}^1}wJ^1+\beta\int_{\Sigma_s}(\partial_t v\cdot\tau)(w\cdot\tau)+(\partial_t\theta,w\cdot\mathcal{N}^1)_{1,\Sigma}+[\partial_t v\cdot\mathcal{N}^1,w\cdot\mathcal{N}^1]_\ell\\&=\int_\Omega(\mu\,\mathrm{div}_{\mathcal{A}^1}\left(\mathbb{D}_{(\partial_t\mathcal{A}^1-\partial_t\mathcal{A}^2)}u^2\right)+\tilde{H}^1)\cdot wJ^1-\int_{\Sigma_s}J^1(w\cdot\tau)(\mu\mathbb{D}_{(\partial_t\mathcal{A}^1-\partial_t\mathcal{A}^2)}u^2\nu\cdot\tau+\tilde{H}^4)\\&\quad-\int_{-\ell}^{\ell}\sigma\partial_t(F^{3,1}-F^{3,2})\partial_1(w\cdot\mathcal{N}^1)+(\mu\mathbb{D}_{(\partial_t\mathcal{A}^1-\partial_t\mathcal{A}^2)}u^2\mathcal{N}^1+\tilde{H}^3)\cdot w-[w\cdot\mathcal{N}^1,\tilde{H}^6]_\ell,\end{aligned}\tag{5.51}$$

for each $w\in\mathcal{V}(t)$ and a.e. $t\in[0,T]$. Then $\partial_t q\in\mathring{H}^0(\Omega)$ might be recovered from Theorem 4.6 in [10]such that

$$\begin{aligned}&\frac{\mu}{2}\int_\Omega\mathbb{D}_{\mathcal{A}^1}v:\mathbb{D}_{\mathcal{A}^1}wJ^1+\beta\int_{\Sigma_s}(v\cdot\tau)(w\cdot\tau)-(\partial_t q,\mathrm{div}_{\mathcal{A}^1}w)_0+(\partial_t\theta,w\cdot\mathcal{N}^1)_{1,\Sigma}\\&\quad+[\partial_t v\cdot\mathcal{N}^1,w\cdot\mathcal{N}^1]_\ell\\&=\int_\Omega(\mu\,\mathrm{div}_{\mathcal{A}^1}\left(\mathbb{D}_{(\partial_t\mathcal{A}^1-\partial_t\mathcal{A}^2)}u^2\right)+\tilde{H}^1)\cdot wJ^1-\int_{\Sigma_s}J^1(w\cdot\tau)(\mu\mathbb{D}_{(\partial_t\mathcal{A}^1-\partial_t\mathcal{A}^2)}u^2\nu\cdot\tau+\tilde{H}^4)\\&\quad-\int_{-\ell}^{\ell}\sigma\partial_t(F^{3,1}-F^{3,2})\partial_1(w\cdot\mathcal{N}^1)+(\mu\mathbb{D}_{(\partial_t\mathcal{A}^1-\partial_t\mathcal{A}^2)}u^2\mathcal{N}^1+\tilde{H}^3)\cdot w-[w\cdot\mathcal{N}^1,\tilde{H}^6]_\ell,\end{aligned}\tag{5.52}$$

for each $w\in\mathcal{W}(t)$ and a.e. $t\in[0,T]$. Moreover,

$$\|\partial_t q\|_{L^2\mathring{H}^0}^2\lesssim\|\partial_t v\|_{L^2H^1}^2+P(\sqrt{\varepsilon})(\|\partial_t\theta\|_{L^2\mathring{H}^{3/2}}^2+\|\theta\|_{L^2W^{5/2}_\delta}^2+\|q\|_{L^2\mathring{W}^1_\delta}^2+\|v\|_{L^2W^2_\delta}^2),\tag{5.53}$$

where the temporal $L^2$ norm is computed on $[0,T]$, and $P(\cdot)$ is a polynomial which would be allowed to change from line to line.

Step 3 – Estimates of the forcing terms.

