Domain formation in membranes near the onset of instability

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Abstract. The formation of microdomains, also called rafts, in biomembranes can be attributed to the surface tension of the membrane. In order to model this phenomenon, a model involving a coupling between the local composition and the local curvature was proposed by Seul and Andelman in 1995. In addition to the familiar Cahn-Hilliard/Modica-Mortola energy, there are additional 'forces' that prevent large domains of homogeneous concentration. This is taken into account by the bending energy of the membrane, which is coupled to the value of the order parameter, and reflects the notion that surface tension associated with a slightly curved membrane influences the localization of phases as the geometry of the lipids has an effect on the preferred placement on the membrane.

The main result of the paper is the study of the Γ-convergence of this family of energy functionals depending on the size of the sample, involving nonlocal as well as negative terms. Since the limiting energy is minimized by a phase function with minimal interfaces, the physical interpretation is that, within a certain parameter range, raft microdomains are not formed.

Keywords: Γ-convergence, nonlocal energies, interpolation.
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1. Introduction

The continuum theory of membranes has been an active area of research in material and biological sciences since the pioneering works of Canham and Helfrich, [6, 16]. Biological cell membranes or biomembranes are complex structures commonly made up of lipids, proteins, and cholesterol. Of recent very widespread interest is the phase separation and domain formation of these compounds forming the cell membrane. The resulting nanoscale microdomains, referred to as 'lipid rafts', are believed to be responsible for membrane trafficking, intracellular signaling, and assembly of specialized structures, [31]. Many important biological processes, such as virus budding, endocytosis, and immune responses, are believed to be linked to membrane rafts, [27]. Ever since the first experimental evidence of raft formation in late 1980's, there has been a growing body of literature on both theoretical and experimental aspects of this phenomenon, [11]. However, due to very small scales associated with raft domains (they are too small to be optically resolved) [27, 5, 23], there are different viewpoints on the precise structure and stability of lipid rafts, [22]. As a result, understanding the conditions for the formation, as well as mechanisms driving stability (and instability), of these microdomains is of great importance.

It has been proposed in [20] that raft formation can be attributed to the surface tension of the membrane. The experimental basis for the theory comes from the work of Rozovsky et al in [29], in which domain formation in a ternary mixture of sphingomyelin, DOPC, and cholesterol is observed for a vesicle adhered to a substrate structure. To study the relation between an increase in surface tension and the morphological transitions on the membrane plane, a coupling between the local composition

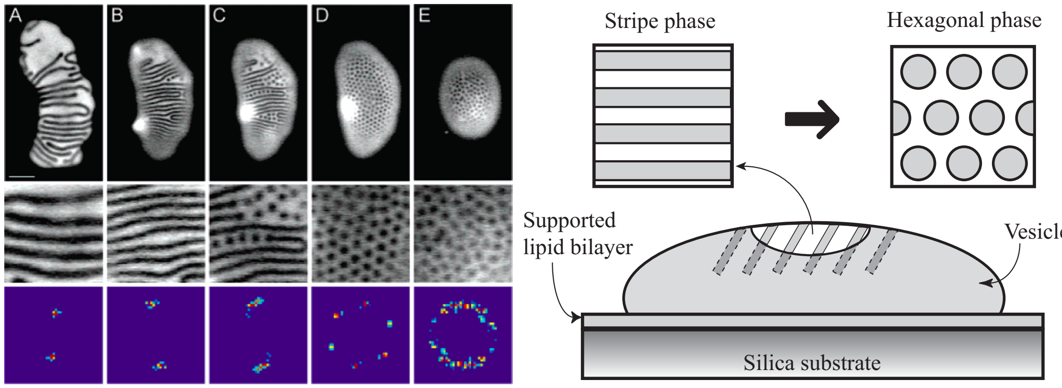

Figure 1. Experimental and schematic representation of rafts. The first picture is reprinted with permission from S. Rozovsky, Y. Kaizuka, and J. T. Groves. Formation and spatio-temporal evolution of periodic structures in lipid bilayers. J. Am. Chem. Soc., 127(1):36–37, 1 2005. Copyright 2005 American Chemical Society. The second picture is reprinted with permission from S. Komura, N. Shimokawa, and D. Andelman. Tension-induced morphological transition in mixed lipid bilayers. Langmuir, 22:6771–6774, 2006. Copyright 2006 American Chemical Society.

and the local curvature was proposed in [20]. The authors consider a free energy framework and use an energy functional first introduced in [30] to model phase separation of a di-block copolymer in a membrane allowing out of plane (bending) distortions (see also [18, 32, 21]).

Similar to the classic Ginzburg-Landau models, the system is described in terms of an order parameter u that may, for instance, model the relative composition of the lipids and cholesterol on the membrane plane. However, in addition to the familiar Cahn-Hilliard/Modica-Mortola energy (see [25]),

$$\mathcal{A}_\varepsilon[u] := \int_\Omega \left(\frac{1}{\varepsilon} W(u) + \varepsilon |\nabla u|^2 \right) dx,$$

that models line tension between domains and represents 'short-range' interactions and whose minimization drives the system to evolve into A rich and B rich phases (corresponding to $u = \alpha$ or $u = \beta$, minima of a double-well potential W), there are additional 'forces' that prevent large domains of homogeneous concentration. In [30] Seul and Andelman proposed a nonlocal contribution to the energy by considering an energy functional that takes into account the bending energy of the membrane, and couples it to the value of the order parameter. The idea is that surface tension associated with a slightly curved membrane influences the localization of phases as the geometry of the lipids has an effect on the preferred placement on the membrane. Similarly, the geometry of the membrane may adapt to that of the molecules. The resulting energy has the form

$$\mathcal{E}[\phi, h] = \int_D \left(f(\phi) + \frac{1}{2} b |\nabla \phi|^2 + \frac{1}{2} \sigma |\nabla h|^2 + \frac{1}{2} k (\Delta h)^2 + \Lambda \phi \Delta h \right) d\bar{x}. \tag{1.1}$$

Here $D := \{Lx : x \in \Omega\}$ is the domain with the characteristic size L, ϕ is the order parameter, h represents the height profile of the membrane, $f(\phi) := \frac{a_2}{2}\phi^2 + \frac{a_4}{4}\phi^4$, where a_2, a_4 are constants, $b > 0$ is related to the line tension between different domains, $\sigma > 0$ and $\kappa > 0$ are the surface tension and bending rigidity of the membrane, respectively, and Λ is the composition-curvature coupling constant, [20]. Since minimizers of $\mathcal{E}$ satisfy the Euler-Lagrange equations, we may consider the minimization problem for $\mathcal{E}[\phi, h]$ under the constraint, $\frac{\delta \mathcal{E}}{\delta h} = 0$. Using the last equation to eliminate h (see the

Appendix) and rescaling

$$u(x) = \phi(Lx), \quad \varepsilon := \sqrt{\frac{k}{L^2\sigma}}, \quad q := 1 - \frac{b\sigma}{\Lambda^2}, \quad W(u) := \frac{2k}{\Lambda^2} f(u), \quad \text{and } \mathcal{F}_\varepsilon^* := \frac{1}{\varepsilon}\frac{2k}{\Lambda^2 L^d}\mathcal{E},$$

one can reduce (1.1) to

$$\mathcal{F}_\varepsilon^*[u] := \frac{1}{\varepsilon}\int_\Omega \left(W(u) - u^2 + (1-q)\varepsilon^2|\nabla u|^2 + u\left(\mathbf{1} - \varepsilon^2\Delta\right)^{-1} u\right) dx. \tag{1.2}$$

Here q is a constant parameter and the second order differential operator $\mathbf{1} - \varepsilon^2\Delta : H^2(\Omega) \to L^2(\Omega)$ is subject to Neumann boundary conditions. A detailed derivation is given in the Appendix. In particular, Table 1 in the Appendix lists typical values for the parameters. One may easily check from the table that the relevant values of the parameter q fall in the interval $(-1, 1)$, and for fixed b and Λ correspond to varying the surface tension.

We approach the question of stability of rafts from the viewpoint of the Calculus of Variations. The main result of the paper is the Γ-convergence of the functional $\mathcal{F}_\varepsilon^*$ to the interfacial perimeter functional for sufficiently small $q > 0$. Since the limiting energy is minimized by a phase function with minimal interfaces, the physical interpretation is that when $L^2 \gg k/\sigma$, ($\varepsilon \ll 1$) raft microdomains are not formed. Finally, we remark that when $q \leq 0$ the functional is nonnegative (this can be seen from the reformulation of the problem presented in (2.1)). This is not the case when $q > 0$, which renders the analysis more complicated.

2. Preliminaries, Notation, and Statement of Results

A natural mathematical framework for studying the asymptotic behavior of the family of functionals (1.2) is the notion of Γ-convergence introduced by De Giorgi in [14] (see also [4, 10]). In a general metric space setting the definition is given below.

Definition 2.1. *Let (Y, d) be a metric space and consider a sequence $\{\mathcal{F}_n\}$ of functionals $\mathcal{F}_n \colon Y \to [-\infty, \infty]$. We say that $\{\mathcal{F}_n\}$ Γ-converges to a functional $\mathcal{F} : Y \to [-\infty, \infty]$ if the following properties hold:*

1. *(Liminf Inequality) For every $y \in Y$ and every sequence $\{y_n\} \subset Y$ such that $y_n \to y$,*
$$\mathcal{F}[y] \leq \liminf_{n\to\infty} \mathcal{F}_n[y_n].$$
2. *(Limsup Inequality) For every $y \in Y$ there exists $\{y_n\} \subset Y$ such that $y_n \to y$ and*
$$\limsup_{n\to\infty} \mathcal{F}_n[y_n] \leq \mathcal{F}[y].$$

The functional $\mathcal{F}$ is called the Γ-limit of the sequence $\{\mathcal{F}_n\}$.

A key property of Γ-convergence is the fact that the sequence of minimizers of the functionals $\mathcal{F}_n$ converge to a minimizer of the limiting functional $\mathcal{F}$. Moreover, one can show that the isolated local minima of the Γ-limit $\mathcal{F}$ persist under small perturbations (see [19, 10]).

The problem of finding a characterization of the Γ-limit of (1.2) has been considered in the one-dimensional setting by Ren and Wei in [28], but in a different parameter regime. Due to the different scaling of the terms, the technique used in that paper is not applicable to our case. Recall that the last term in (1.2) renders the problem nonlocal. A local approximation of (1.2) was studied in [7] and [8]. We refer to the derivation of (5.20) in the Appendix for the precise connection between the models. Qualitative properties of local minimizers of the local approximation model have been studied extensively to explain the formation of periodic layered structures (see [3, 9, 24, 26]).

We now give the precise formulation of our results. Let $\Omega \subset \mathbb{R}^d$ be an open, bounded set of class C^2, and let W be a twice continuously differentiable double-well potential defined on the real line. We make the following hypotheses on W.

Hypotheses 2.2.

1. *$W(s) > 0$ if $s \neq \pm 1$.*
2. *$W(\pm 1) = 0$.*
3. *There exists $c_w > 0$ such that $W(s) \geq c_w(s \mp 1)^2$ for $\pm s \geq 0$.*
4. *There exist constants K_w, $C_w > 0$ such that $|W'(s)| \leq C_w\sqrt{W(s)}$ and $|W''(s)| \leq K_w$ for all $s \in \mathbb{R}$.*

Remark 2.3. *Note that conditions 3 and 4 imply that W has quadratic growth at infinity.*

For the purposes of our analysis it will be convenient to rewrite the functional $\mathcal{F}_\varepsilon^*$ as follows. Given $u \in W^{1,2}(\Omega)$, we define $v \in W^{3,2}(\Omega)$ via

$$-\varepsilon^2\Delta v + v = u \text{ in } \Omega \qquad \text{and} \qquad \frac{\partial v}{\partial n} = 0 \text{ on } \partial\Omega,$$

where n denotes the outward unit normal to $\partial\Omega$, and use the abbreviatory notation $v := (\mathbf{1}-\varepsilon^2\Delta)^{-1}u$. Integrating by parts we obtain (see the Appendix)

$$\begin{aligned}\mathcal{F}_\varepsilon^*[u] &= \int_\Omega \left(\frac{1}{\varepsilon}W(u) - \varepsilon q|\nabla u|^2 + \varepsilon^3(\Delta v)^2 + \varepsilon^5|\nabla\Delta v|^2\right) dx \\ &= \int_\Omega \left(\frac{1}{\varepsilon}W(u) - \varepsilon q|\nabla v|^2 + (1-2q)\varepsilon^3(\Delta v)^2 + (1-q)\varepsilon^5|\nabla\Delta v|^2\right) dx.\end{aligned}$$

Hence, we may also view $\mathcal{F}_\varepsilon^*$ as $\mathcal{F}_\varepsilon[v]$ with $\mathcal{F}_\varepsilon : L^2(\Omega) \to (-\infty, \infty]$ given by

$$\mathcal{F}_\varepsilon[v] := \begin{cases} \mathcal{F}_\varepsilon[v;\Omega] & \text{if } v \in W^{3,2}(\Omega),\ \frac{\partial v}{\partial n} = 0 \text{ on } \partial\Omega, \\ +\infty & \text{otherwise,}\end{cases} \tag{2.1}$$

where

$$\mathcal{F}_\varepsilon[v;A] = \int_A \left(\frac{1}{\varepsilon}W(-\varepsilon^2\Delta v + v) - \varepsilon q|\nabla v|^2 + (1-2q)\varepsilon^3(\Delta v)^2 + (1-q)\varepsilon^5|\nabla\Delta v|^2\right) dx$$

for every open set $A \subset \Omega$.

