LIMIT THEOREMS FOR SMOLUCHOWSKI DYNAMICS ASSOCIATED WITH CRITICAL CONTINUOUS-STATE BRANCHING PROCESSES

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We investigate the well-posedness and asymptotic self-similarity of solutions to a generalized Smoluchowski coagulation equation recently introduced by Bertoin and Le Gall in the context of continuous-state branching theory. In particular, this equation governs the evolution of the Lévy measure of a critical continuous-state branching process which becomes extinct (i.e., is absorbed at zero) almost surely. We show that a nondegenerate scaling limit of the Lévy measure (and the process) exists if and only if the branching mechanism is regularly varying at 0. When the branching mechanism is regularly varying, we characterize nondegenerate scaling limits of arbitrary finite-measure solutions in terms of generalized Mittag-Leffler series.

1. Introduction.

1.1. *Overview.* Recently Bertoin and Le Gall [3] observed a connection between the Smoluchowski coagulation equation and any critical continuous-state branching process (hereafter CSBP) that becomes extinct with probability one. Our general goal in this paper is to establish criteria for the existence of dynamic scaling limits in such branching processes, by extending methods that were recently used to analyze coagulation dynamics in the classically important 'solvable' cases (i.e., cases reduced to PDEs in terms of Laplace transforms).

Substantial progress has been made in recent years understanding the long-time behavior of solutions to solvable Smoluchowski coagulation equations. A rich analogy has been developed between dynamic scaling in these equations and classical limit theorems in probability, including the central

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limit theorem, the classification of stable laws and their domains of attraction [15, 17], and the Lévy-Khintchine representation of infinitely divisible laws [2, 19].

A new challenge in dealing with the coagulation equations that appear in the context of CSBPs is that they typically lack the homogeneity properties which were used extensively in earlier scaling analyses. On the other hand, use of a Laplace exponent transform leads to the study of a rather simple differential equation determined by the branching mechanism of the CSBP. Moreover, these branching mechanisms have a special structure — a Lévy-Khintchine representation formula expressed in terms of a certain measure related to family-size distribution.

To deal with the lack of homogeneity, we will adapt ideas from renormalization-group analysis, studying convergence of rescaled solutions together with the rescaled equations they satisfy. Such methods have been used to study asymptotic limits in a variety of problems including nonlinear parabolic PDE and KAM theory [6, 7]. An important point in this type of analysis, and one featured here, is that nontrivial scaling limits, if they exist, satisfy a homogeneous limiting equation. We describe these features in greater detail below.

1.2. *Continuous-state branching processes.* CSBPs arise as continuous-size, continuous-time limits of scaled Galton-Watson processes, which model the total number in a population of individuals who independently reproduce with identical rates and family-size distributions. A CSBP consists of a two-parameter random process $(t, x) \mapsto Z(t, x) \in [0, \infty)$ $(t \geq 0,\ x > 0)$. For fixed x, the process $t \mapsto Z(t, x)$ is Markov with initial value $Z(0, x) = x$. For fixed t, the process $x \mapsto Z(t, x)$ is an increasing process with independent and stationary increments. The right-continuous version of this process is a Lévy process with increasing sample paths. In particular, the process enjoys the branching property that $Z(t, x+y)$ has the same distribution as the sum of independent copies of $Z(t, x)$ and $Z(t, y)$ for all $t \geqslant 0$.

The structure of the process $Z(t, x)$ has a precise characterization via the Lamperti transform. That is, $t \mapsto Z(t, x)$ can be expressed as a subordinated Markov process with parent process $x + X_t$ where X_t is a spectrally positive Lévy process. More specifically, $Z(t, x) = x + X_{\Theta(t,x)}$ where the process $t \mapsto \Theta(t, x)$ has non-decreasing sample paths and formally solves $\partial_t \Theta = x + X_\Theta$. In this context, the Laplace exponent of X_t, denoted Ψ, is called the *branching mechanism* for $Z(t, x)$ and has Lévy-Khintchine representation

$$\Psi(u) = \alpha u + \beta u^2 + \int_{(0,\infty)} (e^{-ux} - 1 + ux\mathbf{1}_{\{x<1\}})\pi(dx), \tag{1.1}$$

where $\alpha \in \mathbb{R}$, $\beta \geqslant 0$, and $\int_{(0,\infty)}(1 \wedge x^2)\pi(dx) < \infty$. The representation (1.1), having the property $\Psi(0^+) = 0$, assumes no killing for the associated CSBP (cf. [12]).

Due to the nature of the Lamperti transform, $Z(t,x)$ satisfies

$$\mathbb{E}(e^{-qZ(t,x)}) = e^{-x\varphi(t,q)}, \tag{1.2}$$

where the spatial Laplace exponent φ solves the backward equation

$$\partial_t \varphi(t,q) = -\Psi(\varphi(t,q)), \quad q \in (0,\infty),\ t > 0. \tag{1.3}$$

Corresponding to $Z(0,x) = x$, the initial data takes the form $\varphi(0,q) = q$. It follows that $x \mapsto Z(t,x)$ is an increasing process with independent and stationary increments. As the Laplace exponent of a subordinator, φ has the Lévy-Khintchine representation

$$\varphi(t,q) = b_t q + \int_{(0,\infty)} (1 - e^{-qx})\nu_t(dx), \quad q \geq 0, \tag{1.4}$$

where $b_t \geq 0$ and $\int_{(0,\infty)}(1 \wedge x)\nu_t(dx) < \infty$. The quantities b_t and ν_t represent the drift coefficient and the Lévy jump measure, respectively. Taking $q \to \infty$ in (1.2) one sees that the CSBP becomes extinct in time t with positive probability (i.e. $\mathbb{P}[Z(t,x) = 0] > 0$) if and only if $\varphi(t,\infty) < \infty$. This means that $b_t = 0$ and $\rho_t < \infty$, where

$$\rho_t = \langle \nu_t, 1 \rangle \stackrel{\text{def}}{=} \int_{(0,\infty)} \nu_t(dx).$$

(See Proposition 3.7 for a characterization of branching mechanisms of this type.)

In the present work, we restrict our attention to the class of CSBPs for which the branching mechanism Ψ has the property

$$\Psi'(0^+) = \alpha - \int_{[1,\infty)} x\pi(dx) > -\infty. \tag{1.5}$$

That is, we assume Ψ has the representation

$$\Psi(u) = \hat{\alpha}u + \hat{\beta}u^2 + \int_{(0,\infty)} (e^{-ux} - 1 + ux)\pi(dx), \tag{1.6}$$

where $\hat{\alpha} \in \mathbb{R}$, $\hat{\beta} \geqslant 0$, and the *branching measure* $\pi(dx)$ verifies

$$\int_{(0,\infty)} (x \wedge x^2)\pi(dx) < \infty. \tag{1.7}$$

As shown in [10, 12], the CSBP associated to (1.6)-(1.7) is *conservative* in the sense that $\mathbb{P}(Z(t,x)<\infty)=1$ for all $t>0$. Of primary interest is the case of *critical* branching, which is distinguished by the property $\mathbb{E}(Z(t,x))=x$, and corresponds here to the value $\hat{\alpha}=0$.

1.3. *A generalized Smoluchowski coagulation equation.* The connection between branching and coagulation was described by Bertoin and Le Gall in [3] as follows. Informally, the Lévy measure $\nu_t(dx)$ corresponds to the 'size distribution' of the set of descendents of a single individual at the initial time 0. A more precise interpretation, when $b_t=0$, is that $Z(t,x)$ is the sum of atoms of a Poisson measure on $(0,\infty)$ with intensity $x\nu_t(dx)$. Based on the study of the genealogy of CSBPs as in [8] for example, each of these atoms may be interpreted as the size of a clan of individuals at time t that have the same ancestor at the initial time. (It is also possible to interpret $\nu_t(dx)$ as a continuum limit of scaled size distributions of clans descended from a single ancestor in a family of Galton-Watson processes. But precise discussion of this point lies outside the present paper's scope, and is left for future work.)

As shown in [3], the Lévy measure of a critical CSBP which becomes extinct almost surely satisfies a generalized type of *Smoluchowski coagulation equation*. This equation belongs to a general class of coagulation models that account for the simultaneous merging of k clusters with (possibly time-dependent) rate R_k. Specifically, the weak form of this equation is

$$\frac{d\langle \nu_t, f\rangle}{dt} = \sum_{k\geq 2} R_k I_k(\nu_t, f), \quad \text{for all } f\in C([0,\infty]). \tag{1.8}$$

Here

$$I_k(\nu, f) = \int_{(0,\infty)^k} \Big(f(x_1+\ldots+x_k) - \sum_{i=1}^k f(x_i)\Big) \prod_{i=1}^k \frac{\nu(dx_i)}{\langle \nu, 1\rangle}. \tag{1.9}$$

represents the expected change in the moment

$$\langle \nu, f\rangle \stackrel{\text{def}}{=} \int_{(0,\infty)} f(x)\,\nu(dx)$$

upon merger of k clusters with size distribution ν. For the evolution equation of the Lévy measure of a critical CSBP which becomes extinct almost surely, the rate constants R_k have a particular Poissonian structure expressed in terms of the branching mechanism and the total number $\rho_t=\langle \nu_t, 1\rangle$. Namely $R_k=R_k(\rho_t)$ where

$$R_k(\rho) = \frac{(-\rho)^k \Psi^{(k)}(\rho)}{k!} = \int_{(0,\infty)} \frac{(\rho y)^k}{k!} e^{-\rho y}\pi(dy) + \delta_{k2}\hat{\beta}\rho^2. \tag{1.10}$$

Here, $\hat{\beta}$ is the diffusion constant appearing in (1.6), and δ_{k2} is the Kronecker delta function, which is zero for $k \geq 3$. Combining the relations (1.8) and (1.10) gives the coagulation equation

$$\frac{d\langle \nu_t, f\rangle}{dt} = \sum_{k=2}^{\infty} \frac{(-\langle \nu_t, 1\rangle)^k \Psi^{(k)}(\langle \nu_t, 1\rangle)}{k!} I_k(\nu_t, f). \tag{1.11}$$

In the case of the special branching mechanism $\Psi(u) = u^2$, we recover the classical Smoluchowski coagulation equation with rate kernel $K(x, y) = 2$. Also, note that a Lévy measure solution of (1.11) represents a kind of *fundamental solution* for the coagulation equation, having the special property that as $t \to 0$ the measure $x\nu_t(dx)$ converges weakly to a delta function at the origin (see Remark 3.9).

1.4. *Results and organization.*

1.4.1. *Characterization of scaling limits for coagulation.* Our main results relate to long-time scaling limits of measure solutions of the coagulation equation (1.11) where Ψ is a critical branching mechanism for a CSBP which becomes extinct almost surely. That is, we investigate the existence of dynamic scaling limits of the form

$$\alpha(t)\nu_t(\lambda(t)^{-1}dx) \to \hat{\nu}(dx) \qquad \text{as} \quad t \to \infty, \tag{1.12}$$

for functions $\alpha, \lambda > 0$ and a finite measure $\hat{\nu}$. We show that the existence of nondegenerate limits is fundamentally linked to two conditions:

(i) *Regular variation of Ψ at zero with index $\gamma \in (1, 2]$.*
(ii) *Regular variation of the mass distribution function $\int_0^x y\nu_t(dy)$ at infinity with index $1 - \rho$, where $\rho \in (0, 1]$.*

First, assuming condition (i) holds, we prove (Theorem 5.1) that scaling limits of the form (1.12) exist if and only if condition (ii) holds at some initial time $t = t_0 \geqslant 0$. Since initial data satisfying (ii) are easily constructed, condition (i) gives a sufficient condition under which (1.11) admits nontrivial scaling solutions. The remarkable fact (Theorem 6.1) is that condition (i) is both *necessary and sufficient* for the scaling limit (1.12) to exist when ν_t is the *fundamental solution* (defined in Section 3.2).

The theorems cited above also provide a precise characterization of the limiting measure $\hat{\nu}$. Specifically, we show that (i) and (1.12) together imply that there exist constants $c_\lambda > 0$ and $\rho \in (0, 1]$, the latter given by (ii), such that

$$\hat{\nu}(dx) = \langle \hat{\nu}, 1\rangle F_{\gamma,\rho}(\langle \hat{\nu}, 1\rangle^{\frac{1}{\rho}} c_\lambda^{-1} dx), \tag{1.13}$$

where $F_{\gamma,\rho}$ is a generalized Mittag-Leffler probability distribution given by

$$F_{\gamma,\rho}(x) = \sum_{k=1}^{\infty} \frac{(r)_k}{k!} \frac{(-1)^{k+1} x^{sk}}{\Gamma(sk+1)}, \tag{1.14}$$

where $r = (\gamma-1)^{-1}$, $s = \rho(\gamma-1)$, and $(r)_k$ denotes the Pochhammer symbol

$$(r)_k = r(r+1)(r+2)\cdots(r+k-1).$$

Moreover, the corresponding solution ν_t is *asymptotically self-similar* in the sense that for all $t > 0$,

$$\alpha(\tau)\nu_{\tau t}(\lambda(\tau)^{-1}dx) \to t^{\frac{1}{1-\gamma}}\hat{\nu}(t^{\frac{1}{\rho(1-\gamma)}}\,dx) \tag{1.15}$$

as $\tau \to \infty$. In particular, the limiting function in (1.15) belongs to the family of self-similar solutions of (1.11) with homogeneous branching mechanism of the form $\hat{\Psi}(u) = \beta u^{\gamma}$, where $\beta = (\gamma-1)^{-1}\langle\hat{\nu}, 1\rangle^{1-\gamma}$. These solutions have the form

$$\nu_t(dx) = a(t)F_{\gamma,\rho}(a(t)^{\frac{1}{\rho}} c_\lambda^{-1} dx), \qquad a(t) = [\beta(\gamma-1)t]^{\frac{1}{1-\gamma}},$$

which generalizes the one-parameter family obtained in [17] corresponding to the classical Smoluchowski equation, with $\gamma = 2$ and $c_\lambda = 1$.

1.4.2. *Limit theorems for critical CSBPs.* Theorems 5.1 and 6.1 establish a necessary and sufficient condition for the existence of nondegenerate scaling limits of fundamental solutions, namely, Condition (i), above. We now describe two rather direct consequences of this fact in terms of scaling limits of the corresponding CSBP.

First, given a CSBP $Z(t,x)$ for which the corresponding Lévy measure is a fundamental solution of (1.11), we consider scaling limits of the form

$$\lambda(t)Z(t,\alpha(t)x) \xrightarrow{\mathcal{L}} \hat{Z}(x), \tag{1.16}$$

with $\alpha(t) \to \infty$ and $\lambda(t) \to 0$ as $t \to \infty$. That is, we scale by a factor of $\lambda(t)$ the total population at time t descended from an initial population of size $\alpha(t)x$, and we investigate the convergence in law of the rescaled process, with parameter x, to a non-degenerate Lévy process $\hat{Z}$. As above, we prove that such a limit exists if and only if Condition (i) holds. This is Theorem 7.2. In particular, if (1.16) holds, then for each $t > 0$

$$\lambda(\tau)Z(\tau t,\alpha(\tau)x) \xrightarrow{\mathcal{L}} t^{\frac{1}{\gamma-1}}\hat{Z}(t^{\frac{1}{1-\gamma}}x) \tag{1.17}$$

as $\tau \to \infty$, and the right-hand side is equal in law to the CSBP with Lévy measure given by $t^{\frac{1}{1-\gamma}}\hat{\mu}(t^{\frac{1}{1-\gamma}}dx)$, where $\hat{\mu}$ is the Lévy measure of $\hat{Z}$. In this way, we establish the self-similar form of the limiting CSBP.

