A Young measure approach to a nonlinear membrane model involving the bending moment

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Abstract

An integral representation of a relaxed functional arising in the derivation of a nonlinear membrane model accounting for the density of bending moments is obtained in terms of a special class of parametrized probability measures (bending Young measures). A complete characterization of this class is given.

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1 Introduction

Let $\omega \subset \mathbb{R}^2$ be an open bounded domain with Lipschitz boundary, and let I be the interval $(-1/2, 1/2)$ of $\mathbb{R}$. Define $\Omega := \omega \times I$, $\Sigma^{\pm} := \omega \times \{\pm 1/2\}, \Gamma := \partial\omega \times I$, and for $\varepsilon > 0$ set $\Omega_\varepsilon := \omega \times (-\varepsilon/2, \varepsilon/2), \Sigma_\varepsilon^{\pm} := \omega \times \{\pm\varepsilon/2\}$ and $\Gamma_\varepsilon := \partial\omega \times (-\varepsilon/2, \varepsilon/2)$.

In what follows we will regard $\Omega_\varepsilon := \omega \times (-\varepsilon/2, \varepsilon/2)$ as a thin three-dimensional body filled with an elastic material with elastic energy density W and subject to external loading. The rescaled total energy of a deformation $u : \Omega_\varepsilon \to \mathbb{R}^3$ is given by

$$\tilde{E}_\varepsilon(u) := \frac{1}{\varepsilon} \int_{\Omega_\varepsilon} W\left(\left(y_1, y_2, \frac{y_3}{\varepsilon}\right), \nabla u(y)\right) \, dy - \int_{\Omega_\varepsilon} f^\varepsilon(y) \cdot u(y) \, dy - \langle G_\varepsilon, u \rangle.$$

Here $f^\varepsilon := \frac{1}{\varepsilon} f\left(y_1, y_2, \frac{1}{\varepsilon} y_3\right)$ is the bulk density and $G_\varepsilon := g_\varepsilon^+ \mathcal{H}^2 \lfloor \Sigma_\varepsilon^+ - g_\varepsilon^- \mathcal{H}^2 \lfloor \Sigma_\varepsilon^-$ represents a surface density on $\Sigma_\varepsilon^{\pm}$, where, following [11],

$$g_\varepsilon^{\pm} := g_0^{\pm} + \frac{1}{\varepsilon} g,$$

with $f \in L^{p'}(\Omega; \mathbb{R}^3)$, $g_0^{\pm}, g \in L^{p'}(\omega; \mathbb{R}^3)$, where p' stands for the conjugate exponent of p, i.e. $\frac{1}{p} + \frac{1}{p'} = 1$, for a fixed $p \in (1, +\infty)$.

This model differs from the classical approach in membrane theory (see [2], [7], [8], [12], [13], [15], [23], [24]) through the presence of the surface load term $\frac{1}{\varepsilon} g$ which induces in the weak* limit (as $\varepsilon \to 0^+$) of the external loading a first order distribution representing a density of bending moments, in addition to the classical surface load of density $\overline{f} + g_0^+ - g_0^-$.

Let $W^{1,p}_{\Gamma_\varepsilon}(\Omega_\varepsilon;\mathbb{R}^3) := \{\varphi \in W^{1,p}(\Omega_\varepsilon;\mathbb{R}^3) : \varphi = 0 \text{ on } \Gamma_\varepsilon\}$ and $W^{1,p}_{\Gamma}(\Omega;\mathbb{R}^3) := \{\varphi \in W^{1,p}(\Omega;\mathbb{R}^3) : \varphi = 0 \text{ on } \Gamma\}$, and assume that admissible deformations of the body satisfy a boundary condition of place on Γ_ε. The equilibrium problem can then be formulated as a minimization problem

$$\inf\left\{\tilde{E}_\varepsilon(u) : u - y \in W^{1,p}_{\Gamma_\varepsilon}(\Omega_\varepsilon;\mathbb{R}^3)\right\}. \tag{1.1}$$

To study the limit problem in (1.1) as the thickness ε tends to zero, it will be useful to recast energy functionals over varying domains Ω_ε into functionals with a fixed domain of integration Ω by means of the (now standard) change of variables

$$x_1 = y_1,\ x_2 = y_2,\ x_3 = \frac{1}{\varepsilon} y_3. \tag{1.2}$$

With $v(x) := u(y(x))$, $v^\pm(x_\alpha) := v(x_\alpha, \pm\frac{1}{2})$, $v_{0,\varepsilon}(x) := (x_\alpha, \varepsilon x_3)$, we rewrite (1.1) as

$$\inf\left\{E_\varepsilon(v) : v - v_{0,\varepsilon} \in W^{1,p}_{\Gamma}(\Omega;\mathbb{R}^3)\right\}, \tag{1.3}$$

where

$$E_\varepsilon(v) := \int_\Omega W\left(x, \left(\nabla_\alpha v \Big| \frac{1}{\varepsilon}\nabla_3 v\right)\right) dx - \int_\Omega f v dx - \int_\omega (g_0^+ v^+ - g_0^- v^-) dx_\alpha - \int_\omega g\left(\frac{v^+ - v^-}{\varepsilon}\right) dx_\alpha,$$

x_α and (x_α, x_3) stand for (x_1, x_2) and (x_1, x_2, x_3), respectively, $\nabla_\alpha v$ is the 3×2 matrix of partial derivatives $\frac{\partial u_i}{\partial x_\alpha}$, $i \in \{1,2,3\}, \alpha \in \{1,2\}$, and $(A|a) \in \mathbb{M}^{3\times 3}$ denotes a 3×3 matrix whose first two columns are those of the 3×2 matrix $A \in \mathbb{M}^{3\times 2}$ and the last column is the vector $a \in \mathbb{R}^3$.