Remark 2.4. *Observe that if $v \in W^{3,2}(\Omega)$ does not satisfy Neumann boundary conditions on $\partial\Omega$, then $\mathcal{F}_\varepsilon[v;\Omega] < \mathcal{F}_\varepsilon[v] = \infty$.*

Definition 2.5. *Given a vector $\nu \in \mathbb{S}^{d-1}$ ($d-1$ dimensional unit sphere), let $\{\nu_1, \cdots, \nu_{d-1}, \nu\}$ be an orthonormal basis of $\mathbb{R}^d$. We will denote by Q_ν an open unit cube centered at the origin with two of its faces normal to ν, i.e.,*

$$Q_\nu := \left\{x \in \mathbb{R}^d : |x\cdot\nu| < \frac{1}{2},\ |x\cdot\nu_i| < \frac{1}{2},\ i = 1,\ldots,d-1\right\}.$$

If $x_0 \in \mathbb{R}^d$ and $r > 0$, then $Q_\nu(x_0, r) := x_0 + rQ_\nu$. If $\{\nu_1, \cdots, \nu_{d-1}, \nu\}$ is the canonical basis, we drop the dependence on ν, i.e., $Q(x_0, r) := x_0 + r(-1/2, 1/2)^d = x_0 + rQ$, where Q is the open unit cube centered at the origin with faces normal to the coordinate axes.

Define the admissible set to be

$$\begin{aligned}\mathcal{A}_\nu := \{&v \in W^{3,2}_{loc}(\mathbb{R}^d) : v = -1 \text{ in a neighborhood of } x\cdot\nu = -1/2, \\ &v = 1 \text{ in a neighborhood of } x\cdot\nu = 1/2,\ v(x) = v(x+\nu_i) \text{ for all } x \in \mathbb{R}^d,\ i = 1,\ldots,d-1\},\end{aligned}$$

and set

$$m_d := \inf\{\mathcal{F}_\varepsilon[v;Q_\nu] : 0 < \varepsilon \leq 1, v \in \mathcal{A}_\nu\}. \tag{2.2}$$

Remark 2.6. *Since the gradient and Laplacian are invariant with respect to rotations, we can choose the coordinate system in such a way that the standard vector e_d is parallel to ν. It follows that m_d does not depend on ν, and we abbreviate $\mathcal{A} := \mathcal{A}_{e_d}$.*

Remark 2.7. *We will show in Proposition 3.4 that $m_d > 0$ if q is sufficiently small.*

We introduce the functional $\mathcal{F} : L^2(\Omega) \to [0, +\infty]$,

$$\mathcal{F}[v] := \begin{cases} m_d \mathrm{Per}(\{v = 1\}) & \text{if } v \in BV(\Omega; \{-1, 1\}), \\ +\infty & \text{if } v \in L^2(\Omega) \backslash BV(\Omega; \{-1, 1\}). \end{cases} \tag{2.3}$$

Here $BV(\Omega; \{-1, 1\})$ denotes the space of functions of bounded variation taking values in the set $\{-1, 1\}$, (see the discussion at the end of the section). The following theorem establishes the Γ-convergence of $\mathcal{F}_\varepsilon$ to $\mathcal{F}$.

Theorem 2.8. *Assume that $W \in C^2(\mathbb{R})$ satisfies Hypotheses 2.2. There exists $q_0 > 0$, depending only on the potential W and Ω, such that for all $0 < q < q_0$ the following inequalities hold:*

1. *Liminf Inequality: For every sequence of positive real numbers $\varepsilon_n \to 0$, for every $v \in L^2(\Omega)$, and for every $\{v_n\} \subset W^{3,2}(\Omega)$ such that $v_n \to v$ in $L^2(\Omega)$,*
$$\liminf_{n\to\infty} \mathcal{F}_{\varepsilon_n}[v_n; \Omega] \geq \mathcal{F}[v]. \tag{2.4}$$
2. *Limsup Inequality: For every $v \in L^2(\Omega)$ and for every sequence of positive real numbers $\varepsilon_n \to 0$, there exists a sequence $\{v_n\} \subset W^{3,2}(\Omega)$ such that $v_n \to v$ in $L^2(\Omega)$ and*
$$\limsup_{n\to\infty} \mathcal{F}_{\varepsilon_n}[v_n; \Omega] \leq \mathcal{F}[v]. \tag{2.5}$$

Remark 2.9. *We remark that Theorem 2.8 and the compactness property stated in Theorem 3.5 have analogous formulations for the functional $\mathcal{F}^*_\varepsilon$ in (1.2). Since for $v_n := (\mathbf{1} - \varepsilon^2\Delta)^{-1} u_n$, $\mathcal{F}_{\varepsilon_n}[v_n] = \mathcal{F}^*_{\varepsilon_n}[u_n]$, using Theorems 2.8 and 3.5 it is straightforward to show that the family of functionals $\mathcal{F}^*_\varepsilon$ Γ-converges to $\mathcal{F}$.*

We now give a proof of an elliptic regularity result used in the sequel.

Proposition 2.10. *If Ω has a piecewise C^2 boundary, then there exists a constant $C(\Omega)$, depending on Ω, such that*

$$||\nabla^2 v||^2_{L^2(\Omega)} \leq 3||\Delta v||^2_{L^2(\Omega)} + C(\Omega)||v||^2_{L^2(\Omega)} \tag{2.6}$$

for all $v \in W^{2,2}(\Omega)$ such that $\frac{\partial v}{\partial n} = 0$ on $\partial\Omega$.

Proof. Theorem 3.1.1.2 from [15] yields

$$\int_\Omega |\nabla^2 v|^2 dx \leq \int_\Omega |\Delta v|^2 dx + C_1(\Omega) \int_{\partial\Omega} |\nabla v|^2 dx \tag{2.7}$$

for all $v \in W^{2,2}(\Omega)$ with $\frac{\partial v}{\partial n} = 0$ on $\partial\Omega$, where the constant $C_1(\Omega)$ depends only on the curvature of $\partial\Omega$. In turn, applying Theorem 1.5.1.10 from [15] to each component of ∇v we obtain

$$C_1(\Omega) \int_{\partial\Omega} |\nabla v|^2 dx \leq \frac{1}{2} \int_\Omega |\nabla^2 v|^2 dx + C_2(\Omega) \int_\Omega |\nabla v|^2 dx$$

for some $C_2 > 0$ and for all $v \in W^{2,2}(\Omega)$. This, together with (2.7), reduces to

$$\int_\Omega |\nabla^2 v|^2 dx \leq 2 \int_\Omega (\Delta v)^2 dx + 2C_2(\Omega) \int_\Omega |\nabla v|^2 dx. \tag{2.8}$$

Finally, using the Neumann boundary condition and integration by parts we conclude that

$$2C_2(\Omega) \int_\Omega |\nabla v|^2 dx = 2C_2(\Omega) \int_\Omega (-\Delta v) v dx \leq \int_\Omega (\Delta v)^2 dx + C(\Omega) \int_\Omega v^2 dx, \tag{2.9}$$

where in the last step we also used Young's Inequality. Inequalities (2.8) and (2.9) now imply (2.6). □

For the reader's convenience we end the section with a summary of standard measure-theoretic results used in the remainder. A key concept used in the development of the Liminf Inequality in Section 4 is that of a reduced boundary of the set $E := \{x \in \Omega : v(x) = 1\}$ associated to $v \in BV(\Omega; \{-1, 1\})$. We recall that $v \in L^1(\Omega)$ is said to be of bounded variation, $v \in BV(\Omega)$, if the generalized partial derivatives D_i of v in the sense of distributions are bounded Radon measures. In

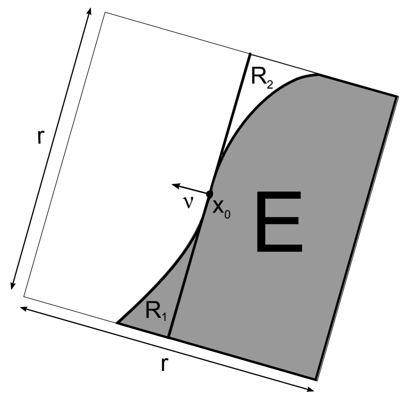

FIGURE 2. The sets E (in grey), $R_1 := \{x \in Q_\nu(x_0,r) \cap E : (x-x_0)\cdot\nu(x_0) > 0\}$, and $R_2 := \{x \in Q_\nu(x_0,r) \backslash E : (x-x_0)\cdot\nu(x_0) < 0\}$.

particular $BV(\Omega;\{-1,1\})$ denotes functions of bounded variation taking values in the set $\{-1,1\}$, and $\mathrm{Per}_\Omega(E) := |D\chi_E|(\Omega) < \infty$.

For sets of finite perimeter the reduced boundary $\partial^* E$ of E is defined as the set of points $x_0 \in \mathrm{spt}|D\chi_E| \cap \Omega$ such that the limit

$$\nu(x_0) := -\lim_{r\to 0^+} \frac{D\chi_E(B_r(x_0))}{|D\chi_E|(B_r(x_0))}$$

exists and satisfies $|\nu(x_0)| = 1$. Here $B_r(x_0)$ is the open ball of radius r centered at x_0. For $x_0 \in \partial^* E$ the vector $\nu(x_0)$ is called the generalized outer unit normal to E. In particular, by Theorem 3.59 from [1], $|D\chi_E| = \mathcal{H}^{d-1} \llcorner \partial^* E$, and for $x_0 \in \partial^* E$,

$$\lim_{r\to 0^+} \frac{\mathcal{H}^{d-1}(Q_\nu(x_0,r) \cap \partial^* E)}{r^{d-1}} = 1,$$

$$\lim_{r\to 0} \frac{1}{r^d} |\{x \in Q_\nu(x_0,r) \cap E : (x-x_0)\cdot\nu(x_0) > 0\}| = 0, \tag{2.10}$$

$$\lim_{r\to 0} \frac{1}{r^d} |\{x \in Q_\nu(x_0,r) \backslash E : (x-x_0)\cdot\nu(x_0) < 0\}| = 0, \tag{2.11}$$

where $|\cdot|$ denotes the Lebesgue measure in $\mathbb{R}^d$.

3. Compactness

In this section we prove the compactness Theorem 3.5. We use the following interpolation inequality.

Proposition 3.1. *Let $A \subset \mathbb{R}^d$ be a bounded open set in $\mathbb{R}^d$. Assume, in addition, that either A has a C^1 boundary or that A can be written as the union of finitely many pairwise disjoint open rectangles and a set of Lebesgue measure zero. Then there exist a constant $q_* \in (0,1)$, independent of A, and $\varepsilon_0 = \varepsilon_0(A, q_*) > 0$ such that*

$$q_* \int_A \varepsilon |\nabla v|^2 dx \le \int_A \left(\frac{W(v)}{\varepsilon} + \varepsilon^3 |\nabla^2 v|^2 \right) dx \tag{3.1}$$

for every $\varepsilon \in (0, \varepsilon_0)$ and $v \in W^{2,2}(A)$.

Proof. See Theorem 1.2 in [7]. □

For every open set $A \subset \Omega$, $v \in W^{3,2}(\Omega)$, and $\varepsilon > 0$, define the functional

$$\mathcal{I}_\varepsilon[v; A] := \int_A \left(\frac{1}{\varepsilon} W(v) + \varepsilon |\nabla v|^2 + \varepsilon^3 |\nabla^2 v|^2 + \varepsilon^5 |\nabla \Delta v|^2 \right) dx.$$

Next, we prove a result that will be useful to bound the energy from below and to obtain compactness of energy bounded sequences (see Theorem 3.5).

Proposition 3.2. *Let $K_w, C_w, c_w, q_*, \varepsilon_0 > 0$ be the constants given in Hypotheses 2.2 and Proposition 3.1. Then there exist $\bar{q} > 0$, depending only on K_w, C_w, q_* (see (3.6)), and $\varepsilon_1 > 0$, depending only on C_w, such that for every $0 < q \leq \bar{q}$, $v \in W^{3,2}(\Omega)$, and $0 < \varepsilon < \varepsilon_1$,*

$$\mathcal{F}_\varepsilon[v] \geq q\, \mathcal{I}_\varepsilon[v; \Omega] - \frac{12q}{q_*} C(\Omega) \varepsilon^3 |\Omega| \tag{3.2}$$

for some constant $C(\Omega) > 0$.

Remark 3.3. *We note that in the energy $\mathcal{F}_\varepsilon[v]$ the potential W acts on u, which is related to v through the condition $u = -\varepsilon^2 \Delta v + v$, while in $\mathcal{I}_\varepsilon[v]$ the potential acts on v. Hence $\mathcal{F}_\varepsilon$ differs from the standard Cahn-Hilliard energies involving solely the potential $W(v)$. In addition, the second order term in $\mathcal{F}_\varepsilon[v]$ involves the Laplacian Δv, while the second order term in $\mathcal{I}_\varepsilon[v]$ involves the Hessian $\nabla^2 v$.*

Proof. If v does not satisfy $\frac{\partial v}{\partial n} = 0$ on $\partial\Omega$ then $\mathcal{F}_\varepsilon[v] = \infty$ and there is nothing to prove. Otherwise, fix $0 < \theta \leq 1$. Using Taylor's formula for W and the fact that W'' is bounded by Hypotheses 2.2, yields

$$\begin{aligned}\mathcal{F}_\varepsilon[v] = \mathcal{F}_\varepsilon[v; \Omega] &= \int_\Omega \left(\frac{1}{\varepsilon} W(-\varepsilon^2 \Delta v + v) - \varepsilon q |\nabla v|^2 + (1 - 2q)\varepsilon^3 (\Delta v)^2 + (1 - q)\varepsilon^5 |\nabla \Delta v|^2 \right) dx \\ &\geq \int_\Omega \left(\frac{\theta}{\varepsilon} W(v) - \theta W'(v) \varepsilon \Delta v - \varepsilon q |\nabla v|^2 + \left(1 - 2q - \frac{\theta}{2} K_w\right) \varepsilon^3 (\Delta v)^2 + (1 - q)\varepsilon^5 |\nabla \Delta v|^2 \right) dx.\end{aligned} \tag{3.3}$$

By Young's Inequality and the condition $|W'(s)| \leq C_w \sqrt{W(s)}$ from Hypotheses 2.2, we have

$$W'(v)\Delta v \leq \frac{1}{2\varepsilon^2 C_w^2} (W'(v))^2 + \frac{\varepsilon^2}{2} C_w^2 (\Delta v)^2 \leq \frac{1}{2\varepsilon^2} W(v) + \frac{\varepsilon^2}{2} C_w^2 (\Delta v)^2. \tag{3.4}$$