Alternatively, one can consider initial population as fixed, and obtain a conditional limit theorem for critical continuous-state branching processes conditioned on non-extinction. In the context of discrete-state branching, several authors [5, 21, 24] have investigated limits of the form

$$\mathbb{P}(\lambda(t)Z_t \leqslant x \mid Z_t > 0) \to F(x) \tag{1.18}$$

as $t \to \infty$, where Z_t is the branching process and F is a nondegenerate distribution function on $(0, \infty)$. By various techniques (our own being most similar to a method of Borovkov [5]), the authors prove that for the special scaling function $\lambda(t) = \mathbb{P}(Z_t > 0)$ a limit of the form (1.18) exists if and only if the process Z_t has an offspring law corresponding to a regularly varying probability generating function. The question of whether the same regular variation condition is implied for a general scaling function $\lambda(t) \to 0$ was left open by Pakes [21]. Theorem 7.1 provides an affirmative answer to the continuous-state analog of the question posed by Pakes as an easy corollary of Theorem 6.1.

Also implied by Theorem 7.1 are the conditional limit theorems obtained by Kyprianou [13] for critical CSBPs with power-law branching mechanism (the so-called α-stable case), and those obtained by Li [16] for critical CSBPs with the property $\Psi''(0^+) < \infty$. In all cases above, including the discrete cases previously mentioned, limiting distributions are characterized by relations of the form (1.14).

Let us note that, by comparison, non-critical CSBPs admit scaling limits of a simpler form. Indeed, a well-known result of Grey [10] states that for any supercritical CSBP with $\Psi'(0^+) > -\infty$, and for any critical or subcritical CSBP with $\Psi'(\infty) < \infty$ (in the latter case, the CSBP remains positive almost surely – see Proposition 3.7(i)), there exists a scaling limit of the form (1.16), where $\alpha(t) = 1$ and $\varphi(t, \lambda(t)) = const.$, with φ solving (1.3). On the other hand, it follows directly from the work of Lambert [14] that any subcritical CSBP which becomes extinct almost surely admits a limit of the form (1.16) with scaling functions given by $\alpha(t) = 1/\varphi(t, \infty)$ and $\lambda(t) = 1$. In contrast with Theorem 7.2, only one nontrivial scaling function is needed in each of the cases above.

1.4.3. *Well-posedness.* For the sake of completeness we also give an account of well-posedness for the coagulation equation. That is, we establish the existence and uniqueness of weak solutions of (1.11) when Ψ is a critical branching mechanism and the initial data is a finite measure (Corollary

3.12). Here, we essentially tie together the ideas of Bertoin and LeGall [3], Norris [20], and Menon and Pego [17] with a few new proofs and observations. In particular, we provide a simple and direct account of well-posedness for the evolution of the Lévy measure ν_t in (1.4) (Proposition 3.7). The point is that equation (1.3) preserves the property that $\varphi(t,\cdot)$ has a completely monotone derivative. For an initial cluster size distribution given by a finite measure, the latter property amounts to a well-posedness result for Smoluchowski dynamics.

1.4.4. *Outline of the paper.* We now give a brief outline of the paper. Section 2 delineates some basic notation and definitions. Section 3 is dedicated to well-posedness results. In Section 4, we derive the family of self-similar solutions to (1.11) associated with generalized Mittag-Leffler laws. Section 5 is dedicated to a study of scaling limits of the form (1.12) in the case of a regularly varying branching mechanism Ψ. In Section 6, we consider scaling limits of fundamental solutions. Finally, in Section 7, we reformulate our scaling results in terms of limit theorems for CSBPs.

2. Preliminaries. We begin with some notation that will be repeatedly used throughout this paper. Let E be the open interval $(0,\infty)$, and $\overline{E}$ denote the extended interval $[0,\infty]$. We use $C(\overline{E})$ to denote the space of continuous functions $f:\overline{E}\to\mathbb{R}$, equipped with the L^∞-norm.

Three spaces of measures that arise often in our context are:

- The space $\mathcal{M}_+$, consisting of positive Radon measures on E equipped with the *vague topology*. We recall that if μ, μ_1, $\mu_2,\ldots$ are measures in $\mathcal{M}_+$, then μ_n *converges vaguely* to μ as $n\to\infty$ (denoted by $\mu_n \xrightarrow{v} \mu$) if $\langle \mu_n,\phi\rangle \to \langle \mu,\phi\rangle$ for all $\phi\in C_c(E)$. Here $C_c(E)$ denotes the space of continuous functions on E with compact support, and $\langle \mu, f\rangle$ denotes the integral of f with respect to the measure μ.
- The space $\mathcal{M}_F$, consisting of finite positive measures on E, equipped with the *weak topology*. That is, if μ, μ_1, $\mu_2,\ldots$ are measures in $\mathcal{M}_F$, then we say μ_n *converges weakly* to μ as $n\to\infty$ (denoted by $\mu_n \xrightarrow{w} \mu$) if $\langle \mu_n,\phi\rangle \to \langle \mu,\phi\rangle$ for all $\phi\in C_b(E)$. Here $C_b(E)$ denotes the space of bounded continuous functions on E.
- The space $\mathcal{M}_{1\wedge x}$, consisting of the set of measures $\mu\in\mathcal{M}_+$ such that

$$\int_{(0,\infty)} (1\wedge x)\mu(dx) < \infty.$$

2.1. *Branching mechanisms and Bernstein functions.*

DEFINITION 2.1. We say a function $\Psi\colon E \to \mathbb{R}$ is a *branching mechanism* if it admits the representation

$$\Psi(u) = \hat{\alpha}u + \hat{\beta}u^2 + \int_E (e^{-ux} - 1 + ux)\,\pi(dx), \tag{2.1}$$

where $\hat{\alpha} \in \mathbb{R}$, $\hat{\beta} \geqslant 0$ and $\pi \in \mathcal{M}_+$ with $\int_E (x \wedge x^2)\pi(dx) < \infty$ (equivalently, $x\pi \in \mathcal{M}_{1\wedge x}$). The branching mechanism is called *critical, subcritical, or supercritical* according to the conditions $\hat{\alpha} = 0$, $\hat{\alpha} > 0$, or $\hat{\alpha} < 0$, respectively.

DEFINITION 2.2. We say that $f \in C^\infty(E)$ is a *Bernstein function* if $f \geqslant 0$ and $(-1)^k f^{(k+1)} \geqslant 0$ for all integers $k \geqslant 0$.

In other words, f is a Bernstein function if f is non-negative and f' is completely monotone. It is well known (see for instance [23]) that a function is Bernstein if and only if it admits the representation

$$f(q) = a + bq + \int_E (1 - e^{-qx})\,\mu(dx), \tag{2.2}$$

where $a, b \geqslant 0$ and $\mu \in \mathcal{M}_{1\wedge x}$. Note that f is strictly positive if and only if $(a, b, \mu) \neq (0, 0, 0)$. On the other hand, a function $\Psi\colon E \to E$ belongs to the set of critical or subcritical branching mechanisms if and only if $\Psi(0^+) = 0$ and Ψ' is a positive Bernstein function. The following lemma, for which we have found no obvious reference, establishes a deeper relation between set of critical or subcritical branching mechanisms and Bernstein functions.

LEMMA 2.3. *Assume $\Psi\colon E \to E$ is a critical or subcritical branching mechanism. Then, the inverse function Ψ^{-1} is a Bernstein function.*

PROOF. Let $f = \Psi^{-1}$ and $g = 1/\Psi'$. Note g is completely monotone, since $x \mapsto 1/x$ is completely monotone and Ψ' is a positive Bernstein function, as observed above. Since f is positive and $f' = g \circ f$, it directly follows that f' is completely monotone, from [19, Lemma 5.5]. □

3. Well-posedness for Smoluchowski dynamics. In this section we define a notion of weak solution for the generalized Smoluchowski equation (1.11). As we will show, the question of existence of weak solutions amounts to a study of (1.3). Several estimates appearing in Sections 3.1 and 3.3 have either been sketched in [3] from a probabilistic point of view, or are straightforward extensions of the well-posedness theory in [17]. The originality of our treatment lies mainly in Lemma 2.3 and its use in the proof of Proposition 3.7. The remaining estimates have been simplified by various degrees and organized for convenience of the reader.

3.1. *Weak solutions.* In this section, we consider a critical branching mechanism Ψ having the representation (2.1) with $\hat{\alpha} = 0$. Following the approach in [17, 20], we associate to each finite, positive measure $\nu \in \mathcal{M}_F$ the continuous linear functional $L(\nu) : C(\overline{E}) \to \mathbb{R}$, defined by

$$\langle L(\nu), f\rangle = \sum_{k\geq 2} R_k(\langle \nu, 1\rangle) I_k(\nu, f) = \sum_{k=2}^{\infty} \frac{(-\langle \nu, 1\rangle)^k \Psi^{(k)}(\langle \nu, 1\rangle)}{k!} I_k(\nu, f) \tag{3.1}$$

where I_k and R_k are defined by (1.9) and (1.10), respectively. To verify continuity of $L(\nu)$, we observe $|I_k(\nu, f)| \leq (k+1)\|f\|_{C(\overline{E})}$. Thus for $m = \langle \nu, 1\rangle$, equations (2.1) and (3.1) give

$$\begin{aligned}
|\langle L(\nu), f\rangle| &= \left| \hat{\beta} m^2 I_2(f) + \sum_{k=2}^{\infty} I_k(f) \int_E \frac{(mx)^k}{k!} e^{-mx}\, \pi(dx) \right| \\
&\leqslant \|f\|_{C(\overline{E})} \left[3\hat{\beta} m^2 + \int_E \left[\sum_{k=2}^{\infty} \frac{m^k x^k}{(k-1)!} + \sum_{k=2}^{\infty} \frac{m^k x^k}{k!} \right] e^{-mx}\, \pi(dx) \right] \\
&= K(m) \|f\|_{C(\overline{E})},
\end{aligned}$$

where $K(m) = 3\hat{\beta} m^2 + 2m\Psi'(m) - \Psi(m) < \infty$, establishing continuity of $L(\nu)$. Observe for future use, that

$$K(m) = \int_0^m \left[\frac{3}{2} \hat{\beta} u + 2u\Psi''(u) + \Psi'(u) \right] du. \tag{3.2}$$

Since $\Psi', \Psi'' \geqslant 0$, the function $m \mapsto K(m)$ is positive and increasing.

With this, the natural notion of weak solutions to (1.11) is as follows.

Definition 3.1. We say that a weakly measurable function $\nu\colon E \to \mathcal{M}_F$ is a *weak solution* of (1.11) if

$$\langle \nu_t, f\rangle = \langle \nu_s, f\rangle + \int_s^t \langle L(\nu_\tau), f\rangle \, d\tau \tag{3.3}$$

for all $t, s > 0$ and for all $f \in C(\overline{E})$. If, additionally, there exists $\nu_0 \in \mathcal{M}_F$ such that ν_t converges weakly to ν_0 as $t \to 0$, then we say $\nu\colon [0, \infty) \to \mathcal{M}_F$ is a weak solution of (1.11) with initial data ν_0.

To any function $\nu\colon E \to \mathcal{M}_{1\wedge x}$, we associate the function

$$\varphi(t, q) \stackrel{\text{def}}{=} \int_E (1 - e^{-qx}) \nu_t(dx). \tag{3.4}$$

Our next result shows that weak solutions to (1.11) are characterized by (1.3) for the associated function φ.

THEOREM 3.2. *Let $\Psi\colon E \to E$ be a critical branching mechanism. Assume $\nu\colon E \to \mathcal{M}_F$ and that φ is related to ν by (3.4). Then, ν is a weak solution of* (1.11) *if and only if φ solves (1.3).*

PROOF. Let ν and φ be as described. First, we claim that φ satisfies (1.3) if and only if (3.3) holds for the family of test functions $f_q(x) \stackrel{\text{def}}{=} 1 - e^{-qx}$, $0 < q \leqslant \infty$. Note, carefully, that we include the function $f_\infty = 1$ in this family. Indeed, since $\nu_t \in \mathcal{M}_F$, we have

$$\varphi(t,q) \to \varphi(t,\infty) \stackrel{\text{def}}{=} \langle \nu_t, 1\rangle < \infty$$

as $q \to \infty$, so that if (1.3) holds for $0 < q < \infty$, it also holds for $q = \infty$.

Note that since Ψ is a critical branching mechanism given by (2.1), it has an analytic extension defined in the right half of the complex plane. Thus, the Taylor series of $\Psi(u)$ expanded about any $m > 0$ converges whenever $0 < u < m$ and gives

$$\Psi(u) = \sum_{k=0}^{\infty} \frac{\Psi^{(k)}(m)}{k!}(u-m)^k, \quad \Psi'(u) = \sum_{k=0}^{\infty} \frac{\Psi^{(k)}(m)}{k!}(u-m)^{k-1}k. \tag{3.5}$$

These formulae hold also for $u = 0$, with $\Psi(0) = 0$, $\Psi'(0) = 0$, due to the consistent sign of the terms for $k \geq 2$. Writing $m = \langle \nu_t, 1\rangle$, we compute that for $0 < q \leqslant \infty$,

$$\begin{aligned} m^k I_k(\nu_t, f_q) &= \int_{E^k} \left[f_q\left(\sum_{i=1}^k x_i\right) - \sum_{i=1}^k f_q(x_i)\right] d\nu_t^k \\ &= \int_{E^k} \left[1 - \prod_{i=1}^k e^{-qx_i} - \sum_{i=1}^k \left(1 - e^{-qx_i}\right)\right] d\nu_t^k \\ &\qquad\qquad = m^k - (m - \varphi(t,q))^k - km^{k-1}\varphi(t,q). \end{aligned}$$

Using this expression (which vanishes for $k = 0$ and 1) in (3.1) and invoking (3.5), since $0 < \varphi(t,q) < m$ we find

$$\begin{aligned} \langle L(\nu_t), f_q\rangle &= \sum_{k=0}^{\infty} \frac{(-1)^k \Psi^{(k)}(m)}{k!} \left[m^k - (m - \varphi(t,q))^k - km^{k-1}\varphi(t,q)\right] \\ &\qquad = \Psi(0) - \Psi(\varphi(t,q)) + \varphi(t,q)\Psi'(0) = -\Psi \circ \varphi(t,q). \end{aligned}$$

Therefore (3.3) holds for $f = f_q$ if and only if

$$\varphi(t,q) - \varphi(s,q) = -\int_s^t \Psi(\varphi(\tau,q))\,d\tau$$

for all $s, t > 0$. This proves the claim.

In particular, if ν is a weak solution of (1.11), then φ solves (1.3). On the other hand, if φ solves (1.3), then (3.3) holds for all test functions f_q, $0 < q \leqslant \infty$. This family of test functions spans a dense subset of the metric space $C(\overline{E})$. Now, given $f \in C(\overline{E})$ and $\varepsilon > 0$, choose $g \in \operatorname{span}\{f_q : 0 < q \leqslant \infty\}$ such that $\|f - g\| < \varepsilon$. By linearity, (3.3) holds for the test function g. Therefore, assuming for definiteness that $t > s$, we have

$$\begin{aligned}
&\left|\langle \nu_t, f\rangle - \langle \nu_s, f\rangle - \int_s^t \langle L(\nu_\tau, f)\rangle\,d\tau\right| \\
&\qquad = \left|\langle \nu_t, f-g\rangle - \langle \nu_s, f-g\rangle - \int_s^t \langle L(\nu_\tau), f-g\rangle\,d\tau\right| \\
&\qquad \leqslant \|f-g\|_{C(\overline{E})}\left(\langle \nu_t, 1\rangle + \langle \nu_s, 1\rangle + \int_s^t K(\langle \nu_\tau, 1\rangle)\,d\tau\right) \\
&\qquad \leqslant \varepsilon[2\langle \nu_s, 1\rangle + (t-s)K(\langle \nu_s, 1\rangle)],
\end{aligned}$$

where the function K is given by (3.2), and we use (1.3) with $q = \infty$ to infer $\langle \nu_t, 1\rangle \leq \langle \nu_s, 1\rangle$. Taking $\varepsilon \to 0$ shows that (3.3) holds for all $f \in C(\overline{E})$. This completes the proof. □

Remark 3.3. Bertoin and LeGall [3] propose a weaker form of Smoluchowski's equation that requires only $\nu_t \in \mathcal{M}_{1\wedge x}$, not $\mathcal{M}_F$, but which still transforms to (1.3). In particular, they show that if $\Psi'(\infty) = \infty$ (see Proposition 3.7, below), then the Lévy measure of the associated CSBP verifies this weak form for the special test functions f_q, $0 < q < \infty$. However, there appear to be no obvious estimates available to deal with a general test function $f \in C(\overline{E})$.