In the sequel we assume that the elastic energy density $W : \Omega \times \mathbb{M}^{3\times 3} \to \mathbb{R}$ is a Carathéodory function satisfying the p-growth and coercivity condition

$$\frac{1}{C}|A|^p - C \le W(x, A) \le C\left(1 + |A|^p\right) \tag{1.4}$$

$\mathcal{L}^3$-a.e $x \in \Omega$ and for all $A \in \mathbb{M}^{3\times 3}$, where $C > 0$ is a real constant and $|A| := \sqrt{\operatorname{tr}(A^T A)}$ is the usual Euclidean norm on the space of real 3×3 matrices. By (1.4), any minimizing sequence in (1.3) satisfies

$$\sup_\varepsilon \int_\Omega \left|\left(\nabla_\alpha v_\varepsilon \Big| \frac{1}{\varepsilon}\nabla_3 v_\varepsilon\right)\right|^p dx < +\infty,$$

and thus, up to a subsequence (not relabelled), $\{v_\varepsilon\}$ and $\left\{\frac{1}{\varepsilon}\nabla_3 v_\varepsilon\right\}$ converge weakly to some functions v and c in $W^{1,p}(\Omega;\mathbb{R}^3)$ and $L^p(\Omega;\mathbb{R}^3)$, respectively (with v depending only on x_α). In particular, $\left\{\frac{1}{\varepsilon}\int_I \nabla_3 v_\varepsilon(\cdot, x_3) dx_3\right\}$ converges weakly to b in $L^p(\omega;\mathbb{R}^3)$, where $b(x_\alpha) := \int_I c(x_\alpha, x_3) dx_3$. This implies that the work of the external loads in the expression of $E_\varepsilon(v_\varepsilon)$ converges to

$$\int_\omega \overline{f} v\ dx_\alpha + \int_\omega (g_0^+ - g_0^-) v\ dx_\alpha + \int_\omega g b\ dx_\alpha,$$

where $\overline{f}(x_\alpha) := \int_I f(x_\alpha, x_3) dx_3$. Consequently, the study of the effective total energy of the limiting system depends on the understanding of the asymptotic behavior of the elastic energies

$$I_\varepsilon := \int_\Omega W\left(x, \left(\nabla_\alpha v_\varepsilon \Big| \frac{1}{\varepsilon}\nabla_3 v_\varepsilon\right)\right) dx,$$

under the weak convergence of $\{v_\varepsilon\}$ to v in $W^{1,p}(\Omega;\mathbb{R}^3)$ and of $\left\{\frac{1}{\varepsilon}\int_I \nabla_3 v_\varepsilon(\cdot,x_3)dx_3\right\}$ to b in $L^p(\omega;\mathbb{R}^3)$, as $\varepsilon \to 0^+$.

In Section 3 of the paper we obtain an integral representation of the relaxed energy functional $E: W^{1,p}(\Omega;\mathbb{R}^3)\times L^p(\omega,\mathbb{R}^3)\to\mathbb{R}$ defined by

$$E(v,b) := \inf\Bigg\{\liminf_{n\to+\infty}\int_\Omega W\left(x,\left(\nabla_\alpha v_n\Big|\frac{1}{\varepsilon_n}\nabla_3 v_n\right)\right)dx : \varepsilon_n\to 0^+,$$
$$v_n \rightharpoonup v \text{ weakly in } W^{1,p}(\Omega;\mathbb{R}^3),\ \frac{1}{\varepsilon_n}\int_I \nabla_3 v_n(\cdot,x_3)dx_3 \rightharpoonup b \text{ weakly in } L^p(\omega,\mathbb{R}^3)\Bigg\}, \tag{1.5}$$

in terms of a suitable class of Young measures, defined as follows

Definition 1.1 *Let $\Omega := \omega\times I$ where $\omega\subset\mathbb{R}^2$ is an open domain and let $1<p<+\infty$. A Young measure $\mu := \{\mu_x\}_{x\in\Omega}$ is called a* bending Young measure *if there exist sequences $\{\varepsilon_n\}\subset(0,+\infty)$ and $\{v_n\}\subset W^{1,p}(\Omega;\mathbb{R}^3)$, with $\varepsilon_n\to 0^+$, and functions $v\in W^{1,p}(\omega;\mathbb{R}^3)$, $b\in L^p(\omega;\mathbb{R}^3)$, such that*

(i) *the sequence $\left\{\left(\nabla_\alpha v_n\big|\frac{1}{\varepsilon_n}\nabla_3 v_n\right)\right\}$ is bounded in $L^p(\Omega,\mathbb{M}^{3\times 3})$ and it generates μ;*

(ii) *$v_n \rightharpoonup v$ weakly in $W^{1,p}(\Omega;\mathbb{R}^3)$;*

(iii) *$\frac{1}{\varepsilon_n}\int_I \nabla_3 v_n(\cdot,x_3)dx_3 \rightharpoonup b$ weakly in $L^p(\omega,\mathbb{R}^3)$.*

The weak limit v of v_n in $W^{1,p}(\Omega;\mathbb{R}^3)$ is called an underlying deformation for *μ while the weak limit b of $\frac{1}{\varepsilon_n}\int_I \nabla_3 v_n(\cdot,x_3)dx_3$ in $L^p(\omega;\mathbb{R}^3)$ is called a* bending moment density associated to *μ. We denote by $Y(v,b)$ the set of bending Young measures with underlying deformation v and associated bending moment density b.*

Our relaxation result is

Theorem 1.2 *Let $1<p<+\infty$, $\Omega:=\omega\times I$ where $\omega\subset\mathbb{R}^2$ is an open bounded Lipschitz domain, and let $W:\Omega\times\mathbb{M}^{3\times 3}\to\mathbb{R}$ be a Carathéodory function satisfying the p-growth and coercivity condition (1.4) for $\mathcal{L}^3$-a.e $x\in\Omega$ and for all $A\in\mathbb{M}^{3\times 3}$. Then, for every $v\in W^{1,p}(\omega;\mathbb{R}^3)$ and $b\in L^p(\omega,\mathbb{R}^3)$,*

$$E(v,b) = \inf_{\nu\in Y(v,b)}\int_\Omega\int_{\mathbb{M}^{3\times 3}} W(x,A)d\nu_x(A)dx. \tag{1.6}$$

Moreover, the infimum on the right hand side is attained, i.e. there exists a bending Young measure $\mu\in Y(v,b)$ such that

$$\inf_{\nu\in Y(v,b)}\int_\Omega\int_{\mathbb{M}^{3\times 3}} W(x,A)d\nu_x(A)dx = \int_\Omega\int_{\mathbb{M}^{3\times 3}} W(x,A)d\mu_x(A)dx. \tag{1.7}$$