Substituting (3.4) into (3.3) implies

$$\mathcal{F}_\varepsilon[v] \geq \int_\Omega \left(\frac{\theta}{2\varepsilon} W(v) - \varepsilon q |\nabla v|^2 + \left(1 - 2q - \frac{\theta}{2} K_w - \frac{\theta}{2} C_w^2\right) \varepsilon^3 (\Delta v)^2 + (1 - q)\varepsilon^5 |\nabla \Delta v|^2 \right) dx.$$

Multiplying (3.1), with $A = \Omega$, by $2q/q_*$ and using it in the previous inequality gives

$$\begin{aligned}\mathcal{F}_\varepsilon[v] \geq \int_\Omega \Bigg(&\left(\frac{\theta}{2} - \frac{2q}{q_*}\right) \frac{1}{\varepsilon} W(v) + \varepsilon q |\nabla v|^2 + \left(1 - 2q - \frac{\theta}{2} K_w - \frac{\theta}{2} C_w^2\right) \varepsilon^3 (\Delta v)^2 \\ &- \frac{2q\varepsilon^3}{q_*} |\nabla^2 v|^2 + (1 - q)\varepsilon^5 |\nabla \Delta v|^2 \Bigg) dx.\end{aligned} \tag{3.5}$$

Fix $\delta > 0$. Using Proposition 2.10 we get

$$\begin{aligned}\mathcal{F}_\varepsilon[v] \geq \int_\Omega \Bigg(&\left(\frac{\theta}{2} - \frac{2q}{q_*}\right) \frac{1}{\varepsilon} W(v) - \left(\delta + \frac{2q}{q_*}\right) \varepsilon^3 C(\Omega) v^2 + \varepsilon q |\nabla v|^2 \\ &+ \left(1 - 2q - \frac{\theta}{2} K_w - \frac{\theta}{2} C_w^2 - \frac{6q}{q_*} - 3\delta\right) \varepsilon^3 (\Delta v)^2 + \delta \varepsilon^3 |\nabla^2 v|^2 + (1 - q)\varepsilon^5 |\nabla \Delta v|^2 \Bigg).\end{aligned}$$

Finally, it follows from Hypotheses 2.2 that $W(s) \geq (c_w/4)s^2$ for $|s| \geq 2$. Hence

$$\mathcal{F}_\varepsilon[v] \geq \int_\Omega \Bigg(\left[\frac{\theta}{2} - \frac{2q}{q_*} - \varepsilon^4 \frac{4C(\Omega)}{c_w} \left(\delta + \frac{2q}{q_*} \right) \right] \frac{1}{\varepsilon} W(v) + \left(1 - 2q - \frac{\theta}{2} K_w - \frac{\theta}{2} C_w^2 - \frac{6q}{q_*} - 3\delta \right) \varepsilon^3 (\Delta v)^2$$

$$+ \varepsilon q |\nabla v|^2 + \delta \varepsilon^3 |\nabla^2 v|^2 + (1-q)\varepsilon^5 |\nabla \Delta v|^2 \Bigg) dx - 4 \left(\delta + \frac{2q}{q_*} \right) \varepsilon^3 C(\Omega) |\Omega|.$$

Choosing $\delta := \frac{q}{q_*}$, $\theta := \frac{8q}{q_*}$, $\varepsilon_1 := \min \left\{ \varepsilon_0, \left(\frac{c_w}{12C(\Omega)} \right)^{1/4} \right\}$ and

$$\bar{q} := \frac{q_*}{2q_* + 4K_w + 4C_w^2 + 10} \tag{3.6}$$

yields (3.2). □

We now prove that for q sufficiently small the "cell" energy is positive.

Proposition 3.4. *Let m_d be defined in* (2.2) *and let $\bar{q}$ be as in Proposition* 3.2. *Then $m_d > 0$ for every $0 < q < \bar{q}$.*

Proof. Without loss of generality we may assume that the infimum in the definition of m_d is taken over $0 < \varepsilon < \varepsilon_0$. The result of the proposition then follows if we show that

$$\inf \left\{ \int_Q \left(\frac{W(v)}{\varepsilon} + \varepsilon |\nabla v|^2 \right) dx : 0 < \varepsilon < \varepsilon_0, v \in \mathcal{A} \right\} > 0. \tag{3.7}$$

Indeed, let $v \in \mathcal{A}$. Since v satisfies periodic boundary conditions on Q, integration by parts yields

$$||\nabla^2 v||^2_{L^2(Q)} = ||\Delta v||^2_{L^2(Q)}. \tag{3.8}$$

Repeating the proof of Proposition 3.2 with Q instead of Ω and using (3.8) in (3.5), we obtain

$$\mathcal{F}_\varepsilon[v; Q] \geq q \mathcal{I}_\varepsilon[v; Q] \geq q \int_Q \left(\frac{W(v)}{\varepsilon} + \varepsilon |\nabla v|^2 \right) dx$$

if $q \leq \bar{q}$. To prove (3.7) we follow [13]. In particular, for $v \in \mathcal{A}$,

$$\int_Q \left(\frac{W(v)}{\varepsilon} + \varepsilon |\nabla v|^2 \right) dx \geq 2 \int_Q \sqrt{W(v)} |\nabla v| dx \geq \int_{Q'} \int_{-1/2}^{1/2} \sqrt{W(v)} \left| \frac{\partial v}{\partial x_d} \right| dx_d dx', \tag{3.9}$$

where $Q' := (-1/2, 1/2)^{d-1}$. Since $v(x', \pm 1/2) = \pm 1$ a change of variables yields

$$\int_{Q'} \int_{-1/2}^{1/2} \sqrt{W(v)} \left| \frac{\partial v}{\partial x_d} \right| dx_d \, dx' \geq \int_{-1}^{1} \sqrt{W(s)} \, ds.$$

Using this lower bound in (3.9) and taking the infimum over $v \in \mathcal{A}$ and $0 < \varepsilon < \varepsilon_0$ gives (3.7). □

Theorem 3.5. *(Compactness) Let $\bar{q}$ be as in Proposition* 3.2. *If $q \leq \bar{q}$, $\varepsilon_n \to 0^+$ and $\{v_n\} \subset W^{3,2}(\Omega)$ satisfies*

$$\sup_n \mathcal{F}_{\varepsilon_n}[v_n; \Omega] < \infty, \tag{3.10}$$

then there exist a subsequence $\{v_{n_k}\}$ of $\{v_n\}$ and $v \in BV(\Omega; \{-1, 1\})$ such that

$$v_{n_k} \to v \quad \text{and} \quad \varepsilon_{n_k}^2 \Delta v_{n_k} \to 0 \quad \text{in } L^2(\Omega). \tag{3.11}$$

Proof. By Proposition 3.2 and (3.10)

$$\sup_n \mathcal{I}_{\varepsilon_n}[v_n; \Omega] < \infty \tag{3.12}$$

and the existence of $v \in BV(\Omega; \{-1, 1\})$ and a subsequence $\{v_{n_k}\}$ converging to v in $L^1(\Omega)$ now follows from standard results for the Modica-Mortola functional (see [25]).

To show the convergence in $L^2(\Omega)$, we recall again that by Hypotheses 2.2, $W(s) \geq (c_w/4)|s|^2$ for $|s| \geq 2$, and hence for every measurable set $E \subset \Omega$,

$$\begin{aligned}\int_E |v_n|^2\, dx &= \int_{\{y\in E:|v_n(y)|<2\}} |v_n|^2 dx + \int_{\{y\in E:|v_n(y)|\geq 2\}} |v_n|^2 dx\\ &\leq 4|E| + \frac{4}{c_w}\int_E W(v_n) \leq 4|E| + C(q)\varepsilon_n,\end{aligned}$$

where in the last step we used (3.12). Therefore $\{|v_{n_k}|^2\}$ is equi-integrable, and convergence of $\{v_{n_k}\}$ to v in $L^2(\Omega)$ is a consequence of Vitali's Convergence Theorem.

To prove $(3.11)_2$, note that (3.12) implies $\varepsilon_n^2\|\Delta v_n\|_{L^2(\Omega)} \leq C(q)\varepsilon_n^{1/2}$. It follows that $\varepsilon_n^2 \Delta v_n \to 0$ in $L^2(\Omega)$. □

The slicing argument in the following proposition uses the notation introduced in Definition 2.5.

Proposition 3.6. *Let $K > 0$, let $k \in \mathbb{N}$, and let $v \in W^{3,2}(Q(x_0, r_0))$ be such that*

$$\mathcal{I}_\varepsilon[v; Q(x_0, r_0)] \leq K \tag{3.13}$$

for some $0 < \varepsilon < \varepsilon_1 := \frac{r_0}{4k\sqrt{C(d)}}$. Then there exist a constant $C(d) > 0$ and $i \in \{1, \ldots, k\}$ (depending on v) such that

$$\mathcal{F}_\varepsilon[v; Q(x_0, r)] \geq q\,\mathcal{I}_\varepsilon[v; Q(x_0, r)] - \frac{q}{q_*}\frac{6K}{k}$$

and

$$\mathcal{I}_\varepsilon[v; L] \leq \frac{K}{k},$$

for all $r \in \left(\frac{r_0}{2}\left(1 + \frac{2i-1}{2k}\right), \frac{r_0}{2}\left(1 + \frac{i}{k}\right)\right)$ and all $0 < q < \bar{q} := \frac{q_*}{2q_* + 4K_w + 4C_w^2 + 3C(d) + 1}$, where

$$L := Q\left(\frac{r_0}{2}\left(1 + \frac{i}{k}\right)\right) \backslash Q\left(\frac{r_0}{2}\left(1 + \frac{i-1}{k}\right)\right).$$

Proof. For simplicity we will use the notation $Q(r) := Q(x_0, r)$. The following estimate is obtained from the proof of Lemma 9.2.3 in [17]. Let $0 < r_1 < r_2 < r_0$. Then,

$$\int_{Q(r_1)} |\nabla^2 v|^2 \leq C(d)\left(\int_{Q(r_2)} |\Delta v|^2\, dx + \frac{1}{(r_2 - r_1)^2}\int_{Q(r_2)\backslash Q(r_1)} |\nabla v|^2\, dx\right). \tag{3.14}$$

Given $k > 0$, we first partition the set $Q(r_0)\backslash Q(r_0/2)$ into k layers

$$L^i := Q\left(\frac{r_0}{2}\left(1 + \frac{i}{k}\right)\right) \setminus Q\left(\frac{r_0}{2}\left(1 + \frac{i-1}{k}\right)\right), \quad i = 1, \ldots, k.$$

Since

$$\sum_{i=1}^k \mathcal{I}_\varepsilon[v; L^i] \leq \mathcal{I}_\varepsilon[v; Q(r_0)],$$

by (3.13) there exists a layer L^{i^*} satisfying

$$\mathcal{I}_\varepsilon[v; L^{i^*}] \leq \frac{1}{k}\mathcal{I}_\varepsilon[v; Q(r_0)] \leq \frac{K}{k}. \tag{3.15}$$

Fix $r \in \left(\frac{r_0}{2}\left(1 + \frac{2i^*-1}{2k}\right), \frac{r_0}{2}\left(1 + \frac{i^*}{k}\right)\right)$. Choosing $r_1 := \frac{r_0}{2}\left(1 + \frac{i^*-1}{k}\right)$, $r_2 := r$ and applying estimate (3.14) we obtain

$$\int_{Q(r_1)} |\nabla^2 v|^2\, dx \leq C(d)\, \left(\int_{Q(r)} |\Delta v|^2\, dx + \frac{16k^2}{r_0^2}\int_{L^{i^*}} |\nabla v|^2\, dx\right).$$

Adding $\int_{L^{i*}} |\nabla^2 v|^2\, dx$ to both sides and multiplying by ε^3 yields, by (3.15),

$$\varepsilon^3 \int_{Q(r)} |\nabla^2 v|^2\, dx \le C(d) \left(\varepsilon^3 \int_{Q(r)} |\Delta v|^2\, dx + \frac{16k^2}{r_0^2}\varepsilon^3 \int_{L^{i*}} |\nabla v|^2\, dx \right) + \varepsilon^3 \int_{L^{i*}} |\nabla^2 v|^2\, dx$$
$$\le C(d) \left(\varepsilon^3 \int_{Q(r)} |\Delta v|^2\, dx + \frac{16k^2}{r_0^2}\varepsilon^2 \frac{K}{k} \right) + \frac{K}{k}.$$

Let $0 < \varepsilon_1^2 \le \frac{r_0^2}{16k^2 C(d)}$. Then for $0 < \varepsilon < \varepsilon_1$ we have

$$\varepsilon^3 \int_{Q(r)} |\nabla^2 v|^2\, dx \le C(d)\, \varepsilon^3 \int_{Q(r)} |\Delta v|^2\, dx + \frac{2K}{k}. \tag{3.16}$$