3.2. *Fundamental solutions.* For any weak solution $\nu\colon E \to \mathcal{M}_F$ of the generalized Smoluchowski equation (1.11), the solution $\varphi(t,q)$ of (1.3) has a finite limit as $t \to 0$ whether or not ν_t has a weak limit as $t \to 0$. Indeed, if $\varphi(t, q_0) \to \infty$ as $t \to 0$ for some $q_0 > 0$, then, by a translation invariance of solutions, one shows that for any $q > q_0$ there exists $t_q > 0$ such that $\varphi(t,q) = \varphi(t - t_q, q_0) \to \infty$ as $t \to t_q$, which contradicts $\nu_{t_q} \in \mathcal{M}_F$.

It follows that φ has the convenient representation

$$\varphi(t,q) = \Phi(t,\varphi(0,q)), \tag{3.6}$$

where $\varphi(0,q) \stackrel{\text{def}}{=} \varphi(0^+,q)$ and where Φ solves the initial value problem

$$\begin{cases} \partial_t \Phi(t,q) = -\Psi(\Phi(t,q)), \qquad q \in E \\ \Phi(0,q) = q. \end{cases} \tag{3.7}$$

The functions $\Phi_t = \Phi(t,\cdot)$ have the semigroup property $\Phi_{t+s} = \Phi_t \circ \Phi_s$ for $t,s > 0$. Because of the composition structure (3.6), we make the following definition.

Definition 3.4. Assume $\Psi\colon E \to E$ is a critical branching mechanism. We say that a function $\mu\colon E \to \mathcal{M}_F$ is the *fundamental solution* of the generalized Smoluchowski equation (1.11) if the function

$$\Phi(t,q) = \int_E (1 - e^{-qx})\mu_t(dx) \tag{3.8}$$

solves the initial value problem (3.7), where $\Phi(0,q) \stackrel{\text{def}}{=} \Phi(0^+,q)$.

The fundamental solution relates solutions of the generalized Smoluchowski equation to their initial data via solutions of a linear problem; see Remark 3.10 below for details. But first we establish necessary and sufficient criteria for the existence of a fundamental solution, and develop the basis for our discussion of well-posedness theory for weak solutions with initial data.

Definition 3.5. We say that a branching mechanism $\Psi : E \to \mathbb{R}$ satisfies *Grey's condition* [10] provided $\Psi(\infty) = \infty$ and

$$\int_a^\infty \frac{1}{\Psi(u)}\,du < \infty \qquad \text{for some } a > 0. \tag{3.9}$$

Remark 3.6. It is well-known that Grey's condition gives a necessary and sufficient condition under which solutions to (1.3) have finite-time blow-up, backward in time. We also mention that Bertoin and LeGall [3] use the term *Condition E* to describe Grey's condition.

Proposition 3.7. *Let Φ be the unique solution of the initial value problem (3.7), where $\Psi\colon E \to \mathbb{R}$ is any branching mechanism of the form (2.1).*

Then, for each fixed $t \geqslant 0$, the map $\Phi(t,\cdot) : E \to E$ is a Bernstein function. More precisely,

$$\Phi(t,q) = b_t q + \int_E (1 - e^{-qx})\mu_t(dx) \tag{3.10}$$

for some $b_t \geqslant 0$ and $\mu_t \in \mathcal{M}_{1\wedge x}$. Furthermore, the following properties hold.

(i) $b_t = 0$ for some (equivalently all) $t > 0$ if and only if $\Psi'(\infty) = \infty$.
(ii) $b_t = 0$ and $\mu_t \in \mathcal{M}_F$ for some (equivalently all) $t > 0$ if and only if Ψ satisfies Grey's condition.

Remark 3.8. While the facts above can be infered from CSBP theory, we summarize them here for convenience of the reader, and give a proof independent of the latter theory. In particular, we recognize equation (3.10) as the Lévy-Khintchine formula for the Laplace exponent of a CSBP with branching mechanism Ψ, as sketched in Section 1.2. In this context, property (ii) states that a CSBP becomes extinct by time t with positive probability ($\Phi(t,\infty) < \infty$) if and only if Grey's condition holds. For critical CSBPs, this is the case if and only if the process becomes extinct almost surely. Thus, property (ii) establishes a one-to-one correspondence between fundamental solutions of (1.11) and Lévy measures for critical CSBPs that become extinct almost surely.

Proof. Our proof is based on the implicit Euler method. First we will show that each iteration of the implicit Euler scheme for (3.7) yields a Bernstein function. Then, since the set of Bernstein functions is closed under composition and pointwise limits [23, pp. 20-21], convergence of the implicit Euler scheme implies that $\Phi(t,\cdot)$ is Bernstein.

By assumption, Ψ has the representation (2.1). Since Ψ' is increasing and $\Psi'(0^+) = \hat{\alpha} \in \mathbb{R}$, it follows that Ψ is Lipshitz on bounded intervals. Hence (3.7) has a unique solution. Furthermore, the solution remains positive for all time since the equation is autonomous and $\Psi(0^+) = 0$. Also, since $\partial_t \Phi = -\Psi(\Phi) \leqslant -\hat{\alpha}\Phi$, we obtain, for all $t, q \geqslant 0$, the bound

$$0 \leqslant \Phi(t,q) \leqslant q e^{-\hat{\alpha} t}. \tag{3.11}$$

For fixed $t > 0$ and $N \in \mathbb{N}$, let $h = t/N$ and consider the iteration scheme

$$\hat{\Phi}_{n+1}(q) = \hat{\Phi}_n(q) - h\Psi(\hat{\Phi}_{n+1}(q)), \quad n = 0, 1, \ldots, N-1. \tag{3.12}$$

Note that for N sufficiently large, the function $F_N : E \to E$ defined by

$$F_N(u) = u + \frac{t}{N}\Psi(u) = u + h\Psi(u) \tag{3.13}$$

is a bijection, since $F_N'(u) = 1 + h\Psi'(u) \geqslant 1 + \hat{\alpha}h > 0$. By consequence, $\hat{\Phi}_{n+1}(q) = F_N^{-1}(\hat{\Phi}_n(q))$ is well-defined and positive for all $q > 0$ and $n = 0, 1, \ldots, N-1$. Since Ψ is locally smooth on E and we have the bound (3.11), the proof of the pointwise convergence $\hat{\Phi}_N(q) \to \Phi(t, q)$ as $N \to \infty$ for each $q > 0$ is standard and we omit it.

Observe now that F_N is a branching mechanism since it has a representation of the form (2.1). Hence by Lemma 2.3, F_N^{-1} is a Bernstein function, provided N is sufficiently large. Since the set of Bernstein functions is closed under composition, and $\hat{\Phi}_0(q) = q$ is a Bernstein function, it follows $\hat{\Phi}_n$ is a Bernstein function for each $n = 0, \ldots, N$. Finally, the pointwise convergence $\hat{\Phi}_N \to \Phi(t, \cdot)$ as $N \to \infty$ implies that $\Phi(t, \cdot)$ is a Bernstein function, by [23, Cor. 3.8]. The representation (2.2) then gives

$$\Phi(t, q) = a_t + b_t q + \int_E (1 - e^{-qx})\mu_t(dx), \tag{3.14}$$

for some $a_t, b_t \geqslant 0$ and $\mu_t \in \mathcal{M}_{1\wedge x}$. Note that (3.11) implies $a_t = \Phi(t, 0^+) = 0$ for all $t \geqslant 0$.

Next we establish (i). Observe that $b_t = \partial_q \Phi(t, \infty)$, and that the relation

$$\partial_q \Phi(t, q) = e^{-\int_0^t \Psi'(\Phi(s,q))\, ds} \tag{3.15}$$

is an easy consequence of (1.3). If $\Psi'(\infty) < \infty$, then since Ψ' and $\Phi(s, \cdot)$ are increasing, for any $t > 0$ we find

$$b_t = e^{-\int_0^t \Psi'(\Phi(s,\infty))\, ds} \geqslant e^{-t\Psi'(\infty)} > 0.$$

Conversely, suppose $b_t > 0$ for some $t > 0$, then (3.14) implies $\Phi(t, \infty) = \infty$ and hence $\Phi_t = \Phi(t, \cdot)$ is a surjection onto E. Since $\Phi_t = \Phi_s \circ \Phi_{t-s}$ for $0 < s < t$, Φ_s is also a surjection and hence $\Phi(s, \infty) = \infty$. Thus $b_t = e^{-t\Psi'(\infty)} > 0$. Hence, $\Psi'(\infty) < \infty$. This completes the proof of (i).

Finally, let us show that (ii) holds. First suppose $b_t = 0$ and $\mu_t \in \mathcal{M}_F$ for some $t > 0$. We claim Grey's condition holds. From (3.10) we have

$$\Phi(t, \infty) = \int_E \mu_t(dx) < \infty. \tag{3.16}$$

Assume for the sake of contradiction that $\Psi(\infty) < \infty$. Then $\Psi'(\infty) \leqslant 0$, and we have by (2.1), $\hat{\beta} = 0$ and $\hat{\alpha} = \Psi'(0^+) \leqslant -\int x\pi(dx)$. In that case, $\Psi(u) < \int_E (e^{-ux} - 1)\pi(dx) < 0$ for all $u \in E$, and $\Phi(\cdot, q)$ is increasing. Hence $\Phi(t, q) \geqslant q \to \infty$ as $q \to \infty$, which contradicts (3.16). This shows $\Psi(\infty) = \infty$.

Now, assume (3.9) fails. As remarked above, failure of this condition ensures that all solutions of (1.3) with finite initial data remain finite backward in time. In particular, by uniqueness and positivity of solutions of (3.7), we have that for all $q > 0$,

$$q = \Phi(0, q) = \Phi(-t, \Phi(t, q)) \leqslant \Phi(-t, \Phi(t, \infty)),$$

which is finite and independent of q. Note that we used monotonicity of Φ in q for the inequality. This is a contradiction. Hence, Grey's condition holds.

Conversely, assume Grey's condition holds, and let

$$q_* \stackrel{\text{def}}{=} \inf\{q \in E : \Psi(q) > 0\}$$

denote the largest equilibrium solution of (1.3). Then, for any $q > q_*$ there exists $t_q < 0$ such that $E \ni \Phi(t, q) \to \infty$ as $t \to t_q^+$. We define the special solution

$$\Phi_*(t) = \Phi(t - t_q, q),$$

which is independent of $q > q_*$ and has the property $E \ni \Phi_*(t) \to \infty$ as $t \to 0^+$. Since $\Phi(0, q) < \Phi_*(0^+) = \infty$, we deduce, by uniqueness of solutions of (1.3), that $\Phi(t, q) < \Phi_*(t)$ for all $t, q > 0$. Therefore, taking $q \to \infty$, shows $\Phi(t, \infty) < \infty$. That is, $b_t = 0$ and $\mu_t \in \mathcal{M}_F$, for all $t > 0$. □

REMARK 3.9. Note that the Bernstein functions $\Phi(t, \cdot)$ converge pointwise to the function $\Phi(0, q) = q$ as $t \to 0$. It follows that

$$\partial_q \Phi(t, q) = b_t + \int_E e^{-qx} x \mu_t(dx)$$

converges pointwise to $\partial_q \Phi(0, q) = 1 = \int_{[0,\infty)} e^{-qx} \delta_0(dx)$ as $t \to 0$ (see, for instance, [23, p. 21]). Therefore, by the continuity theorem (cf. [9, XIII.1]), the measures $\kappa^{(t)}(dx) = b_t \delta_0(dx) + x\mu_t(dx)$ converge vaguely to the measure $\delta_0(dx)$ in the space of positive Radon measures on $[0, \infty)$. In particular, if Ψ is a critical branching mechanism satisfying Grey's condition, then the mass measure, $x\mu_t(dx)$, converges weakly to a delta mass at zero as $t \to 0$. Moreover, the total mass at time t, given by $\partial_q \Phi(t, 0^+)$, is conserved by (3.15).

REMARK 3.10. Formula (3.6) has a standard probabilistic interpretation: For fixed t, the Lévy process with Lévy measure ν_t is subordinated to the Lévy process with Lévy measure ν_0 by the directing process $Z(t, \cdot)$ with Lévy measure μ_t. In terms of generators, this corresponds however to a deterministic formula ((3.20) below) that expresses the weak solution ν_t of the nonlinear Smoluchowski equation in terms of the fundamental solution μ_t and the kernel Q_s of a convolution semigroup (a Lévy diffusion) given by

$$e^{sA}f(x) = \int_{\mathbb{R}} f(x+y)Q_s(dy), \tag{3.17}$$

with generator A determined from $\nu_0(dx)$ by

$$Af(x) = \int_E (f(x+y) - f(x))\nu_0(dy), \tag{3.18}$$

for all smooth $f \in C_c(\mathbb{R})$. Supposing that $f(x) = e_q(x) := e^{-qx}$ for $x \geq 0$, we find that for $x \geq 0$,

$$Af(x) = -\varphi(0,q)e_q(x), \qquad e^{sA}f(x) = e_q(x)\int_{[0,\infty)} e^{-qy}Q_s(dy). \tag{3.19}$$

(Note $Q_s(dx)$ retains an atom at 0 with mass $e^{-s\langle \nu_0,1\rangle}$.) Hence

$$\int_E (1-e^{-qx})\int_E Q_s(dx)\mu_t(ds) = \int_E (1-e^{-s\varphi(0,q)})\mu_t(ds) = \Phi(t,\varphi(0,q)).$$

Consequently, from (3.6) we infer that

$$\nu_t(dx) = \int_E Q_s(dx)\,\mu_t(ds). \tag{3.20}$$

Note that Q_s is determined by solving a linear equation, namely $\partial_t u = Au$.

3.3. *Weak solutions with initial data.* In this section we establish the existence and uniqueness of weak solutions of (1.11) with initial data $\nu_0 \in \mathcal{M}_F$.

