ASYMPTOTIC STABILITY OF SOLITARY WAVES IN THE BENNEY-LUKE MODEL OF WATER WAVES

TETSU MIZUMACHI

Faculty of Mathematics, Kyushu University, Fukuoka 812-8581, Japan

ROBERT L. PEGO

Department of Mathematical Sciences, Carnegie Mellon University, Pittsburgh, PA, USA

JOSÉ RAÚL QUINTERO

Departamento Matemáticas, Universidad del Valle, Colombia

Abstract. We study asymptotic stability of solitary wave solutions in the one-dimensional Benney-Luke equation, a formally valid approximation for describing two-way water wave propagation. For this equation, as for the full water wave problem, the classic variational method for proving orbital stability of solitary waves fails dramatically due to the fact that the second variation of the energy-momentum functional is infinitely indefinite. We establish non-linear stability in energy norm under the spectral stability hypothesis that the linearization admits no non-zero eigenvalues of non-negative real part. We then verify this hypothesis for waves of small energy.

2010 *Mathematics Subject Classification.* 37K40, 35Q35, 35B35, 76B15.

Contents

1. Introduction

In this paper we study the stability of solitary waves for the nonlinear dispersive wave equation

$$\partial_t^2\varphi - \partial_x^2\varphi + a\partial_x^4\varphi - b\partial_x^2\partial_t^2\varphi + (\partial_t\varphi)(\partial_x^2\varphi) + 2(\partial_x\varphi)(\partial_x\partial_t\varphi) = 0. \tag{1.1}$$

This equation is a formally valid approximation for describing small-amplitude, long water waves in water of finite depth. It is the one-dimensional version of an equation originally derived by Benney and Luke [3] as an isotropic model for three-dimensional water waves. (Also see [36].) The parameters $a,\ b > 0$ are such that $a - b = \hat\tau - \frac{1}{3}$, where $\hat\tau$ is the inverse Bond number. We will assume $0 < a < b$ throughout this paper, which corresponds to small or zero surface tension ($\hat\tau < \frac{1}{3}$), and will study solitary waves that travel with any speed c satisfying $c^2 > 1$. These waves are captured in the small-amplitude, long-wave regime of formal validity for speed with c^2 near 1. We remark that (1.1) is an approximation formally valid for describing *two-way* water wave propagation, in contrast to one-way equations such as the KdV, BBM, or KP equations.

There are a considerable number of previous works on model water wave equations and stability of solitary waves—Especially see [5, 6, 8] and references therein regarding a large family of models of Boussinesq type. Our interest in the Benney-Luke equation (1.1) is motivated by a number of features that this equation shares with the full water wave equations with zero surface tension, for which the problem of solitary wave stability remains open.

In particular, many works on model equations employ the variational method originated by Benjamin [2] and Bona [4] for the KdV equation and developed into a powerful abstract tool by Grillakis, Shatah and Strauss [18, 19]. In particular, for (1.1) in the complementary case corresponding to strong surface tension ($a > b$ and $0 \leq c^2 < a/b$), Quintero has used this method to establish orbital stability of solitary waves, in both one and two space dimensions [42, 43]. Techniques related to the GSS variational method have been developed to obtain many further results as well. For the full water wave equations with strong surface tension ($\hat{\tau} > \frac{1}{3}$), Mielke has obtained a conditional orbital stability result for small solitary waves of depression [28]. For generalized KdV equations, a variety of results concerning asymptotic stability with small perturbations of finite energy, even for multiple pulses, have been obtained by Martel and Merle and others, using various methods involving variational, virial, and nonlinear Liouville properties, e.g., see [23, 24, 27, 25, 26, 31, 11].

For the Benney-Luke equation (1.1) in the present case—as for the full water wave equations with zero surface tension [7]—the variational approach fails dramatically, due to the infinite indefiniteness of the energy-momentum functional whose critical points are the solitary-wave profiles. (See Appendix A below for details.) This also happens for the line soliton solution of the KP-II equation with periodically transverse perturbations, which was recently studied by Mizumachi and Tzvetkov [33] using methods based on Miura transformations. But the Benney-Luke and water wave equations are nonintegrable, and such transformation techniques appear to be unavailable.

A salient feature of solitary water waves that is shared by solitary waves of (1.1) in the present case with $0 < a < b$ and $c^2 > 1$ is that they travel at a speed greater than the maximum group velocity of linear waves. This motivates us to study the scattering of localized perturbations by using norms with spatial weights that decay to zero in the direction behind solitary waves. This approach has been used successfully to obtain asymptotic stability results for the KdV equation [39], the BBM equation [29], and Fermi-Pasta-Ulam lattice equations [12, 13, 14], for pertubations small both in energy and in weighted norm with exponential weights e^{ax}. And, in a recent analysis for the Toda lattice [32], Mizumachi established asymptotic stability of solitary waves for arbitrary perturbations of small energy, by combining stability estimates with exponential weights, as used by Friesecke and Pego, with dispersive propagation estimates (virial estimates) related to techniques of Martel and Merle.

In the present paper, we will build on Mizumachi's approach to establish asymptotic stability results for Benney-Luke solitary waves of small amplitude. The analysis comes in two main parts, corresponding to linear and nonlinear analysis.

In the first part, we show that spectral stability—the absence of nonzero eigenvalues with non-negative real part—implies linear stability with exponential decay rate in exponentially weighted norm, for perturbations orthogonal to the adjoint neutral-mode space generated by variations of phase and wave amplitude. Also, we prove that small solitary waves are spectrally stable. This is done by a suitable comparison of a reduced resolvent operator with the corresponding one for KdV solitons. The KdV limit is used to control the reduced resolvent on long length scales at long times, and this is combined with additional estimates to obtain control on all length scales and all time scales. This happens in a manner similar to the spectral stability analysis of small solitary waves in FPU lattices [14] and water waves [38].

In the second part, we prove that nonlinear stability follows from spectral stability. Perturbed solitary waves are studied in terms of i) time-modulated speed and shift parameters, ii) a solution freely propagated from the intial wave perturbation, and iii) the exponentially localized interaction of the two. Key to this analysis is a linear decay estimate based on a discrete-time *recentering* technique reminiscent of the renormalization method developed by Promislow [40] for studying pulse dynamics in parametrically driven nonlinear Schrödinger equations. For the present problem, discrete-time recentering is used to avoid a problem of loss of derivatives in linear stability estimates that may occur in frames translating with time-varying speed.

2. Statement of main results

2.1. **Equations of motion.** In terms of the notation

$$q = \partial_x \varphi, \quad r = \partial_t \varphi, \qquad A = I - a\partial_x^2, \quad B = I - b\partial_x^2,$$

the Benney-Luke equation takes the form of the system

$$\partial_t q - \partial_x r = 0, \qquad B\partial_t r - A\partial_x q + r\partial_x q + 2q\partial_x r = 0. \tag{2.1}$$

We write this system in the abstract form

$$\partial_t u = Lu + f(u), \tag{2.2}$$

with

$$u = \begin{pmatrix} q \\ r \end{pmatrix}, \quad L = \begin{pmatrix} 0 & \partial_x \\ B^{-1}A\partial_x & 0 \end{pmatrix}, \quad f(u) = \begin{pmatrix} 0 \\ -B^{-1}(r\partial_x q + 2q\partial_x r) \end{pmatrix}. \tag{2.3}$$

The energy

$$E(u) = \frac{1}{2}\int_{\mathbb{R}} \left(q^2 + r^2 + a(\partial_x q)^2 + b(\partial_x r)^2\right) dx = \frac{1}{2}\int_{\mathbb{R}} (qAq + rBr)\, dx \tag{2.4}$$

is formally conserved in time for solutions to (2.2):

$$E(u(t)) = E(u_0). \tag{2.5}$$

By standard arguments, the Cauchy problem for (2.1) is globally well-posed for initial data in the Sobolev space $H^s(\mathbb{R})$ for any $s \geq 1$, and (2.5) holds for all t.

2.2. **Solitary waves.** The Benney-Luke system (2.1) admits a two-parameter family of solitary waves

$$(q, r) = (q_c(x - ct - x_0), r_c(x - ct - x_0)), \qquad c^2 > 1, \quad x_0 \in \mathbb{R},$$

whose profiles must satisfy

$$-cq_c' - r_c' = 0, \qquad -cBr_c' - Aq_c' + r_c q_c' + 2q_c r_c' = 0, \tag{2.6}$$

whence

$$r_c = -cq_c, \qquad (bc^2 - a)q_c'' - (c^2 - 1)q_c + \frac{3c}{2}q_c^2 = 0. \tag{2.7}$$

Explicitly,

$$q_c(x) = \frac{c^2 - 1}{c}\operatorname{sech}^2\left(\frac{1}{2}\alpha_c x\right), \qquad \alpha_c = \sqrt{\frac{c^2 - 1}{bc^2 - a}}. \tag{2.8}$$

2.3. **Spectral and linear stability.** We linearize the Benney-Luke system (2.1) about a solitary wave $(q, r) = (q_c, r_c)$ with $c > 1$, after changing to a coordinate frame moving with speed c. The resulting linearized system for the perturbation $z = (q_1, r_1)$ takes the following form, with $\partial = \partial_x$:

$$\partial_t z = \mathcal{L}_c z, \qquad \mathcal{L}_c = \begin{pmatrix} c\partial & \partial \\ -B^{-1}(-A\partial + r_c\partial + 2r_c') & c\partial - B^{-1}(2q_c\partial + q_c') \end{pmatrix}. \tag{2.9}$$

We study equation (2.9) in exponentially weighted function spaces, writing

$$L^p_\alpha = \{g \mid e^{\alpha x} g \in L^p(\mathbb{R})\}, \qquad H^s_\alpha = \{g \mid e^{\alpha x} g \in H^s(\mathbb{R})\},$$

with norms

$$\|g\|_{L^p_\alpha} = \left(\int_{\mathbb{R}} |e^{\alpha x} g(x)|^p \, dx\right)^{1/p}, \quad \|g\|_{H^s_\alpha} = \left(\int_{\mathbb{R}} (1 + |k|^2)^s |\hat{g}(k + i\alpha)|^2 \, \frac{dk}{2\pi}\right)^{1/2}.$$

In these definitions, g may be scalar or vector valued according to context. We normalize the Fourier transform according to the definition

$$\hat{g}(k) = \int_{\mathbb{R}} g(x) e^{-ikx} \, dx.$$

Note that for any $s \geq 0$, the solitary wave profile components $q_c, r_c \in H^s_\alpha$ if and only if $|\alpha| < \alpha_c$. In terms of these spaces, the following basic facts will be established in section 4.

Lemma 2.1. *Assume $0 < a < b$ and $c > 1$. Fix α with $0 < \alpha < \alpha_c$, and consider the operator $\mathcal{L}_c$ from (2.9) in the space L^2_α with domain $D(\mathcal{L}_c) = H^1_\alpha$. Then*

(i) *$\mathcal{L}_c$ is a compact perturbation of $L + c\partial$.*
(ii) *$\mathcal{L}_c$ is the generator of a C^0 semigroup in L^2_α.*
(iii) *The essential spectrum of $\mathcal{L}_c$ is contained strictly in the left half-plane* $\operatorname{Re}\lambda < 0$.
(iv) *The value $\lambda = 0$ is an eigenvalue of $\mathcal{L}_c$ with multiplicity 2. Specifically,*

$$\zeta_{1,c} = \begin{pmatrix} \partial_x q_c \\ \partial_x r_c \end{pmatrix}, \qquad \zeta_{2,c} = -\begin{pmatrix} \partial_c q_c \\ \partial_c r_c \end{pmatrix}, \tag{2.10}$$

satisfy $\mathcal{L}_c \zeta_{1,c} = 0$, $\mathcal{L}_c \zeta_{2,c} = \zeta_{1,c}$.

We let P_c denote the spectral projection onto $\operatorname{span}\{\zeta_{1,c}, \zeta_{2,c}\}$, the generalized eigenspace associated with the eigenvalue 0 for $\mathcal{L}_c$. Our main results in Part I (concerning asymptotic linear stability) are as follows.

Theorem 2.2 (Spectral stability implies linear stability)**.** *Fix $c > 1$ and α with $0 < \alpha < \alpha_c$. Assume that $\mathcal{L}_c$ has no nonzero eigenvalue λ satisfying* $\operatorname{Re}\lambda \geq 0$. *Then there exist positive constants K and β such that for all $z \in L^2_\alpha$ and all $t \geq 0$,*

$$\|e^{\mathcal{L}_c t}(I - P_c) z\|_{L^2_\alpha} \leq K e^{-\beta t} \|z\|_{L^2_\alpha}. \tag{2.11}$$

Theorem 2.3 (Spectral stability for small waves)**.** *Fix $\hat{\alpha} \in (0, (b-a)^{-1/2})$. Then there exists $\epsilon_0 > 0$ such that whenever $0 < \epsilon < \epsilon_0$ and $c = 1 + \frac{1}{2}\epsilon^2$, then $\mathcal{L}_c$ has no nonzero eigenvalue λ satisfying* $\operatorname{Re}\lambda \geq 0$*, in the space $L^2_{\epsilon\hat{\alpha}}$.*

2.4. **Nonlinear stability.** Our basic nonlinear stability result establishes asymptotic orbital stability for the family of solitary waves, with respect to arbitrary small-energy perturbations.

Theorem 2.4 (Spectral stability implies nonlinear stability). *Suppose $c_0 > \sigma > 1$ and $0 < \alpha < \frac{1}{2}\alpha_{c_0}$, and assume that in L^2_α, $\mathcal{L}_{c_0}$ has no nonzero eigenvalue λ satisfying $\operatorname{Re}\lambda \geq 0$. Then there exists $\delta > 0$ satisfying the following: If $u_0(x) = u_{c_0}(x - x_0) + v_0(x)$ where $x_0 \in \mathbb{R}$ and $\|v_0\|_{H^1} < \delta$, then there exist $c_\star > 1$ and a C^1-function $x(t)$ such that*

$$|c_\star - c_0| + \sup_{t>0} |x'(t) - c_0| = O(\|v_0\|_{H^1}), \tag{2.12}$$

$$\lim_{t\to\infty} x'(t) = c_\star, \tag{2.13}$$

$$\sup_{t\geq 0} \|u(t,\cdot) - u_{c_0}(\cdot - x(t))\|^2_{H^1} = O(\|v_0\|_{H^1}), \tag{2.14}$$

$$\lim_{t\to\infty} \|u(t,\cdot) - u_{c_\star}(\cdot - x(t))\|_{H^1(x\geq\sigma t)} = 0. \tag{2.15}$$

If the initial perturbation is sufficiently localized ahead of the main solitary wave, then we obtain local convergence to some fixed solitary wave, and an estimate of the local decay rate.

Theorem 2.5 (Asymptotic phase and local decay rates). *In addition to the assumptions of Theorem 2.4, assume that*

$$\|\omega v_0\|_{L^2(\mathbb{R})} + \|\omega \partial_x v_0\|_{L^2(\mathbb{R})} < \infty\,, \tag{2.16}$$

where ω is an increasing function on $\mathbb{R}$ such that $\omega(x) = 1$ for $x \leq 0$ and $1/\omega(x)$ is integrable on $(0,\infty)$. Then

$$x_\star = \lim_{t\to\infty} (x(t) - c_\star t) \qquad \text{exists.} \tag{2.17}$$

Additionally,

(1) *if $v_0 \in H^1_{\alpha_1}$ for some $\alpha_1 > 0$ small, then there exists $\gamma > 0$ such that as $t \to \infty$,*

$$|x(t) - c_\star t - x_\star| = O(e^{-\gamma t}), \tag{2.18}$$

$$\|u(t,\cdot + c_\star t + x_\star) - u_{c_\star}\|_{H^1_{\alpha_1}} = O(e^{-\gamma t}); \tag{2.19}$$

(2) *if $\int_0^\infty x^{2\rho}(|v_0(x)|^2 + |\partial_x v_0(x)|^2)\,dx < \infty$ for some $\rho > 1$, then as $t \to \infty$,*

$$\|\min(1, e^{\alpha x})(u(t,\cdot + c_\star t + x_\star) - u_{c_\star})\|_{H^1} = O(t^{-\rho+1}). \tag{2.20}$$

A linear stability estimate of particular significance in the proof of nonlinear stability is stated in the following lemma, which is used to deal with time-dependent variations of wave speed and phase. The proof, provided in Appendix D, involves the recentering technique mentioned in the introduction, in order to avoid estimating the time-dependent advection term as a forcing term.

Lemma 2.6. *Let $c_0 > 1$ and $0 < \alpha < \alpha_{c_0}$. Assume that in L^2_α, $\mathcal{L}_c$ has no nonzero eigenvalue λ satisfying $\operatorname{Re}\lambda \geq 0$. Then there exist positive constants $\hat{\delta}$ and $\hat{K}$ with the following property. Suppose $c(t)$ and $\eta(t)$ are continuous functions on $[0,T)$ $(0 < T \leq \infty)$ such that*

$$\sup_{t\in[0,T)} (|c(t) - c_0| + |\eta(t)|) < \hat{\delta}. \tag{2.21}$$

Suppose also that $F \in C([0,T); H^1_\alpha)$, and that $w \in C([0,T); H^1_\alpha)$ is a solution of

$$\partial_t w = \mathcal{L}_{c(t)} w + \eta(t)\partial_y w + F(t), \tag{2.22}$$

satisfying the non-secularity condition $P_{c(t)} w(t) = 0$, $t \in [0,T)$. Then

$$\|w(t)\|_{H^1_\alpha} \le \hat{K} \left(e^{-\beta t/3} \|w(0)\|_{H^1_\alpha} + \int_0^t e^{-\beta(t-s)/3} \|F(s)\|_{H^1_\alpha}\, ds \right), \tag{2.23}$$

where β is the constant given in Theorem 2.2 for $c = c_0$.

Part I. Spectral and linear stability

To prove Theorem 2.2, our plan is to use the characterization of exponential stability provided by the Gearhart-Prüss theorem [15, 41]. (See [9] for a survey of the use of this theorem, and also [1, 20, 21, 22].) By this theorem (in particular see [41, Cor. 4]), exponential stability of a C^0 semigroup in a Hilbert space is equivalent to the uniform boundedness of the resolvent on the right half-plane. We apply this theorem in the space $Z_\alpha := (I - P_c) L^2_\alpha(\mathbb{R}, \mathbb{R}^2)$, the spectral complement of the neutral-mode space. Thus, the conclusion on linear stability in Theorem 2.2 is equivalent to the statement that the restricted resolvent $(\lambda - \mathcal{L}_c)^{-1}|_{Z_\alpha}$ is uniformly bounded on the right half-plane $\mathbb{C}_+ = \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > 0\}$.

Naturally, the restricted resolvent is bounded for λ in a neighborhood of the discrete eigenvalue 0. And due to Lemma 2.1, the main hypothesis of Theorem 2.2 ensures that the resolvent set of $\mathcal{L}_c$ contains all of the closed right half-plane $\bar{\mathbb{C}}_+$ except $\lambda = 0$. The restricted resolvent is therefore bounded on compact subsets of $\bar{\mathbb{C}}_+$. To complete the proof, then, it will suffice to prove that the resolvent is bounded on L^2_α, uniformly outside a bounded set in $\mathbb{C}_+$. That is, there are constants M_1 and M_2 such that

$$\sup_{\operatorname{Re}\lambda > 0, |\lambda| > M_2} \|(\lambda - \mathcal{L}_c)^{-1}\|_\alpha \le M_1. \tag{2.24}$$

(Throughout this part we write $\|\cdot\|_\alpha$ to denote the operator norm in L^2_α.) The proof of (2.24) will be completed in section 6.

The proof of Theorem 2.3 involves analysis of eigenvalues in the KdV scaling limit. The eigenvalue problem for $\mathcal{L}_c$ is reduced to a characteristic value problem (nonlinear eigenvalue problem) for an analytic Fredholm operator bundle $\mathcal{W}(\lambda)$, for which the value $\lambda = 0$ has "null multiplicity" at least 2. In the KdV limit, this bundle converges after scaling to one naturally associated with the KdV soliton, for which the only characteristic value is at the origin, with null multiplicity exactly 2. Theorem 2.3 will be proved using the operator-valued version of Rouché's theorem due to Gohberg and Sigal [17] to conclude that $\mathcal{W}(\lambda)$ can have no nonzero characteristic values $\lambda \ne 0$ satisfying $\operatorname{Re}\lambda \ge 0$.