Repeating the argument of the proof of Proposition 3.2 with $\theta := \frac{8q}{q_*}$ until (3.5) and using (3.16) multiplied by 3 in place of Proposition 2.10 yields

$$\mathcal{F}_\varepsilon[v_n; Q(r)] \ge \int_{Q(r)} \left(\frac{2q}{q_*}\frac{1}{\varepsilon} W(v) + \left(1 - 2q - \frac{4q}{q_*}K_w - \frac{4q}{q_*}C_w^2 - \frac{3q}{q_*}C(d)\right) \varepsilon^3 |\Delta v|^2 \right.$$
$$\left. + q\varepsilon |\nabla v|^2 + \frac{q}{q_*}\varepsilon^3 |\nabla^2 v|^2 + (1-q)\varepsilon^5 |\nabla \Delta v|^2 \right) dx - \frac{q}{q_*}\frac{6K}{k}$$
$$\ge q\, \mathcal{I}_\varepsilon[v; Q(r)] - \frac{q}{q_*}\frac{6K}{k},$$

provided

$$0 < q < \bar{q} := \frac{q_*}{2q_* + 4K_w + 4C_w^2 + 3C(d) + 1}.$$

This completes the proof. □

Proposition 3.7. *Let $k \in \mathbb{N}$, $\varepsilon_n \to 0^+$, $\nu \in \mathbb{S}^{d-1}$, and $\{w_n\} \subset W^{3,2}(Q_\nu(0,1))$ be such that*

$$\lim_{n\to\infty} \int_{Q_\nu(0,1)} |w_n - v_0|^2 dx = 0,$$

and

$$\mathcal{I}_{\varepsilon_n}[w_n; L] \le \frac{C_0}{k} \tag{3.17}$$

for all n and some $C_0 > 0$, where

$$v_0(y) := \begin{cases} -1 & \text{if } y\cdot\nu > 0, \\ 1 & \text{if } y\cdot\nu < 0, \end{cases}$$

and

$$L := Q_\nu(0,1) \backslash Q_\nu(0, 1 - 1/(2k)).$$

Then

$$\mathcal{F}_{\varepsilon_n}[w_n; Q_\nu(0,1)] \ge m_d - \frac{C}{k}.$$

Proof. We modify $\{w_n\}$ to belong to the admissible class $\mathcal{A}_\nu$ without increasing the energy. Given $\Psi \in C_c^\infty(\mathbb{R}^d)$, with $\mathrm{supp}(\Psi) \subset B_1(0)$ and $\int_{\mathbb{R}^d} \Psi(y) dy = 1$, consider the mollifier

$$\Psi_\varepsilon(y) := \frac{1}{\varepsilon^d} \Psi\left(\frac{y}{\varepsilon}\right) \tag{3.18}$$

and

$$\varphi_n := v_0 * \Psi_{\varepsilon_n}.$$

Note that $\varphi_n \in C^\infty(\mathbb{R}^d)$ and

$$||\varphi_n||_{L^\infty(\mathbb{R})} \le 1, \quad ||\nabla\varphi_n||_{L^\infty(\mathbb{R})} \le C\varepsilon_n^{-1}, \quad ||\nabla^2\varphi_n||_{L^\infty(\mathbb{R})} \le C\varepsilon_n^{-2}, \quad ||\nabla^3\varphi_n||_{L^\infty(\mathbb{R})} \le C\varepsilon_n^{-3}. \tag{3.19}$$

In addition,

$$\varphi_n(y) = \begin{cases} -1 & \text{if } y\cdot\nu < -\varepsilon_n, \\ 1 & \text{if } y\cdot\nu > \varepsilon_n, \end{cases}$$

and

$$\nabla^s \varphi_n(y) = 0 \text{ if } |y\cdot\nu| > \varepsilon_n, \quad s = 1,2,3.$$

Hence for ε_n sufficiently small $\varphi_n \in \mathcal{A}_\nu$. We want to define a function z_n to equal φ_n near the boundary of Q_ν and w_n away from the boundary. To be precise, we first partition the set $Q_\nu(0,1)\backslash Q_\nu(0,1-1/(2k))$ into $\lceil \varepsilon_n^{-1}\rceil$ layers of width $\frac{1}{4k\lceil\varepsilon_n^{-1}\rceil}$,

$$L_n^i := Q_\nu\left(0, 1-\frac{i-1}{2k\lceil\varepsilon_n^{-1}\rceil}\right)\Big\backslash Q_\nu\left(0, 1-\frac{i}{2k\lceil\varepsilon_n^{-1}\rceil}\right), \quad i = 1,\dots,\lceil\varepsilon_n^{-1}\rceil,$$

where $\lceil x\rceil$ is defined as the smallest integer not less than x. Since both $w_n \to v_0$ in $L^2(Q_\nu)$ and $\varphi_n \to v_0$ in $L^2(Q_\nu)$, we have

$$||w_n - \varphi_n||^2_{L^2(Q_\nu)} \to 0 \text{ as } n\to\infty.$$

Note that $\cup_i L_n^i = L \subset Q_\nu(0,1)$ and that L_n^i are pairwise disjoint, so the sum over all of the layers is bounded by

$$\sum_i \mathcal{I}_{\varepsilon_n}[w_n; L_n^i] + \frac{\sum_i ||w_n-\varphi_n||^2_{L^2(L_n^i)}}{||w_n-\varphi_n||^2_{L^2(Q_\nu)}} \le \frac{C_0}{k} + 1.$$

Since there are $\lceil\varepsilon_n^{-1}\rceil$ layers, for one of these layers, say $L_n := L_n^{i^*}$, it holds

$$\mathcal{I}_{\varepsilon_n}[w_n; L_n] + \frac{||w_n-\varphi_n||^2_{L^2(L_n)}}{||w_n-\varphi_n||^2_{L^2(Q_\nu)}} \le \left(\frac{C_0}{k}+1\right)\varepsilon_n. \tag{3.20}$$

Define

$$z_n := \eta_n w_n + (1-\eta_n)\varphi_n,$$

where η_n is a smooth function with support in $Q_\nu(0,1)$ such that

$$\eta_n(x) := \begin{cases} 0 & \text{if } x \in Q_n^{out} := Q_\nu(0,1)\backslash Q_\nu\left(0, 1-\frac{i^*-1}{2k\lceil\varepsilon_n^{-1}\rceil}\right), \\ \in (0,1) & \text{if } x\in L_n, \\ 1 & \text{if } x\in Q_n^{in} := Q_\nu\backslash(Q_n^{out}\cup L_n), \end{cases}$$

and

$$||\nabla^s\eta_n||_{L^\infty(Q_\nu)} = \mathcal{O}\left(\frac{k^s}{\varepsilon_n^s}\right), \quad s = 1,2,3. \tag{3.21}$$

Moreover,

$$\mathcal{F}_{\varepsilon_n}[z_n; Q_\nu] = \mathcal{F}_{\varepsilon_n}[\varphi_n; Q_n^{out}] + \mathcal{F}_{\varepsilon_n}[z_n; L_n] + \mathcal{F}_{\varepsilon_n}[w_n; Q_n^{in}].$$

We observe that since $\mathcal{F}_{\varepsilon_n}[w_n; Q_\nu\backslash Q_n^{in}]$ can be negative it is not necessarily true that $\mathcal{F}_{\varepsilon_n}[w_n; Q_n^{in}] \le \mathcal{F}_{\varepsilon_n}[w_n; Q_\nu]$. Instead, we use (3.17) to control the negative terms to obtain

$$\begin{aligned}\mathcal{F}_{\varepsilon_n}[z_n; Q_\nu] &\le \mathcal{F}_{\varepsilon_n}[\varphi_n; Q_n^{out}] + \mathcal{F}_{\varepsilon_n}[z_n; L_n] + \mathcal{F}_{\varepsilon_n}[w_n; Q_\nu] + q\int_L \varepsilon_n|\nabla w_n|^2 dx \\ &\le \mathcal{F}_{\varepsilon_n}[\varphi_n; Q_n^{out}] + \mathcal{F}_{\varepsilon_n}[z_n; L_n] + \mathcal{F}_{\varepsilon_n}[w_n; Q_\nu] + q\frac{C_0}{k}.\end{aligned} \tag{3.22}$$

Note that for $s = 1,2,3$,

$$\varepsilon_n^{2s-1}\int_{Q_n^{out}} |\nabla^s\varphi_n|^2 dx \le \varepsilon_n^{2s-1}\frac{C}{\varepsilon_n^{2s}}|\{x\in Q_n^{out} : \varphi_n \ne \pm 1\}| \le \frac{C}{k}. \tag{3.23}$$

In addition, by the continuity of W,

$$\frac{1}{\varepsilon_n}\int_{Q_n^{out}} W(-\varepsilon_n^2\Delta\varphi_n + \varphi_n)dx \le \frac{C}{\varepsilon_n}|\{x\in Q_n^{out} : \varphi_n\ne\pm1\}| \le \frac{C}{k}. \tag{3.24}$$

Together (3.23) and (3.24) imply

$$\mathcal{F}_{\varepsilon_n}[\varphi_n; Q_n^{out}] \leq \frac{C}{k}. \tag{3.25}$$

To estimate $\mathcal{F}_{\varepsilon_n}[z_n; L_n]$, we first note that

$$\partial_{x_i} z_n = \partial_{x_i}\eta_n(w_n - \varphi_n) + \eta_n \partial_{x_i} w_n + (1-\eta_n)\partial_{x_i}\varphi_n,$$

and

$$\begin{aligned}\partial_{x_i x_k} z_n = {} & \partial_{x_i x_k}\eta_n(w_n - \varphi_n) + \partial_{x_i}\eta_n \partial_{x_k} w_n + \partial_{x_k}\eta_n \partial_{x_i} w_n + \eta_n \partial_{x_i x_k} w_n \\ & - \partial_{x_i}\eta_n \partial_{x_k}\varphi_n - \partial_{x_k}\eta_n \partial_{x_i}\varphi_n + (1-\eta_n)\partial_{x_i x_k}\varphi_n.\end{aligned}$$

We use (3.20) to control the derivatives of w_n in the transition region L_n. From (3.19), (3.20), (3.21), the expressions for the derivatives of z_n and the fact that $||w_n - \varphi_n||_{L^2(Q)} \to 0$, we readily obtain the following bounds on the terms in $\mathcal{F}_{\varepsilon_n}[z_n; L_n]$,

$$\begin{aligned}\varepsilon_n \int_{L_n} |\nabla z_n|^2 dx &\leq C \int_{L_n} \left(\varepsilon_n |\nabla \eta_n|^2 |w_n - \varphi_n|^2 + \varepsilon_n \eta_n^2 |\nabla w_n|^2 + \varepsilon_n (1-\eta_n)^2 |\nabla \varphi_n|^2\right) dx \\ &\leq C \left(\frac{k^2 \varepsilon_n}{\varepsilon_n^2} ||w_n - \varphi_n||^2_{L^2(L_n)} + \left(\frac{C_0}{k} + 1\right)\varepsilon_n + \frac{\varepsilon_n}{\varepsilon_n^2} |\{x \in L_n : \varphi_n \neq \pm 1\}|\right) \\ &\leq C \left(k^2 \left(\frac{C_0}{k} + 1\right) ||w_n - \varphi_n||^2_{L^2(L)} + \left(\frac{C_0}{k} + 1\right)\varepsilon_n + \frac{\varepsilon_n}{k}\right) \leq \frac{C}{k}\end{aligned} \tag{3.26}$$

for n sufficiently large, where we used $|\{x \in L_n : |x \cdot \nu| < \varepsilon_n\}| = \mathcal{O}(\varepsilon_n^2/k)$. Similarly,

$$\begin{aligned}\varepsilon_n^3 \int_{L_n} |\nabla^2 z_n| dx \leq C\varepsilon_n^3 \int_{L_n} \Big(&|\nabla^2 \eta_n|^2 |w_n - \varphi_n|^2 + 2|\nabla \eta_n|^2 |\nabla w_n|^2 + 2|\nabla \eta_n|^2 |\nabla \varphi_n|^2 + \eta_n^2 |\nabla^2 w_n|^2 \\ &+ (1-\eta_n)^2 |\nabla^2 \varphi_n|^2\Big) dx \leq C \Bigg(\frac{\varepsilon_n^3 k^4}{\varepsilon_n^4}\left(\frac{C_0}{k} + 1\right)\varepsilon_n ||w_n - \varphi_n||^2_{L^2(L)} + \frac{\varepsilon_n^2 k^2}{\varepsilon_n^2}\left(\frac{C_0}{k} + 1\right)\varepsilon_n \\ &+ \varepsilon_n^3 \left(\frac{k^2}{\varepsilon_n^2}\frac{1}{\varepsilon_n^2} + \frac{1}{\varepsilon_n^4}\right) |\{x \in L_n : \varphi_n \neq \pm 1\}| + \left(\frac{C_0}{k} + 1\right)\varepsilon_n\Bigg) \leq \frac{C}{k}\end{aligned}$$

for n sufficiently large. To bound the integral involving the potential W we first remark that by Hypotheses 2.2 (and Remark 2.3) W grows quadratically at infinity. Splitting the integral into regions where $|-\varepsilon_n^2 \Delta z_n + z_n| \leq 2$ and $|-\varepsilon_n^2 \Delta z_n + z_n| > 2$, we use the quadratic growth of W to obtain,

$$\begin{aligned}&\left|\frac{1}{\varepsilon_n}\int_{L_n} W(-\varepsilon_n^2 \Delta z_n + z_n) dx\right| \leq \frac{\sup_{|s|\leq 2} W(s)}{\varepsilon_n}|L_n| + \frac{C_w^2}{4\varepsilon_n}\int_{L_n} (-\varepsilon_n^2 \Delta z_n + z_n)^2 dx \\ &\leq \frac{C}{k} + \frac{C_w^2}{2}\int_{L_n} \varepsilon_n^3 |\Delta z_n|^2 dx + \frac{C_w^2}{2\varepsilon_n}\int_{L_n} z_n^2 dx \leq \frac{C}{k} + \frac{C_w^2}{2}\int_{L_n} \varepsilon_n^3 |\Delta z_n|^2 dx + \frac{C_w^2}{\varepsilon_n}\int_{L_n} (w_n^2 + \varphi_n^2) dx \\ &\leq \frac{C}{k} + \frac{C_w^2}{2}\int_{L_n} \varepsilon_n^3 |\Delta z_n|^2 dx + \frac{C}{\varepsilon_n}\int_{L_n} W(w_n) dx + \frac{C}{\varepsilon_n}|L_n| \leq \left(\frac{C_0}{k} + 1\right)\varepsilon_n + \frac{C}{k} \leq \frac{C}{k}\end{aligned} \tag{3.27}$$

for n sufficiently large, where we again used (3.20). Analogous calculations are used to estimate $\varepsilon_n^5 \int_{L_n} |\nabla \Delta z_n|^2 dx$. Combining estimates (3.25), (3.26)-(3.27) with (3.22) completes the proof. □

4. Proof of the Liminf Inequality

In this section we prove the Liminf Inequality of Theorem 2.8. We use the blow-up method to reduce the problem to a unit cube (see [7]). In what follows we assume $q \leq \bar{q}$ (see Theorem 3.5 and Proposition 3.2). Fix $\varepsilon_n \to 0^+$ and $\{v_n\} \in W^{3,2}(\Omega)$, $v_n \to v \in L^2(\Omega)$. We may assume that

$$\liminf_{n\to\infty} \mathcal{F}_{\varepsilon_n}[v_n] < \infty, \tag{4.1}$$

and we extract a subsequence $\{v_{n_k}\}$ of $\{v_n\}$ satisfying

$$\lim_{k\to\infty} \mathcal{F}_{\varepsilon_{n_k}}[v_{n_k}] = \liminf_{n\to\infty} \mathcal{F}_{\varepsilon_n}[v_n] < \infty.$$

By selecting a further subsequence, if necessary, we can assume that $\sup_k \mathcal{F}_{\varepsilon_{n_k}}[v_{n_k}] < \infty$ so that by (3.12)

$$\sup_k \mathcal{I}_{\varepsilon_{n_k}}[v_{n_k}; \Omega] =: K < \infty. \tag{4.2}$$

Since $v_{n_k} \to v$ in $L^2(\Omega)$, Theorem 3.5 implies that $v \in BV(\Omega; \{-1, 1\})$. Therefore,

$$v = \chi_E - \chi_{\Omega\setminus E}, \tag{4.3}$$

where $\mathrm{Per}_\Omega(E) < \infty$. In what follows, to simplify notation we denote the subsequence of $\{v_n\}$ extracted in (4.2) by $\{v_n\}$.