Remark 1.3 a) If we replace (iii) in Definition 1.1 by (iii′) $\frac{1}{\varepsilon_n}\nabla_3 v_n \rightharpoonup c \in L^p(\Omega;\mathbb{R}^3)$ weakly in $L^p(\Omega,\mathbb{R}^3)$,

we recover the definition of *α-gradient p-Young measures* (see [10]) when $1 < p < +\infty$. In this case the weak limit c of $\frac{1}{\varepsilon_n}\nabla_3 v_n$ in $L^p(\Omega;\mathbb{R}^3)$ is called a *Cosserat vector associated to* μ. The set of α-gradient p-Young measures with underlying deformation v and associated Cosserat vector c is denoted by $Y^{\alpha-p}_{v,c}$. Note that if we set $b := \int_I c(\cdot, x_3)dx_3$, then $Y^{\alpha-p}_{v,c} \subset Y(v,b)$.

The corresponding integral representation result (see Theorem 1.1 in [10]) for the relaxed functional $\mathcal{W} : W^{1,p}(\Omega;\mathbb{R}^3) \times L^p(\Omega,\mathbb{R}^3) \to \mathbb{R}$ defined by

$$\mathcal{W}(v,c) := \inf\Bigg\{ \liminf_{n\to+\infty} \int_\Omega W\left(x, \left(\nabla_\alpha v_n \Big| \frac{1}{\varepsilon_n}\nabla_3 v_n\right)\right) dx : \varepsilon_n \to 0^+,$$

$$v_n \rightharpoonup v \text{ weakly in } W^{1,p}(\Omega;\mathbb{R}^3),\ \tfrac{1}{\varepsilon_n}\nabla_3 v_n \rightharpoonup c \text{ weakly in } L^p(\Omega,\mathbb{R}^3)\Bigg\},$$

in terms of α-gradient p-Young measures states that there exists $\mu^0 \in Y^{\alpha-p}_{v,c}$ such that

$$\mathcal{W}(v,c) = \inf_{\nu\in Y^{\alpha-p}_{v,c}} \int_\Omega \int_{\mathbb{M}^{3\times3}} W(x,A)d\nu_x(A)dx = \int_\Omega \int_{\mathbb{M}^{3\times3}} W(x,A)d\mu^0_x(A)dx.$$

b) We refer to Freddi and Paroni [18] for another study involving Young measures in the membrane theory. Their approach is different from ours for several reasons including: (i) they derive a Γ-convergence result in the space of $W^{1,p}(\omega;\mathbb{R}^3)$-gradient Young measures, using a topology that involves a combination of projecting and transversal averaging of the sequences of measures involved. This allows them to deal exclusively with gradient Young measures and the variational limit that they obtain (see Theorem 5.7 in [18]) is defined on the space of two-dimensional gradient Young measures. In the present paper we deal with Young measures (bending Young measures; see Definition 1.1) parametrized on the full $x \in \Omega$ and not generated by sequences of gradients; (ii) an important part of the present work is dedicated to providing a complete characterization of the class of bending Young measures involved in our relaxation result (Theorem 1.2); in the case of gradient Young measures, as in [18], such a characterization already exists [20].

The main effort of the paper is the identification of algebraic and analytical conditions on parametrized probability measures both necessary and sufficient to guarantee that they belong to $Y(v,b)$ (Theorem 1.4). The characterization of Young measures generated by gradients bounded in L^p is due to Kinderlehrer and Pedregal (see [19], [20]), and it was subsequently generalized to the realm of A-quasiconvexity by Fonseca and Müller (see [16]).

Before making our main result precise, we need to introduce some notation. Set

$$\begin{aligned} p-\mathrm{Lip}(I\times\mathbb{M}^{3\times3}) := \big\{ & f : I\times\mathbb{M}^{3\times3} \to \mathbb{R} : \text{ there exists } c>0 \text{ such that } |f(x,X) - f(y,Y)| \\ & \le c(1+|X|^{p-1}+|Y|^{p-1})(|x-y|+|X-Y|) \text{ for all } (x,X),(y,Y)\in I\times\mathbb{M}^{3\times3}\big\}, \end{aligned}$$

and for every $f \in p-\mathrm{Lip}(I\times\mathbb{M}^{3\times3})$ given, let $\mathrm{Lip}(f)$ be defined by

$$\begin{aligned} \mathrm{Lip}(f) := \sup\big\{ & c>0 : |f(x,X)-f(y,Y)| \le c(1+|X|^{p-1}+|Y|^{p-1})(|x-y|+|X-Y|), \\ & \text{for all } (x,X),(y,Y)\in I\times\mathbb{M}^{3\times3}\big\}. \end{aligned}$$

Then $p - \mathrm{Lip}(I \times \mathbb{M}^{3\times 3})$, endowed with the norm

$$\|f\| := |f(0,0)| + \mathrm{Lip}(f),$$

is a Banach space. Define

$$\mathcal{E}_p := \Big\{ f \in p - \mathrm{Lip}(I \times \mathbb{M}^{3\times 3}) : \text{ the map } I \times \mathbb{M}^{3\times 3} \ni (x, X) \longmapsto \tfrac{f(x,X)}{1+|X|^p} \in \mathbb{R} \text{ admits a continuous extension to } [-\tfrac{1}{2}, \tfrac{1}{2}] \times (\mathbb{M}^{3\times 3} \cup \{+\infty\}) \Big\},$$

where $\mathbb{M}^{3\times 3} \cup \{+\infty\}$ stands for the one point compactification of $\mathbb{M}^{3\times 3}$ (function spaces of this type have been previously considered (with $p = 1$) by Kristensen [22]). Note that $(\mathcal{E}_p, \|\cdot\|)$ is a (non-separable) Banach space. As an example, the function $g : I \times \mathbb{M}^{3\times 3} \to \mathbb{R}$ defined by $g(x, X) := |X|^p$ belongs to $\mathcal{E}_p$.