To begin all the analysis, it is convenient to change variables to diagonalize $L + c\partial$, the leading part of the system. Define the Fourier multiplier operators

$$\mathcal{S} = \sqrt{B^{-1}A} = \sqrt{\frac{1 - a\partial^2}{1 - b\partial^2}}, \qquad \mathcal{Q}_\pm = c\partial \pm \mathcal{S}\partial, \tag{2.25}$$

associated with the symbols

$$\hat{\mathcal{S}}(\xi) = \sqrt{\frac{1 + a\xi^2}{1 + b\xi^2}}, \qquad \hat{\mathcal{Q}}_\pm(\xi) = i\xi c \pm i\xi \hat{\mathcal{S}}(\xi), \tag{2.26}$$

and observe

$$\begin{pmatrix} \mathcal{S} & I \\ -\mathcal{S} & I \end{pmatrix} \begin{pmatrix} \lambda - c\partial & -\partial \\ -\mathcal{S}^2\partial & \lambda - c\partial \end{pmatrix} \begin{pmatrix} \mathcal{S} & I \\ -\mathcal{S} & I \end{pmatrix}^{-1} = \begin{pmatrix} \lambda - \mathcal{Q}_+ & 0 \\ 0 & \lambda - \mathcal{Q}_- \end{pmatrix}. \tag{2.27}$$

The Fourier multipliers $\mathcal{S}$, $\mathcal{S}^{-1}$, and $(\lambda - \mathcal{Q}_\pm)^{-1}$ will be seen to be bounded on L^2_α, uniformly for λ of positive real part. Lemma 2.1 concerning the basic properties of $\mathcal{L}_c$ is proved in section 4 (except for the proof of part (iv), which we provide in appendix B). Before that we will develop necessary estimates for various Fourier multipliers associated with the resolvent of $L + c\partial$.

3. General Fourier symbol estimates

Note that for any smooth $g\colon \mathbb{R} \to \mathbb{R}$ with compact support, by Plancherel's theorem,

$$\int_{\mathbb{R}} |e^{\alpha x} g(x)|^2 \, dx = \int_{\mathbb{R}} |\hat{g}(k + i\alpha)|^2 \, \frac{dk}{2\pi}.$$

It follows that if $\mathcal{R}$ is a Fourier multiplier operator with symbol $\hat{\mathcal{R}}$ analytic and bounded on the strip where $0 \le \operatorname{Im} \xi \le \alpha$, then the operator norm of $\mathcal{R}$ acting on L^2_α is

$$\|\mathcal{R}\|_\alpha = \sup_{k\in\mathbb{R}} |\hat{\mathcal{R}}(k + i\alpha)|. \tag{3.1}$$

Lemma 3.1. *Suppose $c > 1$ and $0 < \alpha < \alpha_c$ and $0 < a < b$. For all real $k \neq 0$, with $\xi = k + i\alpha$ we have*

$$k \operatorname{Im} \hat{\mathcal{S}}(\xi) < 0, \tag{3.2}$$

$$\sqrt{\frac{a}{b}} < |\hat{\mathcal{S}}(\xi)| < \hat{\mathcal{S}}(i\alpha) = \sqrt{\frac{1 - a\alpha^2}{1 - b\alpha^2}} < c, \tag{3.3}$$

$$|\hat{\mathcal{S}}(\xi)| < 1 - \frac{1}{2} \frac{(b-a)(k^2 - \alpha^2)}{1 + b(k^2 - \alpha^2)}, \tag{3.4}$$

$$i\xi\hat{\mathcal{S}}(\xi) = -\sqrt{-\xi^2 \frac{1 + a\xi^2}{1 + b\xi^2}}. \tag{3.5}$$

Proof. It suffices to consider $k > 0$. Taking $k \to \pm\infty$ and $k \to 0$, note that due to (2.8b),

$$\hat{\mathcal{S}}(\pm\infty + i\alpha) = \sqrt{\frac{a}{b}} < 1 < \sqrt{\frac{1 - a\alpha^2}{1 - b\alpha^2}} = \hat{\mathcal{S}}(i\alpha) < c.$$

Observe that

$$\hat{\mathcal{S}}(\xi)^2 = \frac{a}{b} + \left(1 - \frac{a}{b}\right) \frac{1}{1 + b\xi^2} = \frac{a}{b} + \left(1 - \frac{a}{b}\right) \frac{1}{1 + b(k^2 - \alpha^2) + 2ibk\alpha}. \tag{3.6}$$

The last term has negative imaginary part and positive real part, which implies (3.2) and the first part of (3.3). By the triangle inequality,

$$|\hat{\mathcal{S}}(\xi)^2| < \frac{a}{b} + \left(1 - \frac{a}{b}\right) \frac{1}{1 + b(k^2 - \alpha^2)} = \frac{1 + a(k^2 - \alpha^2)}{1 + b(k^2 - \alpha^2)} < \frac{1 - a\alpha^2}{1 - b\alpha^2}. \tag{3.7}$$

This gives (3.3), and since $x \le \frac{1}{2} + \frac{1}{2}x^2$ for any $x \ge 0$, taking $x = |\hat{\mathcal{S}}(\xi)|$ gives (3.4).

To prove (3.5), observe that since $\xi^2 = k^2 - \alpha^2 + 2ik\alpha$ and $k > 0$, we have

$$0 < \arg(1 + b\xi^2) < \arg b\xi^2 = \arg \xi^2 < \pi, \qquad 0 < \arg(1 + a\xi^2) < \pi/2.$$

Hence the quantity

$$\frac{-\xi^2}{1+b\xi^2}(1+a\xi^2)$$

is never strictly negative, since its argument is strictly between $-\pi$ and $\pi/2$. Now (3.5) follows by continuation starting at $k = 0$. □

Since (3.5) implies $i\xi\hat{\mathcal{S}}(\xi)$ has negative real part, and since

$$\operatorname{Re}\hat{\mathcal{Q}}_\pm(\xi) = -\alpha c \pm \operatorname{Re}(i\xi\hat{\mathcal{S}}(\xi)) = -\alpha c \mp (\alpha \operatorname{Re}\hat{\mathcal{S}}(\xi) + k \operatorname{Im}\hat{\mathcal{S}}(\xi)),$$

we infer the following by using (3.4) and (3.2) and (3.5), along with (3.1).

Corollary 3.2. *For all real k, with $\xi = k + i\alpha$ we have*

$$-2\alpha c < \operatorname{Re}\hat{\mathcal{Q}}_+(\xi) < -\alpha c,$$
$$-\alpha c < \operatorname{Re}\hat{\mathcal{Q}}_-(\xi) \le -\alpha\left(c - 1 + \frac{1}{2}\frac{(b-a)(k^2-\alpha^2)}{1+b(k^2-\alpha^2)}\right) < 0.$$

Moreover, whenever $\operatorname{Re}\lambda + \alpha(c - \hat{\mathcal{S}}(i\alpha)) \ge 0$ *we have*

$$\|(\lambda - \mathcal{Q}_+)^{-1}\|_\alpha \le (\operatorname{Re}\lambda + \alpha c)^{-1},$$
$$\|(\lambda - \mathcal{Q}_-)^{-1}\|_\alpha \le (\operatorname{Re}\lambda + \alpha(c - \hat{\mathcal{S}}(i\alpha))^{-1}.$$

4. Basic properties of the linearization

Here we provide the proof of Lemma 2.1, parts (i)–(iii). The proof of part (iv) appears in appendix B.

The proof of part (i) is straightforward: Writing $\xi = k + i\alpha$, the Fourier symbol of $B^{-1}\partial^j$ satsifies

$$\frac{(i\xi)^j}{1+b\xi^2} \to 0 \quad \text{as } k \to \pm\infty$$

for $j = 0, 1$. And for $g = q_c$, r_c, q_c', r_c', g is continuous with $g(x) \to 0$ as $x \to \pm\infty$. Hence the operators $B^{-1}\partial^j g$ are all compact on L^2_α, by the convenient compactness criterion of [35], using the isomorphism $g \to e^{\alpha x} g$ from L^2_α to L^2. Since $B^{-1}g\partial = B^{-1}(\partial g - g_x)$, the operator $\mathcal{L}_c - (L + c\partial)$ is a simple linear combination of compact operators, so is compact.

To prove part (ii), we observe that $L + c\partial$ is the generator of a C^0 group on L^2_α. After the change of variables (5.3), this follows from the Hille-Yosida theorem due to the resolvent bounds in Corollary 3.2. Now $\mathcal{L}_c$ generates a C^0 group on L^2_α also, by a standard perturbation theorem [34, Ch. 3, Thm. 1.1].

For part (iii) we note that the spectrum of $L + c\partial$ on L^2_α is the union of the image of the curves

$$k \mapsto \lambda = \hat{\mathcal{Q}}_\pm(k + i\alpha),$$

which lie strictly in the left half-plane $\operatorname{Re}\lambda < 0$ due to Corollary 3.2. Then the essential spectrum of $\mathcal{L}_c$ lies in the left half-plane too, by a standard generalization of Weyl's theorem to non-selfadjoint operators—One applies the analytic Fredholm theorem from [16, I.5.1] or [44, VI.14] in the right half-plane to the factorization

$$I - (\lambda - L - c\partial)^{-1}(\mathcal{L}_c - L - c\partial) = (\lambda - L - c\partial)^{-1}(\lambda - \mathcal{L}_c).$$

5. Reduction of the resolvent

For simplicity we write $(q,r) = (q_c, r_c)$ henceforth. The resolvent equation for the operator $\mathcal{L}_c$ takes the following form:

$$\begin{aligned} (\lambda - c\partial)q_1 - \partial r_1 &= f_1, &(5.1)\\ (-A\partial + r\partial + 2r')q_1 + (B(\lambda - c\partial) + q' + 2q\partial)r_1 &= Bg_1. &(5.2)\end{aligned}$$

We study this system in L^2_α, $0<\alpha<\alpha_c$, by changing variables using the transformation

$$\begin{pmatrix} q_2 \\ r_2 \end{pmatrix} = \begin{pmatrix} \mathcal{S} & I \\ -\mathcal{S} & I \end{pmatrix} \begin{pmatrix} q_1 \\ r_1 \end{pmatrix}, \tag{5.3}$$

which is bounded on L^2_α with bounded inverse, due to (3.3). In the new variables the resolvent system (5.1)-(5.2) is written

$$\begin{pmatrix} \lambda - \mathcal{Q}_+ & 0 \\ 0 & \lambda - \mathcal{Q}_- \end{pmatrix} \begin{pmatrix} q_2 \\ r_2 \end{pmatrix} + \begin{pmatrix} R_r & R_q \\ R_r & R_q \end{pmatrix} \begin{pmatrix} I & -I \\ I & I \end{pmatrix} \begin{pmatrix} q_2 \\ r_2 \end{pmatrix} = \begin{pmatrix} f_2 \\ g_2 \end{pmatrix}, \tag{5.4}$$

with

$$\begin{pmatrix} f_2 \\ g_2 \end{pmatrix} = \begin{pmatrix} \mathcal{S} & I \\ -\mathcal{S} & I \end{pmatrix} \begin{pmatrix} f_1 \\ g_1 \end{pmatrix}, \tag{5.5}$$

$$R_r = \frac{1}{2}B^{-1}(r\partial + 2r')\mathcal{S}^{-1}, \qquad R_q = \frac{1}{2}B^{-1}(q' + 2q\partial). \tag{5.6}$$

Subtracting the second equation from the first, this system becomes

$$\begin{pmatrix} \lambda - \mathcal{Q}_+ & -\lambda + \mathcal{Q}_- \\ R_q + R_r & \lambda - \mathcal{Q}_- + R_q - R_r \end{pmatrix} \begin{pmatrix} q_2 \\ r_2 \end{pmatrix} = \begin{pmatrix} f_2 - g_2 \\ g_2 \end{pmatrix}. \tag{5.7}$$

We can eliminate q_2 by writing

$$q_2 = (\lambda - \mathcal{Q}_+)^{-1}(\lambda - \mathcal{Q}_-)r_2 + (\lambda - \mathcal{Q}_+)^{-1}(f_2 - g_2), \tag{5.8}$$

and using this in the second equation. This reduces the resolvent equation to the form

$$\boxed{\mathcal{W}(\lambda)(\lambda - \mathcal{Q}_-)r_2 = g_3} \tag{5.9}$$

with

$$\mathcal{W}(\lambda) = I + (R_q + R_r)(\lambda - \mathcal{Q}_+)^{-1} + (R_q - R_r)(\lambda - \mathcal{Q}_-)^{-1}, \tag{5.10}$$

and

$$g_3 = g_2 - (R_q + R_r)(\lambda - \mathcal{Q}_+)^{-1}(f_2 - g_2). \tag{5.11}$$

Thus we see that to prove both Theorems 2.2 and 2.3 it will suffice to study the invertibility of the operator bundle $\mathcal{W}(\lambda)$.

Lemma 5.1. *If $\operatorname{Re}\lambda + \alpha(c - \hat{\mathcal{S}}(i\alpha)) \geq 0$, then λ is in the resolvent set of $\mathcal{L}_c$ if and only if $\mathcal{W}(\lambda)$ is invertible.*

For later use, note R_q and R_r are compact (since $B^{-1}\partial q$ and $B^{-1}q'$ are compact), and

$$\begin{aligned} R_q + R_r &= B^{-1}\left(-q'(\tfrac12 + \tfrac12 c\mathcal{S}^{-1}) + \partial q(1 - \tfrac12 c\mathcal{S}^{-1})\right), &(5.12)\\ R_q - R_r &= B^{-1}\left(q'(\tfrac12 + c\mathcal{S}^{-1}) + q\partial(1 + \tfrac12 c\mathcal{S}^{-1})\right) &(5.13)\\ &= \left(q'(\tfrac12 + c\mathcal{S}^{-1}) + q\partial(1 + \tfrac12 c\mathcal{S}^{-1})\right)B^{-1} \\ &\quad + [B^{-1}, q'](\tfrac12 + c\mathcal{S}^{-1}) + [B^{-1}, q]\partial(1 + \tfrac12 c\mathcal{S}^{-1}), \end{aligned}$$

where

$$[B^{-1}, q'] = B^{-1}q' - q'B^{-1}, \qquad [B^{-1}, q] = B^{-1}q - qB^{-1}.$$

6. Spectral implies linear stability

In this section we complete the proof of Theorem 2.2. Fix $c > 1$ and α with $0 < \alpha < \alpha_c$. By Lemma 2.1 and the hypothesis of Theorem 2.2 concerning eigenvalues, we know that the closed right half-plane is in the resolvent set of $\mathcal{L}_c$, except for the origin $\lambda = 0$. We will deduce the conclusion of the theorem by applying the Gearhart-Prüss theorem in the spectral complement $Z_\alpha = (I - P_c)L^2_\alpha$ of the generalized eigenspace for $\lambda = 0$. For this purpose, it suffices to prove the uniform resolvent bound (2.24). Due to the reduction carried out in the previous section, to prove the uniform resolvent bound (2.24), it suffices to prove that

$$\sup_{\operatorname{Re}\lambda \geq 0,\, |\lambda| > R} \|(R_q \pm R_r)(\lambda - \mathcal{Q}_\pm)^{-1}\|_\alpha \to 0 \qquad \text{as } R \to \infty, \tag{6.1}$$

since then $\mathcal{W}(\lambda) \to I$ and g_3 is uniformly bounded in terms of (f_2, g_2).

From Corollary 3.2, we know that $\lambda - \mathcal{Q}_\pm$ has bounded inverse whenever $\operatorname{Re}\lambda \geq 0$. Moreover, we claim that as $|\lambda| \to \infty$ with $\operatorname{Re}\lambda \geq 0$, $(\lambda - \mathcal{Q}_\pm)^{-1} \to 0$ in the strong operator sense on L^2_α. To see this, fix $z \in L^2_\alpha$, and let $w = (\lambda - \mathcal{Q}_\pm)^{-1}z$. Then the Fourier transform

$$\hat{w}(k + i\alpha) = (\lambda - \hat{\mathcal{Q}}_\pm(k + i\alpha))^{-1}\hat{z}(k + i\alpha) \to 0$$

for a.e. k. By dominated convergence it follows $\|w\|_{L^2_\alpha} \to 0$.

Since $R_q \pm R_r$ is compact, (6.1) follows as a consequence of an abstract fact: In a Hilbert space, if a sequence of bounded operators T_n converges strongly to 0, and the operator S is compact, then ST_n converges to 0 in operator norm. (We omit the elementary proof.)

7. Spectral stability in the KdV scaling regime

Our goal in this section is to prove Theorem 2.3, establishing spectral stability for weakly nonlinear waves. Our strategy involves making use of known stability properties of the soliton solution of the KdV equation

$$\partial_t\rho - \partial_x\rho + 3\rho\partial_x\rho + (b - a)\partial_x^3\rho = 0, \tag{7.1}$$

given by $\rho = \theta_0(x)$ where

$$\theta_0(x) = \operatorname{sech}^2\left(\frac{1}{2}\hat{\alpha}_0 x\right), \qquad \hat{\alpha}_0 = \frac{1}{\sqrt{b - a}}. \tag{7.2}$$

The eigenvalue problem for the linearization of (7.1) about θ_0 takes the form

$$(\Lambda - \partial_x + 3\partial_x\theta_0 + (b - a)\partial_x^3)\rho_1 = 0,$$

which we rewrite as

$$\mathcal{W}_0(\Lambda)(\Lambda - \partial + (b - a)\partial^3)\rho_1 = 0, \tag{7.3}$$

in terms of the bundle

$$\mathcal{W}_0(\Lambda) = I + (3\partial\theta_0)(\Lambda - \partial + (b - a)\partial^3)^{-1}. \tag{7.4}$$

Due to known stability properties of the KdV soliton (see Lemma C.2 for a precise characterization), $\mathcal{W}_0(\Lambda)$ is known to be invertible in $L^2_{\hat\alpha}$ whenever $0<\hat\alpha<\hat\alpha_0$ and $\Lambda\neq 0$ with $\operatorname{Re}\Lambda\ge -\hat\beta$, where

$$\hat\beta = \hat\alpha(1-(b-a)\hat\alpha^2). \tag{7.5}$$

(The essential spectrum of $\partial-(b-a)\partial^3$ in $L^2_{\hat\alpha}$ is contained in the half-plane $\operatorname{Re}\Lambda\le -\hat\beta$.)

To see the relevance of this KdV eigenvalue problem for small-energy solitary waves of the Benney-Luke system (2.2), we study the reduced eigenvalue problem from (5.9) using the KdV scaling,

$$c = 1+\frac{\epsilon^2}{2},\quad \lambda=\frac{\epsilon^3}{2}\Lambda,\quad \hat x=\epsilon x. \tag{7.6}$$

The solitary wave profile from (2.8) then takes the form

$$q_c(x)=\epsilon^2\theta_\epsilon(\epsilon x),\quad \theta_\epsilon(\hat x)=\frac{c+1}{2c}\operatorname{sech}^2\left(\frac12\hat\alpha_\epsilon\hat x\right),\quad \hat\alpha_\epsilon=\sqrt{\frac{c+1}{2(bc^2-a)}}. \tag{7.7}$$

Formally, the KdV scaling corresponds to the following:

$$\partial_x\sim\epsilon\partial_{\hat x},\quad \mathcal{S}\sim I+\frac12(b-a)\epsilon^2\partial_{\hat x}^2,\quad q_c\sim -r_c\sim\epsilon^2\theta_0(\hat x).$$

Using this scaling in the reduced resolvent equation (5.9) indicates

$$R_q-R_r\sim\frac{3\epsilon^3}{2}\left(\theta_0'+\theta_0\partial_{\hat x}\right)=\frac{3\epsilon^3}{2}\,\partial_{\hat x}\theta_0\ ,\qquad R_q+R_r\sim\frac{\epsilon^3}{2}\left(-2\theta_0'+\partial_{\hat x}\theta_0\right),$$

$$\lambda-\mathcal{Q}_-\sim\frac{\epsilon^3}{2}(\Lambda-\partial_{\hat x}+(b-a)\partial_{\hat x}^3),\qquad \lambda-\mathcal{Q}_+\sim -2\epsilon\partial_{\hat x},$$

and consequently

$$\mathcal{W}(\lambda)\sim I+(3\partial_{\hat x}\theta_0)(\Lambda-\partial_{\hat x}+(b-a)\partial_{\hat x}^3)^{-1}=\mathcal{W}_0(\Lambda). \tag{7.8}$$

A key step in the proof of Theorem 2.3 is to make the formal limit in (7.8) precise, and invoke the operator-valued Rouché theorem proved by Gohberg and Sigal [17] to deduce that $\mathcal{W}(\lambda)$ is invertible for every nonzero λ in a suitable open half-space that contains the closed right half-plane.