We first note that, due to (4.1) and (4.2), the sequences of functions

$$f_n := \frac{1}{\varepsilon_n} W(-\varepsilon_n^2 \Delta v_n + v_n) - \varepsilon_n q|\nabla v_n|^2 + (1-2q)\varepsilon_n^3 |\Delta v_n|^2 + (1-q)\varepsilon_n^5 |\nabla \Delta v_n|^2,$$

$$g_n := \frac{1}{\varepsilon_n} W(v_n) + \varepsilon_n |\nabla v_n|^2 + \varepsilon_n^3 |\Delta v_n|^2 + \varepsilon_n^5 |\nabla \Delta v_n|^2$$

are bounded in $L^1(\Omega)$. Consider the signed Radon measures defined on Borel subsets of Ω,

$$\lambda_n(E) := \int_E f_n \, dx, \quad \zeta_n(E) := \int_E g_n \, dx.$$

Up to subsequences, not relabeled, we may assume that there exist Radon measures λ, μ, ζ such that

$$\lambda_n \rightharpoonup^* \lambda, \quad |\lambda_n| \rightharpoonup^* \mu, \quad \zeta_n \rightharpoonup^* \zeta$$

in the space $\mathcal{M}_b(\Omega)$ of all bounded signed Radon measures on Ω (see Proposition 1.202 in [12]). We claim that $\lambda \geq 0$.

Indeed, for $r > 0$ sufficiently small $|\lambda|(Q(x_0, r)) > 0$, and by the Besicovitch Derivation Theorem (Theorem 1.155 in [12]), for $|\lambda| - a.e.\ x_0 \in \Omega$

$$\frac{d\lambda}{d|\lambda|}(x_0) = \lim_{r\to 0^+} \frac{\lambda(Q(x_0, r))}{|\lambda|(Q(x_0, r))} \in \mathbb{R}, \tag{4.4}$$

where $|\lambda|$ is the total variation of λ. Fix any x_0 for which (4.4) holds. Let $\eta \in (0, 1)$ and find $\bar{r}_\eta > 0$ such that

$$\frac{d\lambda}{d|\lambda|}(x_0) \geq \frac{\lambda(Q(x_0, r))}{|\lambda|(Q(x_0, r))} - \eta \tag{4.5}$$

for all $0 < r < \bar{r}_\eta$.

Fix $0 < r_0 < \bar{r}_\eta$ and $k \in \mathbb{N}$. By Proposition 3.6 for every n there exists $i_n \in \{1, \dots, k\}$ such that

$$\mathcal{F}_{\varepsilon_n}[v; Q(x_0, r)] \geq q\mathcal{I}_{\varepsilon_n}[v; Q(x_0, r)] - \frac{q}{q^*}\frac{6K}{k} \tag{4.6}$$

for all $r \in \left(\frac{r_0}{2}\left(1 + \frac{2i_n - 1}{2k}\right), \frac{r_0}{2}\left(1 + \frac{i_n}{k}\right)\right)$ where K is given in (4.2). Since $i_n \in \{1, \dots, k\}$ for all n, there exists $i^{(1)} \in \{1, \dots, k\}$ such that $i^{(1)} = i_n$ for countably many n, say n_l, $l \in \mathbb{N}$. Let k be so large that

$$\frac{q}{q_*}\frac{6K}{k} \leq |\lambda|(Q(x_0, r_0/2))\eta \tag{4.7}$$

and take

$$r_1 \in \left(\frac{r_0}{2}\left(1 + \frac{2i^{(1)} - 1}{2k}\right), \frac{r_0}{2}\left(1 + \frac{i^{(1)}}{k}\right)\right)$$

such that $\mu(\partial Q(x_0,r_1))=0$. Then by (4.5), Corollary 1.204 in [12], (4.6), and (4.7)

$$\begin{aligned}\frac{d\lambda}{d|\lambda|}(x_0) &\geq \frac{\lambda(Q(x_0,r_1))}{|\lambda|(Q(x_0,r_1))}-\eta = \lim_{n\to\infty}\frac{\mathcal{F}_{\varepsilon_{n_l}}[v_{n_l};Q(x_0,r_1)]}{|\lambda|(Q(x_0,r_0))}-\eta\\ &\geq \liminf_{n\to\infty}\frac{q\mathcal{I}_{\varepsilon_{n_l}}[v_{n_l};Q(x_0,r_1)]-|\lambda|(Q(x_0,r_0/2)\eta}{|\lambda|(Q(x_0,r_1))}-\eta\\ &\geq -2\eta,\end{aligned}$$

where we used the fact that $r_0/2<r_1$ so that $|\lambda|(Q(x_0,r_1))\geq|\lambda|(Q(x_0,r_0/2))$. Letting $\eta\to 0^+$ we conclude that $\frac{d\lambda}{d|\lambda|}(x_0)\geq 0$.

This shows that $\lambda\geq 0$. In turn, by the Radon-Nikodym and Lebesgue Decomposition theorems ([12] Theorem 1.180) we can decompose

$$\lambda=\lambda_{ac}+\lambda_s,$$

where $\lambda_{ac}\ll\xi$, $\lambda_s\geq 0$, $\lambda_s\perp\xi$, with

$$\xi(B):=\mathcal{H}^{d-1}(B\cap\partial^*E),\quad B\subset\Omega\text{ Borel.}$$

We claim that for $\mathcal{H}^{d-1}$-a.e. $x_0\in\Omega\cap\partial^*E$,

$$\frac{d\lambda_{ac}}{d\xi}(x_0)\geq m_d, \tag{4.8}$$

where m_d is the constant defined in (2.2). Observe that if (4.8) holds, then, since $\lambda_s\geq 0$,

$$\begin{aligned}\lim_{n\to\infty}\mathcal{F}_{\varepsilon_n}[v_n;\Omega] &= \lim_{n\to\infty}\lambda_n(\Omega)\geq\lambda(\Omega)\geq\lambda_{ac}(\Omega)=\int_\Omega\frac{d\lambda_{ac}}{d\xi}d\xi\\ &\geq m_d\mathcal{H}^{d-1}(\Omega\cap\partial^*E)=m_d\mathrm{Per}(E;\Omega),\end{aligned}$$

which gives (2.4) (see (2.3) and (4.3)). In the remainder of the proof we show (4.8).

To this end we first note that by the Besicovitch Derivation Theorem (Theorem 1.155 in [12]), for $\mathcal{H}^{d-1}$-a.e. $x_0\in\Omega\cap\partial^*E$

$$\infty>\frac{d\lambda_{ac}}{d\mathcal{H}^{d-1}}(x_0)=\lim_{r\to 0^+}\frac{\lambda(Q_\nu(x_0,r))}{\mathcal{H}^{d-1}(Q_\nu(x_0,r)\cap\partial^*E)}=\lim_{r\to 0^+}\frac{\lambda(Q_\nu(x_0,r))}{r^{d-1}}, \tag{4.9}$$

$$\infty>\frac{d\zeta_{ac}}{d\mathcal{H}^{d-1}}(x_0)=\lim_{r\to 0^+}\frac{\zeta(Q_\nu(x_0,r))}{\mathcal{H}^{d-1}(Q_\nu(x_0,r)\cap\partial^*E)}=\lim_{r\to 0^+}\frac{\zeta(Q_\nu(x_0,r))}{r^{d-1}}. \tag{4.10}$$

Fix $x_0\in\Omega\cap\partial^*E$ for which (4.9) and (4.10) hold. Then there exists $\bar r>0$ such that

$$\frac{\zeta(Q_\nu(x_0,r))}{r^{d-1}}\leq\frac{d\zeta_{ac}}{d\mathcal{H}^{d-1}}(x_0)+1=:M$$

for all $0<r\leq\bar r$. Let $0<r_0\leq\bar r$ be such that $\zeta(\partial Q_\nu(x_0,r_0))=\mu(\partial Q_\nu(x_0,r_0))=0$. Then by Corollary 1.204 in [12],

$$\lim_{n\to\infty}\frac{\mathcal{I}_{\varepsilon_n}[v_n;Q(x_0,r_0)]}{r_0^{d-1}}=\frac{\zeta(Q_\nu(x_0,r_0))}{r_0^{d-1}}\leq M$$

and so

$$\mathcal{I}_{\varepsilon_n}[v_n;Q(x_0,r_0)]\leq(M+1)\,r_0^{d-1}$$

for all $n\geq n_0=n_0(r_0)$. Let $k\in\mathbb{N}$. By Proposition 3.6 with $K:=(M+1)r_0^{d+1}$, for each $n\geq n_0$ there exists $i_n\in\{1,\dots,k\}$ such that

$$\mathcal{I}_{\varepsilon_n}[v_n;L_n]\leq\frac{(M+1)\,r_0^{d-1}}{k},$$

where $L_n:=Q\left(\frac{r_0}{2}\left(1+\frac{i_n}{k}\right)\right)\setminus Q\left(\frac{r_0}{2}\left(1+\frac{i_n-1}{k}\right)\right)$.

Since $i_n \in \{1,\dots,k\}$ for all $n \geq n_0$, there exists $i^{(1)} \in \{1,\dots,k\}$ such that $i^{(1)} = i_n$ for infinitely many n, say $n_l^{(1)}, l \in \mathbb{N}$. Let $L^{(1)} := L_{i^{(1)}}$. Then

$$\mathcal{I}_{\varepsilon_{n_l^{(1)}}}[v_{n_l^{(1)}}; L^{(1)}] \leq \frac{(M+1)\, r_0^{d-1}}{k}$$

for all $n_l^{(1)}$, $l \in \mathbb{N}$. Let $r_1 \in \left(\frac{r_0}{2}\left(1+\frac{2i^{(1)}-1}{k}\right), \frac{r_0}{2}\left(1+\frac{i^{(1)}}{k}\right)\right)$ be such that $\zeta(\partial Q_\nu(x_0,r_1)) = \mu(\partial Q_\nu(x_0,r_1)) = 0$. Inductively, for every j we find a sequence $\{v_{n_l^{(j)}}\}_{l\in\mathbb{N}} \subset \{v_{n_l^{(j-1)}}\}_{l\in\mathbb{N}}$, a layer

$$L^{(j)} := Q\left(\frac{r_{j-1}}{2}\left(1+\frac{i^{(j-1)}}{k}\right)\right) \backslash Q\left(\frac{r_{j-1}}{2}\left(1+\frac{i^{(j-1)}-1}{k}\right)\right) \tag{4.11}$$

such that

$$\mathcal{I}_{\varepsilon_{n_l^{(j)}}}[v_{n_l^{(j)}}; L^{(j)}] \leq \frac{(M+1)\, r_{j-1}^{d-1}}{k} \tag{4.12}$$

for all $l \in \mathbb{N}$, and we take

$$r_j \in \left(\frac{r_{j-1}}{2}\left(1+\frac{2i^{(j)}-1}{2k}\right), \frac{r_{j-1}}{2}\left(1+\frac{i^{(j)}}{k}\right)\right) \tag{4.13}$$

such that $\zeta(\partial Q_\nu(x_0,r_{j-1})) = \mu(\partial Q_\nu(x_0,r_{j-1})) = 0$. Note that $r_j \to 0^+$.