To handle the term $\partial_t(F^{3,1}-F^{3,2})$, we rewrite it as

$$\begin{aligned}\int_{-\ell}^{\ell}\sigma\partial_tF^3\partial_1(\partial_t v\cdot\mathcal{N}^1)&=\int_{-\ell}^{\ell}\sigma[\partial_z\mathcal{R}^1\partial_1\partial_t\theta+\partial_z(\mathcal{R}^1-\mathcal{R}^2)\partial_1\partial_t\eta^2]\partial_1(\partial_t^2\theta-\tilde{H}^5)\\&=\frac{d}{dt}\left(\int_{-\ell}^{\ell}\partial_z\mathcal{R}^1\frac{|\partial_1\partial_t\theta|^2}{2}-\partial_z(\mathcal{R}^1-\mathcal{R}^2)\partial_1\partial_t\eta^2\partial_1\partial_t\theta\right)-\int_{-\ell}^{\ell}\partial_z^2\mathcal{R}^1\partial_1\partial_t\eta^1\frac{|\partial_1\partial_t\theta|^2}{2}\\&\quad-\int_{-\ell}^{\ell}|\partial_1\partial_t\theta|^2\partial_z^2\mathcal{R}^1\partial_1\partial_t\eta^2+\partial_z^2(\mathcal{R}^1-\mathcal{R}^2)(\partial_1\partial_t\eta^2)^2\partial_1\partial_t\theta\\&\quad+\partial_z(\mathcal{R}^1-\mathcal{R}^2)\partial_1\partial_t^2\eta^2\partial_1\partial_t\theta-\partial_z(\mathcal{R}^1-\mathcal{R}^2)\tilde{H}^5,\end{aligned}\tag{5.54}$$

Then we rewrite (5.47) as

$$\begin{aligned}\frac{d}{dt}\left(\|\partial_t\theta\|_{1,\Sigma}^2+\int_{-\ell}^{\ell}\partial_z\mathcal{R}^1\frac{|\partial_1\partial_t\theta|^2}{2}-\partial_z(\mathcal{R}^1-\mathcal{R}^2)\partial_1\partial_t\eta^2\partial_1\partial_t\theta\right)+\frac{\mu}{2}\int_\Omega|\mathbb{D}_{\mathcal{A}^1}\partial_t v|^2J^1\\ +\beta\int_{\Sigma_s}J^1|\partial_t v\cdot\tau|^2+[\partial_t v\cdot\mathcal{N}^1]_\ell^2\\ =-\frac{\mu}{2}\int_\Omega J^1\mathbb{D}_{(\partial_t\mathcal{A}^1-\partial_t\mathcal{A}^2)}u^2:\mathbb{D}_{\mathcal{A}^1}\partial_t v+\int_\Omega\tilde{H}^1\cdot\partial_t vJ^1+\partial_t q\tilde{H}^2J^1-\int_{\Sigma_s}J^1(\partial_t v\cdot\tau)\tilde{H}^4\\ -\int_{-\ell}^{\ell}|\partial_1\partial_t\theta|^2\partial_z^2\mathcal{R}^1\partial_1\partial_t\eta^2+\partial_z^2(\mathcal{R}^1-\mathcal{R}^2)(\partial_1\partial_t\eta^2)^2\partial_1\partial_t\theta+\partial_z(\mathcal{R}^1-\mathcal{R}^2)(\partial_1\partial_t^2\eta^2\partial_1\partial_t\theta-\tilde{H}^5)\\ -\int_{-\ell}^{\ell}\partial_z^2\mathcal{R}^1\partial_1\partial_t\eta^1\frac{|\partial_1\partial_t\theta|^2}{2}-\int_{-\ell}^{\ell}\left(\tilde{H}^3\cdot\partial_t v-g\partial_t\theta\tilde{H}^5-\sigma\frac{\partial_1\partial_t\theta\partial_1\tilde{H}^5}{(1+|\partial_1\zeta_0|^2)^{3/2}}\right)\\ -[\partial_t v\cdot\mathcal{N}^1,\tilde{H}^6]_\ell.\end{aligned}\tag{5.55}$$

We now estimate the terms on the right hand side of (5.50).

$$\frac{\mu}{2}\int_\Omega J^1\mathbb{D}_{(\partial_t\mathcal{A}^1-\partial_t\mathcal{A}^2)}u^2:\mathbb{D}_{\mathcal{A}^1}\partial_t v\lesssim P(\sqrt{\varepsilon})\|\partial_t v\|_1\|\partial_t\theta\|_{3/2}.\tag{5.56}$$

$$\int_\Omega\tilde{H}^1\cdot\partial_t vJ^1\lesssim P(\sqrt{\varepsilon})\|\partial_t v\|_1(\|\partial_t\theta\|_{3/2}+\|\theta\|_{W_\delta^{5/2}}+\|q\|_{\mathring{W}_\delta^1}+\|v\|_{W_\delta^2}).\tag{5.57}$$

$$\int_{\Sigma_s}J^1(\partial_t v\cdot\tau)\tilde{H}^4\lesssim P(\sqrt{\varepsilon})\|\partial_t v\|_1(\|\theta\|_{W_\delta^{5/2}}+\|\partial_t\theta\|_{3/2}).\tag{5.58}$$

$$\int_\Omega\partial_t q\tilde{H}^2J^1\lesssim P(\sqrt{\varepsilon})\|\partial_t q\|_0(\|\partial_t\theta\|_{3/2}+\|\theta\|_{W_\delta^{5/2}}+\|v\|_{W_\delta^2}).\tag{5.59}$$