We introduce a scaling operator defined by $\mathcal{I}_\epsilon g(x)=g(\epsilon x)\sqrt{\epsilon}$. Then $\mathcal{I}_\epsilon : L^2_{\hat\alpha}\to L^2_{\epsilon\hat\alpha}$ is an isometry, since

$$\int_{\mathbb{R}}|e^{\hat\alpha x}g(x)|^2\,dx=\int_{\mathbb{R}}|e^{\epsilon\hat\alpha x}g(\epsilon x)\sqrt{\epsilon}|^2\,dx.$$

Then any bounded operator Q on $L^2_{\epsilon\hat\alpha}$ induces a bounded operator $\mathcal{I}_\epsilon^{-1}Q\mathcal{I}_\epsilon$ on $L^2_{\hat\alpha}$ with the same norm. We make (7.8) precise in the following sense:

Theorem 7.1 (Bundle convergence)**.** *Fix $\hat\alpha\in(0,\hat\alpha_0)$ and let $\hat\beta=\hat\alpha(1-(b-a)\hat\alpha^2)$. Then with $\mathcal{W}_\epsilon(\Lambda)=\mathcal{I}_\epsilon^{-1}\mathcal{W}(\lambda)\mathcal{I}_\epsilon$ where $\lambda=\frac12\epsilon^3\Lambda$, we have*

$$\sup_{\operatorname{Re}\Lambda\ge-\hat\beta/2}\|\mathcal{W}_\epsilon(\Lambda)-\mathcal{W}_0(\Lambda)\|_{\hat\alpha}\to 0\quad \text{as } \epsilon\to 0. \tag{7.9}$$

7.1. **Estimates for the bundle convergence theorem.** To prove Theorem 7.1 we substitute (5.12)–(5.13) into (5.10). The proof, to be completed in subsection 7.5, follows from three groups of estimates that we detail in this subsection: (a) basic estimates on Fourier multipliers, (b) convergence estimates for certain rescaled Fourier multipliers in the KdV limit, and (c) estimates on junk terms and commutators. We define

$$\Omega_* = \{\Lambda \in \mathbb{C} : \operatorname{Re}\Lambda \geq -\hat{\beta}/2\}. \tag{7.10}$$

Lemma 7.2 (Basic estimates)**.** *Uniformly for small $\epsilon > 0$ and for $\Lambda \in \Omega_*$,*

$$\|\mathcal{S}^{-1}\|_\alpha \leq C, \tag{7.11}$$
$$\|B^{-1}\partial^j\|_\alpha \leq C \quad (j = 0, 1), \tag{7.12}$$
$$\|(\lambda - \mathcal{Q}_+)^{-1}\|_\alpha \leq C\epsilon^{-1}, \tag{7.13}$$
$$\|(\lambda - \mathcal{Q}_-)^{-1}\|_\alpha \leq C\epsilon^{-3}. \tag{7.14}$$

Lemma 7.3 (KdV limit of Fourier multipliers)**.** *For $j, k = 0, 1$,*

$$\|\mathcal{I}_\epsilon^{-1}\left(\epsilon^{3-j}\partial^j\mathcal{S}^{-k}B^{-1}(\lambda - \mathcal{Q}_-)^{-1}\right)\mathcal{I}_\epsilon - 2\partial^j(\Lambda - \partial + (b-a)\partial^3)^{-1}\|_{\hat{\alpha}} \to 0, \tag{7.15}$$

uniformly for $\Lambda \in \Omega_*$.

Lemma 7.3 is key, but its proof turns out not to be very hard, only involving Taylor expansion of symbols at low frequency, and uniform bounds at high frequency. The presence of the smoothing operator B^{-1} simplifies the analysis as compared to the case of water waves treated in [38].

Finally, the junk terms include $(R_q + R_r)(\lambda - \mathcal{Q}_+)^{-1}$, and terms involving the commutators $[B^{-1}, q']$ and $[B^{-1}, q]$ in $R_q - R_r$. The first kind of junk term is handled by noting that since $q = O(\epsilon^2)$ and $q' = O(\epsilon^3)$, the estimates (7.11), (7.12) and (7.13) yield

$$\|(R_q + R_r)(\lambda - \mathcal{Q}_+)^{-1}\|_\alpha \leq C\epsilon, \tag{7.16}$$

where $\alpha = \epsilon\hat{\alpha}$ (here and below). Concerning the commutators, we will establish the following.

Lemma 7.4 (Commutator estimates)**.**

$$\|[B^{-1}, q']\|_\alpha \leq C\epsilon^4, \tag{7.17}$$

$$\|[B^{-1}, q]\partial(\lambda - \mathcal{Q}_-)^{-1}\|_\alpha \to 0 \qquad \text{as } \epsilon \to 0, \tag{7.18}$$

uniformly for $\Lambda \in \Omega_*$.

The first of these estimates is not difficult. However, it turns out that the term $[B^{-1}, q]\partial$ has operator norm $\|[B^{-1}, q]\partial\|_\alpha = O(\epsilon^3)$, which is not small enough to neglect, due to (7.14). Consequently, we have to establish instead the more complicated commutator estimate in (7.18). To establish this we will use the commutator estimate in Lemma 7.5 below (from [38]), and deal separately with the high and the low frequencies.

The result of Theorem 7.1 follows directly from the symbol limits in (7.15) and the estimates in (7.16), (7.17), (7.18) and (7.15), using (5.13) and the fact that $\mathcal{I}_\epsilon$ is an isometry from $L^2_{\hat{\alpha}}$ to L^2_α.

7.2. **Basic estimates.** We now prove Lemma 7.2. From (3.3) and (3.1) we infer

$$\|\mathcal{S}^{-1}\|_\alpha = \sup_{k\in\mathbb{R}} |\hat{\mathcal{S}}(k+i\alpha)^{-1}| = \sqrt{\frac{b}{a}}. \tag{7.19}$$

Moreover,

$$\|B^{-1}\|_\alpha = \sup_{k\in\mathbb{R}} \left| \frac{1}{1+b(k^2-\alpha^2)+2ibk\alpha} \right| = \frac{1}{1-b\alpha^2}, \tag{7.20}$$

and since $2|\xi| \le 1+|\xi|^2$,

$$\|B^{-1}\partial\|_\alpha = \sup_{k\in\mathbb{R}} \left| \frac{k+i\alpha}{1+b(k^2-\alpha^2)+2ibk\alpha} \right| \le \sup_{k\in\mathbb{R}} \frac{1}{2} \left| \frac{1+k^2+\alpha^2}{1+b(k^2-\alpha^2)} \right| \le C. \tag{7.21}$$

Next we invoke Corollary 3.2 with $\Lambda \in \Omega_*$ and $\lambda = \frac{1}{2}\epsilon^3\Lambda$. Since $\hat{\beta} \le \hat{\alpha}$ we have $\operatorname{Re}\Lambda + \hat{\alpha} \ge 0$, so

$$\operatorname{Re}\lambda + \alpha c = \tfrac{1}{2}\epsilon^3 \operatorname{Re}\Lambda + \epsilon\hat{\alpha}(1+\tfrac{1}{2}\epsilon^2) \ge \epsilon\hat{\alpha},$$

hence

$$\|(\lambda - \mathcal{Q}_+)^{-1}\|_\alpha \le \frac{1}{\epsilon\hat{\alpha}}. \tag{7.22}$$

Also, since

$$\hat{\mathcal{S}}(i\alpha) = \sqrt{\frac{1-a\alpha^2}{1-b\alpha^2}} \le 1 + \frac{1}{2}(b-a)\epsilon^2\hat{\alpha}^2,$$

we have

$$\operatorname{Re}\lambda + \alpha(c - \hat{\mathcal{S}}(i\alpha)) \quad \ge \quad \frac{\epsilon^3}{2}(\operatorname{Re}\Lambda + \hat{\alpha}(1-(b-a)\hat{\alpha}^2)) \ge \frac{\epsilon^3\hat{\beta}}{4},$$

hence

$$\|(\lambda - \mathcal{Q}_-)^{-1}\|_\alpha \le \frac{4}{\epsilon^3\hat{\beta}}. \tag{7.23}$$

7.3. **KdV limit of Fourier multipliers.** Next we prove Lemma 7.3. By (3.1), this is equivalent to showing that for $j, \hat{j} = 0, 1$,

$$\left| \frac{\hat{\mathcal{S}}(\epsilon\hat{\xi})^{-\hat{j}}}{1+b\epsilon^2\hat{\xi}^2} \frac{\epsilon^3(i\hat{\xi})^j}{\lambda - \hat{\mathcal{Q}}_-(\epsilon\hat{\xi})} - \frac{2(i\hat{\xi})^j}{\Lambda - i\hat{\xi} - (b-a)i\hat{\xi}^3} \right| \to 0 \quad \text{as } \epsilon \to 0, \tag{7.24}$$

uniformly for $\hat{\xi} = \hat{k} + i\hat{\alpha}$ with $\hat{k} \in \mathbb{R}$, and uniformly for $\Lambda \in \Omega_*$, with $\lambda = \frac{1}{2}\epsilon^3\Lambda$. The factor in (7.24) that corresponds to the symbol of $\mathcal{S}^{-1}B^{-1}$ satisfies

$$\frac{\hat{\mathcal{S}}(\epsilon\hat{\xi})^{-\hat{j}}}{1+b\epsilon^2\hat{\xi}^2} = 1 + O(\epsilon\hat{\xi}), \tag{7.25}$$

and is uniformly bounded. To establish (7.24), we will treat separately the low and high frequencies.

To start, we obtain a basic lower bound on the denominator of the second term in (7.24),

$$m_0 = \Lambda - i\hat{\xi} - (b-a)i\hat{\xi}^3. \tag{7.26}$$

Observe that for $\hat{\xi} = \hat{k} + i\hat{\alpha}$ and $\Lambda \in \Omega_*$ we have the estimate

$$\operatorname{Re} m_0 = \operatorname{Re}\Lambda + \hat{\alpha} + (b-a)(3k^2 - \hat{\alpha}^2)\hat{\alpha} \ge \frac{\hat{\beta}}{2} + 3(b-a)\hat{\alpha}k^2 \ge \frac{\hat{\alpha}|\hat{\xi}|^2}{C}, \tag{7.27}$$

provided $\frac{1}{2}\hat{\beta}C \ge \hat{\alpha}^3$ and $3(b-a)C \ge 1$.

7.3.1. *Low frequency (KdV) regime:* $|\epsilon\hat{\xi}| \le 4\epsilon^p$. We fix $p \in (\frac{1}{3}, \frac{1}{2})$, and let

$$I_0 = \{\hat{\xi} = \hat{k} + i\hat{\alpha} : \hat{k} \in \mathbb{R} \text{ and } |\epsilon\hat{\xi}| \le 4\epsilon^p\}.$$

For frequencies in this regime we carry out a Taylor expansion of the symbols in (7.24), handling the remainder carefully. Observe that

$$\hat{\mathcal{S}}(\xi)^2 = \frac{1 + a\xi^2}{1 + b\xi^2} = 1 - (b-a)\xi^2 + O(\xi^4),$$

so

$$\hat{\mathcal{S}}(\epsilon\hat{\xi}) = 1 - (b-a)\frac{\epsilon^2\hat{\xi}^2}{2} + O(\epsilon^4\hat{\xi}^4). \tag{7.28}$$

Hence

$$\begin{aligned} \lambda - \hat{\mathcal{Q}}_-(\epsilon\hat{\xi}) &= \frac{\epsilon^3}{2}\Lambda - i\epsilon\hat{\xi}(1 + \frac{\epsilon^2}{2}) + i\epsilon\hat{\xi}\hat{\mathcal{S}}(\epsilon\hat{\xi}) \\ &= \frac{\epsilon^3}{2}\left(\Lambda - i\hat{\xi} - (b-a)i\hat{\xi}^3 + \hat{\xi}^3 O(\epsilon^2\hat{\xi}^2)\right). \end{aligned} \tag{7.29}$$

Let us define

$$m_\epsilon = 2\epsilon^{-3}(\lambda - \hat{\mathcal{Q}}_-(\epsilon\hat{\xi})). \tag{7.30}$$

Then by (7.29) we have that

$$E := m_\epsilon - m_0 = \hat{\xi}^3 O(\epsilon^2\hat{\xi}^2) = \hat{\xi}^3 O(\epsilon^{2p}).$$

Then due to the lower bound (7.27), for $\hat{\xi} \in I_0$ we have

$$\left|\frac{E}{m_0}\right| \le \frac{C|\hat{\xi}^3|\epsilon^{2p}}{|m_0|} \le C|\hat{\xi}|\epsilon^{2p} \le C\epsilon^{3p-1}, \tag{7.31}$$

which tends to zero as $\epsilon \to 0$. Then it follows from (7.31) and (7.27) that

$$\left|\frac{(i\hat{\xi})^j}{m_\epsilon} - \frac{(i\hat{\xi})^j}{m_0}\right| = \frac{|\hat{\xi}|^j |E/m_0|}{|m_0||1 + E/m_0|} \le C|\hat{\xi}|^{j-2}\epsilon^{3p-1} \le C\epsilon^{3p-1} \tag{7.32}$$

and consequently (7.24) holds uniformly for $\hat{\xi} \in I_0$ and $\Lambda \in \Omega_*$.

7.3.2. *High frequency regime:* $|\epsilon\hat{k}| \ge 2\epsilon^p$. Consider $\hat{\xi}$ in the set

$$I_1 = \{\hat{\xi} = \hat{k} + i\hat{\alpha} : |\epsilon\hat{k}| \ge 2\epsilon^p\},$$

and note that we have $I_0 \cup I_1 = \mathbb{R} + i\hat{\alpha}$ for sufficiently small $\epsilon > 0$. In this complementary regime we claim that the terms in (7.24) separately go to zero. Consider the second term first. From the lower bound (7.27), we find that this term is bounded by

$$\left|\frac{(i\hat{\xi})^j}{m_0}\right| \le C|\hat{\xi}|^{-1} \le C\epsilon^{1-p} \to 0. \tag{7.33}$$

Now consider the first term in (7.24). With $\xi = k + i\alpha = \epsilon\hat{\xi}$, for small enough ϵ we have $k^2 - \alpha^2 > \frac{1}{2}k^2 \ge 2\epsilon^{2p}$ and

$$\frac{b(k^2 - \alpha^2)}{1 + b(k^2 - \alpha^2)} \ge \frac{2b\epsilon^{2p}}{1 + 2b\epsilon^{2p}} \ge b\epsilon^{2p}.$$

By Corollary 3.2, since $c - 1 = \frac{1}{2}\epsilon^2$ and $\operatorname{Re}\Lambda + \hat{\alpha} \ge 0$ we then get

$$\operatorname{Re}\left(\frac{\epsilon^3}{2}\Lambda - \hat{\mathcal{Q}}_-(\epsilon\hat{\xi})\right) \ge \frac{\epsilon^3}{2}\operatorname{Re}\Lambda + \frac{\epsilon\hat{\alpha}}{2}\left(\epsilon^2 + (b-a)\epsilon^{2p}\right) \ge \frac{\hat{\alpha}}{2}(b-a)\epsilon^{1+2p}. \tag{7.34}$$

By consequence we have that for sufficiently small $\epsilon > 0$,

$$\left| \frac{\epsilon^2}{\lambda - \hat{\mathcal{Q}}_-(\epsilon\hat{\xi})} \right| \le C\epsilon^{1-2p} \to 0. \tag{7.35}$$

Since by (7.20)-(7.21) we have

$$\left| \frac{\epsilon(i\hat{\xi})^j}{1 + b\epsilon^2\hat{\xi}^2} \right| \le C \tag{7.36}$$

for $j = 0, 1$, we see that the first term in (7.24) tends to zero, uniformly for $\hat{\xi} \in I_1$ and $\Lambda \in \Omega_*$.

This finishes the proof of the limit formula (7.15) for Fourier mulitpliers.

7.4. **Commutator estimates.** In this subsection we prove Lemma 7.4. The proof of the following commutator bounds, from [38], is short and is reproduced here for completeness. We write $\langle k \rangle = (1 + |k|^2)^{1/2}$ below.

Lemma 7.5. *Let $\mathcal{P}$, $\mathcal{Q}$ and $\mathcal{R}$ be Fourier multipliers with symbols $\hat{\mathcal{P}}$, $\hat{\mathcal{Q}}$ and $\hat{\mathcal{R}}$ respectively, and let $s \ge 0$. Let $g(x) = \epsilon^2 G(\epsilon x)$ where $G\colon \mathbb{R} \to \mathbb{R}$ is smooth and exponentially decaying, and let $h\colon \mathbb{R} \to \mathbb{R}$ be smooth with compact support. Then*

$$\|\mathcal{P}[\mathcal{Q}, g]\mathcal{R}h\|_{L^2} \le M_\epsilon M_G \|h\|_{L^2},$$

where

$$M_\epsilon = \sup_{k,\hat{k}\in\mathbb{R}} \epsilon^2 \frac{\hat{\mathcal{P}}(\epsilon k)|\hat{\mathcal{Q}}(\epsilon k) - \hat{\mathcal{Q}}(\epsilon\hat{k})|\hat{\mathcal{R}}(\epsilon\hat{k})}{\langle k - \hat{k}\rangle^s}, \qquad M_G = \int_{\mathbb{R}} \langle k \rangle^s |\hat{G}(k)| \, \frac{dk}{2\pi}.$$

Proof. Using the Fourier transform and Young's inequality, since $\hat{g}(k) = \epsilon\hat{G}(k/\epsilon)$, we have

$$\begin{aligned}
\|\mathcal{P}[\mathcal{Q}, g]\mathcal{R}h\|_{L^2}^2 &= \int_{\mathbb{R}} \left| \int_{\mathbb{R}} \hat{\mathcal{P}}(k)(\hat{\mathcal{Q}}(k) - \hat{\mathcal{Q}}(\hat{k}))\epsilon\hat{G}\left(\frac{k-\hat{k}}{\epsilon}\right) \hat{\mathcal{R}}(\hat{k})\hat{h}(\hat{k}) \, \frac{d\hat{k}}{2\pi} \right|^2 \frac{dk}{2\pi} \\
&\le M_\epsilon^2 \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \left\langle \frac{k-\hat{k}}{\epsilon} \right\rangle^s \left| \hat{G}\left(\frac{k-\hat{k}}{\epsilon}\right) \right| |\hat{h}(\hat{k})| \, \frac{d\hat{k}}{2\pi\epsilon} \right)^2 \frac{dk}{2\pi} \\
&\le M_\epsilon^2 M_G^2 \|h\|_{L^2}^2.
\end{aligned}$$

7.4.1. *Main commutator estimate.* Recall the key estimate (7.18) that we need is

$$\|[B^{-1}, q]\partial(\lambda - \mathcal{Q}_-)^{-1}\|_\alpha \to 0 \qquad \text{as } \epsilon \to 0 \tag{7.37}$$

in the L^2_α operator norm. We apply the Lemma with $g = q$ so $G = \theta_\epsilon$ and $M_G = O(1)$, and take the symbols

$$\hat{\mathcal{P}}(k) = 1, \qquad \hat{\mathcal{Q}}(k) = \frac{1}{1 + b(k + i\alpha)^2}, \qquad \hat{\mathcal{R}}(k) = \frac{i(k + i\alpha)}{\lambda - \hat{\mathcal{Q}}_-(k + i\alpha)},$$

with $\alpha = \epsilon\hat{\alpha}$. Taking any $s \geq 1$ should work. Then, writing $\xi = k + i\hat{\alpha}$, $\hat{\xi} = \hat{k} + i\hat{\alpha}$, since $(\epsilon\xi)^2 - (\epsilon\hat{\xi})^2 = \epsilon(k - \hat{k})(\epsilon\xi + \epsilon\hat{\xi})$, we find

$$\begin{aligned} M_\epsilon &= \sup_{k,\hat{k}\in\mathbb{R}} \frac{\epsilon^2}{\langle k - \hat{k}\rangle^s} \left| \frac{1}{1 + b\epsilon^2\xi^2} - \frac{1}{1 + b\epsilon^2\hat{\xi}^2} \right| |\hat{\mathcal{R}}(\epsilon\hat{k})| \\ &\leq \sup_{k,\hat{k}\in\mathbb{R}} \frac{b|\epsilon\xi + \epsilon\hat{\xi}|}{|1 + b\epsilon^2\xi^2||1 + b\epsilon^2\hat{\xi}^2|} \frac{\epsilon^3|\epsilon\hat{\xi}|}{|\lambda - \hat{\mathcal{Q}}_-(\epsilon\hat{\xi})|} \\ &\leq C \sup_{\hat{k}\in\mathbb{R}} \frac{(1 + b|\epsilon\hat{\xi}|)|\epsilon\hat{\xi}|}{|1 + b\epsilon^2\hat{\xi}^2|} \frac{\epsilon^3}{|\lambda - \hat{\mathcal{Q}}_-(\epsilon\hat{\xi})|} \end{aligned}$$

Here we used the bound (7.36) that follows from (7.20)-(7.21). We now treat separately the low and high frequency regimes. In the low frequency regime $|\epsilon\hat{\xi}| \leq 4\epsilon^p$ we get the bounds

$$\frac{(1 + b|\epsilon\hat{\xi}|)|\epsilon\hat{\xi}|}{|1 + b\epsilon^2\hat{\xi}^2|} \leq C\epsilon^p, \qquad \frac{\epsilon^3}{|\lambda - \hat{\mathcal{Q}}_-(\epsilon\hat{\xi})|} \leq C, \tag{7.38}$$

and in the high-frequency regime $|\epsilon\hat{k}| \geq 2\epsilon^p$ we have

$$\frac{(1 + b|\epsilon\hat{\xi}|)|\epsilon\hat{\xi}|}{|1 + b\epsilon^2\hat{\xi}^2|} \leq C, \qquad \frac{\epsilon^3}{|\lambda - \hat{\mathcal{Q}}_-(\epsilon\hat{\xi})|} \leq C\frac{\epsilon^3}{\epsilon^{1+2p}} = C\epsilon^{2-2p}, \tag{7.39}$$

Consequently $M_\epsilon \to 0$ as $\epsilon \to 0$, proving (7.37).