Since by (4.9), Corollary 1.204 in [12],

$$\frac{d\lambda_{ac}}{d\mathcal{H}^{d-1}}(x_0) = \lim_{j\to\infty} \frac{\lambda(Q_\nu(x_0,r_j))}{r_j^{d-1}} = \lim_{j\to\infty}\lim_{n\to\infty} \frac{\mathcal{F}_{\varepsilon_n}[v_n; Q(x_0,r_j)]}{r_j^{d-1}}$$

and by Theorem 3.59 from [1] (see also (2.10) and (2.11))

$$\lim_{j\to\infty}\lim_{n\to\infty} \frac{1}{r_j^d}\int_{Q(x_0,r_j)} |v_n - v_0|^2 dx = \lim_{j\to\infty} \frac{1}{r_j^d}\int_{Q(x_0,r_j)} |v - v_0|^2 dx = 0,$$

also by (4.12) and using the fact that for $j \in \mathbb{N}$, $\varepsilon_{n_l^{(j)}} \to 0$ as $l \to \infty$, we can use a diagonal argument to find $\varepsilon^{(j)} \in \{\varepsilon_{n_l^{(j)}}\}_{l\in\mathbb{N}}$ and $\tilde{v}_j \in \{v_{n_l^{(j)}}\}_{l\in\mathbb{N}}$ such that $\varepsilon^{(j)}/r_j \to 0$,

$$\frac{d\lambda_{ac}}{d\mathcal{H}^{d-1}}(x_0) = \lim_{j\to\infty} \frac{\mathcal{F}_{\varepsilon^{(j)}}[\tilde{v}_j; Q(x_0,r_j)]}{r_j^{d-1}}, \tag{4.14}$$

$$\lim_{j\to\infty} \frac{1}{r_j^d}\int_{Q(x_0,r_j)} |\tilde{v}_j - v_0|^2 dx = 0, \tag{4.15}$$

$$\mathcal{I}_{\varepsilon^{(j)}}[\tilde{v}_j; L^{(j)}] \leq \frac{(M+1)\, r_{j-1}^{d-1}}{k}, \tag{4.16}$$

where v_0 was introduced in Proposition 3.7. Define

$$w_j(y) := \tilde{v}_j(x_0 + r_j y), \quad y \in Q_\nu(0,1),$$

and

$$L := Q_\nu(0,1)\backslash Q_\nu(0, 1-1/(2k)).$$

Since $L^{(j)} \supseteq Q(x_0,r_j)\backslash Q(x_0, r_j(1-1/(2k))) =: L$ by (4.11) and (4.13), by (4.16) we have

$$\mathcal{I}_{\varepsilon^{(j)}/r_j}[w_j, L] = \frac{1}{r_j^{d-1}}\mathcal{I}_{\varepsilon^{(j)}}[\tilde{v}_j, x_0 + r_j L] \leq \frac{1}{r_j^{d-1}}\mathcal{I}_{\varepsilon^{(j)}}[\tilde{v}_j, L^{(j)}] \leq \frac{r_{j-1}^{d-1}}{r_j^{d-1}}\frac{(M+1)}{k} \leq \frac{(M+1)\,2^{d-1}}{k},$$

where we also used $r_j > \frac{r_{j-1}}{2}$. Moreover (4.14) and (4.15) become

$$\frac{d\lambda_{ac}}{d\mathcal{H}^{d-1}}(x_0) = \lim_{j\to\infty} \mathcal{F}_{\varepsilon^{(j)}/r_j}[w_j; Q_\nu(0,1)],$$

and

$$\lim_{j\to\infty} \int_{Q_\nu(0,1)} |w_j - v_0|^2 dy = 0.$$

We can apply Proposition 3.7 to obtain

$$\frac{d\lambda_{ac}}{d\mathcal{H}^{d-1}}(x_0) \geq m_d - \frac{C}{k}.$$

Letting $k \to \infty$ completes the proof.

5. Proof of the Limsup Inequality

We now turn to the proof of (2.5), where we follow closely the argument in [7].

Step 1. Assume first that the target function v has a flat interface orthogonal to a given direction $\nu \in S^{d-1}$, and that Ω has Lipschitz boundary that meets this interface orthogonally. More precisely, without loss of generality (under suitable rigid transformations of the coordinate system), we assume that $v \in BV(\Omega; \{\pm 1\})$ is of the simple form

$$v(x) := \begin{cases} -1 & \text{if } x_d < 0, \\ 1 & \text{if } x_d > 0, \end{cases}$$

where we use the notation $x_d := x \cdot e_d = x \cdot \nu$, and that the normal to $\partial\Omega$ is orthogonal to e_d for all $x \in \partial\Omega$ with $|x_d|$ small enough. Let $\rho > 0$. By definition of m_d (see (2.2) and the remark after), there exist $\varepsilon_0 > 0$ and $w \in \mathcal{A}_\nu$ such that

$$\int_Q \left(\frac{1}{\varepsilon_0} W(-\varepsilon_0^2 \Delta w + w) - \varepsilon_0 q \left|\nabla w\right|^2 + (1-2q)\varepsilon_0^3 |\Delta w|^2 + (1-q)\varepsilon_0^5 |\nabla \Delta w|^2 \right) dx < m_d + \rho. \quad (5.1)$$

Define

$$w_n(x) := \begin{cases} -1 & \text{if } x_d < -\frac{\varepsilon_n}{2\varepsilon_0}, \\ w\left(\frac{\varepsilon_0 x}{\varepsilon_n}\right) & \text{if } |x_d| \leq \frac{\varepsilon_n}{2\varepsilon_0}, \\ 1 & \text{if } x_d > \frac{\varepsilon_n}{2\varepsilon_0}. \end{cases}$$

Note that, for n large enough, $w_n \in W^{3,2}(\Omega)$. Moreover, we claim that $w_n \to v$ in $L^2(\Omega)$. Indeed,

$$\|w_n - v\|_{L^2(\Omega)} = \|w_n - v\|_{L^2(\{x \in \Omega \colon |x_d| < \frac{\varepsilon_n}{2\varepsilon_0}\})} \leq \|w_n\|_{L^2(\{x \in \Omega \colon |x_d| < \frac{\varepsilon_n}{2\varepsilon_0}\})} + \|v\|_{L^2(\{x \in \Omega \colon |x_d| < \frac{\varepsilon_n}{2\varepsilon_0}\})},$$

where for n sufficiently large

$$\|v\|_{L^2(\{x \in \Omega \colon |x_d| < \frac{\varepsilon_n}{2\varepsilon_0}\})} = \left| \left\{ x \in \Omega : |x_d| < \frac{\varepsilon_n}{2\varepsilon_0} \right\} \right| \to 0 \text{ as } n \to \infty.$$

Further, setting $\Omega' := \{x' \in \mathbb{R}^{d-1} : (x', 0) \in \Omega\}$, we have for sufficiently large n, that $\{x \in \Omega : |x_d| \leq \varepsilon_n/(2\varepsilon_0)\} = \Omega' \times [-\varepsilon_n/(2\varepsilon_0), \varepsilon_n/(2\varepsilon_0)]$. Hence, applying the change of variables $t := \frac{\varepsilon_0 x_d}{\varepsilon_n}$ yields

$$\|w_n\|^2_{L^2(\{x \in \Omega \colon |x_d| < \frac{\varepsilon_n}{2\varepsilon_0}\})} = \int_{\left\{x \in \Omega : |x_d| < \frac{\varepsilon_n}{2\varepsilon_0}\right\}} \left| w\left(\frac{\varepsilon_0 x}{\varepsilon_n}\right)\right|^2 dx = \frac{\varepsilon_n}{\varepsilon_0} \int_{-1/2}^{1/2} \int_{\Omega'} \left| w\left(\frac{\varepsilon_0 x'}{\varepsilon_n}, t\right)\right|^2 dx' dt. \quad (5.2)$$

Since w is periodic in the first $d-1$ arguments, applying Fubini's Theorem and the Riemann-Lebesgue Lemma (see for example Lemma 2.85 in [12]) to $\int_{-1/2}^{1/2} \left| w\left(\frac{\varepsilon_0 x'}{\varepsilon_n}, t\right)\right|^2 dt \in L^1_{\text{loc}}(\mathbb{R}^{d-1})$ gives

$$\lim_{n \to \infty} \int_{\Omega'} \int_{-1/2}^{1/2} \left| w\left(\frac{\varepsilon_0 x'}{\varepsilon_n}, t\right)\right|^2 dt\, dx' = \int_{\Omega'} \int_{Q'} \int_{-1/2}^{1/2} |w(y,t)|^2\, dt\, dy\, dx' = \mathcal{L}^{d-1}(\Omega') ||w||^2_{L^2(Q)}.$$

It then follows from (5.2) that

$$\|w_n\|^2_{L^2(\{x \in \Omega \colon |x_d| < \frac{\varepsilon_n}{2\varepsilon_0}\})} \leq \frac{C\varepsilon_n}{\varepsilon_0} ||w||^2_{L^2(Q)} \to 0, \text{ as } n \to \infty.$$

This concludes the proof that $w_n \to v$ in $L^2(\Omega)$.

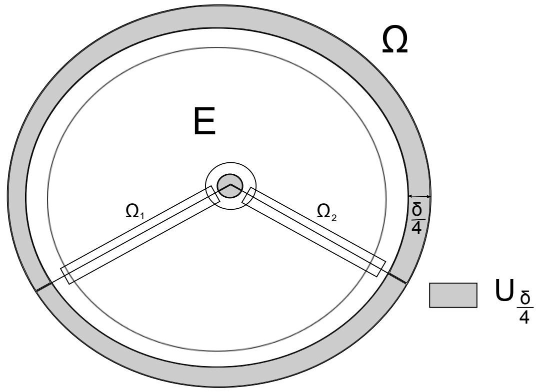

FIGURE 3. Construction in Step 2.

Since $w_n = \pm 1$ on $\{x \in \Omega : |x_d| \geq \frac{\varepsilon_n}{2\varepsilon_0}\}$, the contribution to the energy only comes from the interfacial region $\{x \in \Omega : |x_d| \leq \frac{\varepsilon_n}{2\varepsilon_0}\}$, where we have

$$-\varepsilon_n^2 \Delta w_n(x) + w_n(x) = -\varepsilon_0^2 \Delta w \left(\frac{\varepsilon_0 x}{\varepsilon_n}\right) + w \left(\frac{\varepsilon_0 x}{\varepsilon_n}\right).$$

Setting, as before, $t := \frac{\varepsilon_0 x_d}{\varepsilon_n}$ we have for n sufficiently large

$$\begin{aligned}\mathcal{F}_{\varepsilon_n}[w_n;\Omega] = \int_{\{x\in\Omega:\,|x_d|<\frac{\varepsilon_n}{2\varepsilon_0}\}} &\left\{\frac{1}{\varepsilon_n} W\left(-\varepsilon_0^2\Delta w + w\right) - \frac{\varepsilon_0^2}{\varepsilon_n} q\,|\nabla w|^2 + (1-2q)\frac{\varepsilon_0^4}{\varepsilon_n}|\Delta w|^2 + \right.\\ &\left. + (1-q)\frac{\varepsilon_0^6}{\varepsilon_n}|\nabla\Delta w|^2\right\}\left(\frac{\varepsilon_0 x}{\varepsilon_n}\right)\,dx\\ = \int_{\Omega'}\int_{-\frac{1}{2}}^{\frac{1}{2}} &\left\{\frac{1}{\varepsilon_0} W\Big(\left(-\varepsilon_0^2\Delta w + w\right) - q\varepsilon_0\,|\nabla w|^2 + \right.\\ &\left. + (1-2q)\varepsilon_0^3|\Delta w|^2 + (1-q)\varepsilon_0^5|\nabla\Delta w|^2\right\}\left(\frac{\varepsilon_0 x'}{\varepsilon_n}, t\right)\Big\}\,dt\,dx'.\end{aligned}$$

Since w is periodic in the first $d-1$ arguments, also the functions

$$x' \mapsto \int_{-\frac{1}{2}}^{\frac{1}{2}} W(-\varepsilon_0^2\Delta w + w)(x',t)\,dt, \quad x' \mapsto \int_{-\frac{1}{2}}^{\frac{1}{2}} |\nabla w|^2(x',t)\,dt,$$
$$x' \mapsto \int_{\frac{1}{2}}^{\frac{1}{2}} |\Delta w|^2(x',t)\,dt, \quad \text{and} \quad x' \mapsto \int_{-\frac{1}{2}}^{\frac{1}{2}} |\nabla\Delta w|^2(x',t)\,dt$$

are periodic and locally in L^1, where for the integral involving W we used the quadratic growth assumption from Hypothesis 2.2. Thus, by the Riemann-Lebesgue Lemma and the choice of w (see (5.1)),

$$\begin{aligned}\lim_{n\to\infty}\mathcal{F}_{\varepsilon_n}[w_n;\Omega] = \mathcal{L}^{d-1}(\Omega')\int_Q &\left\{\frac{1}{\varepsilon_0}W(-\varepsilon_0^2\Delta w + w) - q\varepsilon_0\,|\nabla w|^2 + (1-2q)\varepsilon_0^3|\Delta w|^2\right.\\ &\left. + (1-q)\varepsilon_0^5|\nabla\Delta w|^2\right\}dx \leq (m_d+\rho)\mathrm{Per}_\Omega(\{v=1\}),\end{aligned} \tag{5.3}$$

and the limsup inequality follows since $\rho > 0$ is arbitrarily small.

Step 2. Consider now the case in which

$$v = \chi_E - \chi_{\Omega\setminus E},$$

where $\mathrm{Per}_\Omega(E) < \infty$ and E has the form $E = P \cap \Omega$ with P a polyhedron, i.e., there is $L \in \mathbb{N}$ such that $\partial P = H_1 \cup H_2 \cup \cdots \cup H_L \cup F$ with pairwise disjoint relatively open convex polyhedra H_i of dimension $d-1$, $H_i \subset \{x \in \mathbb{R}^d : (x - x_i)\cdot \nu_i = 0\}$ for some $x_i \in \mathbb{R}^d$ and $\nu_i \in S^{d-1}$, $i = 1, \dots, L$, and F is the union of a finite number of convex polyhedra of dimension $d-2$. Finally, we assume that E meets the boundary of Ω transversally, more precisely

$$\partial\Omega \cap \partial P \text{ is the union of a finite number of } C^1 \text{ manifolds of dimension } d-2. \tag{5.4}$$

We extend v to $\mathbb{R}^d$ by setting

$$v(x) := \chi_P(x) - \chi_{\mathbb{R}^d \setminus P}(x),$$

and define

$$\varphi_n := v * \Psi_{\varepsilon_n} \tag{5.5}$$

with mollifiers Ψ_{ε_n} (see (3.18)). For fixed (small) $0 < \delta < 1$ set

$$U_\delta := \{x \in \Omega : \mathrm{dist}(x, \partial\Omega \cup F) \le \delta\}$$

and let H_i' be relatively open subsets of H_i with a $d-2$ dimensional C^∞ boundary such that

$$\left\{x \in H_i \cap \Omega : \mathrm{dist}(x, \partial\Omega \cup F) \ge \frac{\delta}{2}\right\} \subset H_i' \subset \overline{H_i'} \subset H_i \cap \Omega$$

and $\overline{H_i'} \cap U_{\frac{\delta}{4}} = \emptyset$. Fix $0 < \eta < \delta/2$, and set for every $i = 1, 2, \dots, L$,

$$\Omega_i := \{x + t\nu_i : x \in H_i',\ |t| < \eta\}.$$

Taking η sufficiently small we may assume, without loss of generality, that $\Omega_1, \dots, \Omega_L$ are pairwise disjoint and

$$\overline{\Omega_i} \cap U_{\frac{\delta}{4}} = \emptyset. \tag{5.6}$$

We apply *Step 1* to every Ω_i to obtain a sequence $\{w_n^i\} \subset W^{3,2}(\Omega_i)$ such that $w_n^i \to v$ in $L^2(\Omega_i)$, and $\lim_{n\to\infty} \mathcal{F}_{\varepsilon_n}[w_n^i; \Omega_i] \le (m_d + \rho)\mathcal{H}^{d-1}(H_i \cap \Omega_i)$. For every $\delta > 0$ choose cut-off functions $\eta_\delta \in C_c^\infty(\mathbb{R}^d; [0,1])$ such that

$$\eta_\delta = 0 \text{ in } U_\delta, \quad \eta_\delta = 1 \text{ in } \mathbb{R}^d \setminus U_{2\delta}, \quad \|\nabla^k \eta_\delta\|_{L_\infty(\mathbb{R}^d)} \le C/\delta^k \text{ for } k = 1,2,3. \tag{5.7}$$