Set $Q' := (0,1)^2$. For $f \in p\text{-Lip}(I \times \mathbb{M}^{3\times 3})$, $A \in \mathbb{M}^{3\times 2}$, and $a \in \mathbb{R}^3$, define the potential

$$\widehat{Q}f(A|a) := \inf_{(\varphi,\lambda)} \left\{ \int_{Q'\times I} f\left(x_3, \left(A + \nabla_\alpha \varphi \middle| \lambda \nabla_3 \varphi\right)\right) dx : \lambda \in \mathbb{R}, \varphi \in W^{1,p}(\mathbb{R}^2 \times I; \mathbb{R}^3), \right.$$
$$\left. \varphi(\cdot, x_3) \text{ is } Q' - \text{periodic for } \mathcal{L}^1 - \text{a.e. } x_3 \in I,\ \lambda \int_{Q'\times I} \nabla_3 \varphi \, dx = a \right\}. \tag{1.8}$$

We are now ready to state

Theorem 1.4 *Let $1 < p < +\infty$, $\Omega := \omega \times I$ and let $u \in W^{1,p}(\omega;\mathbb{R}^3)$ and $b \in L^p(\omega, \mathbb{R}^3)$ be given. A weak* measurable family $\nu = \{\nu_x\}_{x\in\Omega}$ of probability measures belongs to $Y(u, b)$ if and only if*

$$\int_I \int_{\mathbb{M}^{3\times 3}} \Lambda d\nu_x(\Lambda) dx_3 = \left(\nabla u(x_\alpha)\middle| b(x_\alpha)\right) \ \textit{for } \mathcal{L}^2 - \textit{a.e. } x_\alpha \in \omega, \tag{1.9}$$

$$\int_\Omega \int_{\mathbb{M}^{3\times 3}} |\Lambda|^p d\nu_x(\Lambda) dx < +\infty, \tag{1.10}$$

and

$$\int_I \int_{\mathbb{M}^{3\times 2}\times\mathbb{R}^3} f\left(x_3, \left(\Lambda - \nabla u(x_\alpha) \middle| \lambda - b(x_\alpha)\right)\right) d\nu_x(\Lambda|\lambda) dx_3 \geq \widehat{Q}f(0|0), \tag{1.11}$$

$$\textit{for } \mathcal{L}^2 - \textit{a.e. } x_\alpha \in \omega \textit{ and for all } f \in \mathcal{E}_p.$$

2 Preliminaries

Let Ω be an open subset of $\mathbb{R}^N$ $(N \geq 1)$. The Lebesgue measure in $\mathbb{R}^N$ is designated by $\mathcal{L}^N$, and H^{N-1} stands for the $(N-1)$-dimensional Hausdorff measure in $\mathbb{R}^N$. Let $C(\Omega;\mathbb{R}^d)$ be the space of $\mathbb{R}^d$- valued continuous functions on Ω and define

$$C_0(\Omega;\mathbb{R}^d) := \Big\{ \varphi \in C(\Omega;\mathbb{R}^d) : \text{ for every } \varepsilon > 0 \text{ there exists a compact set } K \subset \Omega \text{ such that } |\varphi(x)| \leq \varepsilon \text{ if } x \in \Omega \setminus K \Big\},$$

where $|\cdot|$ stands for the Euclidian norm in $\mathbb{R}^d$. Endowed with the supremum norm, $C_0(\Omega;\mathbb{R}^d)$ is a separable Banach space. In view of Riesz's Theorem, the dual space $C_0(\Omega;\mathbb{R}^d)'$ can be identified with the space $\mathcal{M}(\Omega;\mathbb{R}^d)$ of bounded $\mathbb{R}^d$- valued Radon measures on Ω with the norm $\|\mu\| := |\mu|(\Omega)$ via the duality pairing

$$\langle \mu, \varphi\rangle = \int_\Omega \varphi(x)\frac{d\mu}{d|\mu|}(x)d|\mu|(x),$$

where $|\mu|$ stands for the total variation of μ and is a non-negative, finite Radon measure on Ω. If $d = 1$ we write $C(\Omega)$ and $C_0(\Omega)$ and $\mathcal{M}(\Omega)$ instead of $C(\Omega;\mathbb{R})$, $C_0(\Omega;\mathbb{R})$, and $\mathcal{M}(\Omega;\mathbb{R})$, respectively.

We say that a sequence $\{y_j\}$ in $L^p(\Omega)$ $(1 \le p < +\infty)$ is *p-equi-integrable* (*equi-integrable* if $p = 1$) if for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\sup_{j\in\mathbb{N}} \int_E |y_j|^p dx < \varepsilon$$

whenever E is a Borel subset of Ω such that $\mathcal{L}^N(E) < \delta$.

Proposition 2.1 *Let $\Omega \subset \mathbb{R}^N$ be an open, bounded domain, and let $\{y_j\} \subset L^p(\Omega)$ $(1 \le p < +\infty)$.*

(i) (Dunford-Pettis) *The sequence $\{y_j\}$ is p-equi-integrable if and only if $\{|y_j|^p\}$ is relatively weakly compact in $L^1(\Omega)$;*

(ii) *the sequence $\{y_j\}$ is p-equi-integrable if and only if*

$$\lim_{R\to\infty} \limsup_{j\to+\infty} \int_{\{x\in\Omega:|y_j(x)|>R\}} |y_j|^p dx = 0;$$

(iii) (De la Valée-Poussin) *the sequence $\{y_j\}$ is p-equi-integrable if and only if there exists $\Theta : [0,+\infty) \to [0,+\infty)$ such that*

$$\lim_{t\to\infty} \frac{\Theta(t)}{t} = +\infty \text{ and } \sup_{j\in\mathbb{N}} \int_\Omega \Theta(|y_j|^p)dx < +\infty.$$

A map $\nu : \Omega \to \mathcal{M}(\mathbb{R}^d)$ is said to be *weak* measurable* if $x \mapsto \langle \nu(x), \varphi\rangle$ is measurable for every $\varphi \in C_0(\Omega)$. In the sequel we will denote $\nu(x)$ by ν_x.