Lemma 3.11. *Assume $\Psi\colon E \to \mathbb{R}$ is a branching mechanism, and let $\nu_0 \in \mathcal{M}_{1\wedge x}$. Then, there exists a unique vaguely continuous map $\nu\colon [0,\infty) \to \mathcal{M}_+$ such that φ, defined by (3.4), is a solution of (1.3) with initial data*

$$\varphi_0(q) = \int_E (1-e^{-qx})\nu_0(dx) \tag{3.21}$$

Furthermore, if $\nu_0 \in \mathcal{M}_F$, then $\nu_t \in \mathcal{M}_F$ for all $t > 0$, and $\nu\colon [0,\infty) \to \mathcal{M}_F$ is weakly continuous. In this case, we have for $t > 0$

$$\frac{d\langle \nu_t, 1\rangle}{dt} = -\Psi(\langle \nu_t, 1\rangle). \tag{3.22}$$

PROOF. Note that (3.6) represents the unique solution of (1.3) with initial data $\varphi(0,q)=\varphi_0(q)$. By Proposition 3.7, the map $\Phi(t,\cdot)$ is a Bernstein function for all $t\geqslant 0$. Also, φ_0 is a Bernstein function as it admits a representation of the form (2.2). Therefore the composite function, given by (3.6), is a Bernstein function for all $t\geqslant 0$. Furthermore, we have $\varphi(t,0^+)=0$ and

$$\partial_q\varphi(t,\infty)=\lim_{q\to\infty}\partial_q\Phi(t,\varphi_0(q))\cdot\lim_{q\to\infty}\varphi_0'(q).$$

By assumption, the latter limit vanishes, and since $\partial_q\Phi(t,\cdot)$ is decreasing, the former limit is finite. Hence, $\partial_q\varphi(t,\infty)=0$ and the representation (2.2) for φ reduces to

$$\varphi(t,q)=\int_E\frac{1-e^{-qx}}{x}\mu_t(dx), \tag{3.23}$$

for some $\mu_t\in\mathcal{M}_+$ with $x^{-1}\,\mu_t\in\mathcal{M}_{1\wedge x}$. Note that the measure μ_t is determined uniquely by its Laplace transform $\partial_q\varphi(t,q)$. Further, $\partial_q\varphi(t,q)$ is continuous in t, since $\partial_t\partial_q\varphi(t,q)=\Psi'(\varphi(t,q))\Psi(\varphi(t,q))$. Therefore, viewing μ_t as a measure on $[0,\infty)$ (which assigns measure zero to the point $\{0\}$), it follows from the continuity theorem (cf. [9, XIII.1]) that the map $\mu\colon[0,\infty)\to\mathcal{M}_+([0,\infty))$ is vaguely continuous, where $\mathcal{M}_+([0,\infty))$ is the space of Radon measures on $[0,\infty)$. In particular, for all $f\in C_c(E)\subset C_c([0,\infty))$, we have $\langle f,\mu_s\rangle\to\langle f,\mu_t\rangle$ as $s\to t$. That is, the map $\mu:[0,\infty)\to\mathcal{M}_+$ is vaguely continuous. Hence, the map $t\mapsto\nu_t\stackrel{\text{def}}{=}x^{-1}\mu_t\in\mathcal{M}_+$ is also vaguely continuous (since $\mu_s\xrightarrow{v}\mu_t$ implies $\phi\cdot\mu_s\xrightarrow{v}\phi\cdot\mu_t$ for any $\phi\in C_c(E)$). This establishes the first part of the lemma. Finally, observe

$$\langle\nu_t,1\rangle=\lim_{q\to\infty}\varphi(t,q)=\lim_{q\to\infty}\Phi(t,\varphi_0(q))=\Phi(t,\langle\nu_0,1\rangle).$$

Thus if $\nu_0\in\mathcal{M}_F$, equation (3.22) follows from (3.7) by taking $q=\langle\nu_0,1\rangle$. Since (3.22) implies $t\mapsto\langle\nu_t,1\rangle$ is continuous on $[0,\infty)$, we conclude that $\nu\colon[0,\infty)\to\mathcal{M}_F$ is weakly continuous (see, for instance, [1, Theorem 30.8]). □

COROLLARY 3.12. *Assume $\Psi\colon E\to E$ is a critical branching mechanism, and let $\nu_0\in\mathcal{M}_F$. Then, there exists a unique weak solution of* (1.11) *with initial data ν_0.*

PROOF. First, by Lemma 3.11, there exists a weakly continuous map $\nu\colon[0,\infty)\to\mathcal{M}_F$ such that φ, defined by (3.4), satisfies (1.3) with initial data (3.21). In particular, $\nu_t\in\mathcal{M}_F$ converges weakly to ν_0 as $t\to 0$. By Theorem 3.2, ν restricted to E verifies (3.3). Hence, by definition, ν is weak solution of (1.11) with initial data ν_0. Uniqueness of the solution follows from Lemma 3.11. □

4. Self-similarity and Generalized Mittag-Leffler functions. Recall from [17], that the classical Smoluchowski equation, which corresponds here to the special branching mechanism $\Psi(u) = u^2$, admits a one-parameter family of self-similar solutions of the form

$$\nu_t(dx) = t^{-1} F_\rho(t^{-\frac{1}{\rho}}\, dx), \qquad t > 0, \quad 0 < \rho \leqslant 1,$$

where F_ρ is given by the classical Mittag-Leffler distribution function, satisfying

$$F_\rho(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^{\rho k}}{\Gamma(\rho k + 1)}, \qquad \int_E (1 - e^{-qx}) F_\rho(dx) = \frac{1}{1 + q^{-\rho}}. \tag{4.1}$$

We now discuss the existence of self-similar solutions for homogeneous branching mechanisms of the form

$$\Psi(u) = \beta u^\gamma, \qquad 1 < \gamma \leqslant 2, \quad \beta > 0. \tag{4.2}$$

As in [17], we look for self-similar solutions of the form

$$\nu_t(dx) = \alpha(t) F(\lambda(t)^{-1} dx), \tag{4.3}$$

where F is a probability distribution and $\alpha, \lambda > 0$ are differentiable. In this case, the function φ, defined by (3.4), takes the form

$$\varphi(t, q) = \alpha(t) \bar{\varphi}(q\lambda(t)), \qquad \bar{\varphi}(s) = \int_E (1 - e^{-sx}) F(dx). \tag{4.4}$$

Furthermore, by Theorem 3.2, φ satisfies the equation

$$\partial_t \varphi(t, q) = -\beta \varphi(t, q)^\gamma \tag{4.5}$$

for all $0 \leqslant q \leqslant \infty$, where $\varphi(t, \infty) \stackrel{\text{def}}{=} \int_E \nu_t(dx)$. By (4.4), $\varphi(t, \infty) = \alpha(t)$. Hence, up to the normalization $\alpha(0^+) = \infty$, (4.5) gives

$$\alpha(t) = [(\gamma - 1)\beta t]^{\frac{1}{1-\gamma}}. \tag{4.6}$$

Now, given (4.4), we rewrite (4.5) as

$$\frac{\alpha'(t)}{\alpha(t)^\gamma} \bar{\varphi}(q\lambda(t)) + q\alpha(t)^{1-\gamma} \lambda'(t) \bar{\varphi}'(q\lambda(t)) = -\beta \bar{\varphi}(q\lambda(t))^\gamma.$$

In terms of the variable $s \stackrel{\text{def}}{=} q\lambda(t)$, separation of variables yields

$$\frac{(\gamma - 1) t \lambda'(t)}{\lambda(t)} = \frac{\bar{\varphi}(s) - \bar{\varphi}(s)^\gamma}{s \bar{\varphi}'(s)} = \frac{1}{\rho},$$

where we label the separation constant as $1/\rho$ for convenience. The constant β disappears thanks to (4.6). Solving for the general solution in each case, we obtain

$$\lambda(t) = c_1 t^{\frac{1}{\rho(\gamma-1)}}, \qquad \bar{\varphi}(s) = \left[\frac{1}{1 + c_2 s^{-\rho(\gamma-1)}}\right]^{\frac{1}{\gamma-1}}, \tag{4.7}$$

where $c_1, c_2 > 0$ are arbitrary constants. Taking into account (4.4), we have $\bar{\varphi}(0^+) = 0$. Therefore, $\rho > 0$. Furthermore, the fact that $\bar{\varphi}$ is a Bernstein function implies that $0 < \rho \leqslant 1$, otherwise $\bar{\varphi}''$ takes positive values near $s = 0$. We obtain the following proposition.

PROPOSITION 4.1. *Assume Ψ is given by (4.2). Then (1.11) admits a one-parameter family of self-similar solutions, indexed by $\rho \in (0, 1]$, of the form*

$$\mu_t^{\beta,\gamma,\rho}(dx) = \alpha(t) F_{\gamma,\rho}(\alpha(t)^{\frac{1}{\rho}} dx), \qquad \alpha(t) = [(\gamma-1)\beta t]^{\frac{1}{1-\gamma}}, \tag{4.8}$$

where $F_{\gamma,\rho}$ is a probability measure determined by the relation

$$\int_E (1 - e^{-qx}) F_{\gamma,\rho}(dx) = \left[\frac{1}{1 + q^{-\rho(\gamma-1)}}\right]^{\frac{1}{\gamma-1}}. \tag{4.9}$$

More precisely, the function

$$\varphi^{\beta,\gamma,\rho}(t, q) = \int_E (1 - e^{-qx}) \mu_t^{\beta,\gamma,\rho}(dx) \tag{4.10}$$

solves (4.5) with initial data $\varphi^{\beta,\gamma,\rho}(0, q) = q^\rho$. In particular, $\mu_t^{\beta,\gamma,1}$ is the fundamental solution of (1.11). That is, $\mu_t^{\beta,\gamma,1}$ is the Lévy measure for $Z_{\beta,\gamma}(t, x)$, the continuous-state branching process with branching mechanism (4.2).

PROOF. We set $c_1 = [(\gamma-1)\beta]^{\frac{1}{\rho(\gamma-1)}}$ and $c_2 = 1$ in (4.7), so that (4.4) takes the form

$$\varphi^{\beta,\gamma,\rho}(t, q) = \left[\frac{1}{(\gamma-1)\beta t + q^{-\rho(\gamma-1)}}\right]^{\frac{1}{\gamma-1}}. \tag{4.11}$$

By construction, this function solves (4.5) and has initial data $\varphi^{\beta,\gamma,\rho}(0, q) = q^\rho$, which is a Bernstein function. Hence, by Proposition 3.7, formula (3.6), and Theorem 3.2, $\varphi^{\beta,\gamma,\rho}$ has the representation in (4.10) where $\mu_t^{\beta,\gamma,\rho}$ solves (1.11). The remaining statements regarding the case $\rho = 1$ follow easily from definitions – see Sections 1.2 and 3.2. $\square$

Finally, we show that the distribution function for $F_{\gamma,\rho}$ has a generalized Mittag-Leffler structure analogous to (4.1).

LEMMA 4.2. *Suppose F is a probability measure on E such that for some fixed $r,\ s>0$,*

$$\int_E (1-e^{-qx})F(dx) = \left[\frac{1}{1+q^{-s}}\right]^r, \tag{4.12}$$

for all $q>0$. Then the distribution function of F takes the form

$$F(x) = \sum_{k=1}^{\infty} \frac{(r)_k}{k!} \cdot \frac{(-1)^{k+1} x^{sk}}{\Gamma(sk+1)}, \tag{4.13}$$

where $(r)_k$ denotes the Pochhammer symbol, or "rising factorial" function

$$(r)_k = r(r+1)(r+2)\cdots(r+k-1).$$

REMARK 4.3. A study of generalized Mittag-Leffler distribution functions of the form (4.13) is given by Prabhakar [22]. In particular, we may define, as in [22], the family of generalized Mittag-Leffler functions

$$E^{\rho}_{\alpha,\beta}(x) = \sum_{k=0}^{\infty} \frac{(\rho)_k\, x^k}{k!\,\Gamma(\alpha k+\beta)}, \qquad \alpha,\beta,\rho>0, \tag{4.14}$$

in which case (4.13) has the particular form

$$F(x) = 1 - E^{r}_{s,1}(-x^s).$$

PROOF. By series expansion of $(1-x)^{-r}$ at $x=0$, one easily computes that

$$\left[\frac{1}{1+q^{-s}}\right]^r = \sum_{k=0}^{\infty} \frac{r(r+1)\dots(r+k-1)}{k!}(-q^{-s})^k$$

for $|q|>1$. Next, note that

$$q^{-sk} = \frac{sk}{\Gamma(sk+1)} \int_0^{\infty} e^{-qx} x^{sk-1}\, dx,$$

for $k \geqslant 1$. Since

$$1 - \int_0^{\infty} e^{-qx} F(dx) = \sum_{k=0}^{\infty} \frac{(r)_k}{k!}(-1)^k q^{-sk} = 1 - \sum_{k=1}^{\infty} \frac{(r)_k}{k!}(-1)^{k+1} q^{-sk},$$

we conclude, formally, that

$$F(dx) = \sum_{k=1}^{\infty} \frac{(r)_k}{k!} \cdot \frac{(-1)^{k+1} sk}{\Gamma(sk+1)} x^{sk-1}\, dx. \tag{4.15}$$

Indeed, the previous series converges for all $x > 0$, has a (probability) distribution function given by (4.13), and satisfies (4.12) for all $\mathrm{Re}(q) > 1$. It follows by the identity theorem, that (4.12) holds for all $q > 0$, since both the left and right hand sides of (4.12) are analytic for $\mathrm{Re}(q) > 0$. □

5. Scaling limits with regularly varying Ψ. Proposition 4.1 establishes the existence of a family of self-similar solutions of (1.11) with power-law branching mechanisms $\Psi(u) = \beta u^\gamma$, $1 < \gamma \leqslant 2$. These solutions have a scaling invariance given by

$$\mu_t^{\beta,\gamma,\rho}(dx) = s^{\frac{1}{\gamma-1}} \mu_{st}^{\beta,\gamma,\rho}(s^{\frac{1}{\rho(\gamma-1)}}\, dx),$$

for any $s > 0$. While self-similarity arises in this case due to homogeneity of Ψ, we will show that branching mechanisms with an asymptotic power-law structure admit solutions which are asymptotically self-similar.

Recall that a function $f > 0$ is said to be *regularly varying* at zero (respectively, infinity) with index $\rho \in \mathbb{R}$ if

$$\frac{f(tx)}{f(t)} \to x^\rho$$

as $t \to 0$ (respectively, $t \to \infty$) for all $x > 0$. If $\rho = 0$, then f is said to be *slowly varying.*

THEOREM 5.1. *Let $\nu\colon E \to \mathcal{M}_F$ be a weak solution of equation (1.11) where Ψ is a critical branching mechanism which is regularly varying at zero with index $\gamma \in (1, 2]$.*

(i) Suppose there exists a nonzero $\hat\nu \in \mathcal{M}_F$ and functions $\alpha, \lambda > 0$ such that

$$\alpha(t)\nu_t(\lambda(t)^{-1}\, dx) \xrightarrow{w} \hat\nu(dx) \quad \text{as } t \to \infty. \tag{5.1}$$

Then, there exists $\rho \in (0, 1]$ such that for all $t > 0$,

$$\int_0^x y\nu_t(dy) \sim x^{1-\rho} L(t, x) \qquad \text{as } x \to \infty, \tag{5.2}$$

where $L(t,\cdot)$ is slowly varying at infinity. Furthermore, there exists $c_\lambda > 0$, given by (5.16), such that

$$\hat{\nu}(dx) = \langle \hat{\nu}, 1\rangle \, F_{\gamma,\rho}(\langle \hat{\nu}, 1\rangle^{\frac{1}{\rho}} c_\lambda^{-1} dx). \tag{5.3}$$

Here $F_{\gamma,\rho}$ is the generalized Mittag-Leffler distribution defined by (4.9). Moreover, for all $t > 0$,

$$\alpha(s)\nu_{st}(\lambda(s)^{-1}dx) \xrightarrow{w} t^{\frac{1}{1-\gamma}}\hat{\nu}(t^{\frac{1}{\rho(1-\gamma)}}dx) \quad \text{as } s \to \infty, \tag{5.4}$$

and the limit in (5.4) is a self-similar solution of (1.11) with branching mechanism

$$\hat{\Psi}(u) = \beta u^\gamma, \qquad \beta = \frac{\langle \hat{\nu}, 1\rangle^{1-\gamma}}{\gamma - 1}. \tag{5.5}$$

(ii) Conversely, suppose there exist $t_0 > 0$, $\rho \in (0,1]$, and L slowly varying at infinity such that (5.2) holds for $t = t_0$. Then, there exists a function $\lambda(t) \to 0$, implicitly defined by (5.17), such that (5.1) holds with $\alpha(t) = \langle \nu_t, 1\rangle^{-1}$ and $\hat{\nu} = F_{\gamma,\rho}$.