Define V_n by

$$V_n := \begin{cases} \eta_\delta w_n^i + (1-\eta_\delta)\varphi_n & \text{in } \overline{\Omega_i}, \quad i = 1, \dots, L, \\ \eta_{\frac{\delta}{8}} \varphi_n & \text{in } A := \Omega \setminus (\overline{\Omega_1} \cup \cdots \cup \overline{\Omega_L}). \end{cases} \tag{5.8}$$

We claim that $V_n \in W^{3,2}(\Omega)$ and satisfies Neumann boundary conditions on $\partial\Omega$. Indeed, considering V_n in the neighborhood of ∂A, we observe that by construction of w_n^i in *Step 1*

$$w_n^i(x) = v(x) \quad \text{ for } x \in \overline{\Omega_i} \text{ and } \mathrm{dist}(x, H_i) \ge \frac{\varepsilon_n}{2\varepsilon_0}.$$

Hence, from (5.5), for sufficiently large n we have $w_n^i = \varphi_n$ in a neighborhood of $\{x \in \partial\Omega_i : \mathrm{dist}(x, H_i) = \eta\}$ (the part of $\partial\Omega_i$ parallel to H_i), and by (5.6) in that region both $\eta_\delta w_n^i + (1-\eta_\delta)\varphi_n$ and $\eta_{\frac{\delta}{8}}\varphi_n$ are equal to φ_n. In addition, $\{x \in \partial\Omega_i : \mathrm{dist}(x, H_i) < \eta\}$ (the part of $\partial\Omega_i$ orthogonal to H_i) is contained in $U_\delta \setminus U_{\delta/4}$ and both $\eta_\delta w_n^i + (1-\eta_\delta)\varphi_n$ and $\eta_{\frac{\delta}{8}}\varphi_n$ are equal to φ_n also in that region. Finally, V_n is identically zero in a neighborhood of $U_{\frac{\delta}{8}}$ so the Neumann boundary conditions are satisfied.

Furthermore, $\lim_{n\to\infty} ||V_n - v||_{L^2(\Omega)} \le C\delta$, since $w_n^i \to v$ in $L^2(\Omega_i)$ and $\varphi_n \to v$ in $L^2(\Omega \setminus U_{\frac{\delta}{8}})$. It remains to estimate the energies. By (5.5), V_n is possibly different from ± 1 only on $U_{\frac{\delta}{4}}$ and on

$$R_n := \{x \in \Omega : \mathrm{dist}(x, \partial P) \le \max\{\varepsilon_n/(2\varepsilon_0), \varepsilon_n\}\}.$$

Using the notation from (5.8), $V_n = \eta_{\frac{\delta}{8}}\varphi_n$ on $U_{\frac{\delta}{4}}$, $V_n = \varphi_n$ on $A\backslash U_{\frac{\delta}{4}}$, $A\cap R_n \subset U_\delta$ and $\mathcal{H}^{d-1}(\partial P\cap U_\delta) \leq C\delta$. Thus, for n sufficiently large,

$$\mathcal{F}_{\varepsilon_n}[V_n;A] \leq \left|\mathcal{F}_{\varepsilon_n}[\eta_{\frac{\delta}{8}}\varphi_n;U_{\frac{\delta}{4}}]\right| + \int_{A\cap R_n}\Big(\frac{1}{\varepsilon_n}W(-\varepsilon_n^2\Delta\varphi_n+\varphi_n)+\varepsilon_n|q||\nabla\varphi_n|^2+(1-2q)\varepsilon_n^3|\Delta\varphi_n|^2 + (1-q)\varepsilon_n^5|\nabla\Delta\varphi_n|^2\Big)dx \leq C\delta,$$

where we also used (3.19) and (5.7) to bound the derivatives of φ_n and $\eta_{\frac{\delta}{8}}$, respectively. Next we estimate the energy in Ω_i. In $\Omega_i\cap U_\delta$, $V_n=\varphi_n$ and using (3.19) yields

$$\mathcal{F}_{\varepsilon_n}[V_n;\Omega_i\cap U_\delta] = \mathcal{F}_{\varepsilon_n}[V_n;\Omega_i\cap U_\delta\cap R_n] \leq C\delta. \tag{5.9}$$

To obtain estimates inside $T := \Omega_i\cap(U_{2\delta}\backslash U_\delta)$ we first observe that

$$\partial_{x_i}V_n = w_n^i\partial_{x_i}\eta_\delta + \eta_\delta\partial_{x_i}w_n^i - \varphi_n\partial_{x_i}\eta_\delta + (1-\eta_\delta)\partial_{x_i}\varphi_n,$$

and arguing as in (5.3),

$$\lim_{n\to\infty}\varepsilon_n^{2k-1}||\nabla^k w_n^i||^2_{L^2(\Omega_i\cap U_{2\delta})} \leq C(\rho)\,\mathcal{H}^{d-1}(H_i\cap U_{2\delta}) \leq C(\rho)\,\delta \text{ for } k=0,\dots 3, \tag{5.10}$$

where we also used the fact that $w\in W^{3,\infty}_{loc}(\mathbb{R}^d)$. Combined with the bounds on φ_n from (3.19), it follows that,

$$\int_T \varepsilon_n|\nabla V_n|^2dx = \int_{T\cap R_n}\varepsilon_n|\nabla V_n|^2dx \leq C(\rho)\Big(\frac{\varepsilon_n}{\delta^2}||w_n^i||^2_{L^2(T)} + \varepsilon_n||\nabla w_n^i||^2_{L^2(T)} + \frac{\varepsilon_n}{\delta^2}||\varphi_n||^2_{L^2(T)} + + \varepsilon_n||\nabla\varphi_n||^2_{L^2(T)}\Big) \leq C(\rho)\left(\delta+\frac{\varepsilon_n}{\delta^2}\right).$$

Analogous calculations for the higher derivatives of V_n, yield the bound

$$\mathcal{F}_{\varepsilon_n}[V_n;\Omega_i\cap(U_{2\delta}\setminus U_\delta)] \leq C(\rho)\,\delta \tag{5.11}$$

for n sufficiently large. Next, by (5.8), (5.10) and (3.19), we have

$$\lim_{n\to\infty}\int_{\Omega_i\cap U_{2\delta}}\varepsilon_n|\nabla V_n|^2\,dx \leq C(\rho)\,\delta,$$

and hence

$$\begin{aligned}&\int_{\Omega_i\backslash U_{2\delta}}\left\{\frac{1}{\varepsilon_n}W(V_n)-\varepsilon_n q|\nabla V_n|^2+(1-2q)\varepsilon_n^3|\Delta V_n|^2+(1-q)\varepsilon_n^5|\nabla\Delta V_n|^2\right\}dx\\ &\leq \int_{\Omega_i}\left\{\frac{1}{\varepsilon_n}W(V_n)-\varepsilon_n q|\nabla V_n|^2+(1-2q)\varepsilon_n^3|\Delta V_n|^2+(1-q)\varepsilon_n^5|\nabla\Delta V_n|^2\right\}dx + C(\rho)\,\delta.\end{aligned} \tag{5.12}$$

Combining (5.3), (5.9), (5.11), and (5.12), we obtain for δ sufficiently small a sequence $V_n\in W^{3,2}(\Omega)$, with Neumann boundary conditions on $\partial\Omega$, satisfying

$$\lim_{n\to\infty}||V_n - v||_{L^2(\Omega)} \leq \rho$$

and

$$\begin{aligned}\limsup_{n\to\infty}\mathcal{F}_{\varepsilon_n}[V_n] = \limsup_{n\to\infty}\mathcal{F}_{\varepsilon_n}[V_n;\Omega] &\leq \limsup_{n\to\infty}\sum_{i=1}^L\mathcal{F}_{\varepsilon_n}[V_n;\Omega_i] + \limsup_{n\to\infty}\mathcal{F}_{\varepsilon_n}[V_n;A]\\ &\leq (m_d+\rho)\sum_{i=1}^L\mathcal{H}^{d-1}(\Omega_i\cap H_i)+C(\rho)\,\delta\\ &\leq (m_d+\rho)\mathcal{H}^{d-1}(\Omega\cap\partial P)+\rho,\end{aligned}$$

and the Limsup Inequality (2.4) follows by a standard diagonalizing argument.

Symbol	Description	Value
a_4		$10^{-5}J/m^2$
b	line tension	$5 \times 10^{-19}J$
σ	surface tension	5×10^{-6} to $10^{-4}J/m^2$
κ	bending rigidity of the membrane	$10^{-19}J$
Λ	composition-curvature coupling constant	$4.9 \times 10^{-12}J/m$

TABLE 1. Parameter descriptions and characteristic values, [20].

Step 3. Lastly we consider the case in which the target function is

$$v = \chi_E - \chi_{\Omega\setminus E},$$

where E is an arbitrary set of finite perimeter in Ω. Since Ω is bounded and has C^2 boundary, we can approximate E with smooth sets (see Remark 3.43 in [1]) and then with polyhedral sets. In particular, we may find sets $E_k \subset \Omega$ of the form $E_k = P_k \cap \Omega$, where P_k are polyhedral sets satisfying (5.4) such that $\mathcal{H}^{d-1}(\partial E_k \cap \partial\Omega) = 0$, $\chi_{E_k} \to \chi_E$ in $L^2(\Omega)$, and $\mathrm{Per}_\Omega(E_k) \to \mathrm{Per}_\Omega(E)$ as $k \to +\infty$. We apply *Step 2* to each function $v_k := \chi_{E_k} - \chi_{\Omega\setminus E_k}$ to find a sequence

$$V_n^k \to v_k$$

satisfying

$$\limsup_{n\to\infty} \mathcal{F}_{\varepsilon_n}[V_n^k;\Omega] \le m_d \mathcal{H}^{d-1}(E_k \cap \partial P_k)$$

and

$$\limsup_{k\to\infty} \limsup_{n\to\infty} \mathcal{F}_{\varepsilon_n}[V_n^k] \le \limsup_{k\to\infty} \left(m_d \mathcal{H}^{d-1}(E_k \cap \partial P_k)\right) = m_d \mathrm{Per}_\Omega(E).$$

The general result now follows by a diagonalizing argument.

Appendix

We derive the energy functional (1.2) from (1.1). To eliminate the dependence on h we assume that ϕ and h satisfy the Euler-Lagrange equation

$$\frac{\delta\mathcal{E}}{\delta h}(\phi, h) = 0. \tag{5.13}$$

After changing variables, $x := \bar{x}/L$, $u(x) := \phi(\bar{x})$ in (1.1) we have

$$\frac{1}{L^d}\mathcal{E}[u,h] = \int_\Omega \left(f(u) + \frac{b}{2L^2}|\nabla u|^2 + \frac{\sigma}{2L^2}|\nabla h|^2 + \frac{k}{2L^4}[\Delta h]^2 + \frac{\Lambda}{L^2} u\Delta h \right) dx, \tag{5.14}$$

where $\Omega := \{x/L : x \in D\}$. Assuming natural boundary conditions, the Euler-Lagrange equation (5.13) takes the form

$$\begin{cases} \Delta\left(\frac{k}{L^4}\Delta h - \frac{\sigma}{L^2}h + \frac{\Lambda}{L^2}u\right) = 0 & \text{in } \Omega, \\ \frac{\partial h}{\partial n} = 0, \frac{\partial \Delta h}{\partial n} = 0, \frac{\partial u}{\partial n} = 0, \frac{\partial \Delta u}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.15}$$

Consider the Fourier Series expansions of h and u,

$$h = \sum_{i=0}^\infty h_i\psi_i, \quad u = \sum_{i=0}^\infty u_i\psi_i,$$

where ψ_i are the eigenfunctions of $-\Delta$ on $H^1(\Omega)$ with Neumann boundary conditions. Denote the corresponding nonnegative eigenvalues by λ_i^2. Then, since $\psi_0 = const$ (due to Neumann boundary conditions), we have

$$\Delta h = -\sum_{i=1}^\infty \lambda_i^2 h_i\psi_i, \quad \text{and} \quad \Delta^2 h = \sum_{i=1}^\infty \lambda_i^4 h_i\psi_i,$$

and thus by (5.15)

$$\sum_{i=1}^{\infty} \lambda_i^2 \left(\frac{k}{L^2} \lambda_i^2 h_i + \sigma h_i - \Lambda u_i \right) \psi_i = 0.$$

Taking the L^2 inner product with ψ_j, and noting that $\langle \psi_i, \psi_j \rangle_{L^2(\Omega)} = \delta_{ij}$, we obtain

$$\lambda_j^2 \left(\frac{k}{L^2} \lambda_j^2 h_j + \sigma h_j - \Lambda u_j \right) = 0 \text{ for } j = 1, \ldots, \infty.$$

Solving for h_j yields

$$h_j = \frac{\Lambda u_j}{\sigma + (k/L^2)\lambda_j^2} \text{ for } j = 1, \ldots, \infty,$$

and

$$h(x) = \sum_{i=0}^{\infty} h_i \psi_i(x) = const + \sum_{i=1}^{\infty} h_i \psi_i(x) = const + \sum_{i=1}^{\infty} \frac{\Lambda u_i \psi_i(x)}{\sigma + (k/L^2)\lambda_i^2}.$$