By the direct computation for derivatives of (1.13), we may employ the Sobolev embedding theory to derive that

$$\begin{aligned}-\int_{-\ell}^{\ell}|\partial_1\partial_t\theta|^2\partial_z^2\mathcal{R}^1\partial_1\partial_t\eta^2+\partial_z^2(\mathcal{R}^1-\mathcal{R}^2)(\partial_1\partial_t\eta^2)^2\partial_1\partial_t\theta+\partial_z(\mathcal{R}^1-\mathcal{R}^2)(\partial_1\partial_t^2\eta^2\partial_1\partial_t\theta-\tilde{H}^5)\\ -\int_{-\ell}^{\ell}\partial_z^2\mathcal{R}^1\partial_1\partial_t\eta^1\frac{|\partial_1\partial_t\theta|^2}{2}\lesssim P(\sqrt{\varepsilon})(\|\partial_t\theta\|_{3/2}^2+\|\theta\|_{W_\delta^{5/2}}^2+\|\theta\|_{W_\delta^{5/2}}+\|v\|_{W_\delta^2}),\end{aligned}\tag{5.60}$$

and

$$\int_{-\ell}^{\ell}\partial_z\mathcal{R}^1\frac{|\partial_1\partial_t\theta|^2}{2}-\partial_z(\mathcal{R}^1-\mathcal{R}^2)\partial_1\partial_t\eta^2\partial_1\partial_t\theta\lesssim P(\sqrt{\varepsilon})(\|\partial_t\theta\|_{3/2}^2+\|\theta\|_{W_\delta^{5/2}}).\tag{5.61}$$

$$\int_{-\ell}^{\ell}\sigma\tilde{H}^3\cdot\partial_t v\lesssim P(\sqrt{\varepsilon})\|\partial_t v\|_1(\|\theta\|_{W_\delta^{5/2}}+\|\partial_t\theta\|_{3/2}+\|q\|_{\mathring{W}_\delta^1}+\|v\|_{W_\delta^2}).\tag{5.62}$$

Due to the fact that $v_1^1=v_1^2=0$ at the endpoints $x_1=\pm\ell$, after integrating by parts,

$$\begin{aligned}\int_{-\ell}^{\ell}-g\partial_t\theta\tilde{H}^5-\sigma\frac{\partial_1\partial_t\theta\partial_1\tilde{H}^5}{(1+|\partial_1\zeta_0|^2)^{3/2}}\lesssim P(\sqrt{\varepsilon})\left[\|\partial_t\theta\|_1(\|\theta\|_{W_\delta^{5/2}}+\|\partial_t\theta\|_1+\|v\|_{W_\delta^2})+\|\partial_t\theta\|_{3/2}^2\right]\\ +\|\partial_t\eta^1\|_{W_\delta^{5/2}}\|\partial_t\theta\|_{3/2}\|v\|_1.\end{aligned}\tag{5.63}$$

$$[\partial_t v\cdot\mathcal{N}^1,\tilde{H}^6]_\ell=[\partial_t^2\theta,\tilde{H}^6]_\ell\lesssim P(\sqrt{\varepsilon})\|\partial_t\theta\|_1^2.\tag{5.64}$$

Then combining all the above estimates (5.56)–(5.64), we can derive that

$$\begin{aligned}&\frac{d}{dt}\left(\|\partial_t\theta\|_{1,\Sigma}^2+\int_{-\ell}^{\ell}\partial_z\mathcal{R}^1\frac{|\partial_1\partial_t\theta|^2}{2}-\partial_z(\mathcal{R}^1-\mathcal{R}^2)\partial_1\partial_t\eta^2\partial_1\partial_t\theta\right)+\|\partial_t v\|_1^2+[\partial_t v\cdot\mathcal{N}^1]^2\\ &\le CP(\sqrt{\varepsilon})\left(\|\partial_t\theta\|_{1,\Sigma}^2+\int_{-\ell}^{\ell}\partial_z\mathcal{R}^1\frac{|\partial_1\partial_t\theta|^2}{2}-\partial_z(\mathcal{R}^1-\mathcal{R}^2)\partial_1\partial_t\eta^2\partial_1\partial_t\theta\right)\\ &\quad+CP(\sqrt{\varepsilon})\|\theta\|_{W_\delta^{5/2}}^2+\|\partial_t\theta\|_{3/2}^2+\|q\|_{\mathring{W}_\delta^1}^2+\|v\|_{W_\delta^2}^2.\end{aligned}\tag{5.65}$$