Remark 5.2. Note that if (6.1) holds for a weak solution $\nu\colon [0,\infty) \to \mathcal{M}_F$ with initial data ν_0, then (5.2) holds for $t = 0$. Similarly, if (5.2) holds for $t = t_0 = 0$, then the converse result holds (cf. [17]). Indeed, the proof below extends easily to these cases.

We begin our analysis leading to the proof of Theorem 5.1 with the following pair of useful lemmas.

Lemma 5.3. *(Uniform Convergence Lemma) Assume $f > 0$ is monotone and regularly varying at $x = \infty$ with index $\rho \neq 0$. Assume $h > 0$. Then, for any $0 \leqslant \lambda \leqslant \infty$,*

$$\frac{f(xh(x))}{f(x)} \to \lambda^\rho \tag{5.6}$$

as $x \to \infty$ if and only if $h(x) \to \lambda$ as $x \to \infty$.

Proof. The result essentially follows from the Uniform Convergence Theorem of Karamata (see, for instance, [4, Theorem 1.5.2]). In particular, if f satisfies the hypotheses above, then the convergence $f(\lambda x)/f(x) \to \lambda^\rho$ as $x \to \infty$ is uniform in λ on compact subsets of E. Therefore, if $h(x) \to \lambda$ as $x \to \infty$ for $0 < \lambda < \infty$, then (5.6) holds. The cases $\lambda = 0$ and $\lambda = \infty$ then follow from the monotonicity of f.

Conversely, suppose (5.6) holds for some $0 \leqslant \lambda \leqslant \infty$. Then, if $h(x) \nrightarrow \lambda$, there exists a subsequence $x_n \to \infty$ such that $h(x_n) \to \nu$ for some $0 \leqslant \nu \leqslant \infty$ with $\nu \neq \lambda$. We deduce that $f(x_n h(x_n))/f(x_n) \to \nu^\rho$, which contradicts (5.6). This completes the proof. $\square$

Lemma 5.4. *Assume $\Psi\colon E \to E$ is continuous and regularly varying at $u = 0$ with index $\gamma > 1$. Further, assume $u : E \to E$ solves the ordinary differential equation*

$$u' = -\Psi(u). \tag{5.7}$$

Then u is regularly varying at infinity with index $(1-\gamma)^{-1}$.

Proof. First assume u is invertible. Then it suffices to show that the function $u^{-1} : (0, u(0^+)) \to E$ is regularly varying at $s = u(\infty) = 0$ with index $1/(1-\gamma)^{-1} = 1-\gamma$. In that case, we apply Lemma 5.3 to the identity

$$\left(x^{\frac{1}{1-\gamma}}\right)^{1-\gamma} = \frac{u^{-1}(u(tx))}{u^{-1}(u(t))} = \frac{u^{-1}\left(\frac{u(tx)}{u(t)}u(t)\right)}{u^{-1}(u(t))},$$

to obtain

$$\lim_{t\to\infty} \frac{u(tx)}{u(t)} = x^{\frac{1}{1-\gamma}},$$

for all $x > 0$. Hence, u is regularly varying at infinity with index $(1-\gamma)^{-1}$.

Obviously, u is decreasing when $u > 0$. Therefore, to show that u is invertible, we must show that u does not vanish in finite time. Writing (5.7) in integral form, we have

$$t - t_0 = \int_{u(t)}^{u(t_0)} \frac{1}{\Psi(w)} dw. \tag{5.8}$$

Thus, u vanishes in finite time if and only if $\int_0^1 \frac{1}{\Psi(w)} dw < \infty$. Note that $\Psi(s) = s^\gamma L(s)$ where L is slowly varying. Also,

$$\int_s^1 \frac{1}{\Psi(w)} dw = \frac{s}{\Psi(s)} \int_1^{\frac{1}{s}} \frac{\Psi(s)}{\Psi(sw)}\, dw = \frac{1}{s^{\gamma-1}L(s)} \int_1^{\frac{1}{s}} \frac{\Psi(s)}{\Psi(sw)}\, dw. \tag{5.9}$$

Since Ψ is regularly varying at zero, the integral term on the right-hand side is bounded away from zero for s sufficiently small. Also, $s^r L(s) \to 0$ as $s \to 0$ for all $r > 0$ (see, for instance, [9, VIII.8]). Hence the left-hand side of (5.9) diverges as $s \to 0$, and we conclude that u is invertible.

It remains to show that u^{-1} is regularly varying at zero with index $1-\gamma$. We consider any fixed $0 < s_0 < u(0^+)$. By a change of variables, (5.8) implies

$$u^{-1}(s) = u^{-1}(s_0) - \int_{s_0}^{s} \frac{1}{\Psi(w)} dw$$

for all $0 < s < u(0^+)$. Therefore, by L'Hôpital's rule, we obtain for any $s > 0$

$$\lim_{\tau\to 0} \frac{u^{-1}(\tau s)}{u^{-1}(\tau)} = \lim_{\tau\to 0} \frac{s\left(u^{-1}\right)'(\tau s)}{\left(u^{-1}\right)'(\tau)} = \lim_{\tau\to 0} \frac{\dfrac{-s}{\Psi(\tau s)}}{\dfrac{-1}{\Psi(\tau)}} = \lim_{\tau\to 0} \frac{s\Psi(\tau)}{\Psi(\tau s)} = s^{1-\gamma}.$$

This completes the proof. □

Solutions of the autonomous equation (5.7) have a translation invariance which plays an important role in our analysis. Specifically, if $\Psi > 0$ is continuous and $\Psi(0^+) = 0$ (for instance, any critical branching mechanism), and if $u \geqslant v > 0$ are solutions of (5.7) defined on E, then

$$v(t) = u(t - \tau + u^{-1}(v(\tau)))$$

for all $t, \tau > 0$. Recall that if $\nu : E \to \mathcal{M}_F$ is a weak solution of (1.11), then the function $\varphi(\cdot, q)$, defined by (3.4), solves (5.7) for all $0 \leqslant q \leqslant \infty$. In particular, the function

$$\eta(t) \stackrel{\text{def}}{=} \langle \nu_t, 1\rangle = \varphi(t, \infty) \tag{5.10}$$

solves (5.7). Since $\varphi(t,\infty) \geqslant \varphi(t,q) > 0$, we obtain the identity

$$\varphi(t, q) = \eta(t - \tau + \eta^{-1}(\varphi(\tau, q))) \tag{5.11}$$

for all $q > 0$. Thanks to this identity, the characterization of scaling limits in the case of regularly varying branching mechanisms is relatively straightforward.

Proof of Theorem 5.1. Let φ and η be defined by (3.4) and (5.10), respectively. Assuming (5.1) holds, we have

$$\alpha(t)\eta(t) = \big\langle \alpha(t)\nu_t(\lambda(t)^{-1}\, dx),\, 1\big\rangle \to \big\langle \hat{\nu},\, 1\big\rangle$$

Moreover, taking into account (5.11), we have

$$
\begin{aligned}
\left\langle \hat{\nu},\ 1-e^{-qx}\right\rangle &= \lim_{t\to\infty} \left\langle \alpha(t)\nu_t(\lambda(t)^{-1}\,dx),\ 1-e^{-qx}\right\rangle \\
&= \lim_{t\to\infty} \alpha(t)\varphi(t,q\lambda(t)) \\
&= \lim_{t\to\infty} \alpha(t)\eta(t)\cdot \lim_{t\to\infty} \frac{\eta(t[1-\frac{\tau}{t}+\frac{1}{t}\eta^{-1}(\varphi(\tau,q\lambda(t)))])}{\eta(t)} \\
&= \left\langle \hat{\nu},\ 1\right\rangle \left(\lim_{t\to\infty}\left[1+\frac{1}{t}\eta^{-1}(\varphi(\tau,q\lambda(t)))\right]\right)^{\frac{1}{1-\gamma}} \qquad (5.12)
\end{aligned}
$$

where the last equality follows from Lemmas 5.3 and 5.4. Since the left-hand side is finite and independent of τ, we conclude that there exists $\chi(q)<\infty$ such that

$$
\frac{1}{t}\eta^{-1}(\varphi(\tau,q\lambda(t))) \to \chi(q), \qquad (5.13)
$$

for all $\tau>0$. Since $\Psi>0$ and $\Psi(0^+)=0$, the function η, which solves (5.7), is decreasing and $\eta(\infty)=0$. Also, by the analysis of Section 3.2, $\varphi(\tau,\cdot)$ is increasing with $\varphi(\tau,0)=0$ for all $\tau>0$. Hence, $\eta^{-1}(\varphi(\tau,\cdot))$ is decreasing with $\eta^{-1}(\varphi(\tau,0))=+\infty$. Since the limit in (5.12) is non-constant in q, we must have $\chi(q)>0$ and $\lambda(t)\to 0$ (otherwise, χ vanishes on an unbounded interval). Therefore,

$$
\frac{\frac{1}{t}\eta^{-1}(\varphi(\tau,q\lambda(t)))}{\frac{1}{t}\eta^{-1}(\varphi(\tau,\lambda(t)))} \to \frac{\chi(q)}{\chi(1)} > 0.
$$

as $t\to\infty$. A standard rigidity lemma (see, for instance, [9, VIII.8]) implies $\chi(q)=\chi(1)q^{\hat{\rho}}$ for some $\hat{\rho}$, and implies $\eta^{-1}(\varphi(\tau,\cdot))$ is regularly varying at $q=0$ with index $\hat{\rho}$. Note by (5.13) that χ is decreasing because $\varphi(t,\cdot)$ is decreasing and η^{-1} is increasing. Further χ is not constant and so $\hat{\rho}<0$. Also, since η^{-1} is regularly varying at $q=0$ with index $1-\gamma$ (see the proof of Lemma 5.4), it follows that $\varphi(\tau,\cdot)$ is regularly varying at $q=0$ with index $\rho=\hat{\rho}/(1-\gamma)$ for all $\tau>0$. Therefore,

$$
\hat{\varphi}(q) \overset{\text{def}}{=} \left\langle \hat{\nu},\, 1-e^{-qx}\right\rangle = \left\langle \hat{\nu},\, 1\right\rangle \left[1+\chi(1)q^{\rho(1-\gamma)}\right]^{\frac{1}{1-\gamma}}. \qquad (5.14)
$$

As a pointwise limit of Bernstein functions, $\hat{\varphi}$ is a Bernstein function. Hence, we must have $0<\rho\leqslant 1$, otherwise $\hat{\varphi}''$ takes positive values near $q=0$.

Now let us show that (5.2) holds. For $t>0$, we write

$$
\varphi(t,q) = \int_0^\infty (1-e^{-qx})\nu_t(dx) \sim q^\rho L(t,q^{-1}) \quad \text{as } q\to 0,
$$

where $L(t,\cdot)$ is slowly varying at infinity. Next, we claim that $q\partial_q\varphi(t,q) \sim \rho\varphi(t,q)$ as $q \to 0$ for all $t > 0$. Indeed, since $\partial_q^2\varphi \leqslant 0$, we have

$$\frac{q\partial_q\varphi(t,q)}{\varphi(t,q)} \geqslant \frac{\frac{\varphi(t,xq)}{\varphi(t,q)} - 1}{x-1}$$

for all $x > 1$. Hence,

$$\liminf_{q\to 0} \frac{q\partial_q\varphi(t,q)}{\varphi(t,q)} \geqslant \liminf_{q\to 0} \frac{\frac{\varphi(t,xq)}{\varphi(t,q)} - 1}{x-1} = \frac{x^\rho - 1}{x-1}.$$

Also, the reverse inequality holds if we consider $x < 1$ and take the limit supremum instead. Thus, as $x \to 1$, we recover the limit $q\partial_q\varphi(t,q)/\varphi(t,q) \to \rho$.

Therefore, we have

$$\partial_q\varphi(t,q) = \int_0^\infty e^{-qx} x\nu_t(dx) \sim \rho q^{\rho-1} L(t,q^{-1}) \quad \text{as } q \to 0.$$

This establishes a regular variation condition on the Laplace transform of the measure $x\nu_t(dx)$. By a classical Tauberian result (see, for instance, [17, Theorem 3.2]) we obtain the following equivalent condition on the distribution function:

$$\int_0^x y\nu_t(dy) \sim x^{1-\rho} L(t,x) \cdot \frac{\rho}{\Gamma(2-\rho)} \qquad \text{as } x \to \infty.$$

Hence, redefining L by a multiplicative factor, we obtain (5.2).

Finally, let us verify (5.4). A slight variation of estimate (5.12) gives, for all $0 \leqslant q \leqslant \infty$,

$$\begin{aligned}
&\lim_{s\to\infty} \left\langle \alpha(s)\nu_{st}(\lambda(s)^{-1}\,dx), 1 - e^{-qx} \right\rangle \\
&\qquad = \lim_{s\to\infty} \alpha(s)\varphi(st, q\lambda(s)) \\
&\qquad = \left\langle \hat{\nu},\, 1 \right\rangle \left(\lim_{s\to\infty} \left[t + \frac{1}{s}\eta^{-1}(\varphi(\tau, q\lambda(s))) \right] \right)^{\frac{1}{1-\gamma}} \\
&\qquad = \left\langle \hat{\nu},\, 1 \right\rangle \left[t + \chi(1) q^{\rho(1-\gamma)} \right]^{\frac{1}{1-\gamma}} \\
&\qquad = \left[\langle \hat{\nu},\, 1 \rangle^{1-\gamma} t + \left(\langle \hat{\nu},\, 1 \rangle^{\frac{1}{\rho}} \chi(1)^{\frac{1}{\rho(1-\gamma)}} q \right)^{\rho(1-\gamma)} \right]^{\frac{1}{1-\gamma}} \\
&\qquad = \left\langle \mu_t^{\beta,\gamma,\rho}(c_\lambda^{-1}\,dx),\; 1 - e^{-qx} \right\rangle
\end{aligned} \tag{5.15}$$

where $\mu^{\beta,\gamma,\rho}$ is defined by (4.8), with $\beta = (\gamma-1)^{-1}\langle\hat\nu,\,1\rangle^{1-\gamma}$ and

$$c_\lambda = \langle\hat\nu,\,1\rangle^{\frac{1}{\rho}}\chi(1)^{\frac{1}{\rho(1-\gamma)}} = \left[\frac{d}{dq}\bigg|_{q=0}\hat\varphi(q^{\frac{1}{\rho}})\right]^{\frac{1}{\rho}} \tag{5.16}$$

chosen according with (4.11). The last equality, which is by no means obvious, follows from (5.14). In particular, when $\rho = 1$, we obtain the relation $c_\lambda = \hat\varphi'(0) = \langle x\hat\nu, 1\rangle$ (cf. Theorem 6.1).

Since (5.15) is valid for all $0 \leqslant q \leqslant \infty$ (note, carefully, that we include $q = \infty$) the continuity theorem (see, for instance, [9, XIII.1]) implies that $\alpha(s)\nu_{st}(\lambda(s)^{-1}\,dx)$ converges vaguely to $\mu_t^{\beta,\gamma,\rho}(c_\lambda^{-1}\,dx)$. Also, the case $q = \infty$ implies convergence in total measure. We therefore obtain convergence in the weak topology (see, for instance, [1, Theorem 30.8]). Hence, taking into account (4.8), we obtain (5.3)–(5.5). This completes the proof of part (i) of the theorem.