7.4.2. *Simple commutator estimate.* In order to prove

$$\|[B^{-1}, q']\|_\alpha \leq C\epsilon^4, \tag{7.40}$$

we take $g(x) = q'(x) = \epsilon^3\theta_\epsilon(\epsilon x)$, so $G(x) = \epsilon\theta_\epsilon(x)$ and $M_G \leq C\epsilon$, and take $\hat{\mathcal{Q}}(k)$ as above, and $\hat{\mathcal{P}}(k) = \hat{\mathcal{R}}(k) = 1$. Then the Lemma now yields

$$M_\epsilon \leq \sup_{k,\hat{k}\in\mathbb{R}} \frac{\epsilon^3 b|\epsilon\xi + \epsilon\hat{\xi}|}{|1 + b\epsilon^2\xi^2||1 + b\epsilon^2\hat{\xi}^2|} \leq C\epsilon^3,$$

whence (7.40) follows since $M_\epsilon M_G \leq C\epsilon^4$.

This finishes the proof of the bundle convergence theorem 7.1.

7.5. **Proof of Theorem 2.3.**

Lemma 7.6. *$\|\mathcal{W}_0(\Lambda) - I\|_{\hat{\alpha}} \to 0$ as $|\Lambda| \to \infty$ with $\operatorname{Re}\Lambda \geq -\hat{\beta}/2$.*

Proof. This follows from the estimate (7.33) for $|\epsilon\hat{k}| \geq 2\epsilon^p$, together with the estimate

$$\left| \frac{(i\hat{\xi})^j}{m_0} \right| \leq \frac{C\epsilon^{-1}}{|\Lambda| - C\epsilon^{-3}} \leq C\epsilon^{1-p}$$

for $|\epsilon\hat{k}| \leq 1$, $j = 0, 1$ and for $|\Lambda|$ sufficiently large depending on ϵ. □

As a consequence of Lemma 7.6, there exists $M_0 > 0$ such that for $\epsilon > 0$ sufficiently small, $\|\mathcal{W}_0(\Lambda) - I\|_{\hat{\alpha}} < \frac{1}{4}$. Applying the bundle convergence theorem 7.1, we infer that for small enough ϵ, $\mathcal{W}_\epsilon(\Lambda)$ is invertible for $\operatorname{Re}\Lambda \geq -\hat{\beta}/2$ and $|\Lambda| \geq M_0$. This implies $\mathcal{L}_c$ has no eigenvalue satisfying $\operatorname{Re}\lambda \geq -\frac{1}{4}\epsilon^3\hat{\beta}$ and $|\lambda| \geq \frac{1}{2}\epsilon^3 M_0$.

Moreover, with $\hat{\Omega} = \{\Lambda : |\Lambda| \leq M_0,\ \operatorname{Re}\Lambda \geq -\hat{\beta}/2\}$, then for small enough $\epsilon > 0$,

$$\|(\mathcal{W}_\epsilon(\Lambda) - \mathcal{W}_0(\Lambda))\mathcal{W}_0(\Lambda)^{-1}\|_{\hat{\alpha}} < 1 \tag{7.41}$$

for all $\Lambda \in \partial\hat{\Omega}$. By the Rouché theorem of Gohberg and Sigal [17], it follows that the total null multiplicity of characteristic values of the bundle $\mathcal{W}_\epsilon(\Lambda)$ for $\Lambda \in \hat{\Omega}$ agrees with that of $\mathcal{W}_0(\Lambda)$. Denoting these multiplicities respectively by $m(\hat{\Omega}, \mathcal{W}_\epsilon)$ and $m(\hat{\Omega}, \mathcal{W}_0)$, we have

$$m(\hat{\Omega}, \mathcal{W}_\epsilon) = m(\hat{\Omega}, \mathcal{W}_0). \tag{7.42}$$

As discussed in Appendix C, the null multiplicity of the characteristic value 0 is at least 2 for $\mathcal{W}$, and $m(\hat{\Omega}, \mathcal{W}_0) \leq 2$. Hence $m(\hat{\Omega}, \mathcal{W}_\epsilon) = 2$, so $\Lambda = 0$ is the only characteristic value of $\mathcal{W}_\epsilon$ in $\hat{\Omega}$.

This implies that for all nonzero λ satisfying $\operatorname{Re}\lambda \geq -\frac{1}{4}\epsilon^3\hat{\beta}$, $\mathcal{W}(\lambda)$ is invertible and so λ is not an eigenvalue of $\mathcal{L}_c$. This concludes the proof of Theorem 2.3.

Part II. **Nonlinear stability**

8. Decomposition of perturbed solitary waves

In this part we prove Theorems 2.4 and 2.5. Let

$$\mathcal{M} = \{u_c(\cdot - x_0) \mid c^2 > 1,\ x_0 \in \mathbb{R}\}$$

denote the two-dimensional manifold of solitary-wave states for the Benney-Luke system (2.2). To describe the behavior of solutions near $\mathcal{M}$, we will represent them using the ansatz

$$u(t,x) = u_{c(t)}(y) + v(t,y), \qquad y = x - x(t). \tag{8.1}$$

Here $u_{c(t)}$ comprises the main solitary-wave part of the solution and v is a remainder. The modulating parameters $c(t)$ and $x(t)$ describe the speed and phase of the main solitary wave at time t. Substituting (8.1) into (2.2) and noting $cu_c' + Lu_c + f(u_c) = 0$, we require

$$\partial_t v = \mathcal{L}_{c(t)} v + (\dot{x}(t) - c(t))\partial_y v + l(t) + f(v), \tag{8.2}$$

where $\mathcal{L}_c = L + c\partial_y + f'(u_c)$ and $\dot{x} = dx/dt$ and

$$l(t) = (\dot{x}(t) - c(t))\partial_y u_{c(t)}(y) - \dot{c}(t)\partial_c u_{c(t)}(y).$$

If we were only going to consider initial data that is exponentially well-localized, we could impose the nonsecularity condition $P_{c(t)}v(t) = 0$ at this point and study (8.2) in an exponentially weighted space H^1_α, using the exponential decay estimate supplied by Lemma 2.6. However, this is not feasible for arbitrary small-energy perturbations of solitary waves. The reason is that the spectral projection P_c is not continuous on the energy space H^1, due to the fact that an element of the generalized kernel of the adjoint $\mathcal{L}_c^*$ does not decay as $x \to \infty$.

To deal with this difficulty, as in [32] we split the remainder $v(t)$ into a part generated by free propagation from the initial perturbation, and a well-localized part arising from interaction with the main solitary wave. We write

$$v(t,y) = v_1(t,x) + v_2(t,y), \tag{8.3}$$

where $v_1(t,x)$ is the solution to

$$\begin{cases} \partial_t v_1 = Lv_1 + f(v_1) & \text{for } (t,x) \in \mathbb{R}^2, \\ v_1(0,x) = v_0(x) & \text{for } x \in \mathbb{R}. \end{cases} \tag{8.4}$$

The freely propagating perturbation v_1 will decay locally in a coordinate frame following the main solitary wave, due to the viral estimates that we establish in section 11. The remainder v_2 satisfies

$$\begin{cases} \partial_t v_2 = \mathcal{L}_{c(t)} v_2 + (\dot{x} - c)\partial_y v_2 + l + k_1 + k_2, \\ v_2(0, y) = 0, \end{cases} \tag{8.5}$$

where

$$k_1 = f'(u_{c(t)})\tilde{v}_1(t), \qquad k_2 = f(v(t)) - f(\tilde{v}_1(t)), \qquad \tilde{v}_1(t, y) = v_1(t, y + x(t)). \tag{8.6}$$

This part will be 'slaved' to v_1 via the estimates in exponentially weighted norm that are provided in Lemma 2.6. To enable the use of that Lemma and fix the decomposition, we will impose the constraint $P_{c(t)} v_2(t) = 0$. In terms of the elements $\zeta^*_{1,c}$, $\zeta^*_{2,c}$ described in Appendix B, that span the generalized kernel of $\mathcal{L}^*_c$, this means

$$\langle v_2(t), \zeta^*_{1,c(t)} \rangle = 0, \quad \langle v_2(t), \zeta^*_{2,c(t)} \rangle = 0\,. \tag{8.7}$$

Notation. Some additional notation to be used in Part II is as follows. We will write $g \lesssim h$ to mean that there exists a positive constant such that $g \leq Ch$. For $\mathbb{R}^2$-valued functions $g = (g_1, g_2)$ and $h = (h_1, h_2)$ let

$$\langle g, h \rangle = \int_{\mathbb{R}} (g_1(x)h_1(x) + g_2(x)h_2(x))\, dx.$$

For a Banach space X we denote by $B(X)$ the space of all continuous linear operators on X.

9. Local existence and continuation of the decomposition

In this section we establish the validity of the representation described above in (8.1)–(8.7). We first show that $u(t) - v_1(t)$ remains in H^1_α whenever $0 \leq \alpha < \alpha_{c_0}$. Recall α_c from (2.8) is the exponential decay rate of the wave profile u_c, and $\alpha_c < b^{-1/2}$.

Lemma 9.1. *Let $c_0 > 1$, $x_0 \in \mathbb{R}$ and $v_0 \in H^1$. Let $u(t)$ be a solution to (2.2) satisfying $u(0) = u_{c_0}(\cdot - x_0) + v_0$ and let v_1 be a solution to (8.4). Then for every $\alpha \in (-\alpha_{c_0}, \alpha_{c_0})$,*

$$u(t) - v_1(t) \in C([0,\infty); H^1_\alpha(\mathbb{R}; \mathbb{R}^2)) \cap C^1([0,\infty); L^2_\alpha(\mathbb{R}; \mathbb{R}^2)). \tag{9.1}$$

Proof. By standard well-posedness arguments, u, v_1, and $w = u - v_1$ lie in $C(\mathbb{R}; H^1)$. Writing

$$v_1 = \begin{pmatrix} q_1 \\ r_1 \end{pmatrix}, \quad w = \begin{pmatrix} \tilde{q} \\ \tilde{r} \end{pmatrix},$$

we find w satisfies a linear equation

$$\begin{cases} \partial_t w = Lw + F(t)w, \\ w(0) = u_{c_0}(\cdot - x_0), \end{cases} \tag{9.2}$$

where

$$F(t)w = -B^{-1} \begin{pmatrix} 0 \\ \partial_x(r\tilde{q} + 2q\tilde{r}) + \tilde{q}\partial_x(2r_1 - r) + \tilde{r}\partial_x(q_1 - 2q) \end{pmatrix}.$$

Since B^{-1} and $B^{-1}\partial_x$ are bounded on L^2_α, we have

$$\|F(t)w(t)\|_{H^1_\alpha} \lesssim (1 + \|u(t)\|_{H^1} + \|v_1(t)\|_{H^1})\|w\|_{H^1_\alpha},$$

and $F(t) \in C(\mathbb{R}; B(H^1_\alpha))$. Since e^{tL} is a C^0-semigroup on both spaces H^1 and $H^1 \cap H^1_\alpha$, and u_{c_0} lies there, it follows that (9.2) has a solution in $C([0,\infty); H^1 \cap H^1_\alpha)$ which agrees with $u - v_1$ by uniqueness in H^1. This proves (9.1). □

Next, we associate a unique phase/speed pair to each u near u_{c_0} in L^2_α. Here and below we will make use of the following pointwise estimates for the neutral and adjoint neutral modes $\zeta_{j,c}$, $\zeta^*_{j,c}$, satisfied uniformly for c in a neighborhood of c_0:

$$|\zeta_{1,c}| + |\zeta^*_{2,c}| \lesssim e^{-\alpha_c|y|}, \qquad |\zeta_{2,c}| + |\partial_c \zeta^*_{2,c}| \lesssim e^{-\alpha_c|y|}(1+|y|), \tag{9.3}$$

$$|\zeta^*_{1,c}| \lesssim \min(1, e^{\alpha_c y}(1+|y|)), \qquad |\partial_c \zeta^*_{1,c}| \lesssim \min(1, e^{\alpha_c y}(1+|y|^2)). \tag{9.4}$$

Lemma 9.2. *Let $c_0 > 1$ and $\alpha \in (0, \alpha_{c_0})$. Then there exist positive constants δ_0, δ_1 such that with*

$$U_0 = \{w \in L^2_\alpha : \|w - u_{c_0}\|_{L^2_\alpha} < \delta_0\}, \quad U_1 = \{(\gamma, c) \in \mathbb{R}^2 : |\gamma| + |c - c_0| < \delta_1\},$$

then for each $w \in U_0$ there is a unique $(\gamma, c) \in U_1$ satisfying

$$\langle w(\cdot + \gamma) - u_c, \zeta^*_{1,c}\rangle = \langle w(\cdot + \gamma) - u_c, \zeta^*_{2,c}\rangle = 0\,.$$

Further, the mapping $w \mapsto \Phi(w) = (\gamma, c)$ is smooth.

Proof. The map $G : L^2_\alpha \times \mathbb{R} \times (0,\infty) \to \mathbb{R}^2$ defined by

$$G(w,\gamma,c) = \begin{pmatrix} \langle w - u_c(\cdot - \gamma), \zeta^*_{1,c}(\cdot - \gamma)\rangle \\ \langle w - u_c(\cdot - \gamma), \zeta^*_{2,c}(\cdot - \gamma)\rangle \end{pmatrix} \tag{9.5}$$

is smooth since $(\gamma, c) \mapsto \zeta^*_{j,c}(\cdot - \gamma)$ is smooth with values in $L^2_{-\alpha} = (L^2_\alpha)^*$, due to the definitions in (B.5) and Lemma B.1. Moreover, $G(u_c, 0, c) = 0$ and

$$\frac{\partial G}{\partial(\gamma, c)}(u_c, 0, c) = \begin{pmatrix} \langle \partial_y u_c, \zeta^*_{1,c}\rangle & \langle -\partial_c u_c, \zeta^*_{1,c}\rangle \\ \langle \partial_y u_c, \zeta^*_{2,c}\rangle & \langle -\partial_c u_c, \zeta^*_{2,c}\rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

due to (B.6). Thus the result follows immediately from the implicit function theorem. □

Now we establish the local existence of the desired representation of solutions, and we provide a continuation principle that ensures its existence as long as a suitable distance to $\mathcal{M}$ and the wave-speed variation remain small. Since the manifold $\mathcal{M}$ is translation invariant, we need only to use the local coordinates in Lemma 9.2, without needing to study the global geometry of $\mathcal{M}$ as in [12].

Proposition 9.3. *Make the assumptions of Lemma 9.1, let $0 < \alpha < \alpha_{c_0}$, and let δ_0, δ_1 be given by Lemma 9.2. Then there exist $T > 0$ and C^1 functions $x(t)$, $c(t)$ on $[0,T)$ satisfying*

$$x(0) = x_0, \quad c(0) = c_0, \quad |c(t) - c_0| < \delta_1, \tag{9.6}$$

such that if v_2 is defined by the decomposition

$$u(t,x) = u_{c(t)}(y) + v_1(t,x) + v_2(t,y), \qquad y = x - x(t), \tag{9.7}$$

then the orthogonality relations (8.7) hold for all $t \in [0,T)$.

Moreover, if $T < \infty$ and

$$\sup_{t\in[0,T)} \|u_{c(t)} + v_2(t) - u_{c_0}\|_{L^2_\alpha} < \delta_0, \tag{9.8}$$

then T is not maximal.

Proof. Define $w(t;\hat{x}) = (u - v_1)(t, \cdot + \hat{x})$ for $\hat{x} \in \mathbb{R}$. Since $w(0;x_0) = u_{c_0} \in U_0$ by assumption, there exists $T_1 > 0$ such that $t \mapsto w(t;x_0)$ is C^1 with values in U_0 for $t \in [0,T_1)$. Then $(\gamma(t), c(t)) := \Phi(w(t;x_0))$ are the unique points in U_1 such that $G(w(t;x_0+\gamma),0,c) = 0$. It follows that with $x(t) := x_0 + \gamma(t)$, and with $v_2(t) = w(t;x(t)) - u_{c(t)}$ given by (9.7), (9.6) and (8.7) hold for $t \in [0,T_1)$. Moreover $\gamma(t)$ and $c(t)$ are C^1 because Φ is C^1 on U_0.

Suppose now that C^1 functions $x(t)$, $c(t)$ exist on $[0,T)$ such that (9.6) and (8.7) hold with v_2 given by (9.7), which means such that for $0 \le t < T$, we have (9.6) and

$$G(w(t;x(t)),0,c(t)) = 0. \tag{9.9}$$

Suppose further that (9.8) holds. Then there is a closed ball $\hat{U}_0 \subset U_0$ such that for all $t \in [0,T)$, $w(t;x(t)) \in \hat{U}_0$. Since $w\colon [0,T+1] \to L^2_\alpha$ is uniformly continuous, by enlarging $\hat{U}_0$ if necessary we can say there exists $\tau_0 > 0$ such that whenever $\hat{t} \in [0,T)$ and $\tau \in [0,2\tau_0]$,

$$w(\hat{t}+\tau;x(\hat{t})) \in \hat{U}_0. \tag{9.10}$$

Fix $\hat{t} = T - \tau_0$. Applying Lemma 9.2, we infer that $(\hat{\gamma}(\tau), \hat{c}(\tau)) := \Phi(w(\hat{t}+\tau;x(\hat{t})))$ are the unique points in U_1 such that for $\tau \in [0,2\tau_0]$,

$$G(w(\hat{t}+\tau;x(\hat{t})+\hat{\gamma}),0,\hat{c}) = 0. \tag{9.11}$$

Also, $\hat{\gamma}(\tau)$ and $\hat{c}(\tau)$ are C^1, and the values $(\hat{\gamma}(\tau),\hat{c}(\tau))$ lie in a compact $\hat{U}_1 \subset U_1$ for $\tau \in [0,2\tau_0]$.