Definition 2.2 (i) *A function $f : \Omega \times \mathbb{R}^d \to \mathbb{R}\cup\{+\infty\}$ is called a* normal integrand *if f is Borel measurable and $f(x,\cdot) : \mathbb{R}^d \to \mathbb{R}\cup\{+\infty\}$ is lower semicontinuous for every $x \in \Omega$.*

(ii) *A real-valued function $f : \Omega \times \mathbb{R}^d \to \mathbb{R}$ is called a* Carathéodory integrand *if both f and $-f$ are normal integrands.*

Theorem 2.3 (The Fundamental Theorem on Young measures, [5], [6], [28]) *Let $\Omega \subset \mathbb{R}^N$ be a measurable set of finite measure and let $\{y_j\}$ be a sequence of measurable functions, $y_j : \Omega \to \mathbb{R}^d$. Then there exist a subsequence $\{y_{j_k}\} \subset \{y_j\}$ and a weak* measurable map $\nu : \Omega \to \mathcal{M}(\mathbb{R}^d)$ such that the following hold:*

(i) $\nu_x \ge 0$, $\|\nu_x\|_{\mathcal{M}(\mathbb{R}^d)} := \int_{\mathbb{R}^d} d\nu_x \le 1$, *for $\mathcal{L}^N$ - a.e. $x \in \Omega$;*

(ii) $\|\nu_x\|_{\mathcal{M}(\mathbb{R}^d)} = 1$ *if and only if either*

$$\lim_{R\to\infty} \sup_{n\in\mathbb{N}} \mathcal{L}^N\left(\{x \in \Omega : |y_{j_k}(x)| \geq R\}\right) = 0,$$

or

$$\text{there exists a Borel function } g : \mathbb{R}^d \to [0, +\infty] \text{ such that } \lim_{|A|\to+\infty} g(A) = +\infty,$$

$$\text{and } \sup_{n\in\mathbb{N}} \int_\Omega g(y_{j_k}(x))dx < +\infty,$$

hold. In this case we say that ν is a Young measure generated by the sequence $\{y_{j_k}\}$;

(iii) *if $K \subset \mathbb{R}^d$ is compact and* $\mathrm{dist}(y_{j_k}, K) \to 0$ *in measure, then* $\mathrm{supp}\nu_x \subset K$ *for $\mathcal{L}^N$-a.e. $x \in \Omega$. The converse also holds provided that $\|\nu_x\|_{\mathcal{M}(\mathbb{R}^d)} = 1$;*

(iv) *if $f : \Omega \times \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ is a normal integrand and if $\{f^-(\cdot, y_{j_k})\}$ is equi-integrable then*

$$\int_\Omega \int_{\mathbb{R}^d} f(x, \Lambda)d\nu_x(\Lambda)dx \leq \liminf_{k\to+\infty} \int_\Omega f(x, y_{j_k}(x))dx,$$

where $f^- := -\min\{f, 0\}$;

(v) *if ν is generated by $\{y_{j_k}\}$ and if f is a Carathéodory integrand then $\{f(\cdot, y_{j_k})\}$ is equi-integrable if and only if*

$$\int_\Omega \int_{\mathbb{R}^d} f(x, \Lambda)d\nu_x(\Lambda)dx = \lim_{k\to+\infty} \int_\Omega f(x, y_{j_k}(x))dx.$$

Young measures have been originally introduced in Optimal Control Theory by L.C. Young in connection to nonconvex problems, thus providing the appropriate framework for the description of generalized minimizers in the Calculus of Variations (see [31], [32]). Later, Tartar developed the use of Young measures in the PDE framework (see [28], [29], [30]). For more details regarding the study of Young measures we refer the reader to [4], [5], [6], [17], [19], [20], [21], [22], [27], among others.

Proposition 2.4 *If a sequence $\{v_n\}$ of measurable functions $v_n : \Omega \to \mathbb{R}^d$ generates a Young measure ν and if $w_n \to w$ in measure, then the sequence $\{v_n + w_n\}$ generates the translated Young measure $\overline{\nu} = \{\nu_x\}_{x\in\Omega}$ defined by $\overline{\nu}_x := \Gamma_{w(x)}\nu_x$, where, for $a \in \mathbb{R}^d$, $\varphi \in C_0(\mathbb{R}^d)$ and $\mu \in \mathcal{M}(\mathbb{R}^d)$, we define $\langle\Gamma_a\mu, \varphi\rangle := \langle\mu, \varphi(\cdot + a)\rangle$. In particular, if $\{u_n\}$ is a sequence of measurable mappings from Ω into $\mathbb{R}^d$ such that $v_n - u_n \to 0$ in measure then $\{u_n\}$ also generates ν.*

We recall that a Borel function $f : \mathbb{M}^{m\times N} \to \mathbb{R}$ is said to be *quasiconvex* (in the sense of Morrey) if for every $A \in \mathbb{M}^{m\times N}$

$$f(A) \leq \int_{(0,1)^N} f(A + \nabla\varphi(x))dx \text{ for all } \varphi \in W_0^{1,\infty}((0,1)^N; \mathbb{R}^m). \tag{2.1}$$

It is well known (see, e.g. [14]) that in (2.1) we can replace $(0,1)^N$ by any bounded domain D of $\mathbb{R}^N$. Moreover, if f is quasiconvex then (2.1) still holds when we test with φ in the class of $(0,1)^N$-periodic functions in $W^{1,\infty}(\mathbb{R}^N; \mathbb{R}^m)$. It has been established by Morrey [25], [26] (see also

[1]) that under suitable p-growth conditions on f (no growth condition is necessary if $p = \infty$) quasiconvexity of f is necessary and sufficient for the sequential weak (weak * if $p = \infty$) lower semicontinuity in $W^{1,p}(\Omega;\mathbb{R}^m)$ of $I(u) := \int_\Omega f(\nabla u)dx$.

A Young measure ν is called a $W^{1,p}$*-gradient Young measure* if it is generated by $\{\nabla u_n\}$, with $u_n \to u$ in $W^{1,p}(\Omega;\mathbb{R}^m)$. The following characterization of $W^{1,p}$-gradient Young measures is due to Kinderlehrer and Pedregal [19], [20] (see also [16], [17], [21], [22]).