Now suppose there exists $t_0 > 0$, $\rho \in (0,1]$, and L slowly varying at infinity such that (5.2) holds for $t = t_0$. Again, by the Tauberian theorem, we have $\partial_q\varphi(t_0,\cdot)$ regularly varying at $q = 0$ with index $\rho - 1$. The regular variation of $\varphi(t_0,\cdot)$ at $q = 0$ with index ρ then follows from the observation:

$$\frac{\varphi(t_0,q)}{q\partial_q\varphi(t_0,q)} = \frac{\varphi(t_0,q)-\varphi(t_0,0)}{q\partial_q\varphi(t_0,q)} = \int_0^1 \frac{\partial_q\varphi(t_0,qz)}{\partial_q\varphi(t_0,q)}\,dz \to \frac{1}{\rho} \quad \text{as } q \to 0.$$

The convergence of the integral term is easy to verify, see for instance [19, Lemma 3.3].

Finally, for $s > 0$, let $\alpha(s) = \eta(s)^{-1}$ and define $\lambda(s)$ by the relation

$$\frac{1}{s}\eta^{-1}(\varphi(t_0,\lambda(s))) = 1. \tag{5.17}$$

It follows that

$$\frac{1}{s}\eta^{-1}(\varphi(t_0,q\lambda(s))) \to q^{\rho(\gamma-1)},$$

and we conclude, as above, that

$$\begin{aligned}\lim_{s\to\infty}\left\langle\eta(s)^{-1}\nu_{st}(\lambda(s)^{-1}\,dx),\,1-e^{-qx}\right\rangle &= \left[t + q^{\rho(1-\gamma)}\right]^{\frac{1}{1-\gamma}}\\ &= \left\langle\mu_t^{\beta,\gamma,\rho}(dx),\;1-e^{-qx}\right\rangle\end{aligned}$$

for all $t > 0$ and for all $0 \leqslant q \leqslant \infty$, where $\beta = (\gamma-1)^{-1}$. Weak convergence of the measures follows as before. This completes the proof. □

6. Scaling limits for fundamental solutions. In this section we show that a necessary condition for asymptotic self-similarity of fundamental solutions is regular variation of the branching mechanism Ψ. In view of Definition (3.4) and property (ii) of Theorem (3.7), we consider critical branching mechanisms Ψ for which Grey's condition holds. Our main result is the following:

THEOREM 6.1. *Let $\mu\colon E \to \mathcal{M}_F$ be the fundamental solution of* (1.11)*, where Ψ is a critical branching mechanism verifying Grey's condition. Further, assume there exists a nonzero probability measure $\hat{\mu} \in \mathcal{M}_F$ and a function $\lambda > 0$ such that*

$$\frac{\mu_t(\lambda(t)^{-1}\,dx)}{\langle \mu_t, 1\rangle} \xrightarrow{w} \hat{\mu} \tag{6.1}$$

as $t \to \infty$. Then Ψ is regularly varying at $u = 0$ with index $\gamma \in (1, 2]$. Furthermore, $x\hat{\mu} \in \mathcal{M}_F$ and $\lambda(t)/\langle \mu_t, 1\rangle \to \langle x\hat{\mu}, 1\rangle$ as $t \to \infty$. Moreover, we have the representation

$$\hat{\mu} = F_{\gamma,1}(\langle x\hat{\mu}, 1\rangle^{-1}\,dx) \tag{6.2}$$

where $F_{\gamma,1}$ is the generalized Mittag-Leffler distribution defined by (4.9)*.*

Before proving Theorem 6.1, we discuss a few basic properties of fundamental solutions. Let us define, as before, the total measure function

$$\eta(t) \stackrel{\text{def}}{=} \langle \mu_t, 1\rangle = \Phi(t, \infty),$$

where Φ is given by (3.8). Since $\Phi(t, \cdot)$ is increasing, we have

$$\lim_{t\to 0^+} \eta(t) > \lim_{t\to 0^+} \Phi(t, q) = q$$

for all $q > 0$. Hence, $\eta(t) \to \infty$ as $t \to 0$. Moreover, η solves (5.7), where $\Psi > 0$ and $\Psi(0^+) = 0$. Hence, $\eta(t)$ decreases to zero as $t \to \infty$. It follows that $\eta : E \to E$ is bijective, and it is straightforward to check that its inverse is given by

$$\zeta(\tau) \stackrel{\text{def}}{=} \eta^{-1}(\tau) = \int_\tau^\infty \frac{1}{\Psi(u)}\,du. \tag{6.3}$$

With this notation, Φ in (3.8) has the representation

$$\Phi(t, q) = \eta(t + \zeta(q)), \tag{6.4}$$

which is a special case of (5.11). From this it follows easily that Φ satisfies the forward equation

$$\partial_t \Phi + \Psi(q)\partial_q \Phi = 0. \tag{6.5}$$

Finally, we note the following useful estimates.

LEMMA 6.2. *Assume Ψ is a critical branching mechanism that satisfies Grey's condition, and assume ζ is defined by (6.3). Then the following hold for all $s > 0$:*

(i) $\dfrac{d}{ds}\left[\dfrac{\Psi(s)}{s^2}\right] \leqslant 0 \leqslant \dfrac{d}{ds}\left[\dfrac{\Psi(s)}{s}\right]$,

(ii) $\dfrac{d^2}{ds^2}\left[\dfrac{1}{\Psi(s)}\right] \geqslant 0$,

(iii) $\dfrac{d}{ds}[s\zeta(s)] \geqslant 0$.

PROOF. Part (i) is equivalent to the estimate

$$\frac{\Psi(s)}{s} \leqslant \Psi'(s) \leqslant \frac{2\Psi(s)}{s}. \tag{6.6}$$

The first inequality in (6.6) follows from the convexity of Ψ. That is,

$$\frac{\Psi(s)}{s} = \frac{\Psi(s) - \Psi(0)}{s - 0} \leqslant \Psi'(s).$$

Similarly, the concavity of Ψ' gives the estimate $\Psi''(s) \leqslant \Psi'(s)/s$, which implies

$$s\Psi'(s) = \int_0^s [\tau\Psi''(\tau) + \Psi'(\tau)]\, d\tau \leqslant \int_0^s 2\Psi'(\tau)\, d\tau = 2\Psi(s).$$

For the proof of (ii), we compute

$$\frac{d^2}{ds^2}\left[\frac{1}{\Psi(s)}\right] = \frac{2\Psi'(s)^2 - \Psi''(s)\Psi(s)}{\Psi(s)^3},$$

which is nonnegative by the estimate

$$\frac{\Psi''(s)}{\Psi'(s)} \leqslant \frac{1}{s} \leqslant \frac{\Psi'(s)}{\Psi(s)} \leqslant \frac{2\Psi'(s)}{\Psi(s)}.$$

Finally, for the proof of (iii), observe that

$$\frac{d}{ds}[s\zeta(s)] = \int_s^\infty \frac{1}{\Psi(u)}\,du - \frac{s}{\Psi(s)} = \int_1^\infty \frac{s}{\Psi(su)}\,du - \int_1^\infty \frac{s}{\Psi(s)}\cdot\frac{1}{u^2}\,du \tag{6.7}$$

For $u \geqslant 1$, part (i) implies

$$\frac{\Psi(su)}{(su)^2} \leqslant \frac{\Psi(s)}{s^2}.$$

Hence (6.7) is nonnegative, and the proof is complete. □

Proof of Theorem 6.1. Assuming (6.1), we have for all $0 < q \leqslant \infty$ that

$$\frac{\eta(t+\zeta(q\lambda(t)))}{\eta(t)} = \left\langle \frac{\mu_t(\lambda(t)^{-1}\,dx)}{\langle \mu_t, 1\rangle},\ 1-e^{-qx} \right\rangle \to \hat{\varphi}(q) \stackrel{\text{def}}{=} \langle \hat{\mu}, 1-e^{-qx}\rangle$$

as $t \to \infty$. Equivalently, in terms of the variables

$$\tau \stackrel{\text{def}}{=} \eta(t), \qquad \ell(\tau) \stackrel{\text{def}}{=} \lambda(\zeta(\tau)),$$

we have

$$\tilde{\varphi}(\tau, q) \stackrel{\text{def}}{=} \frac{\eta(\zeta(\tau)+\zeta(q\ell(\tau)))}{\tau} \quad \to \hat{\varphi}(q) \tag{6.8}$$

as $\tau \to 0$. Note that $\hat{\varphi}\colon [0,\infty) \to [0,\infty)$ is increasing from 0 to 1, since $\hat{\mu}$ is a probability measure. Also, note that $\tilde{\varphi}$ is implicitly determined by the relation

$$\zeta(q\ell(\tau)) = \zeta(\tau\tilde{\varphi}(\tau,q)) - \zeta(\tau), \tag{6.9}$$

Claim: $\ell(\tau) \to 0$ as $\tau \to 0$.

Proof of claim: Consider

$$F(\tau) \stackrel{\text{def}}{=} \frac{\tau}{\Psi(\tau)}. \tag{6.10}$$

By part (i) of Lemma 6.2, F is nonincreasing, and $\tau \mapsto \tau F(\tau)$ is nondecreasing, hence

$$1 \leqslant \frac{F(\tau u)}{F(\tau)} \leqslant \frac{1}{u}, \quad \text{for } u \leqslant 1, \qquad \frac{1}{u} \leqslant \frac{F(\tau u)}{F(\tau)} \leqslant 1, \quad \text{for } u > 1. \tag{6.11}$$

Therefore, equations (6.3) and (6.9) imply

$$\frac{\zeta(q\ell(\tau))}{F(\tau)} = \int_{\tilde\varphi(\tau,q)}^{1} \frac{F(\tau v)}{F(\tau)}\frac{dv}{v} \geqslant \int_{\tilde\varphi(\tau,q)}^{1} \frac{dv}{v}\ .$$

As $\tau \to 0$, the right-hand side is bounded away from zero for fixed $q > 0$. Also, $F(\tau) \to \infty$ since $\Psi(0^+) = 0$ and $\Psi'(0^+) = 0$. It follows that $\zeta(q\ell(\tau)) \to \infty$ as $\tau \to 0$. Since ζ is decreasing on E and $\zeta(0^+) = \infty$, the claim follows.

We now consider the rescaled equation

$$\zeta_s(q\ell_s(\tau)) = \zeta_s(\tau\tilde\varphi(s\tau, q)) - \zeta_s(\tau), \tag{6.12}$$

where

$$\zeta_s(\tau) \stackrel{\text{def}}{=} \frac{\zeta(s\tau)}{\zeta(s)}, \qquad \ell_s(\tau) \stackrel{\text{def}}{=} \frac{\ell(s\tau)}{s}. \tag{6.13}$$

We will show that as $s \to 0$ a nontrivial limiting version of (6.12) holds. That is,

$$\hat\zeta(q\hat\ell(\tau)) = \hat\zeta(\tau\hat\varphi(q)) - \hat\zeta(\tau) \tag{6.14}$$

holds where

$$\ell_s(\tau) \xrightarrow{s\to 0} \hat\ell(\tau) = \hat\varphi'(0)\tau, \qquad \zeta_s(\tau) \xrightarrow{s\to 0} \hat\zeta(\tau) = \tau^{-r}, \tag{6.15}$$

for some $r \in (0, 1]$. We will then show that the previous limits imply that Ψ is regularly varying at zero with index $\gamma = r+1$ and that $\hat\varphi$ has a generalized Mittag-Leffler form determined by

$$\hat\varphi(q) = \left[\frac{1}{1 + (\hat\varphi'(0)q)^{-r}}\right]^{\frac{1}{r}}. \tag{6.16}$$

The main idea is to show that subsequential limits of (6.12) exist and are unique. We divide the proof into three main steps.

Step 1. (*Existence of subsequential limits.*)

First, we write

$$\zeta_s(\tau) - 1 = \int_1^\tau \zeta_s'(u)\,du = \int_1^\tau \frac{s\zeta'(su)}{\zeta(s)}\,du = \frac{-s\zeta'(s)}{\zeta(s)} \int_\tau^1 \frac{\Psi(s)}{\Psi(su)}\,du. \tag{6.17}$$

Note that for fixed $s > 0$, the function

$$\Psi_s(u) \overset{\text{def}}{=} \frac{\Psi(su)}{\Psi(s)}$$

is increasing and convex. Furthermore, by part (i) of Lemma 6.2, we have

$$u^2 \leqslant \Psi_s(u) \leqslant u \quad \text{for } u \leqslant 1, \qquad u \leqslant \Psi_s(u) \leqslant u^2 \quad \text{for } u > 1. \tag{6.18}$$

On the other hand, by part (iii) of Lemma 6.2, for all $s > 0$ we have

$$\xi(s) \overset{\text{def}}{=} \frac{-s\zeta'(s)}{\zeta(s)} \in (0, 1]. \tag{6.19}$$

Claim: $\limsup_{s\to 0} \xi(s) > 0$.

Proof of claim: Assume for the sake of contradiction that $\xi(s) \to 0$ as $s \to 0$. Then, by (6.17) and (6.18), ζ is slowly varying at $u = 0$. By Helly's selection theorem, there exists a sequence $\tau_j \to 0$ and a function $1 \leqslant f(u) \leqslant 1/u$ such that

$$\frac{F(\tau_j u)}{F(\tau_j)} \to f(u)$$

pointwise for $u \in (0, 1)$. Since ζ is slowly varying and $\ell(\tau_j) \to 0$, we have

$$\int_{\hat\varphi(q)}^1 \frac{f(u)}{u}\, du = \lim_{j\to\infty} \frac{\zeta(q\ell(\tau_j))}{F(\tau_j)} = \lim_{j\to\infty} \frac{\zeta(q\ell(\tau_j))}{\zeta(\ell(\tau_j))} \cdot \frac{\zeta(\ell(\tau_j))}{F(\tau_j)} = \lim_{j\to\infty} \frac{\zeta(\ell(\tau_j))}{F(\tau_j)}.$$

This gives a contradiction since $\hat\varphi$ is nonconstant and the right-hand side is independent of q. Therefore, the claim holds.

Now we may apply Helly's selection theorem to find a sequence $s_k \to 0$ and a function $\hat\Psi > 0$ for which

$$\xi(s_k) \to \hat\xi \quad \text{for some } \hat\xi \in (0, 1], \qquad \Psi_{s_k}(u) \to \hat\Psi(u) \qquad \text{for all } u > 0. \tag{6.20}$$

Furthermore, as a pointwise limit of convex functions, $\hat\Psi$ is convex. By dominated convergence,

$$\zeta_{s_k}(\tau) \to \hat\zeta(\tau) \overset{\text{def}}{=} 1 + \hat\xi \int_\tau^1 \frac{1}{\hat\Psi(u)}\, du. \tag{6.21}$$

Since $\hat{\Psi}$ is convex and positive, $\hat{\zeta} \in C^1(E)$ and is strictly decreasing, and the convergence in (6.21) occurs locally uniformly for $\tau \in E$. Now by assumption (6.8), the right-hand side of (6.12) converges to

$$R(\tau, q) \stackrel{\text{def}}{=} \hat{\zeta}(\tau\hat{\varphi}(q)) - \hat{\zeta}(\tau) > 0 \tag{6.22}$$

for all $\tau, q > 0$. Hence, the left-hand side of (6.12) also converges, and if $\hat{\zeta}^{-1}$ is defined on E, then

$$\ell_{s_k}(\tau) \to \hat{\ell}(\tau) \stackrel{\text{def}}{=} q^{-1}\hat{\zeta}^{-1}(R(\tau, q)). \tag{6.23}$$

Claim: $\hat{\zeta} \colon E \to E$ is a bijection.