Now, note that by (9.9) and the definition of G, for $\tau \in [0,\tau_0)$ we have $\hat{t}+\tau < T$ and

$$G(w(\hat{t}+\tau;x(\hat{t}+\tau)),0,c(\hat{t}+\tau)) = 0. \tag{9.12}$$

For $\tau \in [\tau_0, 2\tau_0]$ we have $\hat{t}+\tau \in [T, T+\tau_0]$, and we *define*

$$(x(\hat{t}+\tau), c(\hat{t}+\tau)) = (x(\hat{t}) + \hat{\gamma}(\tau), \hat{c}(\tau)). \tag{9.13}$$

Then (9.6) and (9.9) hold for $0 \le t \le T+\tau_0$, due to (9.11) for $\hat{t}+\tau \in [T,T+\tau_0]$. We *claim* that (9.13) holds for all $\tau \in [0,\tau_0)$ also, hence for all $\tau \in [0,2\tau_0]$. From this claim it follows that $x(t)$ and $c(t)$ are C^1 and (9.6) and (9.9) hold on $[0,T+\tau_0]$, so T is not maximal.

To prove the claim, note that by (9.10)–(9.12) and the local uniqueness statement in Lemma 9.2 applied with $w = w(\hat{t}+\tau;x(\hat{t})) \in U_0$, we have the following implication. For $\tau \in [0,\tau_0)$,

$$\text{if } \; z(\tau) := (x(\hat{t}+\tau) - x(\hat{t}), c(\hat{t}+\tau)) \in U_1, \quad \text{then } \; z(\tau) = (\hat{\gamma}(\tau),\hat{c}(\tau)) \in \hat{U}_1. \tag{9.14}$$

Since indeed $z(0) = (0, c(\hat{t})) \in U_1$ and $\hat{U}_1$ is a compact subset of U_1, however, we infer by continuity that $\sup\{\tau \in [0,\tau_0) : z(\tau) \in U_1\} = \tau_0$. This proves the claim, and finishes the proof of the Proposition. □

Remark 9.1. We remark that this Lemma implies that the decomposition (9.7) can be continued as long as $|c(t) - c_0|$ and $\|v_2\|_{L^2_\alpha}$ remain sufficiently small, since $\|u_c - u_{c_0}\|_{L^2_\alpha} \lesssim |c - c_0|$.

10. Modulation equations and energy estimates

The decomposition described in (8.1)–(8.7) yields a system of ordinary differential equations that govern the modulating speed $c(t)$ and phase shift $x(t)$ of the main solitary wave. In this section we describe these modulation equations, and we provide estimates that control the energy norm of the combined perturbation v in terms of initial data and the modulation of the wave speed. For $u = (q, r)$ we denote the energy density by

$$\mathcal{E}(u) = \frac{1}{2}\left(q^2 + r^2 + a(\partial_x q)^2 + b(\partial_x r)^2\right). \tag{10.1}$$

10.1. Modulation equations.

Differentiate (8.7) with respect to t and substitute (8.5) into the resulting equation. Using the fact from (B.6) that $\mathcal{L}_c^*\zeta_{2,c}^* = 0$ and $\mathcal{L}_c^*\zeta_{1,c}^* = \zeta_{2,c}^*$ are both orthogonal to v_2, it follows that for $i = 1$ and 2,

$$\begin{aligned} 0 &= \frac{d}{dt}\langle v_2(t), \zeta_{i,c(t)}^*\rangle - \langle v_2, \mathcal{L}_{c(t)}^*\zeta_{i,c(t)}^*\rangle \\ &= \dot{c}\langle v_2, \partial_c\zeta_{i,c}^*\rangle + (\dot{x} - c)\langle \partial_y v_2, \zeta_{i,c}^*\rangle + \langle l + k_1 + k_2, \zeta_{i,c}^*\rangle. \end{aligned}$$

Since $l = (\dot{x} - c)\zeta_{1,c} + \dot{c}\zeta_{2,c}$, by the biorthogonality relations $\langle \zeta_{i,c}, \zeta_{j,c}^*\rangle = \delta_{ij}$ from (B.6) we obtain that $\dot{x}$ and $\dot{c}$ are determined by the *modulation equations*

$$\begin{pmatrix} 1 + \langle \partial_y v_2, \zeta_{1,c}^*\rangle & \langle v_2, \partial_c\zeta_{1,c}^*\rangle \\ \langle \partial_y v_2, \zeta_{2,c}^*\rangle & 1 + \langle v_2, \partial_c\zeta_{2,c}^*\rangle \end{pmatrix} \begin{pmatrix} \dot{x} - c \\ \dot{c} \end{pmatrix} + \begin{pmatrix} \langle k_1 + k_2, \zeta_{1,c}^*\rangle \\ \langle k_1 + k_2, \zeta_{2,c}^*\rangle \end{pmatrix} = 0. \tag{10.2}$$

Our next lemma provides estimates for these modulation equations in terms of the space W_ν with localized energy norm defined by

$$\|v\|_{W_\nu} = \left(\int_{\mathbb{R}} e^{-2\nu|y|}\mathcal{E}(v(y))\,dy\right)^{1/2}. \tag{10.3}$$

Lemma 10.1. *Let $c_0 > 1$, $x_0 \in \mathbb{R}$ and suppose $0 < \nu \le \alpha < \frac{1}{2}\alpha_{c_0}$. Then there exist positive constants δ_2 and C with the following property. Suppose the decomposition in Proposition 9.3 holds on $[0, T]$ and suppose*

$$\sup_{t\in[0,T]}\left(|c(t) - c_0| + \|v(t)\|_{H^1} + \|v_1(t)\|_{H^1} + \|v_2(t)\|_{H^1_\alpha}\right) \le \delta_2.$$

Then for $t \in [0, T]$,

$$|\dot{x}(t) - c(t)| \le C\|\tilde{v}_1(t)\|_{W_\nu} + C\|v_2(t)\|_{H^1_\alpha}(\|v_1(t)\|_{H^1} + \|v(t)\|_{H^1} + \|v_2(t)\|_{H^1_\alpha}), \tag{10.4}$$

$$|\dot{c}(t)| \le C\|\tilde{v}_1(t)\|_{W_\nu} + C\|v_2(t)\|_{H^1_\alpha}(\|\tilde{v}_1(t)\|_{W_\nu} + \|v_2(t)\|_{H^1_\alpha}). \tag{10.5}$$

Furthermore,

$$\frac{d}{dt}\left(c(t) + \langle \tilde{v}_1(t), \zeta_{2,c(t)}^*\rangle\right) = O\left(\|\tilde{v}_1(t)\|_{W_\nu}^2 + \|v_2(t)\|_{H^1_\alpha}^2\right). \tag{10.6}$$

Proof. Note that for δ_2 small enough, $2\alpha < \alpha_{c(t)}$ for all $t \in [0, T]$. And due to the estimates

$$|\langle \partial_y v_2, \zeta_{j,c}^*\rangle| + |\langle v_2, \partial_c\zeta_{j,c}^*\rangle| \lesssim \|v_2\|_{H^1_\alpha} \le \delta_2$$

for $j = 1, 2$, the matrix in (10.2) is invertible with inverse $I + O(\|v_2\|_{H^1_\alpha})$.

To estimate terms involving $k_1 = f'(u_c)\tilde v_1$, note $\zeta^*_{1,c}$ and $\zeta^*_{2,c}$ are uniformly bounded in $L^2_{-\alpha}$, so $|\langle k_1, \zeta_{j,c}\rangle| \lesssim \|k_1\|_{L^2_\alpha}$ for $j = 1, 2$. Since

$$f'(u_c)\tilde v_1 = -(B^{-1}\partial_x)\begin{pmatrix}0 & 0\\ r_c & 2q_c\end{pmatrix}\tilde v_1 - B^{-1}\begin{pmatrix}0 & 0\\ r_c' & -q_c'\end{pmatrix}\tilde v_1,$$

and B^{-1} and $B^{-1}\partial$ are bounded from L^2_α to H^1_α, we may deduce

$$\|k_1\|_{H^1_\alpha} \lesssim \|q_c\tilde v_1\|_{L^2_\alpha} \lesssim \|e^{-\alpha_c|y|/2}\tilde v_1\|_{L^2} \lesssim \|\tilde v_1\|_{W_\nu}, \tag{10.7}$$

using $\alpha < \frac{1}{2}\alpha_c$. (This estimate will be used also in section 12.)

Next we estimate terms involving k_2. Since f is quadratic,

$$k_2 = f(v) - f(\tilde v_1) = f'(\tilde v_1)v_2 + f(v_2) = f'(v)v_2 - f(v_2).$$

As for k_1, we find $|\langle k_2, \zeta_{1,c}\rangle| \lesssim \|k_2\|_{L^2_\alpha}$ and

$$\|k_2\|_{H^1_\alpha} \lesssim \|v_2\|_{H^1_\alpha}(\|v\|_{H^1} + \|v_2\|_{H^1}) \lesssim \|v_2\|_{H^1_\alpha}(\|v\|_{H^1} + \|v_1\|_{H^1}). \tag{10.8}$$

Since $|\zeta^*_{2,c}| \lesssim e^{-\alpha_c|y|} \le e^{-2\alpha|y|}$, however, from Lemma 10.2 below we find the tighter estimate

$$|\langle k_2, \zeta^*_{2,c}\rangle| \lesssim \|e^{-2\alpha|y|}(f'(\tilde v_1)v_2 + f(v_2))\|_{L^1} \lesssim \|\tilde v_1\|_{W_\nu}\|v_2\|_{H^1_\alpha} + \|v_2\|^2_{H^1_\alpha}.$$

Then directly we obtain (10.4) and (10.5).

Next we prove (10.6). Using (8.4) and the fact that $\langle \mathcal{L}_c\tilde v_1, \zeta^*_{2,c}\rangle = \langle \tilde v_1, \mathcal{L}^*_c\zeta^*_{2,c}\rangle = 0$, we have

$$\begin{aligned}\frac{d}{dt}\langle \tilde v_1, \zeta^*_{2,c(t)}\rangle &= \langle \dot x(t)\partial_y\tilde v_1 + L\tilde v_1 + f(\tilde v_1), \zeta^*_{2,c}\rangle + \dot c\,\langle \tilde v_1, \partial_c\zeta^*_{2,c}\rangle\\ &= -\langle f'(u_c)\tilde v_1, \zeta^*_{2,c}\rangle + O\left((|\dot x(t) - c(t)| + |\dot c(t)|)\|\tilde v_1(t)\|_{W_\nu} + \|\tilde v_1(t)\|^2_{W_\nu}\right)\\ &= -\langle k_1, \zeta^*_{2,c}\rangle + O(\|\tilde v_1(t)\|^2_{W_\nu} + \|v_2(t)\|^2_{H^1_\alpha}).\end{aligned}$$

Combining this with (10.2) and (10.5), we obtain (10.6). This completes the proof. □

For later use, we also note here that we have

$$|\langle \tilde v_1(t), \zeta^*_{2,c(t)}\rangle| \lesssim \|\tilde v_1(t)\|_{W_\nu}. \tag{10.9}$$

Lemma 10.2. *Let $|\alpha| < 1/\sqrt{b}$. Then for $p \in [1, \infty]$,*

$$\|e^{-\alpha|x|}B^{-1}g\|_{L^p} + \|e^{-\alpha|x|}B^{-1}\partial_x g\|_{L^p} \le C\|e^{-\alpha|x|}g\|_{L^p}, \tag{10.10}$$

where C is a positive constant depending only on α.

Proof. Observe $B^{-1}g(x) = \int_{\mathbb{R}} \frac{1}{2\sqrt{b}}e^{-|x-y|/\sqrt{b}}g(y)\,dy$. Then for $j = 0, 1$,

$$|e^{-\alpha|x|}(\partial_x^j B^{-1}g)(x)| \lesssim \int_{\mathbb{R}} \kappa(x,y)e^{-\alpha|y|}|g(y)|\,dy,$$

where $\kappa(x,y) = e^{-\alpha|x|}e^{-|x-y|/\sqrt{b}}e^{\alpha|y|}$. Using the fact that $\sup_y \int_{\mathbb{R}} \kappa(x,y)\,dx + \sup_x \int_{\mathbb{R}} \kappa(x,y)\,dy < \infty$ we have (10.10). □

10.2. **Energy norm estimates on v.** We will estimate the energy norm of $v(t)$ by using the convexity of the energy functional as was done for the case of FPU lattice models in [12]. Since the solitary wave is not a critical point of the energy functional $E(u)$, the estimate of $v(t)$ depends on the modulation of the speed $c(t)$.

Lemma 10.3. *Let $c_0 > 1$ and $u(0) = u_{c_0}(\cdot - x_0) + v_0$ for some $x_0 \in \mathbb{R}$. Let δ_3 be a sufficiently small positive number and $T \in [0, \infty]$. Suppose that the decomposition of Proposition 9.3 exists for $t \in [0, T)$ and that*

$$\|v_0\|_{H^1} + \sup_{t\in[0,T)} (|c(t) - c_0| + \|v(t)\|_{H^1}) \le \delta_3.$$

Then

$$\|v(t)\|_{H^1}^2 \le C(\|v_0\|_{H^1} + |c(t) - c_0|) \quad \text{for } t \in [0, T),$$

where C is a positive constant depending only on δ_3 and c_0.

Proof. Since the energy $E(u)$ is invariant under time evolution and spatial translation,

$$E(u(t)) = E(u_{c_0}(\cdot - x_0) + v_0) = E(u_{c_0}) + O(\|v_0\|_{H^1}).$$

Expanding $E(u(t)) = E(u_{c(t)} + v(t))$ in a Taylor series about $u_{c(t)}$, we have

$$E(u(t)) = E(u_{c(t)}) + \langle E'(u_{c(t)}), v\rangle + \frac{1}{2}\langle E''(u_{c(t)})v, v\rangle + O(\|v(t)\|_{H^1}^3).$$

Since $E'(u_c) = (Aq_c, Br_c) =: \eta_{1,c}$ is a multiple of $\zeta_{2,c}^*$ from Appendix B, by (8.7) we have

$$\langle E'(u_{c(t)}), v(t)\rangle = \langle \tilde{v}_1(t), \eta_{1,c}\rangle = O(\|v_1(t)\|_{H^1}).$$

Since $E''(u_c) = \operatorname{diag}(A, B)$ is positive definite, there exist a positive constant C' such that

$$\|v(t)\|_{H^1}^2 \le C'(|E(u_{c(t)}) - E(u_{c_0})| + \|v_0\|_{H^1} + \|v_1(t)\|_{H^1} + \|v(t)\|_{H^1}^3).$$

If δ_3 is sufficiently small, it follows

$$\|v(t)\|_{H^1}^2 \le C(\|v_0\|_{H^1} + |c(t) - c_0|),$$

where C is a positive constant depending only on δ_3 and c_0. Note that from (2.5) it follows $\|v_1(t)\|_{H^1}^2 \lesssim E(v_1(t)) = O(\|v_0\|_{H^1}^2)$, because $v_1(t)$ is a solution of (8.4). This completes the proof of Lemma 10.3. □

11. Virial transport estimate

In this section, we prove a virial lemma for small-energy solutions of (8.4) — solutions of the 'free' Benney-Luke system. This kind of result involves bounds on the transport of energy density, measured using weighted integral quantities. Essentially, this provides nonlinear estimates that correspond to the fact that the solitary wave speed exceeds the group velocity of linear waves in the present case ($0 < a < b$ and $c > 1$) for the Benney-Luke equation (1.1).

We start by observing that by a straightforward calculation, we find that the energy density $\mathcal{E}(v_1)$ from (10.1) satisfies a conservation law

$$\partial_t \mathcal{E}(v_1) = \partial_x \mathcal{F}(v_1), \tag{11.1}$$

with the flux $\mathcal{F} = \mathcal{F}_2 + \mathcal{F}_3$ where

$$\mathcal{F}_2(v_1) = r_1 B^{-1} A q_1 + a(\partial_x q_1)(\partial_x r_1),$$
$$\mathcal{F}_3(v_1) = -r_1^2 q_1 - b r_1 \partial_x B^{-1}(r_1 \partial_x q_1 + 2q_1 \partial_x r_1).$$

Let ν be a positive constant and $\tilde{x}(t)$ be a C^1-function. We introduce a smoothed Heaviside function and a corresponding weighted energy by

$$\chi_\nu(x) = 1 + \tanh \nu x, \qquad \mathcal{V}(t) = \int_{\mathbb{R}} \chi_\nu(x - \tilde{x}(t))\mathcal{E}(v_1(t,x))\, dx. \tag{11.2}$$

Note

$$\chi_\nu'(x) = \nu\psi_\nu(x)^2 \le 4\nu e^{-2\nu|x|}, \quad \text{where} \quad \psi_\nu(x) = \operatorname{sech} \nu x.$$

Lemma 11.1 (Virial Lemma)**.** *For any constant $c_1 > 1$, there exist positive numbers ν_0, δ_4 and μ with the following property. Given any $\nu \in (0, \nu_0)$, any C^1 function $\tilde{x}(t)$ satisfying $\partial_t \tilde{x}(t) \ge c_1$ for all t, and any solution $v_1(t)$ to (8.4) with $\|v_0\|_{H^1} < \delta_4$, we have*

$$\mathcal{V}(t) + \mu\nu \int_0^t \int_{\mathbb{R}} \psi_\nu(x - \tilde{x}(s))^2 \mathcal{E}(v_1(s,x))\, dx\, ds \le \mathcal{V}(0). \tag{11.3}$$

Lemma 11.1 yields that $v_1(t)$ locally tends to 0 as $t \to \infty$, with respect to any coordinate frame that moves at a speed strictly greater than one, provided the energy is sufficiently small. Before providing the proof, we establish two claims.