Using this expansion and $\Delta \psi_i = -\lambda_i^2 \psi_i$ gives

$$-\Delta h(x) = \sum_{i=1}^{\infty} \frac{\Lambda \lambda_i^2 u_i \psi_i(x)}{\sigma + (k/L^2)\lambda_i^2}. \tag{5.16}$$

In addition, multiplying (5.15) by h and integrating by parts, we obtain

$$\int_\Omega \left((k/L^2)(\Delta h)^2 + \sigma |\nabla h|^2 + \Lambda u \Delta h \right) \, dx = 0,$$

and consequently

$$\frac{1}{2} \int_\Omega \left((k/L^2)(\Delta h)^2 + \sigma |\nabla h|^2 \right) \, dx = -\frac{1}{2} \int_\Omega \Lambda u \Delta h \, dx. \tag{5.17}$$

Substituting (5.17) into (5.14) yields

$$\frac{1}{L^d} \mathcal{E}[u, h] = \int_\Omega \left(f(u) + \frac{b}{2L^2} |\nabla u|^2 + \frac{\Lambda}{2L^2} u \Delta h \right) \, dx. \tag{5.18}$$

To eliminate the dependence on h observe that since $\langle \psi_i, \psi_j \rangle_{L^2(\Omega)} = \delta_{ij}$, (5.16) implies that

$$\int_\Omega u \Delta h \, dx = -\int_\Omega \left(\sum_{i=0}^{\infty} u_i \psi_i(x) \right) \left(\sum_{j=1}^{\infty} \frac{\Lambda \lambda_j^2 u_j \psi_j(x)}{\sigma + (k/L^2)\lambda_j^2} \right) dx = -\sum_{i=1}^{\infty} \frac{\Lambda \lambda_i^2 u_i^2}{\sigma + (k/L^2)\lambda_i^2}.$$

Substituting this expression into the energy functional (5.18) yields

$$\begin{aligned} \frac{1}{L^d} \mathcal{E}[u] &= \int_\Omega \left(f(u) + \frac{b}{2L^2} |\nabla u|^2 \right) dx - \frac{1}{2L^2} \sum_{i=1}^{\infty} \frac{\Lambda^2 \lambda_i^2}{\sigma + (k/L^2)\lambda_i^2} u_i^2 \\ &= \int_\Omega \left(f(u) + \frac{b}{2L^2} |\nabla u|^2 \right) dx - \frac{\Lambda^2}{2k} \sum_{i=1}^{\infty} \left(\frac{(k/L^2)\lambda_i^2 + \sigma - \sigma}{\sigma + (k/L^2)\lambda_i^2} \right) u_i^2 \\ &= \int_\Omega \left(f(u) + \frac{b}{2L^2} |\nabla u|^2 \right) dx - \frac{\Lambda^2}{2k} \sum_{i=1}^{\infty} u_i^2 + \frac{\Lambda^2}{2k} \sum_{i=1}^{\infty} \left(\frac{\sigma}{\sigma + (k/L^2)\lambda_i^2} \right) u_i^2. \end{aligned} \tag{5.19}$$

At this point one can use a long-wavelength approximation as suggested for example in [20] resulting in

$$\frac{\sigma}{\sigma + (k/L^2)\lambda_i^2} \sim 1 - \frac{k}{L^2 \sigma} \lambda_i^2 + \frac{k^2}{L^4 \sigma^2} \lambda_i^4,$$

and an approximation energy

$$\begin{aligned}\frac{1}{L^d}\mathcal{E}_{ap}[u] &:= \int_\Omega \left(f(u) + \frac{b}{2L^2}|\nabla u|^2 \right) dx - \frac{\Lambda^2}{2L^2\sigma}\sum_{i=1}^\infty \lambda_i^2 u_i^2 + \frac{\Lambda^2 k}{2L^4\sigma^2}\sum_{i=1}^\infty \lambda_i^4 u_i^2 \\ &= \int_\Omega \left(f(u) + \frac{1}{2L^2}\left(b - \frac{\Lambda^2}{\sigma}\right)|\nabla u|^2 + \frac{\Lambda^2 k}{2L^4\sigma^2}(\Delta u)^2 \right) dx, \end{aligned} \tag{5.20}$$

which was studied in [7, 8]. Returning to the full energy in (5.19), we have

$$\begin{aligned}\frac{1}{L^d}\mathcal{E}[u] &= \int_\Omega \left(f(u) + \frac{b}{2L^2}|\nabla u|^2 - \frac{\Lambda^2}{2k}u^2 \right) dx + \frac{\Lambda^2 L^2\sigma}{2k^2}\sum_{i=1}^\infty \frac{1}{\frac{L^2\sigma}{k} + \lambda_i^2}u_i^2 \\ &= \int_\Omega \left(f(u) + \frac{b}{2L^2}|\nabla u|^2 - \frac{\Lambda^2}{2k}u^2 + \frac{\Lambda^2 L^2\sigma}{2k^2}u\left(\frac{L^2\sigma}{k} - \Delta\right)^{-1} u \right) dx \\ &= \int_\Omega \left(f(u) - \frac{\Lambda^2}{2k}u^2 + \frac{b}{2L^2}|\nabla u|^2 + \frac{\Lambda^2}{2k}u\left(\mathbf{1} - \frac{k}{L^2\sigma}\Delta\right)^{-1} u \right) dx \\ &= \frac{\Lambda^2}{2k}\int_\Omega \left(\frac{2k}{\Lambda^2}f(u) - u^2 + \frac{kb}{L^2\Lambda^2}|\nabla u|^2 + u\left(\mathbf{1} - \frac{k}{L^2\sigma}\Delta\right)^{-1} u \right) dx.\end{aligned}$$

Setting

$$\varepsilon := \sqrt{\frac{k}{L^2\sigma}}, \quad q := 1 - \frac{b\sigma}{\Lambda^2}, \quad W(u) := \frac{2k}{\Lambda^2}f(u), \quad \text{and } \mathcal{F}_\varepsilon^* := \frac{1}{\varepsilon}\frac{2k}{\Lambda^2 L^d}\mathcal{E},$$

yields

$$\mathcal{F}_\varepsilon^*[u] := \frac{1}{\varepsilon}\int_\Omega \left(W(u) - u^2 + (1-q)\varepsilon^2|\nabla u|^2 + u\left(\mathbf{1} - \varepsilon^2\Delta\right)^{-1} u \right) dx.$$

We now outline the derivation of (2.1). Given $u \in W^{1,2}(\Omega)$, we define $v \in W^{3,2}(\Omega)$ via

$$-\varepsilon^2\Delta v + v = u \text{ in } \Omega \qquad \text{and} \qquad \frac{\partial v}{\partial n} = 0 \text{ on } \partial\Omega,$$

where n denotes the outward unit normal to $\partial\Omega$, and use the abbreviatory notation $v := (\mathbf{1} - \varepsilon^2\Delta)^{-1}u$. Integrating by parts we obtain

$$\int_\Omega u^2\, dx = \int_\Omega (\varepsilon^4(\Delta v)^2 + 2\varepsilon^2|\nabla v|^2 + v^2)\, dx,$$

$$\int_\Omega u(\mathbf{1} - \varepsilon^2\Delta)^{-1}u = \int_\Omega uv\, dx = \int_\Omega (\varepsilon^2|\nabla v|^2 + v^2)\, dx,$$

and

$$\int_\Omega |\nabla u|^2\, dx = \int_\Omega \left|-\varepsilon^2\nabla\Delta v + \nabla v\right|^2 dx = \int_\Omega \left(\varepsilon^4|\nabla\Delta v|^2 + 2\varepsilon^2(\Delta v)^2 + |\nabla v|^2\right) dx,$$

which allows us to rewrite the functional as

$$\begin{aligned}\mathcal{F}_\varepsilon^*[u] &= \int_\Omega \left(\frac{1}{\varepsilon}W(u) + (1-q)\varepsilon|\nabla u|^2 - \varepsilon|\nabla v|^2 - \varepsilon^3(\Delta v)^2 \right) dx \\ &= \int_\Omega \left(\frac{1}{\varepsilon}W(u) - \varepsilon q|\nabla u|^2 + \varepsilon^3(\Delta v)^2 + \varepsilon^5|\nabla\Delta v|^2 \right) dx \\ &= \int_\Omega \left(\frac{1}{\varepsilon}W(u) - \varepsilon q|\nabla v|^2 + (1-2q)\varepsilon^3(\Delta v)^2 + (1-q)\varepsilon^5|\nabla\Delta v|^2 \right) dx.\end{aligned}$$

This is the form that appears in (2.1).

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References

[1] L. Ambrosio, N. Fusco, and D. Pallara. *Functions of bounded variation and free discontinuity problems.* Clarendon Press, Oxford, 2000.

[2] M. Baía, A. C. Barroso, M. Chermisi, and J. Matias. Coupled second order singular perturbations for phase transitions. *Nonlinearity*, 26(5):1271–1311, 2013.

[3] D. Bonheure, L. Sanchez, M. Tarallo, and S. Terracini. Heteroclinic connections between nonconsecutive equilibria of a fourth order differential equation. *Calc. Var. Partial Differential Equations*, 17(4):341–356, 2003.

[4] A. Braides. *Γ-convergence for beginners.* Oxford University Press, Oxford, 2002.

[5] D. A. Brown and E. London. Functions of lipid rafts in biological membranes. *Annual review of cell and developmental biology*, 14(1):111–136, 1998.

[6] P. B. Canham. The minimum energy of bending as a possible explanation of the biconcave shape of the human red blood cell. *J. Theor. Biol.*, 26(1):61–81, 1 1970.

[7] M. Chermisi, G. Dal Maso, I. Fonseca, and G. Leoni. Singular perturbation models in phase transitions for second order materials. *Indiana Univ. Math. J.*, 60(2):367–410, 2011.

[8] M. Cicalese, E. Spadaro, and C. Zeppieri. Asymptotic analysis of a second-order singular perturbation model for phase transitions. *Calc. Var. Partial Differential Equations*, 41(1-2):127–150, 2011.

[9] B.D. Coleman, M. Marcus, and V.J. Mizel. On the thermodynamics of periodic phases. *Arch. Rational Mech. Anal.*, 117(4):321–347, 1992.

[10] G. Dal Maso. *An introduction to Γ-convergence.* Progress in Nonlinear Differential Equations and their Applications. Birkhuser Boston, Inc., Boston, MA, 1993.

[11] E. L. Elson, E. Fried, J. E. Dolbow, and G. M. Genin. Phase separation in biological membranes: integration of theory and experiment. *Annu. Rev. Biophys.*, 39:207, 2010.

[12] I. Fonseca and G. Leoni. *Modern methods in the calculus of variations: L^p-spaces.* Springer Monographs in Mathematics. Springer, New York, 2007.

[13] I. Fonseca and L. Tartar. The gradient theory of phase transitions for systems with two potential wells. *Proc. Roy. Soc. Edinburgh Sect. A*, 111(1-2):89–102, 1989.

[14] E. De Giorgi. Sulla convergenza di alcune successioni d'integrali del tipo dell'area. *Rend. Mat.*, 8(6):277–294, 1975.

[15] P. Grisvard. *Elliptic problems in nonsmooth domains.* Classics in Applied Mathematics. Society for Industrial and Applied Mathematics, 2011.

[16] W. Helfrich. Elastic properties of lipid bilayers: theory and possible experiments. *Z. Naturforsch C*, 28(11):693–703, 1973.

[17] J. Jost. *Partial differential equations.* Graduate Texts in Mathematics. Springer, 2007.

[18] T. Kawakatsu, D. Andelman, K. Kawasaki, and T. Taniguchi. Phase transitions and shapes of two-component membranes and vesicles I: strong segregation limit. *J. Phys. II (France)*, 3(7):971–997, 1993.

[19] R. V. Kohn and P. Sternberg. Local minimisers and singular perturbations. *Proc. Roy. Soc. Edinburgh Sect. A*, 111(1-2):69–84, 1989.

[20] S. Komura, N. Shimokawa, and D. Andelman. Tension-induced morphological transition in mixed lipid bilayers. *Langmuir*, 22(16):6771–6774, 2006.

[21] S. Leibler and D. Andelman. Ordered and curved meso-structures in membranes and amphiphilic films. *J. Phys. France*, 48(11):2013–2018, 1987.

[22] M. Leslie. Do lipid rafts exist? *Science*, 334(6059):1046–1047, 2011.

[23] S. Meinhardt, R. L. C. Vink, and F. Schmid. Monolayer curvature stabilizes nanoscale raft domains in mixed lipid bilayers. *Proc. Natl. Acad. Sci. USA*, 110(12):4476–81, 3 2013.

[24] V.J. Mizel, L.A. Peletier, and W.C. Troy. Periodic phases in second-order materials. *Arch. Rational. Mech. Anal.*, 145(4):343–382, 1998.

[25] L. Modica. The gradient theory of phase transitions and the minimal interface criterion. *Arch. Rational Mech. Anal.*, 98(2):123–142, 1987.

[26] L.A. Peletier and W.C. Troy. Spatial patterns described by the extended Fisher-Kolmogorov equation: periodic solutions. *SIAM J. Math. Anal.*, 28(6):1317–1353, 1997.

[27] L. Rajendran and K. Simons. Lipid rafts and membrane dynamics. *J. Cell. Sci.*, 118(6):1099–1102, 2005.

[28] X. Ren and J. Wei. The soliton-stripe pattern in the Seul-Andelman membrane. *Phys. D*, 188(3-4):277–291, 2004.

[29] S. Rozovsky, Y. Kaizuka, and J. T. Groves. Formation and spatio-temporal evolution of periodic structures in lipid bilayers. *J. Am. Chem. Soc.*, 127(1):36–7, 1 2005.

[30] M. Seul and D. Andelman. Domain shapes and patterns: the phenomenology of modulated phases. *Science*, 267(5197):476–483, 1995.

[31] K. Simons and E. Ikonen. Functional rafts in cell membranes. *Nature*, 387(6633):569–72, 6 1997.

[32] T. Taniguchi, K. Kawasaki, D. Andelman, and T. Kawakatsu. Phase transitions and shapes of two-component membranes and vesicles II: weak segregation limit. *J. Phys. II (France)*, 4:1333–1362, 1994.

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