Proof of claim: Recall $\hat{\zeta}$ is strictly decreasing. For $\tau \leqslant 1$, estimate (6.18) implies

$$\hat{\zeta}(\tau) \geqslant 1 + \hat{\xi} \int_\tau^1 \frac{1}{u}\, du = 1 - \hat{\xi}\, \ln \tau \to \infty \quad \text{as } \tau \to 0.$$

It remains to show that $\hat{\zeta}(\tau) \to 0$ as $\tau \to \infty$. Assume, for the sake of contradiction, that $\hat{\zeta}(\tau) \to L > 0$ as $\tau \to \infty$. Since $\hat{\varphi}(q) \to 1$ as $q \to \infty$, we may choose for any $\tau > 0$ a value $\hat{q} > 0$ sufficiently large so that $R(\tau, \hat{q}) < L$. It follows

$$\ell_{s_k}(\tau) \to \infty \qquad \text{as } k \to \infty, \tag{6.24}$$

for otherwise, along some bounded subsequence, the left-hand side of (6.12) would have a subsequential limit with value larger than L, which is a contradiction. But now (6.24) implies that for all $q > 0$

$$R(\tau, q) = \lim_{k\to\infty} \zeta_{s_k}(q\ell_{s_k}(\tau)) \leqslant 1,$$

since $\zeta_s(u) \leqslant 1$ for any $s > 0$ and $u \geqslant 1$. However, by (6.22), $R(\tau, q) \to \infty$ as $q \to 0$, since $\hat{\zeta}$ is unbounded above and $\hat{\varphi} \to 0$ as $q \to 0$. This is a contradiction, which gives the claim.

We conclude that, along the sequence $s_k \to 0$, equation (6.12) has a well-defined limit of the form (6.14) for all $\tau, q > 0$. Furthermore, by (6.12) and (6.17) we have

$$1 + \xi(s_k) \int_{q\ell_{s_k}(\tau)}^1 \frac{1}{\Psi_{s_k}(u)}\, du = \xi(s_k) \int_{\tau\tilde{\varphi}(s_k\tau, q)}^\tau \frac{1}{\Psi_{s_k}(u)}\, du.$$

In particular, fixing $q>0$ and taking into account (6.18) shows that

$$\ell_{s_k}(\tau) \to \hat{\ell}(\tau) \qquad \text{locally uniformly for } \tau \in E. \tag{6.25}$$

This fact will play a role in the uniqueness proof to follow.

Step 2. (*Uniqueness of subsequential limits.*)

We now show that subsequential limits obtained as in Step 1 are unique. First, equations (6.14) and (6.21) imply

$$\hat{\zeta}(\tau) = \hat{\xi} \int_{\tau\hat{\varphi}(q)}^{q\hat{\ell}(\tau)} \hat{F}(s)\, \frac{ds}{s} = \hat{\xi} \int_{\tau\frac{\hat{\varphi}(q)}{q}}^{\hat{\ell}(\tau)} \hat{F}(qs)\, \frac{ds}{s}, \qquad \hat{F}(u) \stackrel{\text{def}}{=} \frac{u}{\hat{\Psi}(u)}. \tag{6.26}$$

Further, since $\hat{\varphi}$ is concave, it follows that

$$\hat{\varphi}'(q) \leqslant \frac{\hat{\varphi}(q)}{q} \leqslant \frac{\hat{\ell}(\tau)}{\tau} \qquad \text{for all } \tau, q > 0.$$

Therefore, $\hat{\varphi}'(q) = \int_E e^{-qx} x \hat{\mu}(dx)$ is decreasing and bounded above, and we deduce that

$$0 < \hat{\varphi}'_0 \stackrel{\text{def}}{=} \hat{\varphi}'(0^+) < \infty.$$

Furthermore, taking $q \to 0$ in (6.26) gives

$$\hat{\zeta}(\tau) = \hat{\xi}\hat{F}_0 \ln \frac{\hat{\ell}(\tau)}{\hat{\varphi}'_0 \tau}, \qquad \text{with} \quad \hat{F}_0 \stackrel{\text{def}}{=} \hat{F}(0^+). \tag{6.27}$$

Formally, we have $\hat{F}_0 = \infty$ if and only if $\hat{\ell}(\tau) = \hat{\varphi}'_0\tau$ for all $\tau > 0$. More precisely, note that the left-hand side of (6.26) is positive, so that $\hat{F}(qs)$ has a finite limit as $q \to 0$ if and only if $\hat{\ell}(\tau) > \hat{\varphi}'_0 \tau$ for each $\tau > 0$.

On the other hand, equations (6.14) and (6.21) also imply

$$\hat{\zeta}(q\hat{\ell}(\tau)) = \hat{\xi} \int_{\tau\hat{\varphi}(q)}^{\tau} \hat{F}(s)\, \frac{ds}{s} = \hat{\xi} \int_{\hat{\varphi}(q)}^{1} \hat{F}(\tau s)\, \frac{ds}{s}. \tag{6.28}$$

Taking $\tau \to 0$ implies

$$\hat{\zeta}(q\hat{\ell}_0) = \hat{\xi}\hat{F}_0 \ln \frac{1}{\hat{\varphi}(q)}, \qquad \text{with} \quad \hat{\ell}_0 \stackrel{\text{def}}{=} \hat{\ell}(0^+). \tag{6.29}$$

In particular, $\hat{\ell}_0 = 0$ if and only if $\hat{F}_0 = \infty$. Note also $\hat{\ell}_0 < \infty$, since $\hat{F}_0 \geqslant 1$.

We now consider two cases.

Case 1: ($\hat{\ell}_0 = 0$). As noted above, $\hat{\ell}_0 = 0$ if and only if $\hat{\ell}(\tau) = \hat{\varphi}_0'\tau$ for all $\tau > 0$. Hence, (6.14) reduces to

$$\hat{\zeta}(\tau q\hat{\varphi}_0') = \hat{\zeta}(\tau\hat{\varphi}(q)) - \hat{\zeta}(\tau). \tag{6.30}$$

Differentiating in q and τ gives the relations

$$\frac{\hat{\varphi}(q)}{q\hat{\varphi}'(q)} = \frac{\hat{F}(\tau\hat{\varphi}(q))}{\hat{F}(\tau q\hat{\varphi}_0')} = 1 + \frac{\hat{F}(\tau)}{\hat{F}(\tau q\hat{\varphi}_0')}. \tag{6.31}$$

Therefore, $\hat{F}(\tau q\hat{\varphi}_0')/\hat{F}(\tau)$ is constant in τ and we deduce that $\hat{F}$ is a power law: $\hat{F}(u) = u^{-r}$, since $\hat{F}(1) = 1$. Note that $r \neq 0$, since $\hat{F}_0 = \infty$. It then follows from (6.18) and (6.26) that $0 < r \leqslant 1$. The second equality above reduces to

$$\frac{(q\hat{\varphi}_0')^r}{\hat{\varphi}(q)^r} = 1 + (q\hat{\varphi}_0')^r, \tag{6.32}$$

which gives (6.16). On the other hand, $\hat{\Psi}(u) = u^{r+1}$, so that (6.21) implies

$$0 = \hat{\zeta}(\infty) = 1 - \hat{\xi}\int_1^\infty \frac{1}{u^{r+1}}\,du = 1 - \frac{\hat{\xi}}{r}.$$

Hence, $\hat{\xi} = r$. In summary, we obtain in this case:

$$\hat{\ell}(\tau) = \hat{\varphi}_0'\tau, \quad \hat{\Psi}(\tau) = \tau^{r+1}, \quad \hat{\zeta}(\tau) = \tau^{-r}, \quad \hat{\xi} = r, \tag{6.33}$$

where $0 < r \leqslant 1$ and $\hat{\varphi}$ is given by (6.16).

Case 2: ($\hat{\ell}_0 > 0$). We will show that the remaining case, $\hat{\ell}_0 > 0$, leads to a contradiction. We divide this case into three parts.

(i) First, let us show that if $\hat{\ell}_0 > 0$, then $\hat{\varphi}$ has the form (6.16), and

$$\hat{\ell}(\tau)^r = \hat{\ell}_0^r + (\tau\hat{\varphi}_0')^r. \tag{6.34}$$

The idea is to consider a rescaling of (6.14), of the same form as (6.12); namely,

$$\hat{\zeta}_s(q\hat{\ell}_s(\tau)) = \hat{\zeta}_s(\tau\hat{\varphi}(q)) - \hat{\zeta}_s(\tau) \tag{6.35}$$

where

$$\hat{\zeta}_s(\tau) \stackrel{\text{def}}{=} \frac{\hat{\zeta}(s\tau)}{\hat{\zeta}(s)}, \qquad \hat{\ell}_s(\tau) \stackrel{\text{def}}{=} \frac{\hat{\ell}(s\tau)}{s}. \tag{6.36}$$

Since $\hat{\zeta}(\tau) \to 0$ as $\tau \to 0$, we deduce from (6.27) that

$$\hat{\ell}_s(\tau) = \tau \cdot \frac{\hat{\ell}(s\tau)}{s\tau} \to \tau \hat{\varphi}_0' \qquad \text{as} \quad s \to \infty. \tag{6.37}$$

Furthermore, since Lemma 6.2 applies to the functions $\hat{\Psi}$ and $\hat{\zeta}$, we can use Helly's selection principle, as in Step 1, to pass to the limit in (6.35) along some sequence $\hat{s}_k \to \infty$. Up to relabeling, the limit equation matches exactly the form (6.30). In particular, (6.37) implies that $\hat{\zeta}_{\hat{s}_k}$ has a nonconstant limit, and we obtain, as before, (6.16) from the relations (6.31). Note that the constant r in (6.16) is the same as in the previous case, since $\hat{\varphi}$ is fixed.

Now, substituting $q = \tau/\hat{\ell}_0$ in (6.29) and comparing with (6.27), we obtain

$$\frac{\hat{\ell}(\tau)}{\tau \hat{\varphi}_0'} = \frac{1}{\hat{\varphi}(\tau/\hat{\ell}_0)} \tag{6.38}$$

for all $\tau > 0$. Using (6.16) in the previous relation gives (6.34). In particular,

$$\frac{\hat{\ell}(\tau)}{\tau} = \left[\frac{\hat{\ell}_0^r}{\tau^r} + (\hat{\varphi}_0')^r\right]^{\frac{1}{r}} \tag{6.39}$$

is decreasing as a function of $\tau > 0$.

(ii) Next, let us show that if (6.39) holds, then

$$\limsup_{\tau \to 0} \frac{\ell(\tau)}{\tau} = \infty \qquad \text{and} \qquad \liminf_{\tau \to 0} \frac{\ell(\tau)}{\tau} = \hat{\varphi}_0'. \tag{6.40}$$

Recall that $\hat{\ell}(\tau) = \lim_{k\to\infty} \ell_{s_k}(\tau)$ for some $s_k \to 0$. Therefore, (6.39) implies

$$\lim_{k\to\infty} \frac{\ell(s_k\tau)}{s_k\tau} = \frac{\hat{\ell}(\tau)}{\tau} \to \infty$$

as $\tau \to 0$, and the first statement in (6.40) follows.

On the other hand, by (6.9)

$$\frac{\ell(\tau)}{\tau} > \frac{\tilde{\varphi}(\tau, q)}{q}, \qquad \text{for all } \tau, q > 0.$$

Hence, for all $t > 0$ and for all $q > 0$,

$$\frac{\hat{\ell}(t)}{t} = \lim_{k\to\infty} \frac{\ell(s_k t)}{s_k t} \geqslant \liminf_{\tau\to 0} \frac{\ell(\tau)}{\tau} > \liminf_{\tau\to 0} \frac{\tilde{\varphi}(\tau, q)}{q} = \frac{\hat{\varphi}(q)}{q}$$

Taking into account (6.39) and passing to the limit $t \to \infty$ on the left and $q \to 0$ on the right yields the last statement in (6.40).

(iii) Finally, we show that (6.39) and (6.40) lead to a contradiction. Fix $M > m > \hat{\varphi}_0'$, and choose a sequence of disjoint intervals $[a_k, b_k]$ as follows.

1. Choose $b_k \to 0$ such that $\dfrac{\ell(b_k)}{b_k} > M$.
2. Define $c_k = \sup\{\tau < b_k : \dfrac{\ell(\tau)}{\tau} < m\}$.
3. Choose a_k such that $1 < \dfrac{c_k}{a_k} < 1 + \dfrac{1}{k}$ and $\dfrac{\ell(a_k)}{a_k} < m$.

Taking $s = a_k$ and $\tau = 1$ in (6.12), we have

$$\zeta_{a_k}(q\ell_{a_k}(1)) = \zeta_{a_k}(\tilde{\varphi}(a_k, q)) - 1. \tag{6.41}$$

Since

$$\hat{\varphi}_0' \leqslant \liminf_{k\to\infty} \frac{\ell(a_k)}{a_k} \leqslant \limsup_{k\to\infty} \frac{\ell(a_k)}{a_k} \leqslant m$$

and $\tilde{\varphi}(a_k, q) \to \hat{\varphi}(q)$ as $k \to \infty$, it follows that the sequence $\xi(a_k)$, defined by (6.19), is bounded away from zero; otherwise, there exists a subsequence $\zeta_{a_{k_j}}(\tau) \to 1$, which contradicts (6.41). Therefore, as in Step 1, (6.12) has a nontrivial limit along a subsequence $a_{k_j} \to 0$, $j \geqslant 1$. In particular, the local uniform convergence of $\ell_{a_{k_j}}$ implies

$$\frac{\ell_{a_{k_j}}(\tau)}{\tau} \to \frac{\hat{\Lambda}(\tau)}{\tau} \qquad \text{locally uniformly for } \tau \in E, \tag{6.42}$$

where $\hat{\Lambda}$ satisfies (6.34), or, equivalently,

$$\frac{\hat{\Lambda}(\tau)}{\tau} = \left[\frac{\hat{\Lambda}_0^r}{\tau^r} + (\hat{\varphi}_0')^r\right]^{\frac{1}{r}}. \tag{6.43}$$

If $\hat{\Lambda}_0 > 0$, then (6.43) is strictly decreasing in τ, and we have, for all $\tau > 1$,

$$\frac{\hat{\Lambda}(\tau)}{\tau} < \hat{\Lambda}(1) = \lim_{j\to\infty} \ell_{a_{k_j}}(1) \leqslant m.$$

On the other hand, if $\hat{\Lambda}_0 = 0$, then $\hat{\Lambda}(\tau)/\tau = \hat{\Lambda}(1) = \hat{\varphi}_0' < m$ for all $\tau > 0$. Hence, in either case, we have $\hat{\Lambda}(\tau)/\tau < m$ is non-increasing for all $\tau > 1$.

Next, choose $\varepsilon < \min\{M - m,\ m - \frac{\hat{\Lambda}(2)}{2}\}$, and choose J large enough so that

$$\left| \frac{\ell_{a_{k_j}}(\tau)}{\tau} - \frac{\hat{\Lambda}(\tau)}{\tau} \right| < \varepsilon \qquad \forall\, j \geqslant J, \quad \forall\, \tau \in [1, 3]. \tag{6.44}$$

Since $r_j \stackrel{\text{def}}{=} b_{k_j}/a_{k_j} > 1$ and

$$\left| \frac{\ell_{a_{k_j}}(r_j)}{r_j} - \frac{\hat{\Lambda}(r_j)}{r_j} \right| = \left| \frac{\ell(b_{k_j})}{b_{k_j}} - \frac{\hat{\Lambda}(r_j)}{r_j} \right| \geqslant |M - m| > \varepsilon,$$

it follows from (6.44) that $r_j > 3$ for all $j \geqslant J$. Therefore,

$$\frac{\ell_{a_{k_j}}(\tau)}{\tau} \geqslant m \qquad \forall\, j \geqslant J, \quad \forall\, \tau \in\ [2, 3] \subset \left(c_{k_j}/a_{k_j},\ r_j\right], \tag{6.45}$$

since, by construction, $\ell(\tau)/\tau \geqslant m$ for all $\tau \in (c_k, b_k]$. Hence, (6.45) implies

$$\left| \frac{\ell_{a_{k_j}}(\tau)}{\tau} - \frac{\hat{\Lambda}(\tau)}{\tau} \right| \geqslant \left| m - \frac{\hat{\Lambda}(\tau)}{\tau} \right| \geqslant \left| m - \frac{\hat{\Lambda}(2)}{2} \right| > \varepsilon \tag{6.46}$$

for all $j \geqslant J$ and for all $\tau \in [2, 3]$. This contradicts (6.44). Therefore, the hypothesis of Case 2, $\hat{\ell}_0 > 0$, is never satisfied, and we obtain in Step 1 unique subsequential limits of the form (6.33).