Claim 11.1. *For every $u \in H^1(\mathbb{R})$ we have $\int_{\mathbb{R}}(\mathcal{E}(u) + \mathcal{F}_2(u))\, dx \ge 0$.*

Proof. Due to Plancherel's identity, $\int_{\mathbb{R}}(\mathcal{E}(u) + \mathcal{F}_2(u))\, dx = \frac{1}{2}\int_{\mathbb{R}} \hat{u}(\xi)^t D(\xi)\overline{\hat{u}(\xi)}\, d\xi$ where

$$D(\xi) = \begin{pmatrix} 1 + a\xi^2 & S(\xi)^2 + a\xi^2 \\ S(\xi)^2 + a\xi^2 & 1 + b\xi^2 \end{pmatrix}.$$

But by Gerschgorin's circle theorem, both eigenvalues $\kappa_1(\xi)$ and $\kappa_2(\xi)$ of $D(\xi)$ satisfy

$$\kappa_j(\xi) > (1 + a\xi^2) - (S(\xi)^2 + a\xi^2) \ge 0,$$

for $\xi \ne 0$ since $0 < a < b$. Thus $D(\xi)$ is positive definite and Claim 11.1 follows. □

Claim 11.2. *Let $0 < \nu_0 < 1/\sqrt{b}$. Then for $\nu \in (0, \nu_0)$ we have*

$$\|[\psi_\nu, \partial_x]g\|_{L^2} + \|[\psi_\nu, B^{-1}]g\|_{L^2} = O(\nu\|\psi_\nu g\|_{L^2}).$$

Proof. The bound on the first commutator holds because $|\psi_\nu'/\psi_\nu| \le \nu$ uniformly. Note

$$[\psi_\nu, B^{-1}] = B^{-1}[B, \psi_\nu]B^{-1} = -bB^{-1}(2\psi_\nu'\partial_x + \psi_\nu'')B^{-1}.$$

Then since $|\psi_\nu''| \le 2\nu^2\psi_\nu$ and $e^{-\nu|x|} \le \psi_\nu(x) \le 2e^{-\nu|x|}$, the bound on the second commutator follows by Lemma 10.2. □

Proof of Lemma 11.1. Let $v_1 = (q_1, r_1)$. By (11.1), we compute

$$\begin{aligned} \frac{d}{dt}\int_{\mathbb{R}} \chi_\nu(x - \tilde{x}(t))\mathcal{E}(v_1(t,x))\, dx &= \int_{\mathbb{R}} \chi_\nu'(x - \tilde{x}(t))(-(\partial_t\tilde{x})\mathcal{E}(v_1) - \mathcal{F}(v_1))\, dx \\ &\le \int_{\mathbb{R}} \chi_\nu'(x - \tilde{x}(t))(-c_1\mathcal{E}(v_1) - \mathcal{F}(v_1))\, dx. \end{aligned} \tag{11.4}$$

Now $\chi_\nu'(x-\tilde{x}(t)) = \nu\tilde{\psi}^2$ where $\tilde{\psi} = \psi_\nu(x-\tilde{x}(t))$. Due to the commutator estimates of Claim 11.2, and then by Claim 11.1,

$$\begin{aligned}\int_{\mathbb{R}} \tilde{\psi}^2(-c_1\mathcal{E}(v_1) - \mathcal{F}_2(v_1))\,dx &= \int_{\mathbb{R}} (-c_1\mathcal{E}(\tilde{\psi}v_1) - \mathcal{F}_2(\tilde{\psi}v_1))\,dx + O(\nu\|\tilde{\psi}v_1\|_{L^2}^2) \\ &\le (-c_1+1+O(\nu))\int_{\mathbb{R}} \mathcal{E}(\tilde{\psi}v_1)\,dx \le (-c_1+1+O(\nu))\int_{\mathbb{R}} \tilde{\psi}^2\mathcal{E}(v_1)\,dx. \quad (11.5)\end{aligned}$$

Moreover, $\int_{\mathbb{R}} \tilde{\psi}^2|r_1^2q_1|\,dx \le \|q_1\|_{L^\infty}\int_{\mathbb{R}} |\tilde{\psi}r_1|^2\,dx$. Writing $h(x) = r_1\partial_x q_1 + 2q_1\partial_x r_1$, we have the estimates

$$\|\tilde{\psi}h\|_{L^2} \lesssim \|v_1\|_{L^\infty}\|\tilde{\psi}v_1\|_{H^1}, \quad (11.6)$$

and by Lemma 10.2,

$$\int_{\mathbb{R}} \tilde{\psi}^2|r_1\partial_x B^{-1}h|\,dx \le \|\tilde{\psi}r_1\|_{L^2}\|\tilde{\psi}\partial_x B^{-1}h\|_{L^2} \lesssim \|\tilde{\psi}r_1\|_{L^2}\|\tilde{\psi}h\|_{L^2}.$$

It follows

$$\int_{\mathbb{R}} \tilde{\psi}^2|\mathcal{F}_3(v_1)|\,dx \lesssim \|v_1\|_{L^\infty}\int_{\mathbb{R}} \mathcal{E}(\tilde{\psi}v_1)\,dx \lesssim \|v_1\|_{L^\infty}\int_{\mathbb{R}} \tilde{\psi}^2\mathcal{E}(v_1)\,dx. \quad (11.7)$$

By energy conservation (2.5) and the Sobolev imbedding theorem, $\|v_1(t,\cdot)\|_{L^\infty} \lesssim E(v_0)^{1/2}$. Thus, with $\mu = \frac{1}{2}(c_1-1)$, say, if we choose ν_0 and $\delta_4 > 0$ sufficiently small then it follows

$$\frac{d}{dt}\int_{\mathbb{R}} \chi_\nu(x-\tilde{x}(t))\mathcal{E}(v_1(t,x))\,dx + \mu\int_{\mathbb{R}} \chi_\nu'(x-\tilde{x}(t))\mathcal{E}(v_1(t,x))\,dx \le 0. \quad (11.8)$$

Integrating this on $[0,t]$, we have (11.3). □

Corollary 11.2. *Under the conditions of Lemma 11.1, if there is a positive constant $\hat{\sigma}$ such that $\partial_t\tilde{x}(t) \ge c_1 + \hat{\sigma}$ for all t, then*

$$\int_{\mathbb{R}} \chi_\nu(x-\tilde{x}(t))\mathcal{E}(v_1(t,x))\,dx \le \int_{\mathbb{R}} \chi_\nu(x-\tilde{x}(0)-\hat{\sigma}t)\mathcal{E}(v_0(x))\,dx,$$

and this tends to 0 as $t\to\infty$.

Proof. Given any $t_1 > 0$ define $\tilde{x}_1(t) = \tilde{x}(t) - \hat{\sigma}(t-t_1)$. Then $\tilde{x}_1(t_1) = \tilde{x}(t_1)$ and $\partial_t\tilde{x}_1(t) \ge c_1$ for all t. Using $\tilde{x}_1$ in place of $\tilde{x}$ in Lemma 11.1, we find

$$\int_{\mathbb{R}} \chi_\nu(x-\tilde{x}(t_1))\mathcal{E}(v_1(t_1,x))\,dx \le \int_{\mathbb{R}} \chi_\nu(x-\tilde{x}_1(0))\mathcal{E}(v_0(x))\,dx.$$

□

12. Stability estimates with a priori smallness assumptions

For the remaining three sections, we fix $c_0 > \sigma > c_1 > 1$. Also fix $\alpha \in (0, \frac{1}{2}\alpha_{c_0})$ and suppose that in L^2_α, $\mathcal{L}_{c_0}$ has no nonzero eigenvalue satisfying $\operatorname{Re}\lambda \ge 0$. Let ν_0

be given by the Virial Lemma, and fix $\nu \in (0, \nu_0)$ with $\nu \le \alpha$. Define

$$\begin{aligned}
\mathbb{M}_1(T) &= \sup_{t\in[0,T]} \|v_1(t)\|_{H^1} + \|\tilde v_1\|_{L^2(0,T;W_\nu)},\\
\mathbb{M}_2(T) &= \sup_{t\in[0,T]} \|v_2(t)\|_{H^1_\alpha} + \|v_2\|_{L^2(0,T;H^1_\alpha)},\\
\mathbb{M}_v(T) &= \sup_{0\le t\le T} \|v(t)\|^2_{H^1},\\
\mathbb{M}_c(T) &= \sup_{t\in[0,T]} |c(t)-c_0|, \qquad \mathbb{M}_x(T) = \sup_{t\in[0,T]} |\dot x(t) - c(t)|,\\
\mathbb{M}_{\mathrm{tot}}(T) &= \mathbb{M}_v(T) + \mathbb{M}_1(T) + \mathbb{M}_2(T) + \mathbb{M}_c(T) + \mathbb{M}_x(T).
\end{aligned}$$

We shall first deduce a priori bounds on $\mathbb{M}_c$, $\mathbb{M}_x$, $\mathbb{M}_v$ and $\mathbb{M}_1$ in terms of $\|v_0\|_{H^1}$ and $\mathbb{M}_2$.

Lemma 12.1. *There exists a positive constant δ_5 such that if $\|v_0\|_{H^1} + \mathbb{M}_{\mathrm{tot}}(T) \le \delta_5$, then*

$$\mathbb{M}_1(T) \lesssim \|v_0\|_{H^1}, \tag{12.1}$$

$$\mathbb{M}_v(T) \lesssim \|v_0\|_{H^1} + \mathbb{M}_2(T)^2, \tag{12.2}$$

$$\mathbb{M}_c(T) \lesssim \|v_0\|_{H^1} + \mathbb{M}_2(T)^2, \tag{12.3}$$

$$\mathbb{M}_x(T) \lesssim \|v_0\|_{H^1} + \mathbb{M}_2(T)^2. \tag{12.4}$$

Proof. Energy conservation and the Virial Lemma imply (12.1). Lemma 10.3 implies

$$\mathbb{M}_v(T) \lesssim \|v_0\|_{H^1} + \mathbb{M}_c(T). \tag{12.5}$$

Integrating (10.6) and using that $c(0) = c_0$, we find

$$|c(t) - c_0| \lesssim \mathbb{M}_1(T) + \mathbb{M}_1(T)^2 + \mathbb{M}_2(T)^2. \tag{12.6}$$

Combining this with (12.1) we obtain (12.3). Then (12.2) follows from (12.3) and (12.5). Finally, by (10.4) we have

$$\begin{aligned}
\mathbb{M}_x(T) &\lesssim \mathbb{M}_1(T) + (\mathbb{M}_1(T) + \mathbb{M}_v(T)^{1/2} + \mathbb{M}_2(T))\mathbb{M}_2(T)\\
&\lesssim \mathbb{M}_1(T) + \mathbb{M}_1(T)^2 + \mathbb{M}_v(T) + \mathbb{M}_2(T)^2,
\end{aligned}$$

and combining this with (12.1) and (12.2) we obtain (12.4). □

Now we will estimate $\mathbb{M}_2(T)$, making use of the recentering lemma (Lemma 2.6).

Lemma 12.2. *Let δ_5 be as in Lemma 12.1. If δ_5 is sufficiently small, then $\mathbb{M}_2(T) \lesssim \|v_0\|_{H^1}$.*

Proof. We will prove Lemma 12.2 by applying Lemma 2.6 to (8.5). By Lemma 10.1 and the definition of $l(s)$, we have for $s \in [0, T]$,

$$\begin{aligned}
\|l(s)\|_{H^1_\alpha} &\lesssim |\dot x(s) - c(s)| + |\dot c(s)|\\
&\lesssim \|\tilde v_1(s)\|_{W_\nu} + (\mathbb{M}_v(T)^{1/2} + \mathbb{M}_1(T) + \mathbb{M}_2(T))\|v_2(s)\|_{H^1_\alpha}.
\end{aligned}$$

And by (10.7) and (10.8), we have

$$\|k_1(s)\|_{H^1_\alpha} \lesssim \|\tilde v_1(s)\|_{W_\nu}, \qquad \|k_2(s)\|_{H^1_\alpha} \lesssim (\mathbb{M}_v(T)^{1/2} + \mathbb{M}_1(T))\|v_2(s)\|_{H^1_\alpha}.$$

Since $v_2(0)=0$, Lemma 2.6 implies that there is a constant C_1 such that

$$\|v_2(t)\|_{H^1_\alpha} \le C_1 \int_0^t e^{-\beta(t-s)/3} \left(\|\tilde v_1(s)\|_{W_\nu} + (\delta_5+\sqrt{\delta_5})\|v_2(s)\|_{H^1_\alpha}\right) ds\,,$$

for $t\in[0,T]$. For δ_5 small enough, $C_1(\delta_5+\sqrt{\delta_5}) \le \beta/12$. Then by Gronwall's inequality,

$$\|v_2(t)\|_{H^1_\alpha} \le C_1 \int_0^t e^{-\beta(t-s)/4}\|\tilde v_1(s)\|_{W_\nu}\, ds \tag{12.7}$$

for $t\in[0,T]$. Using Young's inequality and $\mathbb{M}_1(T)\lesssim \|v_0\|_{H^1}$, we infer

$$\mathbb{M}_2(T) = \sup_{t\in[0,T]} \|v_2(t)\|_{H^1_\alpha} + \|v_2\|_{L^2(0,T;H^1_\alpha)} \lesssim \|v_0\|_{H^1}.$$

This completes the proof of Lemma 12.2. □

13. Proof of asymptotic stability

Now we are in position to complete the proof of Theorem 2.4 concerning the stability of solitary wave solutions.

Proof. Let δ_5 be a positive constant given by Lemma 12.2. Since $u(0)=\varphi_{c_0}(\cdot - x_0)+v_0$, $v_1(0)=v_0$, it follows from Proposition 9.3 that if $\|v_0\|_{H^1}$ is small enough, then there exists a $T>0$ such that

$$\mathbb{M}_{\mathrm{tot}}(T) \le \delta_5. \tag{13.1}$$

Lemmas 12.1 and 12.2 imply that

$$\mathbb{M}_{\mathrm{tot}}(T) \lesssim \|v_0\|_{H^1} \le \frac{\delta_5}{2}, \tag{13.2}$$

provided $\|v_0\|_{H^1}$ is sufficiently small. Let $T_1\in(0,\infty]$ be the maximal time so that the decomposition in Proposition 9.3 persists for $t\in[0,T_1]$ and (13.1) holds for any $T<T_1$. If $T_1<\infty$, then by (13.2) and Proposition 9.3, there exists $T_2>T_1$ such that the decomposition in Proposition 9.3 exists for $t\in[0,T_2]$ and (13.1) holds for $T=T_2$, which is a contradiction. Thus $T_1=\infty$ and (13.1) holds for $T=\infty$. It follows

$$\|u(t)-u_{c_0}(\cdot - x(t))\|_{H^1} = \|u_{c(t)}+v(t)-u_{c_0}\|_{H^1} \lesssim \|v(t)\|_{H^1} + |c(t)-c_0| \lesssim \|v_0\|_{H^1}.$$

Thus we obtain (2.14).

Next we will prove (2.12) and (2.13). By Corollary 11.2, since $c_1<\sigma<\inf_t \dot x(t)$ for $\|v_0\|_{H^1}$ small enough, we have

$$\|\tilde v_1(t)\|_{W_\nu} \lesssim \int_{\mathbb{R}} \chi_\nu(x-\sigma t - x_0)\mathcal{E}(v_1(t,x))\,dx \to 0 \quad \text{as } t\to\infty. \tag{13.3}$$

Integrating (10.6) and combining (13.3) with (10.9) and the estimate

$$\int_0^\infty (\|\tilde v_1(s)\|^2_{W_\nu} + \|v_2(s)\|^2_{H^1_\alpha})\,ds \le \mathbb{M}_1(\infty)^2 + \mathbb{M}_2(\infty)^2 \lesssim \|v_0\|^2_{H^1}, \tag{13.4}$$

we conclude that $c_\star = \lim_{t\to\infty} c(t)$ exists and $|c_\star - c_0| \lesssim \|v_0\|_{H^1}$, whence (2.12). Then, using (13.3) with (12.7) we find

$$\|v_2(t)\|_{H^1_\alpha} \to 0 \quad \text{as } t\to\infty, \tag{13.5}$$

so by (10.4) we obtain (2.13). For use below, we note that since $\nu = \theta\alpha$ where $\theta \in (0,1]$, interpolating by Hölder's inequality we also have

$$\|v_2(t)\|_{H^1_\nu} \lesssim \|v_2(t)\|_{H^1}^{1-\theta} \|v_2(t)\|_{H^1_\alpha}^{\theta} \to 0 \quad \text{as } t \to \infty. \tag{13.6}$$

It remains to prove (2.15). For this we will use a monotonicity argument as in [25], applying virial estimates to $v(t,y)$. First, observe that since $\|u_{c(t)} - u_{c_\star}\|_{H^1} \to 0$, it suffices to show

$$\|u(t) - u_{c(t)}(\cdot - x(t))\|_{H^1(x\ge\sigma t)} = \|v(t)\|_{H^1(y\ge \sigma t - x(t))} \to 0 \qquad \text{as } t\to\infty. \tag{13.7}$$

We will follow the arguments in the proof of the Virial Lemma up to (11.7). Note that (13.3) and (13.5) already imply that

$$\|v(t)\|_{H^1(y>0)} \lesssim \int_{\mathbb{R}} \chi_\nu(y)\mathcal{E}(v(t,y))\,dy \to 0 \quad \text{as } t\to\infty. \tag{13.8}$$

By (8.2), v satisfies

$$\partial_t v = Lv + f(v) + \dot{x}(t)\partial_y v + l + f'(u_c)v, \tag{13.9}$$

hence

$$\partial_t \mathcal{E}(v) = \partial_y \mathcal{F}(v) + \dot{x}\partial_y \mathcal{E}(v) + \mathcal{E}'(v)(l + f'(u_c)v), \tag{13.10}$$

where with $v = (\bar{q}, \bar{r})$, $\tilde{v} = (\tilde{q}, \tilde{r})$ we write

$$\mathcal{E}'(v)(\tilde{v}) = \bar{q}\tilde{q} + \bar{r}\tilde{r} + a(\partial_x \bar{q})(\partial_x \tilde{q}) + b(\partial_x \bar{r})(\partial_x \tilde{r}).$$

Next, for any $t_1 > 0$ given, let $\tilde{y}(t) = c_1 t - x(t) + x(t_1) - c_1 t_1$, and compute

$$\frac{d}{dt}\int_{\mathbb{R}} \chi_\nu(y - \tilde{y}(t))\mathcal{E}(v(t,y))\,dy = \int_{\mathbb{R}} \chi_\nu'(y-\tilde{y}(t))(-c_1\mathcal{E}(v) - \mathcal{F}(v))\,dy + I_1(t), \tag{13.11}$$

$$I_1(t) = \int_{\mathbb{R}} \chi_\nu(y - \tilde{y}(t))\mathcal{E}'(v)(l + f'(u_c)v)\,dy.$$

As in the proof of Lemma 11.1, writing $\chi_\nu'(y - \tilde{y}(t)) = \nu\tilde{\psi}^2$, for $\|v_0\|_{H^1}$ sufficiently small, we are guaranteed that

$$\begin{aligned}\int_{\mathbb{R}} &\chi_\nu'(y-\tilde{y}(t))(-c_1\mathcal{E}(v) - \mathcal{F}(v))\,dy \\ &\le (-c_1 + 1 + O(\nu) + O(\|v(t)\|_{H^1}))\int_{\mathbb{R}} \nu\tilde{\psi}^2\mathcal{E}(v) \le 0.\end{aligned} \tag{13.12}$$

Moreover, due to the localized nature of $l = (\dot{x} - c)\zeta_{1,c} + \dot{c}\zeta_{2,c}$ and $f'(u_c)v$, using Lemmas 10.1 and 10.2 we have

$$\begin{aligned}\|\mathcal{E}'(v)(l)\|_{L^1} &\lesssim (|\dot{x} - c| + |\dot{c}|)\|v\|_{W_\alpha} \lesssim \|\tilde{v}_1\|_{W_\nu}^2 + \|v_2\|_{H^1_\alpha}^2, \\ \|\mathcal{E}'(v)(f'(u_c)v)\|_{L^1} &\le \|e^{-\alpha|y|}\mathcal{E}'(v)\|_{L^2}\|e^{\alpha|y|}f'(u_c)v\|_{L^2} \\ &\lesssim \|v\|_{W_\alpha}\|e^{\alpha|y|}q_c v\|_{L^2} \lesssim \|v\|_{W_\alpha}^2 \lesssim \|\tilde{v}_1\|_{W_\nu}^2 + \|v_2\|_{H^1_\alpha}^2,\end{aligned}$$

thus $\int_0^\infty I_1(t)\,dt \lesssim \|v_0\|_{H^1}^2$ by (13.4). Integrating (13.11) for $t_1 \le t$ we find that since $\tilde{y}(t_1) = 0$,

$$\int_{\mathbb{R}} \chi_\nu(y - \tilde{y}(t))\mathcal{E}(v(t,y))\,dy \le \int_{\mathbb{R}} \chi_\nu(y)\mathcal{E}(v(t_1,y))\,dy + \int_{t_1}^\infty I_1(t)\,dt. \tag{13.13}$$

The right hand side tends to zero as $t_1 \to \infty$. Since $\sigma t - x(t) \geq \tilde{y}(t)$ provided $(\sigma - c_1)t \geq x(t_1) - c_1 t_1$, we can conclude that (13.7) holds. This completes the proof of Theorem 2.4. □

14. Asymptotic stability in weighted spaces

It remains to prove Theorem 2.5. In addition to the assumptions of Theorem 2.4, assume (2.16) where $\omega : \mathbb{R} \to \mathbb{R}$ is increasing with $\omega(x) = 1$ for $x \leq 0$ and $\int_0^\infty \omega(x)^{-1}\,dx < \infty$.

14.1. **Convergence of the phase shift.** In this subsection, we will prove (2.17) by using the Virial Lemma.