Since

$$\sup_{0\le t\le T}\int_{-\ell}^{\ell}\partial_z\mathcal{R}^1\frac{|\partial_1\partial_t\theta|^2}{2}-\partial_z(\mathcal{R}^1-\mathcal{R}^2)\partial_1\partial_t\eta^2\partial_1\partial_t\theta\lesssim P(\sqrt{\varepsilon})(\|\partial_t\theta\|^2_{L^\infty\mathring{H}^1}+\|\theta\|^2_{L^\infty\mathring{H}^{3/2}})$$
$$\lesssim P(\sqrt{\varepsilon})(\|\partial_t\theta\|^2_{L^\infty\mathring{H}^1}+\|\partial_t\theta\|^2_{L^2\mathring{H}^{3/2}}+\|\theta\|^2_{L^2W^{5/2}_\delta}),$$

Gronwall's lemma together with the smallness of $\varepsilon$ implies that

$$\begin{aligned}\|\partial_t\theta\|^2_{L^\infty\mathring{H}^1}+\|\partial_t v\|^2_{L^2H^1}+\int_0^T[\partial_t v\cdot\mathcal{N}^1]^2&\lesssim e^{CP(\sqrt{\varepsilon})T_1}CP(\sqrt{\varepsilon})(\|\theta\|^2_{L^2W^{5/2}_\delta}+\|\partial_t\theta\|^2_{L^2\mathring{H}^{3/2}}\\&\quad+\|\partial_t q\|^2_{L^2\mathring{H}^0}+\|q\|^2_{L^2\mathring{W}^1_\delta}+\|v\|^2_{L^2W^2_\delta})\\&\lesssim e^{CP(\sqrt{\varepsilon})T_1}CP(\sqrt{\varepsilon})(\|\theta\|^2_{L^2W^{5/2}_\delta}+\|\partial_t\theta\|^2_{L^2\mathring{H}^{3/2}}+\|q\|^2_{L^2\mathring{W}^1_\delta}+\|v\|^2_{L^2W^2_\delta}),\end{aligned}\tag{5.66}$$

where the temporal $L^\infty$ and $L^2$ norms are computed over $[0,T]$ and $0<t<T\le T_1$. We assume that $\varepsilon_1$ and $T_1$ are sufficiently small for $e^{CP(\sqrt{\varepsilon})T_1}\le e^{CP(\sqrt{\varepsilon_1})T_1}\le 2$. Then we deduce the bound

$$\|\partial_t\theta\|^2_{L^\infty\mathring{H}^1}+\|\partial_t v\|^2_{L^2H^1}+\int_0^T[\partial_t v\cdot\mathcal{N}^1]^2\lesssim P(\sqrt{\varepsilon})(\|\theta\|^2_{L^2W^{5/2}_\delta}+\|\partial_t\theta\|^2_{L^2\mathring{H}^{3/2}}+\|q\|^2_{L^2\mathring{W}^1_\delta}+\|v\|^2_{L^2W^2_\delta}).\tag{5.67}$$

Since $\partial_t\theta\in\mathring{H}^1((-\ell,\ell))$ and (5.51), with $\varepsilon$ sufficient small, Theorem 4.11 in [10] reveals that

$$\|\partial_t\theta\|^2_{L^2\mathring{H}^{3/2}}\lesssim P(\sqrt{\varepsilon})(\|\theta\|^2_{L^2W^{5/2}_\delta}+\|q\|^2_{L^2\mathring{W}^1_\delta}+\|v\|^2_{L^2W^2_\delta}).\tag{5.68}$$

Step 4 – Elliptic estimates for $v$, $q$ and $\theta$.