Theorem 2.5 *Let $1 \le p \le +\infty$. A weak* measurable map $\nu : \Omega \to \mathcal{M}(\mathbb{M}^{m\times N})$ is a $W^{1,p}$-gradient Young measure if and only if $\nu_x \ge 0$ for $\mathcal{L}^N$-a.e. $x \in \Omega$, and*

(i) *there exists $u \in W^{1,p}(\Omega;\mathbb{R}^m)$ such that $\langle \nu_x, id\rangle = \nabla u(x)$ for $\mathcal{L}^N$-a.e. $x \in \Omega$;*

(ii) $\int_\Omega \int_{\mathbb{M}^{m\times N}} |\Lambda|^p d\nu_x(\Lambda)dx < +\infty$ (supp $\nu_x \subset K$ *for $\mathcal{L}^N$-a.e. $x \in \Omega$, where $K \subset \mathbb{M}^{m\times N}$ is a compact set, if $p = \infty$*);

(iii) *$\langle \nu_x, f\rangle \ge f(\langle \nu_x, id\rangle)$ for $\mathcal{L}^N$-a.e. $x \in \Omega$ and for all quasiconvex functions $f : \mathbb{M}^{m\times N} \to \mathbb{R}$, with $|f(A)| \le C(1 + |A|^p)$ for some $C > 0$ and all $A \in \mathbb{M}^{m\times N}$, if $1 \le p < +\infty$.*

The following decomposition result for scaled gradients (see [9] for a proof) will play an important role in the sequel.

Theorem 2.6 *Let $\Omega := \omega \times I$, where $\omega \subset \mathbb{R}^2$ is an open, bounded Lipschitz domain, let $\{\varepsilon_n\}$ be a sequence of positive real numbers converging to zero, and let $\{v_n\}$ be a bounded sequence in $W^{1,p}(\Omega,\mathbb{R}^3)$ $(1 < p < +\infty)$ satisfying*

$$\sup_{n\in\mathbb{N}} \int_\Omega \left|\left(\nabla_\alpha v_n \Big| \frac{1}{\varepsilon_n}\nabla_3 v_n\right)\right|^p dx < +\infty.$$

Suppose further that $v_n \rightharpoonup v$ in $W^{1,p}(\Omega,\mathbb{R}^3)$ and $\frac{1}{\varepsilon_n}\nabla_3 v_n \rightharpoonup c$ in $L^p(\Omega,\mathbb{R}^3)$. Then there exists a subsequence $\{v_{n_k}\}$ of $\{v_n\}$ and a sequence $\{w_k\} \subset W^{1,\infty}(\Omega,\mathbb{R}^3)$ such that

(i) $\lim_{k\to\infty} \mathcal{L}^3\left(\{x \in \Omega : w_k(x) \ne v_{n_k}(x)\}\right) = 0$;

(ii) $\left\{\left(\nabla_\alpha w_k \big| \frac{1}{\varepsilon_{n_k}}\nabla_3 w_k\right)\right\}$ *is p-equi-integrable;*

(iii) *$w_k \rightharpoonup v$ weakly in $W^{1,p}(\Omega,\mathbb{R}^3)$;*

(iv) *$\frac{1}{\varepsilon_{n_k}}\nabla_3 w_k \rightharpoonup c$ weakly in $L^p(\Omega,\mathbb{R}^3)$.*

Remark 2.7 In view of (i) and (ii) it follows immediately that

$$\limsup_{k\to\infty} \int_\Omega \left|\left(\nabla_\alpha w_k \Big| \frac{1}{\varepsilon_{n_k}}\nabla_3 w_k\right)\right|^p dx \le \limsup_{n\to\infty} \int_\Omega \left|\left(\nabla_\alpha v_n \Big| \frac{1}{\varepsilon_n}\nabla_3 v_n\right)\right|^p dx < +\infty. \tag{2.2}$$

Remark 2.8 If we assume from the beginning that $\frac{1}{\varepsilon_n}\int_I \nabla_3 v_n(\cdot, x_3)dx_3 \rightharpoonup b \in L^p(\omega;\mathbb{R}^3)$ weakly in $L^p(\omega,\mathbb{R}^3)$ then the subsequence $\{n_k\} \subset \{n\}$ and the sequence $\{w_k\}$ can be chosen such that, in addition to (i)-(iv) of Theorem 2.6, we have $\frac{1}{\varepsilon_{n_k}}\int_I \nabla_3 w_k(\cdot,x_3)dx_3 \rightharpoonup b$ weakly in $L^p(\omega,\mathbb{R}^3)$. Moreover, we can further assume, without loss of generality, that

$$w_k = v \text{ on } \partial\omega\times I,$$

and

$$\begin{aligned}&\limsup_{k\to\infty}\int_\Omega \left|\left(\nabla_\alpha w_k\Big|\frac{1}{\varepsilon_{n_k}}\nabla_3 w_k\right)\right|^p dx\\ &\le C\left(\int_\Omega |\nabla_\alpha v|^p dx + \limsup_{n\to\infty}\int_\Omega\left|\left(\nabla_\alpha v_n\Big|\frac{1}{\varepsilon_n}\nabla_3 v_n\right)\right|^p dx\right) < +\infty, \end{aligned}\tag{2.3}$$

where C is a positive constant that only depends on p.

Indeed, for each $j\in\mathbb{N}$ define $\omega_j := \left\{x\in\omega : \text{dist }(x,\partial\omega) < \frac{1}{j}\right\}$, and consider cut-off functions $\varphi_j \in C_c^\infty(\omega;[0,1])$ such that φ_j vanishes in a neighborhood of $\partial\omega$, $\varphi_j\equiv 1$ in $\omega\backslash\omega_j$, and $\|\nabla\varphi_j\|_{L^\infty(\omega;\mathbb{R}^2)} \le Cj$, for some constant $C>1$. Extract a subsequence $\{k(j)\}$ of $\{k\}$ such that, for each $j\in\mathbb{N}$,

$$\|w_{k(j)} - v\|_{L^p(\Omega;\mathbb{R}^3)} \le \frac{1}{j^2}, \tag{2.4}$$

and define

$$\overline{w}_j(x_\alpha,x_3) := v(x_\alpha) + \varphi_j(x_\alpha)\left(w_{k(j)}(x_\alpha,x_3) - v(x_\alpha)\right).$$