Proof of (2.17). With $3\hat{\sigma} = \sigma - c_1$ we have $\dot{x}(t) \geq c_1 + 3\hat{\sigma}$, so by Corollary 11.2 we have

$$\|\tilde{v}_1(t)\|_{W_\nu}^2 \leq \int_{\mathbb{R}} \chi_\nu(x - x(t))\mathcal{E}(v_1(t,x))\,dx \leq \int_{\mathbb{R}} \chi_\nu(x - 2\hat{\sigma}t)\mathcal{E}(v_0(x))\,dx$$

for t so large that $\hat{\sigma}t + x_0 \geq 0$. Since $\chi_\nu(x) \leq \min(2, 2e^{2\nu x})$ for all x, and ω is increasing, we have

$$\chi_\nu(x - 2\hat{\sigma}t) \leq \begin{cases} 2\omega(x)^2/\omega(\hat{\sigma}t)^2 & \text{if } x \geq \hat{\sigma}t, \\ 2e^{-2\nu\hat{\sigma}t} & \text{if } x \leq \hat{\sigma}t. \end{cases}$$

Therefore, we have

$$\int_{\mathbb{R}} \chi_\nu(y)\mathcal{E}(\tilde{v}_1(t,y))\,dy \leq 2\int_{\mathbb{R}} \left(e^{-2\nu\hat{\sigma}t} + \frac{\omega(x)^2}{\omega(\hat{\sigma}t)^2}\right)\mathcal{E}(v_0(x))\,dx \lesssim \Theta(t)^2, \tag{14.1}$$

where

$$\Theta(t) = (e^{-\nu\hat{\sigma}t} + \omega(\hat{\sigma}t)^{-1})(\|\omega v_0\|_{L^2} + \|\omega\partial_x v_0\|_{L^2}).$$

Due to (12.7) and Young's inequality, since $\int_0^\infty \Theta(t)\,dt < \infty$, (14.1) implies

$$\|\tilde{v}_1\|_{L^1(0,\infty;W_\nu)} + \|v_2\|_{L^1(0,\infty;H^1_\alpha)} < \infty. \tag{14.2}$$

In view of (14.2) and the modulation estimates (10.4)–(10.5), $\dot{x}(t) - c(t)$ and $\dot{c}(t)$ are integrable on $(0,\infty)$. To show convergence of

$$\lim_{t\to\infty}(x(t) - c_\star t) = x_0 + \int_0^t (\dot{x}(s) - c(s))\,ds + \int_0^t (c(s) - c_\star)\,ds\,, \tag{14.3}$$

it suffices to prove that $c(t) - c_\star \in L^1(0,\infty)$.

By (12.7) and (14.1) and the fact that Θ is decreasing, we have

$$\|v_2(t)\|_{H^1_\alpha} \leq \left(\int_0^{t/2} + \int_{t/2}^t\right) e^{-\beta(t-s)/4}\Theta(s)\,ds \leq \frac{4}{\beta}\left(e^{-\beta t/8}\Theta(0) + \Theta(t/2)\right). \tag{14.4}$$

Then for large $t \geq 0$, since $\int_0^\infty \Theta(t)\,dt < \infty$,

$$\int_t^\infty \left(\|\tilde{v}_1(s)\|_{W_\nu}^2 + \|v_2(s)\|_{H^1_\alpha}^2\right)\,ds \lesssim e^{-\beta t/8} + \Theta(t/2). \tag{14.5}$$

Since $\|\tilde{v}_1(t)\|_{W_\nu}$ is integrable, it follows by integrating (10.6) and using (10.9) that

$$c(t) - c_\star = \int_\infty^t \dot{c}(s)\,ds = O\left(\|\tilde{v}_1(t)\|_{W_\nu} + \int_t^\infty \left(\|\tilde{v}_1(s)\|_{W_\nu}^2 + \|v_2(s)\|_{H^1_\alpha}^2\right)\,ds\right) \tag{14.6}$$

is integrable on $(0,\infty)$. Now letting

$$x_\star := x_0 + \int_0^\infty (\dot{x}(s) - c(s))\,ds + \int_0^\infty (c(s) - c_\star)\,ds, \tag{14.7}$$

we obtain (2.17). □

14.2. **Exponentially localized data.** In this subsection, we will prove (2.18)–(2.19). To begin with, we will prove exponential decay of $v_1(t)$.

Lemma 14.1. *There is a positive constants $\hat{C}$ such that if $\|v_0\|_{H^1} \le \delta_4$ and $v_0 \in H^1_{\alpha_1}$ for some $\alpha_1 \in (0,\nu_0)$, then for all $t \ge 0$,*

$$\left\| e^{\alpha_1(x-c_1t)} v_1(t,\cdot) \right\|_{H^1} \le \hat{C}\|v_0\|_{H^1_{\alpha_1}}. \tag{14.8}$$

Proof. Observe that $\bar{\chi}^n(t,x) := e^{2\alpha_1 n}\chi_{\alpha_1}(x - c_1t - n) \to e^{2\alpha_1(x-c_1t)}$ monotonically as $n \to \infty$. Then Lemma 11.1 implies that for every $n \in \mathbb{N}$,

$$\int_{\mathbb{R}} \bar{\chi}^n(t,x)\mathcal{E}(v_1(t,x))dx \le \int_{\mathbb{R}} \bar{\chi}^n(0,x)\mathcal{E}(v_0(x))dx\,.$$

Letting $n \to \infty$, by using Beppo Levi's theorem we obtain

$$\int_{\mathbb{R}} e^{2\alpha_1(x-c_1t)}\mathcal{E}(v_1(t,x))dx \le \int_{\mathbb{R}} e^{2\alpha_1 x}\mathcal{E}(v_0(x))dx\,. \tag{14.9}$$

Eq. (14.8) immediately follows. □

Proof of (2.18)–(2.19). We suppose $v_0 \in H^1_{\alpha_1}$ where $0 < \alpha_1 < \min(\nu_0,\alpha)$. Let $\gamma_1 = \alpha_1(\sigma - c_1)$. Lemma 14.1 implies that since $\dot{x}(t) > \sigma$,

$$\|\tilde{v}_1(t)\|_{W_\nu} \lesssim \|\tilde{v}_1(t)\|_{H^1_{\alpha_1}} \lesssim e^{-\alpha_1(x(t)-c_1t)} \lesssim e^{-\gamma_1 t}. \tag{14.10}$$

By (14.10) and (12.7), we have

$$\|v_2(t)\|_{H^1_\alpha} \lesssim e^{-\gamma_2 t} \tag{14.11}$$

for γ_2 satisfying $0 < \gamma_2 < \min(\gamma_1, \beta/4)$. Interpolating as we did in (13.6) then yields

$$\|v_2(t)\|_{H^1_{\alpha_1}} \lesssim e^{-\gamma t}, \qquad \gamma = \gamma_2\alpha_1/\alpha.$$

Thus by Lemma 10.1,

$$|c(t) - c_\star| = O(e^{-\gamma t}) \quad \text{and} \quad |x(t) - c_\star t - x_\star| = O(e^{-\gamma t}) \quad \text{as } t \to \infty. \tag{14.12}$$

Let $x_0(t) = x(t) - c_\star t - x_\star$. Combining (14.10) and (14.11) with (14.12), we obtain

$$\begin{aligned} \|u_{c_\star} - u(t,\cdot + c_\star t + x_\star)\|_{H^1_{\alpha_1}} &= e^{\alpha_1 x_0(t)}\|u_{c_\star}(\cdot + x_0(t)) - u(t,\cdot + x(t))\|_{H^1_{\alpha_1}} \\ &= e^{\alpha_1 x_0(t)}\|u_{c_\star}(\cdot + x_0(t)) - u_{c(t)} - \tilde{v}_1(t) - v_2(t)\|_{H^1_{\alpha_1}} \\ &\lesssim |x_0(t)| + |c(t) - c_\star| + \|\tilde{v}_1(t)\|_{H^1_{\alpha_1}} + \|v_2(t)\|_{H^1_{\alpha_1}} = O(e^{-\gamma t}). \end{aligned} \tag{14.13}$$

Thus we prove (2.18) and (2.19). □

14.3. **Polynomially localized data.** In this subsection, we will prove (2.20), essentially as an immediate consequence of the arguments of subsection 14.1. Below, let $x_+ = \max(0, x)$, and let $\rho > 1$ be constant.

First, we remark that for a localized perturbation with $\omega v_0 \in H^1$ for $\omega(x) = (1 + x_+)^\rho$, Eqs. (14.1) and (14.4) imply

$$\|\chi_{\nu/2}\tilde{v}_1(t)\|_{H^1} + \|v_2(t)\|_{H^1_\alpha} \lesssim (1+t)^{-\rho}. \tag{14.14}$$

By (14.6) and (14.14), and the fact that $\|\tilde{v}_1(t)\|_{W_\nu} \lesssim \|\chi_{\nu/2}\tilde{v}_1(t)\|_{H^1}$,

$$|c(t) - c_\star| \lesssim (1+t)^{-\rho} + \int_t^\infty (1+s)^{-2\rho}\, ds \lesssim (1+t)^{-\rho} \tag{14.15}$$

provided $\rho > 1$. Since $x(0) = x_0$ and $\dot{x}(t) - c(t) = O(\|\tilde{v}_1(t)\|_{W_\nu} + \|v_2(t)\|_{H^1_\alpha})$ by Lemma 10.1,

$$x(t) - c_\star t - x_\star = \int_\infty^t (\dot{x}(s) - c(s))\, ds + \int_\infty^t (c(s) - c_\star)\, ds = O\left((1+t)^{-\rho+1}\right) \tag{14.16}$$

follows from (14.14), (14.15), and (14.7). Now (2.20) follows from (14.14), (14.15) and (14.16) in a manner very similar to (14.13), using the fact that $\chi_{\nu/2}(x) \lesssim \chi_{\alpha/2}(x) \sim \min(1, e^{\alpha x})$.

Appendix A. Hamiltonian and variational structure

The Benney-Luke equation (1.1) has a Hamiltonian structure which we now describe. Also we show that the solitary-wave profile is an infinitely indefinite critical point of the naturally associated energy-momentum functional.

We modify slightly the form in [36] to use q in place of φ. In terms of the conjugate momentum variable

$$p = r + B^{-1}\left(\frac{1}{2}q^2\right), \tag{A.1}$$

the Hamiltonian is given by

$$\begin{aligned}\mathcal{H} &= \frac{1}{2}\int_{\mathbb{R}} rBr + qAq\, dx \\ &= \frac{1}{2}\int_{\mathbb{R}} \left(p - B^{-1}\left(\frac{1}{2}q^2\right)\right) B \left(p - B^{-1}\left(\frac{1}{2}q^2\right)\right) + qAq\, dx.\end{aligned} \tag{A.2}$$

We find that formally, taking variations with respect to (q, p),

$$\delta\mathcal{H} = \begin{pmatrix} -rq + Aq \\ Br \end{pmatrix},$$

and that (2.1) is equivalent to the following system in Hamiltonian form,

$$\partial_t \begin{pmatrix} q \\ p \end{pmatrix} = \mathcal{J}\,\delta\mathcal{H}, \qquad \mathcal{J} = \begin{pmatrix} 0 & \partial_x B^{-1} \\ \partial_x B^{-1} & 0 \end{pmatrix}. \tag{A.3}$$

Due to the translation invariance of the Hamiltonian, Noether's Theorem assures the existence of the conserved momentum functional

$$\mathcal{N} = \int_{\mathbb{R}} qBp\, dx,$$

with the property that $\mathcal{J}\delta\mathcal{N} = \partial_x$. Solitary waves with speed $c > 0$ have profiles given as the stationary points of the energy-momentum functional

$$\mathcal{H}_c = \mathcal{H} + c\mathcal{N}. \tag{A.4}$$

Noting that

$$\delta\mathcal{H}_c = \begin{pmatrix} -rq + Aq + cBp \\ Br + cBq \end{pmatrix}$$

one checks that solutions of $\delta\mathcal{H}_c = 0$ satisfy (2.6), and (2.7)-(2.8) yields the localized solutions for $c^2 > 1$.

The classic variational approach to proving orbital stability for solitary waves [2, 18] is based on showing that the second variation $\delta^2\mathcal{H}_c$ has definite sign when subject to a finite number of constraints induced by time-conserved quantities. Here, at a critical point $(q, p) = (q_c, p_c)$, in terms of a variation $(\dot{q}, \dot{r}) = (\dot{q}, \dot{p} - B^{-1}(q\dot{q}))$ we can express the second variation as

$$\begin{aligned} \left\langle \begin{pmatrix} \dot{q} \\ \dot{r} \end{pmatrix}, \delta^2\mathcal{H}_c \begin{pmatrix} \dot{q} \\ \dot{r} \end{pmatrix} \right\rangle &= \int_{\mathbb{R}} \begin{pmatrix} \dot{q} \\ \dot{r} \end{pmatrix}^T \begin{pmatrix} A + 3cq & cB \\ cB & B \end{pmatrix} \begin{pmatrix} \dot{q} \\ \dot{r} \end{pmatrix} dx \\ &= \int_{\mathbb{R}} \dot{q}(A - c^2B + 3cq)\dot{q} + (\dot{r} + c\dot{q})B(\dot{r} + c\dot{q})\, dx. \end{aligned} \tag{A.5}$$

The operator $B = I - b\partial_x^2$ is positive. However, since $c^2 > 1$ and $b > a$, the operator

$$L_c = A - c^2B + 3cq = (1 - c^2) + (bc^2 - a)\partial_x^2 + 3cq$$

has the interval $(-\infty, 1 - c^2]$ as continuous spectrum. Zero is an eigenvalue, with eigenfunction $\partial_x q$ due to (2.7). Since this eigenfunction changes sign exactly once, oscillation theory implies that L_c has exactly one positive eigenvalue. Thus, L_c is strictly negative except for two directions which are associated with the two degrees of freedom of the solitary wave, while B is a positive operator. It follows that $\delta^2\mathcal{H}_c$ is infinitely indefinite. This situation also occurs in the full water wave equations [7] and in other Boussinesq-type nonlinear wave equations having two-way wave propagation [45, 37].

Appendix B. Multiplicity of the zero eigenvalue

Here our aim is to prove part (iv) of Lemma 2.1, and determine the generalized kernels of both $\mathcal{L}_c$ and $\mathcal{L}_c^*$. We will show that $\lambda = 0$ is an eigenvalue of the operator $\mathcal{L}_c$ with algebraic multiplicity two and geometric multiplicity one, in the space L^2_α, $0 < \alpha < \alpha_c$.

Let us write (q, r) for (q_c, r_c) below for simplicity. By differentiating the solitary wave equations (2.6) with respect to x and c it follows directly that the functions

$$\zeta_{1,c} = \begin{pmatrix} \partial_x q \\ \partial_x r \end{pmatrix}, \qquad \zeta_{2,c} = -\begin{pmatrix} \partial_c q \\ \partial_c r \end{pmatrix}, \tag{B.1}$$

satisfy $\mathcal{L}_c\zeta_{1,c} = 0$, $\mathcal{L}_c\zeta_{2,c} = \zeta_{1,c}$.

By basic asymptotic theory for ODEs (after multiplying the second component by B), solutions of $\mathcal{L}_c z = 0$ satisfy $z(x) \sim ve^{\mu x}$ as $x \to \infty$, where $v \in \mathbb{R}^2$ and where μ is an eigenvalue of the characteristic matrix, satisfying

$$\det \begin{pmatrix} c\mu & \mu \\ (1 - a\mu^2)\mu & c\mu(1 - b\mu^2) \end{pmatrix} = \mu^2((c^2 - 1) - (bc^2 - a)\mu^2) = 0.$$

The roots are $\mu = \pm\alpha_c$ and the double root $\mu = 0$, so any solution of $\mathcal{L}_c z = 0$ that lies in the space L^2_α decays exponentially to zero as $x \to \infty$ and must be a constant multiple of $\zeta_{1,c}$. Thus $\lambda = 0$ has geometric multiplicity one.

Next we treat the adjoint $\mathcal{L}_c^*$. In the following lemma, ∂^{-1} denotes a right inverse for ∂ on the space $L^2_{-\alpha}$ dual to L^2_α, defined by $\partial^{-1}g(x) = \int_{-\infty}^x g(y)\,dy$.

Lemma B.1. *Suppose $0 < \alpha < \alpha_c$, and let*

$$\eta_{1,c} = \begin{pmatrix} Aq \\ Br \end{pmatrix}, \qquad \eta_{2,c} = -c\begin{pmatrix} q(\partial^{-1}\partial_c q) + B\partial^{-1}\partial_c p \\ B\partial^{-1}\partial_c q \end{pmatrix}, \tag{B.2}$$

where $p = r + B^{-1}(\frac{1}{2}q^2)$. Then $\eta_{1,c}$ and $\eta_{2,c}$ lie in $H^1_{-\alpha}$ and satisfy $\mathcal{L}_c^*\eta_{1,c} = 0$, $\mathcal{L}_c^*\eta_{2,c} = \eta_{1,c}$. Moreover,

$$\begin{pmatrix} \langle\zeta_{1,c},\eta_{1,c}\rangle & \langle\zeta_{2,c},\eta_{1,c}\rangle \\ \langle\zeta_{1,c},\eta_{2,c}\rangle & \langle\zeta_{2,c},\eta_{2,c}\rangle \end{pmatrix} = \begin{pmatrix} 0 & -\beta_0 \\ -\beta_0 & \beta_1 \end{pmatrix}, \tag{B.3}$$

with

$$\beta_0 = \frac{d}{dc}E(u_c) > 0, \qquad \beta_1 = c\left(\frac{d}{dc}\int_{\mathbb{R}} q\right)\left(\frac{d}{dc}\int_{\mathbb{R}} p\right). \tag{B.4}$$

For use in Part II, we define vectors biorthogonal to $\zeta_{1,c}$, $\zeta_{2,c}$ via

$$\begin{pmatrix} \zeta^*_{1,c} \\ \zeta^*_{2,c} \end{pmatrix} = \begin{pmatrix} \theta_1 & \theta_0 \\ \theta_0 & 0 \end{pmatrix}\begin{pmatrix} \eta_{1,c} \\ \eta_{2,c} \end{pmatrix}, \qquad \begin{pmatrix} \theta_0 \\ \theta_1 \end{pmatrix} = -\frac{1}{\beta_0^2}\begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix}. \tag{B.5}$$

Then for $i, j = 1, 2$,

$$\langle\zeta_{i,c},\zeta^*_{j,c}\rangle = \delta_{ij}, \quad \mathcal{L}_c^*\zeta^*_{1,c} = \zeta^*_{2,c}, \quad \mathcal{L}_c^*\zeta^*_{2,c} = 0. \tag{B.6}$$

Taking the lemma for granted temporarily, we claim it follows that $\lambda = 0$ has algebraic multiplicity exactly equal to two. Suppose the multiplicity is higher. Then there exists $\zeta_3 \in L^2_\alpha$ with $\mathcal{L}_c\zeta_3 = \zeta_{2,c}$. But then, by Lemma B.1, since $\eta_{1,c} \in H^1_{-\alpha}$,

$$0 = \langle\zeta_3, \mathcal{L}_c^*\eta_{1,c}\rangle = \langle\mathcal{L}_c\zeta_3, \eta_{1,c}\rangle = \langle\zeta_{2,c},\eta_{1,c}\rangle = -\beta_0 \neq 0.$$

This contradiction shows ζ_3 cannot exist, hence the multiplicity is exactly two.