In order to close our estimates, we must be able to estimate $v$, $q$ and $\theta$. The elliptic estimates imply that

$$\begin{aligned}\|v\|^2_{W^2_\delta}+\|q\|^2_{\mathring{W}^1_\delta}+\|\theta\|^2_{W^{5/2}_\delta}&\lesssim\|\operatorname{div}_{\mathcal{A}^1}(\mathbb{D}_{\mathcal{A}^1-\mathcal{A}^2}u^2)+H^1\|^2_{W^0_\delta}+\|H^2\|^2_{W^1_\delta}\\&\quad+\|\partial_t\theta-H^5\|^2_{W^{3/2}_\delta}+\|H^3\|^2_{W^{1/2}_\delta}+\|\mathbb{D}_{\mathcal{A}^1-\mathcal{A}^2}u^2\nu\cdot\tau\|^2_{W^{1/2}_\delta}\\&\quad+\|\partial_1(F^{3,1}-F^{3,2})\|^2_{W^{1/2}_\delta}+[\partial_t\theta\pm H^6]^2_\ell.\end{aligned}\tag{5.69}$$

Then after integrating temporally from 0 to $T$, we have that

$$\begin{aligned}\|v\|^2_{L^2W^2_\delta}+\|q\|^2_{L^2\mathring{W}^1_\delta}+\|\theta\|^2_{L^2W^{5/2}_\delta}&\lesssim P(\sqrt{\varepsilon})\|\theta\|^2_{L^2W^{5/2}_\delta}+\|\partial_t\theta\|^2_{L^2W^{3/2}_\delta}\\&\le CP(\sqrt{\varepsilon})(\|\theta\|^2_{L^2W^{5/2}_\delta}+\|q\|^2_{L^2\mathring{W}^1_\delta}+\|v\|^2_{L^2W^2_\delta}),\end{aligned}\tag{5.70}$$

where $P(0)=0$. Since $\varepsilon$ is sufficiently small, we might restrict $\varepsilon_1$ such that $CP(\sqrt{\varepsilon})<1$. Thus

$$\|v\|^2_{L^2W^2_\delta}+\|q\|^2_{L^2\mathring{W}^1_\delta}+\|\theta\|^2_{L^2W^{5/2}_\delta}=0.\tag{5.71}$$

□

5.5. **Diffeomorphism of $\Phi$.** From the definition of $J$ and restrict theory in Sobolev spaces, we can derive that

$$\|J\|_{L^\infty}\ge 1-C(\|\bar\eta\|_{L^\infty}+\|\partial_2\bar\eta\|_{L^\infty})\ge 1-C\|\eta\|_{W^{5/2}_\delta}.$$

The smallness of $\mathfrak{K}(\eta)$ sufficiently guarantees that $\Phi$, defined in (1.6), is a $C^1$ diffeomorphism for each $t\in[0,T]$. For more details, one can see [8] in $3D$ domains.

## Appendix A. Properties involving $\mathcal{A}$

We now record some useful properties involving $\mathcal{A}$.

**Lemma A.1.** *The following identities hold.*
*(1)* $\partial_j(J\mathcal{A}_{ij})=0$ *for* $j=1,2$ *and each* $i=1,2$.
*(2)* $J\mathcal{A}\mathcal{N}_0=\mathcal{N}$ *on* $\Sigma$,
*(3)* $R^\top\mathcal{N}=-\partial_t\mathcal{N}$ *on* $\Sigma$, *where* $R$ *is defined by* (4.11).

*Proof.* The first equality comes from Lemma A.3 in [9]. On $\Sigma$,

$$\begin{aligned} J\mathcal{A}\mathcal{N}_0 &= \begin{pmatrix} J & -A \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -\partial_1\zeta_0 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} -(1+\partial_2\bar{\eta}+\frac{\eta}{\zeta_0})\partial_1\zeta_0 - \partial_1\eta + \partial_2\bar{\eta}\partial_1\zeta_0 + \frac{1}{\zeta_0}\partial_1\zeta_0\eta \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} -\partial_1\zeta_0 - \partial_1\eta \\ 1 \end{pmatrix} = \mathcal{N}. \end{aligned}$$

It is easily to compute that $R^\top = J\partial_t K I_{2\times 2} - \partial_t\mathcal{A}\mathcal{A}^{-1}$. Since $J\mathcal{A}\mathcal{N}_0 = \mathcal{N}$,

$$\begin{aligned} R^\top\mathcal{N} &= (J\partial_t K - \partial_t\mathcal{A}\mathcal{A}^{-1})J\mathcal{A}\mathcal{N}_0 \\ &= (-K\partial_t J - \partial_t\mathcal{A}\mathcal{A}^{-1})J\mathcal{A}\mathcal{N}_0 \\ &= (-\partial_t J\mathcal{A} - J\partial_t\mathcal{A})\mathcal{N}_0 = -\partial_t(J\mathcal{A}\mathcal{N}_0) = -\partial_t\mathcal{N}. \end{aligned}$$

□

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