Step 3. (*Limit as $s \to 0$.*)

To finish the proof of the Theorem, note that we must have $\xi(s) \to \hat{\xi} = r$ as $s \to 0$. Otherwise, by Step 1, it is possible to extract subsequential limits with distinct values of $\hat{\xi}$, contradicting (6.33). Similarly, the full limit of each of the rescaled functions ζ_s, Ψ_s, and ℓ_s exists as $s \to 0$, since given any sequence $s_k \to 0$, there exist unique subsequential limits by Steps 1 and 2. In particular, (6.33) shows that Ψ is regularly varying with index $\gamma = r + 1 \in (1, 2]$. Also, (6.32) implies

$$\langle \hat{\mu}, 1 - e^{-qx} \rangle = \hat{\varphi}(q) = \frac{1}{[1 + (\hat{\varphi}'(0)q)^{-r}]^{1/r}},$$

where $\hat{\varphi}'(0) = \langle x\hat{\mu}, 1 \rangle$. This gives (6.2). Finally, (6.15) implies that

$$\frac{\lambda(\zeta(s\tau))}{\eta(\zeta(s\tau))}\tau = \frac{\lambda(\zeta(s\tau))}{s} = \ell_s(\tau) \to \hat{\varphi}'(0)\tau.$$

Hence $\lambda(t) \sim \hat{\varphi}'(0)\eta(t) = \langle x\hat{\mu}, 1 \rangle\ \langle \mu_t, 1 \rangle$ as $t \to \infty$, and the proof is complete. $\square$

REMARK 6.3. The conclusions of the Theorem follow much more quickly if one *assumes* that the scaling function $\lambda(t) \sim \langle \mu_t, 1\rangle$ in (6.1), based on the arguments of Pakes [21] which make use of the forward equation (6.5). Testing (6.1) with xe^{-qx} it follows

$$\partial_q \Phi(t, \lambda q) = \frac{\Psi(\Phi(t, \lambda q))}{\Psi(\lambda q)} \to \hat{\varphi}'(q), \quad q > 0. \tag{6.47}$$

Writing $u = \lambda q$ and noting $\theta = \hat{\varphi}(q)/q$ is a monotonic function of q, we have that $\Phi(t, \lambda q) = u\theta(1 + o(1))$ and thus

$$\frac{\Psi(u\theta(1 + o(1)))}{\Psi(u)} \to h(\theta) \tag{6.48}$$

as $u \to 0$. By simple estimates based on the continuity and monotonicity of Ψ, one can eliminate the $1 + o(1)$ factor and conclude that Ψ is regularly varying by the standard rigidity lemma in [9, VIII.8].

7. Limit theorems for critical CSBPs. We conclude this paper by applying the results in Sections 5 and 6 to derive limit theorems for critical CSBPs that become extinct almost surely. First, we obtain a conditional limit theorem for fixed initial population x. In particular this solves the continuous-state analog of the open question posed by Pakes in [21, Remark 6.1].

THEOREM 7.1. *Assume $Z(t, x)$ is a continuous-state branching process with critical branching mechanism Ψ verifying Grey's condition. Further, assume that for some (equivalently all) $x > 0$, there exists a function $\lambda > 0$ and a probability measure $\hat{\mu} \in \mathcal{M}_F$ such that*

$$\mathbb{P}(\lambda(t) Z(t, x) \leqslant z \mid Z(t, x) > 0) \to \int_{(0,z)} \hat{\mu}(du) \tag{7.1}$$

holds for all points z for which $\hat{\mu}(\{z\}) = 0$. Then, there exists $1 < \gamma \leqslant 2$ such that Ψ is regularly varying at $u = 0$ with index γ. Furthermore, $x\hat{\mu} \in \mathcal{M}_F$ and $\lambda(t) \sim \langle x\hat{\mu}, 1\rangle \mathbb{P}(Z(t, 1) > 0)$ as $t \to \infty$.

Conversely, suppose Ψ is regularly varying at $u = 0$ with index $1 < \gamma \leqslant 2$. Then, (7.1) holds with $\lambda(t) = \mathbb{P}(Z(t, x) > 0)$ and $\hat{\mu} = F_{\gamma,1}(dz)$.

PROOF. It follows from (1.2) that

$$\mathbb{P}(Z(t, x) = 0) = \lim_{q \to \infty} \mathbb{E}(e^{-qZ(t,x)}) = e^{-x\varphi(t,\infty)} = e^{-x\langle \mu_t, 1\rangle}, \tag{7.2}$$

with μ_t the Lévy measure for $Z(t,x)$. By the continuity theorem [9, XIII.1], (7.1) implies

$$\int_E e^{-qy}\hat{\mu}(dy) = \lim_{t\to\infty} \frac{\mathbb{E}(e^{-q\lambda(t)Z(t,x)}) - \mathbb{P}(Z(t,x)=0)}{\mathbb{P}(Z(t,x)>0)} = \lim_{t\to\infty} \frac{e^{-x\varphi(t,\lambda(t)q)} - e^{-x\langle\mu_t,1\rangle}}{1-e^{-x\langle\mu_t,1\rangle}}$$

Hence,

$$\lim_{t\to\infty}\int_E (1-e^{-qy})\frac{\mu_t(\lambda(t)^{-1}dy)}{\langle\mu_t,1\rangle} = \lim_{t\to\infty}\frac{\varphi(t,\lambda(t)q)}{\langle\mu_t,1\rangle} = \lim_{t\to\infty}\frac{1-e^{-x\varphi(t,\lambda(t)q)}}{1-e^{-x\langle\mu_t,1\rangle}} = \int_E (1-e^{-qy})\hat{\mu}(dy),$$

where the second equality follows by Taylor expansion and the fact that $0 < \varphi(t,\lambda(t)q) < \langle\mu(t),1\rangle \to 0$ as $t\to\infty$. Since μ_t is the fundamental solution of the associated equation (1.11), we conclude, by Theorem 6.1, that there exists $1<\gamma\leqslant 2$ such that Ψ is regularly varying at $u=0$ with index γ. Also, by Theorem 6.1,

$$\lambda(t) \sim \langle x\hat{\mu},1\rangle\langle\mu_t,1\rangle \sim \langle x\hat{\mu},1\rangle(1-e^{-\langle\mu_t,1\rangle}) = \langle x\hat{\mu},1\rangle\mathbb{P}(Z(t,1)>0)$$

as $t\to\infty$. The converse follows easily from Theorem 5.1. This completes the proof. □

Next, based on the same results on scaling limits of fundamental solutions, we study scaling limits as $t\to\infty$ of CSBPs with initial population scaled to obtain non-degenerate Lévy process limits $x\mapsto\hat{Z}(x)$. As in [11, Ch. VI], let $\mathbb{D}$ to denote the space of càdlàg paths equipped with the Skorokhod topology. We use the notation $\overset{\mathcal{L}}{=}$ to denote equality in law (i.e. both processes define the same measure on the Skorokhod space $\mathbb{D}$), and the notation $\overset{\mathcal{L}}{\to}$ to denote convergence in law for these processes (i.e. weak convergence of the induced distributions on the Skorokhod space).

For convenience, we introduce a notation for rescaled processes. If $\lambda,\alpha>0$, and $x\mapsto X(x)$ is a process, then we define the rescaled process $\delta_{\lambda,\alpha}X$ by $\delta_{\lambda,\alpha}X(x) \overset{\text{def}}{=} \lambda X(\alpha x)$.

Theorem 7.2. *Let $Z(t,x)$ be a continuous state branching process with critical branching mechanism Ψ satisfying Grey's condition.*

(i) Assume there exists a Lévy process $\hat{Z} = \hat{Z}(x)$ and functions $\alpha, \lambda > 0$ such that

$$\delta_{\lambda(t),\alpha(t)} Z(t,\cdot) \xrightarrow[t\to\infty]{\mathcal{L}} \hat{Z}(\cdot). \tag{7.3}$$

Further, assume the non-degeneracy condition

$$\lim_{t\to\infty} \mathbb{P}(Z(t,\alpha(t)x) = 0) = \mathbb{P}(\hat{Z}(x) = 0) \in (0,1) \tag{7.4}$$

for some $x > 0$. Then, there exists $1 < \gamma \leqslant 2$ such that Ψ is regularly varying at $u = 0$ with index γ, and there exist constants $c_\alpha, c_\lambda > 0$ such that

$$\frac{c_\alpha}{\alpha(t)} \sim \frac{\lambda(t)}{c_\lambda} \sim \mathbb{P}(Z(t,1) > 0) \qquad \text{as} \;\; t \to \infty. \tag{7.5}$$

Furthermore, for all (fixed) $t > 0$

$$\delta_{\lambda(s),\alpha(s)} Z(st,\cdot) \xrightarrow[s\to\infty]{\mathcal{L}} \delta_{t^{\gamma^*},t^{-\gamma^*}} \hat{Z}, \qquad \text{where} \quad \gamma^* \stackrel{def}{=} \frac{1}{\gamma-1}. \tag{7.6}$$

Also, for all $t > 0$,

$$\delta_{1,c_\alpha c_\lambda} Z_{\beta,\gamma}(t,\cdot) \stackrel{\mathcal{L}}{=} \delta_{t^{\gamma^*},t^{-\gamma^*}} \hat{Z} \tag{7.7}$$

where $Z_{\beta,\gamma}(t,\cdot)$ is the continuous-state branching process with branching mechanism $\hat{\Psi}(u) = \beta u^\gamma$ with $\beta = \gamma^* c_\lambda^{\gamma-1}$.

(ii) Conversely, assume Ψ is regularly varying at zero with index $1 < \gamma \leqslant 2$. Then (7.6) holds with $\lambda(s) = \alpha(s)^{-1} = \mathbb{P}(Z(s,1) > 0)$, where $\hat{Z}(x)$ is the Lévy process with Lévy measure $F_{\gamma,1}(dx)$ defined by (4.9).

Proof. Since we are dealing with increasing Lévy processes, the process convergence in (7.3) is equivalent to the pointwise convergence of Laplace exponents

$$\alpha(t)\varphi(t,\lambda(t)q) \xrightarrow[t\to\infty]{} \hat{\varphi}(q), \quad \text{for all } q \in [0,\infty), \tag{7.8}$$

where $x\hat{\varphi}(q) \stackrel{\text{def}}{=} -\ln \mathbb{E}(e^{-q\hat{Z}(x)})$ is the Laplace exponent of $\hat{Z}(x)$. (See, for instance [11, XV.3.2,XIII.1.2] or the proof of Theorem 1 in [18].) Furthermore, by (7.4) we must have

$$\begin{aligned}
\lim_{t\to\infty}\lim_{q\to\infty} x\alpha(t)\varphi(t,\lambda(t)q) &= \lim_{t\to\infty}\lim_{q\to\infty} -\ln \mathbb{E}(e^{-qZ^{(t)}(x)}) \\
&= \lim_{t\to\infty} -\ln \mathbb{P}(\lambda(t)Z(t,\alpha(t)x) = 0) = -\ln \mathbb{P}(\hat{Z}(x) = 0) = \lim_{q\to\infty} x\hat{\varphi}(q),
\end{aligned}$$

and hence

$$\lim_{t\to\infty} \alpha(t)\varphi(t,\infty) = \hat{\varphi}(\infty) \in (0,\infty). \tag{7.9}$$

Then, denoting the Lévy measures of $Z(t,x)$ and $\hat{Z}(x)$ by μ_t and $\hat{\mu}$, respectively, we deduce from (7.8) and (7.9) that

$$\frac{1}{\langle \mu_t, 1\rangle}\mu_t(\lambda(t)^{-1}dx) \xrightarrow{w} \frac{1}{\langle \hat{\mu}, 1\rangle}\hat{\mu}(dx). \tag{7.10}$$

Therefore, by Theorem 6.1, there exists $1 < \gamma \leqslant 2$ such that Ψ is regularly varying at $u = 0$ with index γ. Moreover,

$$\lambda(t) \sim \frac{\langle x\hat{\mu}, 1\rangle}{\langle \hat{\mu}, 1\rangle}\langle \mu_t, 1\rangle \sim \frac{\langle x\hat{\mu}, 1\rangle}{\langle \hat{\mu}, 1\rangle}(1 - e^{-\langle \mu_t, 1\rangle}) = \frac{\langle x\hat{\mu}, 1\rangle}{\langle \hat{\mu}, 1\rangle}\mathbb{P}(Z(t,1) > 0).$$

Hence, together with (7.9), we obtain (7.5) with $c_\alpha = \langle \hat{\mu}, 1\rangle$ and $c_\lambda = \frac{\langle x\hat{\mu},1\rangle}{\langle \hat{\mu},1\rangle}$. Also, by Theorem 6.1,

$$\hat{\mu}(dx) = c_\alpha F_{\gamma,1}(c_\lambda^{-1}dx) = c_\alpha c_\lambda \left[c_\lambda^{-1} F_{\gamma,1}(c_\lambda^{-1}dx)\right] = c_\alpha c_\lambda\, \mu_1^{\beta,\gamma,1}(dx)$$

where $\mu^{\beta,\gamma,1}$ is defined by (4.8) with $\beta = \frac{c_\lambda^{\gamma-1}}{\gamma-1}$. Therefore, by Theorem 5.1, we have

$$\alpha(s)\varphi(st,\lambda(s)q) \to t^{-\gamma^*}\hat{\varphi}(t^{\gamma^*}q) = c_\alpha c_\lambda\, \varphi^{\beta,\gamma,1}(t,q) \tag{7.11}$$

as $s \to \infty$ for all $0 \leqslant q \leqslant \infty$, where $\varphi^{\beta,\gamma,1}$ is defined by (4.10). Since

$$\begin{aligned}\mathbb{E}(e^{-q\lambda(s)Z(st,\alpha(s)x)}) &= e^{-x\alpha(s)\varphi(st,\lambda(s)q)} \\ &\xrightarrow{s\to\infty} e^{-xt^{-\gamma^*}\hat{\varphi}(t^{\gamma^*}q)} = \mathbb{E}(e^{-qt^{\gamma^*}\hat{Z}(t^{-\gamma^*}x)}),\end{aligned}$$

we obtain (7.6). Similarly, we obtain (7.7) from (7.11).

For the converse, we observe that the convergence in (7.3) holds if and only if the Laplace exponent converge pointwise as in (7.8) (see, for instance [11, XV.3.2,XIII.1.2] or the proof of Theorem 1 in [18]). The converse now follows by a similar argument. □

REMARK 7.3. The non-degeneracy condition (7.4) has the following interpretation. The spatial process $x \mapsto Z_t^{\alpha,\lambda}(x)$ is a compound Poisson process with jump measure $\alpha(t)\mu_t(\lambda(t)^{-1}dx)$ and scaled intensity $\alpha(t)\langle \mu_t, 1\rangle$. One thing that (7.4) means is that we assume the scaled intensity converges to the intensity of jumps $\langle \hat{\mu}, 1\rangle = 1$ in the limiting process $\hat{Z}$. In particular this presumes there are no small jumps with finite intensity being lost in the limit.

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