Proof of Lemma B.1. It is straightforward to check that

$$\mathcal{L}_c^*\eta_{1,c} = \begin{pmatrix} -c\partial & (-A\partial - c\partial q + 2cq')B^{-1} \\ -\partial & (-cB\partial + 2\partial q - q')B^{-1} \end{pmatrix}\begin{pmatrix} Aq \\ -cBq \end{pmatrix} = 0. \tag{B.7}$$

Next, we compute that

$$\langle\zeta_{1,c},\eta_{1,c}\rangle = \int_{\mathbb{R}} ((\partial_x q)Aq + (\partial_x r)Br)\,dx = 0, \tag{B.8}$$

$$\langle\zeta_{2,c},\eta_{1,c}\rangle = -\int_{\mathbb{R}} ((\partial_c q)Aq + (\partial_c r)Br)\,dx = -\frac{dE}{dc},$$

where

$$E = E(u_c) = \frac{1}{2}\int_{\mathbb{R}} (qAq + rBr)\,dx = \frac{1}{2}\int_{\mathbb{R}} \left((1+c^2)q^2 + (a+bc^2)(\partial_x q)^2\right)\,dx. \tag{B.9}$$

Using the facts that

$$\int_{\mathbb{R}} \operatorname{sech}^4\frac{x}{2}\,dx = \frac{8}{3}, \qquad \int_{\mathbb{R}} \operatorname{sech}^4\frac{x}{2}\tanh^2\frac{x}{2}\,dx = \frac{8}{15},$$

from the explicit expression (2.8) for q, we find that

$$\begin{aligned} E &= \frac{4}{3}(1+c^2)\frac{(c^2-1)^2}{c^2}\alpha_c^{-1} + \frac{4}{15}(a+bc^2)\frac{(c^2-1)^2}{c^2}\alpha_c \qquad (B.10)\\ &= \frac{4\rho^2}{15(\rho+1)}\left(5(\rho+2)\sqrt{b+\frac{b-a}{\rho}} + (b\rho+(b+a))\sqrt{\frac{\rho}{b\rho+b-a}}\right), \end{aligned}$$

where $\rho = c^2-1$. From this expression it is evident that $dE/dc > 0$ for $c > 1$.

Next we find some $\eta_{2,c} \in H^1_{-\alpha}$ such that $\mathcal{L}_c^*\eta_{2,c} = \eta_{1,c}$. Writing $\eta_{2,c} = (\tilde{q}, \tilde{r})$, this means

$$\mathcal{L}_c^*\eta_{2,c} = \begin{pmatrix} -c\partial\tilde{q} + (-A\partial - cq\partial + cq')B^{-1}\tilde{r} \\ -\partial\tilde{q} + (-cB\partial + 2q\partial + q')B^{-1}\tilde{r} \end{pmatrix} = \begin{pmatrix} Aq \\ -cBq \end{pmatrix}$$

Eliminating $\tilde{q}$ and comparing with the equation obtained by differentiating (2.7) in c,

$$(A - c^2B + 3cq)\partial_c q = 2cBq - \frac{3}{2}q^2 = \frac{1}{c}(Aq + c^2Bq), \qquad (B.11)$$

we may choose $\tilde{r} = -cB\partial^{-1}\partial_c q$. Then since $2q\partial + q' = \partial q + q\partial$,

$$\partial(\tilde{q} - qB^{-1}\tilde{r}) = (-cB + q)\partial B^{-1}\tilde{r} + cBq = cB(q + c\partial_c q) - cq\partial_c q = -cB\partial_c p.$$

Hence $\eta_{2,c}$ is given by (B.2), and $\eta_{2,c} \in H^1_{-\alpha}$. Moreover, we find

$$\langle\zeta_{1,c}, \eta_{2,c}\rangle = \langle\mathcal{L}_c\zeta_{2,c}, \eta_{2,c}\rangle = \langle\zeta_{2,c}, \mathcal{L}_c^*\eta_{2,c}\rangle = \langle\zeta_{2,c}, \eta_{1,c}\rangle = -\frac{dE}{dc} = -\beta_0. \qquad (B.12)$$

Finally, we compute $\langle\zeta_{2,c}, \eta_{2,c}\rangle$. Since $\partial_c p = \partial_c r + B^{-1}(q\partial_c q)$, we can write

$$\begin{pmatrix} \partial_c q \\ \partial_c p \end{pmatrix} = \mathcal{T}\begin{pmatrix} \partial_c q \\ \partial_c r \end{pmatrix}, \quad \mathcal{T} = \begin{pmatrix} I & 0 \\ B^{-1}q & I \end{pmatrix}, \quad \mathcal{T}^{-*} = \begin{pmatrix} I & -qB^{-1} \\ 0 & I \end{pmatrix},$$

and we note

$$\eta_{2,c} = -c\begin{pmatrix} I & qB^{-1} \\ 0 & I \end{pmatrix} B\partial^{-1}\begin{pmatrix} \partial_c p \\ \partial_c q \end{pmatrix} = -c\mathcal{T}^*\mathcal{J}^{-1}\begin{pmatrix} \partial_c q \\ \partial_c p \end{pmatrix},$$

with $\mathcal{J}$ as in (A.3). Hence we find

$$\begin{aligned} \langle\zeta_{2,c}, \eta_{2,c}\rangle &= \langle\mathcal{T}\zeta_{2,c}, \mathcal{T}^{-*}\eta_{2,c}\rangle = c\left\langle \begin{pmatrix} \partial_c q \\ \partial_c p \end{pmatrix}, \mathcal{J}^{-1}\begin{pmatrix} \partial_c q \\ \partial_c p \end{pmatrix} \right\rangle \\ &= c\int_{\mathbb{R}} (\partial_c q)B\partial^{-1}(\partial_c p) + (\partial_c p)B\partial^{-1}(\partial_c q) = c\left(\int_{\mathbb{R}} \partial_c q\right)\left(\int_{\mathbb{R}} \partial_c p\right). \qquad (B.13) \end{aligned}$$

□

Appendix C. Null multiplicity of the zero characteristic value

In order to apply the Gohberg-Sigal theory, we need to show that (i) for the bundle $\mathcal{W}(\lambda)$, $\lambda = 0$ is a *characteristic value of null multiplicity at least two*, and (ii) for the KdV bundle $\mathcal{W}_0(\Lambda)$, $\Lambda = 0$ is the only characteristic value satisfying $\operatorname{Re}\Lambda > -\hat{\beta}$, and has null multiplicity *no more than two*. According to what this means in the terminology of [17], we need to prove the following.

Lemma C.1. *For some nontrivial analytic map* $\lambda \mapsto \psi(\lambda) \in L^2_\alpha$, $\|\mathcal{W}(\lambda)\psi(\lambda)\|_\alpha = o(\lambda)$ *as* $|\lambda| \to 0$.

Lemma C.2. *Suppose* $\hat\alpha \in (0, (b-a)^{-1/2})$ *and* $\hat\beta = \hat\alpha(1-(b-a)\hat\alpha^2)$. *Then* $\mathcal{W}_0(\Lambda)$ *is invertible in* $L^2_{\hat\alpha}$ *whenever* $\operatorname{Re}\Lambda > -\hat\beta$ *and* $\Lambda \neq 0$. *Moreover,* $\mathcal{W}_0(0)$ *has one-dimensional kernel, and for no nontrivial analytic map* $\Lambda \mapsto \psi(\Lambda)$ *do we have* $\|\mathcal{W}_0(\Lambda)\psi(\Lambda)\|_\alpha = o(\Lambda^2)$ *as* $|\Lambda| \to 0$.

For the proof of Lemma C.2 see the proof of Proposition 12.3 in Appendix C of [38]. (The KdV bundle $W_0(\lambda)$ there differs from $\mathcal{W}_0(\Lambda)$ here by a simple scaling.)

To prove Lemma C.1 is a simple calculation when done in the right way. The trick is to apply the transformation in (5.3) to the original solitary-wave equations (2.1), *then* differentiate with respect to x and c. Using (2.27) with $\lambda = 0$ we find that

$$-\mathcal{Q}_+(Sq+r) + B^{-1}(rq' + 2qr') = 0, \tag{C.1}$$
$$-\mathcal{Q}_-(-Sq+r) + B^{-1}(rq' + 2qr') = 0. \tag{C.2}$$

Thus, with

$$\rho := -\mathcal{Q}_+(Sq+r) = -\mathcal{Q}_-(-Sq+r)$$

we have

$$r = \frac{1}{2}(-\mathcal{Q}_+^{-1} - \mathcal{Q}_-^{-1})\rho, \qquad q = \frac{1}{2}S^{-1}(-\mathcal{Q}_+^{-1} + \mathcal{Q}_-^{-1})\rho, \tag{C.3}$$
$$\rho + B^{-1}(rq' + 2qr') = 0. \tag{C.4}$$

Differentiating these equations with respect to x, we find

$$\partial_x r = \frac{1}{2}(-\mathcal{Q}_+^{-1} - \mathcal{Q}_-^{-1})\partial_x\rho, \qquad \partial_x q = \frac{1}{2}S^{-1}(-\mathcal{Q}_+^{-1} + \mathcal{Q}_-^{-1})\partial_x\rho, \tag{C.5}$$
$$\partial_x\rho + B^{-1}(q' + 2q\partial_x)\partial_x r + B^{-1}(r\partial_x + 2r')\partial_x q = 0. \tag{C.6}$$

This yields

$$(I + (R_q + R_r)(-\mathcal{Q}_+^{-1}) + (R_q - R_r)(-\mathcal{Q}_-^{-1}))\partial_x\rho = \mathcal{W}(0)\partial_x\rho = 0. \tag{C.7}$$

Next we differentiate with respect to c. Since $\mathcal{Q}_\pm = c\partial_x \pm S\partial_x$ we have

$$\partial_c(Q_\pm^{-1}\rho) = Q_\pm^{-1}\partial_c\rho - Q_\pm^{-2}\partial_x\rho.$$

Hence

$$\partial_c r = \frac{1}{2}(-\mathcal{Q}_+^{-1} - \mathcal{Q}_-^{-1})\partial_c\rho + \frac{1}{2}(\mathcal{Q}_+^{-2} + \mathcal{Q}_-^{-2})\partial_x\rho, \tag{C.8}$$
$$\partial_c q = \frac{1}{2}S^{-1}(-\mathcal{Q}_+^{-1} + \mathcal{Q}_-^{-1})\partial_c\rho + \frac{1}{2}S^{-1}(\mathcal{Q}_+^{-2} - \mathcal{Q}_-^{-2})\partial_x\rho. \tag{C.9}$$

and therefore

$$\begin{aligned} 0 \;=\; & (I + (R_q + R_r)(-\mathcal{Q}_+^{-1}) + (R_q - R_r)(-\mathcal{Q}_-^{-1}))\partial_c\rho \\ & + ((R_q + R_r)\mathcal{Q}_+^{-2} + (R_q - R_r)\mathcal{Q}_-^{-2})\partial_x\rho. \end{aligned} \tag{C.10}$$

Since

$$W'(\lambda) = -(R_q + R_r)(\lambda - \mathcal{Q}_+)^{-2} - (R_q - R_r)(\lambda - \mathcal{Q}_-)^{-2},$$

this means

$$\mathcal{W}(0)\partial_x\rho - W'(0)\partial_c\rho = 0. \tag{C.11}$$

Since $\partial_x\rho$ and $\partial_c\rho$ belong to the weighted space L^2_α, combining (C.7) and (C.11) we obtain

$$\mathcal{W}(\lambda)(\partial_x\rho - \lambda\partial_c\rho) = o(\lambda) \qquad \text{as } |\lambda| \to 0,$$

in L^2_α, and this finishes the proof of Lemma C.1.

Appendix D. Exponential linear stability via recentering

To begin, we extend the linear stability estimate from Theorem 2.2 to Sobolev spaces $H^n(\mathbb{R})$ spaces of arbitrary order.

Proposition D.1. *Fix $c > 1$ and α with $0 < \alpha < \alpha_c$, and let $n \geq 0$ be an integer. Assume that $\mathcal{L}_c$ has no nonzero eigenvalue λ satisfying $\operatorname{Re}\lambda \geq 0$. Then there exist positive constants K_n and β such that for all $t \geq 0$,*

$$\|e^{\mathcal{L}_c t} Q_c z\|_{H^n_\alpha} \leq K_n e^{-\beta t} \|z\|_{H^n_\alpha}, \tag{D.1}$$

where $Q_c = I - P_c$ is the spectral projection complementary to the generalized kernel of $\mathcal{L}_c$.

Proof. Since 1 is in the resolvent set of $\mathcal{L}_c$ by Lemma 2.1, and Q_c commutes with $\mathcal{L}_c$, we see that $(1 - \mathcal{L}_c)^n$ is an isomorphism from $Q_c H^n_\alpha$ to $Q_c L^2_\alpha$. Applying Theorem 2.2, we find

$$\|e^{\mathcal{L}_c t} Q_c z\|_{H^n_\alpha} \lesssim \|(1 - \mathcal{L}_c)^n e^{\mathcal{L}_c t} Q_c z\|_{L^2_\alpha} \lesssim e^{-\beta t} \|z\|_{H^n_\alpha}.$$

This completes the proof. □

Our main goal in this appendix is to prove Lemma 2.6 by a recentering argument. Such arguments were used to analyze pulse dynamics by Ei [10] for reaction-diffusion systems and Promislow [40] for damped Schrödinger equations. See also [30] for a result for gKdV equations. Although here we merely analyze stability of a single solitary wave, we need these arguments because a general perturbation in the energy space may create a divergent phase shift of solitary waves.

To prove Lemma 2.6, we need to compare weighted norms of w in recentered moving coordinates. Recall that τ_h is a translation operator defined by $(\tau_h f)(x) := f(x - h)$.

Claim D.1. *Let $c_0 > 1$ and $\alpha \in (0, \alpha_{c_0})$. There exists positive constants δ_6, δ_7 and C_0 such that if $|c - c_0| < \delta_6$ and $|h| < \delta_7$, then for any $v \in H^1_\alpha$ with $P_c v = 0$,*

$$C_0^{-1} \|v\|_{H^1_\alpha} \leq \|Q_{c_0} \tau_h v\|_{H^1_\alpha} \leq C_0 \|v\|_{H^1_\alpha}.$$

Proof. Since $P_c v = 0$ we have

$$\|P_{c_0} \tau_h v\|_{H^1_\alpha} \leq \|P_{c_0}(\tau_h v - v)\|_{H^1_\alpha} + \|(P_{c_0} - P_c) v\|_{H^1_\alpha} \lesssim (|c - c_0| + |h|) \|v\|_{L^2_\alpha}.$$

Combining this with $\|\tau_h v\|_{H^1_\alpha} = e^{\alpha h} \|v\|_{H^1_\alpha}$ and

$$\big| \|Q_{c_0} \tau_h v\|_{H^1_\alpha} - \|\tau_h v\|_{H^1_\alpha} \big| \leq \|P_{c_0} \tau_h v\|_{H^1_\alpha},$$

we have Claim D.1. □

Proof of Lemma 2.6. To handle the time dependent advection term $\eta(t)\partial_y w$, we use a sequence of coordinate frames moving with the constant speed c_0, changing the phase from time to time so that the center of coordinates remains close to the solitary wave position $x(t)$ for all time.

Let the constants K_1 and β be as given by Proposition D.1, and let δ_6, δ_7 and C_0 be from Claim D.1. We fix $T_1 > 0$ such that

$$C_0^2 K_1 e^{-\beta T_1/6} \leq 1, \tag{D.2}$$

and let $t_j = jT_1$ for $j \geq 0$. Also let $h_j(t) := \int_{t_j}^t \big(c(s) - c_0 + \eta(s)\big)\, ds$. Under assumption (2.21), for $\hat\delta$ small enough we have that for every j,

$$\sup_{t\in[t_j,t_{j+1}]} |h_j(t)| \leq T_1\hat\delta \leq \delta_7. \tag{D.3}$$

Now let $w_j(t) = Q_{c_0}\tau_{h_j(t)}w(t)$. We rewrite (2.22) as

$$\partial_t w_j = \mathcal{L}_{c_0} w_j + Q_{c_0}\tau_{h_j(t)}(F(t) + \widetilde{F}(t)),$$

where $\widetilde{F}(t) = (f'(u_{c(t)}) - \tau_{-h_j(t)}f'(u_{c_0})\tau_{h_j(t)})w(t)$. Using the variation of constants formula, we have

$$w_j(t) = e^{(t-t_j)\mathcal{L}_{c_0}}Q_{c_0}w_j(t_j) + \int_{t_j}^t e^{(t-s)\mathcal{L}_{c_0}}Q_{c_0}\tau_{h_j(s)}(F(s) + \widetilde{F}(s))\, ds. \tag{D.4}$$

By (D.3) and the definition of H^1_α, the operator norm $\|\tau_{h_j(s)}\|_\alpha \leq e^{\alpha\delta_7}$. Since $P_{c(t)}w(t) = 0$, Claim D.1 implies that for $t \in [t_j, t_{j+1}]$,

$$C_0^{-1}\|w(t)\|_{H^1_\alpha} \leq \|w_j(t)\|_{H^1_\alpha} \leq C_0\|w(t)\|_{H^1_\alpha}, \tag{D.5}$$

whence

$$\|\widetilde{F}(s)\|_{H^1_\alpha} \lesssim (|c(t) - c_0| + |h_j(t)|)\|w_j(t)\|_{H^1_\alpha} \leq (1 + T_1)\hat\delta\|w_j(t)\|_{H^1_\alpha}.$$

Applying Proposition D.1 to (D.4) and using the estimates above, we have

$$\begin{aligned} e^{\beta t}\|w_j(t)\|_{H^1_\alpha} &\leq K_1 e^{\beta t_j}\|w_j(t_j)\|_{H^1_\alpha} + K_1 e^{\alpha\delta_7}\int_{t_j}^t e^{\beta s}\|F(s) + \widetilde{F}(s)\|_{H^1_\alpha}\, ds \\ &\leq K_1 e^{\beta t_j}\|w_j(t_j)\|_{H^1_\alpha} + C_1\int_{t_j}^t e^{\beta s}\left(\|F(s)\|_{H^1_\alpha} + \hat\delta\|w_j(s)\|_{H^1_\alpha}\right) ds \end{aligned}$$

for $t \in [t_j, t_{j+1}]$, where C_1 is a constant independent of j. Supposing $C_1\hat\delta \leq \beta/2$, by Gronwall's inequality we infer that for $t \in [t_j, t_{j+1}]$,

$$e^{\beta t/2}\|w_j(t)\|_{H^1_\alpha} \leq K_1 e^{\beta t_j/2}\|w_j(t_j)\|_{H^1_\alpha} + C_1\int_{t_j}^t e^{\beta s/2}\|F(s)\|_{H^1_\alpha}\, ds. \tag{D.6}$$

By using (D.5) and then using (D.6) with j replaced by $j-1$, we find

$$\begin{aligned} e^{\beta t_j/2}\|w_j(t_j)\|_{H^1_\alpha} &\leq e^{\beta t_j/2}C_0^2\|w_{j-1}(t_j)\|_{H^1_\alpha} \\ &\leq C_0^2 K_1 e^{\beta t_{j-1}/2}\|w_{j-1}(t_{j-1})\|_{H^1_\alpha} + C_0^2 C_1\int_{t_{j-1}}^{t_j} e^{\beta s/2}\|F(s)\|_{H^1_\alpha}\, ds. \end{aligned} \tag{D.7}$$

Now $C_0^2 K_1 \leq e^{\beta(t_j - t_{j-1})/6}$ due to (D.2), and $e^{\beta s/2} \leq e^{\beta s/3}e^{\beta t_j/6}$ for $s \in [0, t_j]$, hence

$$\begin{aligned} e^{\beta t_j/3}\|w_j(t_j)\|_{H^1_\alpha} &\leq e^{\beta t_{j-1}/3}\|w_{j-1}(t_{j-1})\|_{H^1_\alpha} + C_0^2 C_1\int_{t_{j-1}}^{t_j} e^{\beta s/3}\|F(s)\|_{H^1_\alpha}\, ds \\ &\leq \|w_0(0)\|_{H^1_\alpha} + C_0^2 C_1\int_0^{t_j} e^{\beta s/3}\|F(s)\|_{H^1_\alpha}\, ds, \end{aligned} \tag{D.8}$$

by induction. Combining (D.8) with (D.6) and (D.5) yields the conclusion of the Lemma. □

Acknowledgments: This material is based upon work supported by the National Science Foundation under grant nos. DMS 0604420 and DMS 0905723, and partially supported by the Center for Nonlinear Analysis (CNA) under the National Science Foundation Grant no. 0635983 and PIRE Grant no. OISE-0967140. TM is supported by Grant-in-Aid for Scientific Research no. 21540220. The work of JRQ is supported by the Universidad del Valle (Cali, Colombia).

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