It is clear that $\overline{w}_j = v$ on $\partial\omega\times I$. Since

$$\{x\in\Omega : \overline{w}_j(x)\neq v_{n_{k(j)}}(x)\} \subset (\omega_j\times I)\cup\{x\in\Omega : w_{k(j)}(x)\neq v_{n_{k(j)}}(x)\},$$

we infer that

$$\lim_{j\to\infty}\mathcal{L}^3\left(\{x\in\Omega : \overline{w}_j(x)\neq v_{n_{k(j)}}(x)\}\right) = 0.$$

Let A be a Borel subset of Ω. In view of (2.4) we have

$$\begin{aligned}&\int_A\left|\left(\nabla_\alpha\overline{w}_j\Big|\frac{1}{\varepsilon_{n_{k(j)}}}\nabla_3\overline{w}_j\right)\right|^p dx\\ &\le C\left(\int_A|\nabla_\alpha v|^p dx + j^p\int_A |w_{k(j)}-v|^p dx + \int_A\left|\left(\nabla_\alpha w_{k(j)}\Big|\frac{1}{\varepsilon_{n_{k(j)}}}\nabla_3 w_{k(j)}\right)\right|^p dx\right)\\ &\le C\left(\int_A|\nabla_\alpha v|^p dx + \left(\int_A |w_{k(j)}-v|^p dx\right)^{\frac{1}{2}} + \int_A\left|\left(\nabla_\alpha w_{k(j)}\Big|\frac{1}{\varepsilon_{n_{k(j)}}}\nabla_3 w_{k(j)}\right)\right|^p dx\right),\end{aligned}$$

which implies that the sequence $\left\{\left(\nabla_\alpha\overline{w}_j\Big|\frac{1}{\varepsilon_{n_{k(j)}}}\nabla_3\overline{w}_j\right)\right\}$ is p-equi-integrable. Note that we also have

$$\limsup_{j\to\infty}\int_\Omega \left|\left(\nabla_\alpha \overline{w}_j \Big| \frac{1}{\varepsilon_{n_{k(j)}}}\nabla_3 \overline{w}_j\right)\right|^p dx$$
$$\le C\left(\int_\Omega |\nabla_\alpha v|^p dx + \limsup_{n\to\infty}\int_\Omega \left|\left(\nabla_\alpha v_n \Big| \frac{1}{\varepsilon_n}\nabla_3 v_n\right)\right|^p dx\right) < +\infty,$$

where we have used (2.2). Next, relabel $\{k(j)\}$ by $\{k\}$ and $\{\overline{w}_j\}$ by $\{w_k\}$ and we have obtained the sequence $\{w_k\}$ with the desired properties.

3 Proof of Theorem 1.2

We will first prove that

$$E(v,b) \ge \inf_{\nu\in Y(v,b)} \int_\Omega \int_{\mathbb{M}^{3\times 3}} W(x,A)d\nu_x(A)dx. \tag{3.1}$$

If $E(v,b) = +\infty$ then there is nothing to prove. Assume now that $E(v,b) < +\infty$, and for every $k \in \mathbb{N}$ let $\{\varepsilon_n^k\} \subset (0,+\infty)$ and $\{v_n^k\} \subset W^{1,p}(\Omega;\mathbb{R}^3)$ be such that $\varepsilon_n^k \to 0^+$, $v_n^k \rightharpoonup v$ weakly in $W^{1,p}(\Omega;\mathbb{R}^3)$, $\dfrac{1}{\varepsilon_n^k}\displaystyle\int_I \nabla_3 v_n^k(\cdot,x_3)dx_3 \rightharpoonup b$ weakly in $L^p(\omega;\mathbb{R}^3)$ (as $n\to\infty$), and

$$E(v,b) \le \liminf_{n\to+\infty}\int_\Omega W\left(x,\left(\nabla_\alpha v_n^k \Big| \frac{1}{\varepsilon_n^k}\nabla_3 v_n^k\right)\right)dx < E(v,b)+\frac{1}{k}. \tag{3.2}$$

Extract an increasing subsequence $\{n(j,k)\}_j$ of $\{n\}$ so that

$$\lim_{j\to+\infty}\int_\Omega W\left(x,\left(\nabla_\alpha v_{n(j,k)}^k \Big| \frac{1}{\varepsilon_{n(j,k)}^k}\nabla_3 v_{n(j,k)}^k\right)\right)dx = \liminf_{n\to+\infty}\int_\Omega W\left(x,\left(\nabla_\alpha v_n^k \Big| \frac{1}{\varepsilon_n^k}\nabla_3 v_n^k\right)\right)dx$$
$$< E(v,b)+\tfrac{1}{k} < +\infty. \tag{3.3}$$

The coercivity condition in (1.4) guarantees that the sequence $\left\{\left(\nabla_\alpha v_{n(j,k)}^k \Big| \frac{1}{\varepsilon_{n(j,k)}^k}\nabla_3 v_{n(j,k)}^k\right)\right\}_j$ is bounded in $L^p(\Omega;\mathbb{M}^{3\times 3})$ and, consequently, we can extract a subsequence (not relabelled) generating a Young measure $\nu^{(k)} = \left\{\nu_x^{(k)}\right\}_{x\in\Omega} \in Y(v,b)$. By Theorem 2.3 (iv) and taking into account (3.3), we have

$$\int_\Omega \int_{\mathbb{M}^{3\times 3}} W(x,A)d\nu_x^{(k)}(A)dx \le \lim_{j\to+\infty}\int_\Omega W\left(x,\left(\nabla_\alpha v_{n(j,k)}^k \Big| \frac{1}{\varepsilon_{n(j,k)}^k}\nabla_3 v_{n(j,k)}^k\right)\right)dx.$$

Since $\nu^{(k)} \in Y(v,b)$, and in view of (3.2), (3.3), we obtain that

$$\inf_{\nu\in Y(v,b)}\int_\Omega \int_{\mathbb{M}^{3\times 3}} W(x,A)d\nu_x(A)dx \le E(v,b)+\frac{1}{k}.$$

Letting $k \to \infty$ we deduce (3.1).

Conversely, let $\nu \in Y(v,b)$. There exist sequences $\{\varepsilon_n\} \subset (0,+\infty)$ and $\{v_n\} \subset W^{1,p}(\Omega;\mathbb{R}^3)$ such that $\varepsilon_n \to 0^+$, $v_n \rightharpoonup v$ weakly in $W^{1,p}(\Omega;\mathbb{R}^3)$, $\displaystyle\frac{1}{\varepsilon_n}\int_I \nabla_3 v_n(\cdot,x_3)dx_3 \rightharpoonup b$ weakly in $L^p(\Omega;\mathbb{R}^3)$, and ν is generated by the sequence $\left\{\left(\nabla_\alpha v_n \big| \frac{1}{\varepsilon_n}\nabla_3 v_n\right)\right\}$, bounded in $L^p(\Omega;\mathbb{M}^{3\times 3})$. By Theorem 2.6 and in view of Remark 2.8, there exists a subsequence $\{n_k\}$ of $\{n\}$ and a sequence $\{w_k\} \subset W^{1,\infty}(\Omega,\mathbb{R}^3)$ such that

$$\lim_{k\to\infty} \mathcal{L}^3\left(\{x \in \Omega : w_k(x) \neq v_{n_k}(x)\}\right) = 0, \tag{3.4}$$

$$\left\{\left(\nabla_\alpha w_k \Big| \frac{1}{\varepsilon_{n_k}}\nabla_3 w_k\right)\right\} \text{ is } p-\text{equi}-\text{integrable}, \tag{3.5}$$

$$w_k \rightharpoonup v \text{ weakly in } W^{1,p}(\Omega,\mathbb{R}^3), \tag{3.6}$$

and

$$\frac{1}{\varepsilon_{n_k}}\int_I \nabla_3 w_k(\cdot,x_3)dx_3 \rightharpoonup b \text{ weakly in } L^p(\omega,\mathbb{R}^3). \tag{3.7}$$

Taking into account (3.4) and Proposition 2.4, we deduce that the sequence $\left\{\left(\nabla_\alpha w_k \big| \frac{1}{\varepsilon_{n_k}}\nabla_3 w_k\right)\right\}$ generates ν. Theorem 2.3 (v), together with (1.4) and (3.5), yields

$$\int_\Omega \int_{\mathbb{M}^{3\times 3}} W(x,A)d\nu_x(A)dx = \lim_{k\to+\infty}\int_\Omega W\left(x,\left(\nabla_\alpha w_k \Big| \frac{1}{\varepsilon_{n_k}}\nabla_3 w_k\right)\right)dx \geq E(v,b),$$

where the last inequality follows by (3.6) and (3.7). Passing to the infimum over all $\nu \in Y(v,b)$ we obtain that

$$E(v,b) \leq \inf_{\nu\in Y(v,b)} \int_\Omega \int_{\mathbb{M}^{3\times 3}} W(x,A)d\nu_x(A)dx,$$

which, together with (3.1), asserts (1.6). It remains to prove (1.7).

For every $n \in \mathbb{N}$, let $\{\varepsilon_{k,n}\} \subset (0,+\infty)$ and $\{v_{k,n}\} \subset W^{1,p}(\Omega;\mathbb{R}^3)$ be such that $\varepsilon_{k,n} \to 0^+$, $v_{k,n} \rightharpoonup v$ weakly in $W^{1,p}(\Omega;\mathbb{R}^3)$, $\displaystyle\frac{1}{\varepsilon_{k,n}}\int_I \nabla_3 v_{k,n}(\cdot,x_3)dx_3 \rightharpoonup b$ weakly in $L^p(\omega;\mathbb{R}^3)$ (as $k \to +\infty$), and

$$E(v,b) \leq \liminf_{k\to+\infty}\int_\Omega W\left(x,\left(\nabla_\alpha v_{k,n} \Big| \frac{1}{\varepsilon_{k,n}}\nabla_3 v_{k,n}\right)\right)dx \leq E(v,b) + \frac{1}{n}.$$

Extract an increasing subsequence $\{k(j,n)\}_j$ of $\{k\}$ so that

$$\begin{aligned}&\liminf_{k\to+\infty}\int_\Omega W\left(x,\left(\nabla_\alpha v_{k,n} \Big| \frac{1}{\varepsilon_{k,n}}\nabla_3 v_{k,n}\right)\right)dx \\ &= \lim_{j\to+\infty}\int_\Omega W\left(x,\left(\nabla_\alpha v_{k(j,n),n} \Big| \frac{1}{\varepsilon_{k(j,n),n}}\nabla_3 v_{k(j,n),n}\right)\right)dx.\end{aligned}$$

A diagonalization procedure (see [10] for details in a similar context) yields sequences $\varepsilon_n \to 0^+$ and $\{v_n\} \subset W^{1,p}(\Omega;\mathbb{R}^3)$ such that $v_n \rightharpoonup v$ weakly in $W^{1,p}(\Omega;\mathbb{R}^3)$, $\frac{1}{\varepsilon_n}\int_I \nabla_3 v_n(\cdot,x_3)dx_3 \rightharpoonup b$ weakly in $L^p(\omega;\mathbb{R}^3)$, and

$$\lim_{n\to+\infty}\int_\Omega W\left(x,\left(\nabla_\alpha v_n\Big|\frac{1}{\varepsilon_n}\nabla_3 v_n\right)\right)dx = E(v,b). \tag{3.8}$$

Since we may assume without loss of generality that $E(v,b) < +\infty$, the coercivity condition in (1.4) implies that the sequence $\left\{\left(\nabla_\alpha v_n\Big|\frac{1}{\varepsilon_n}\nabla_3 v_n\right)\right\}$ is bounded in $L^p(\Omega;\mathbb{M}^{3\times3})$. Let $\mu \in Y(v,b)$ be the Young measure generated by a subsequence (not relabelled) of $\left\{\left(\nabla_\alpha v_n\Big|\frac{1}{\varepsilon_n}\nabla_3 v_n\right)\right\}$ and let $\{n_k\} \subset \{n\}$ and $\{w_k\} \subset W^{1,\infty}(\Omega,\mathbb{R}^3)$ be the sequences provided by Theorem 2.6 (again, we need to take into account Remark 2.8), so that (3.4)-(3.7) hold. In view of Theorem 2.3 we have that

$$\begin{aligned} E(v,b) &\le \liminf_{k\to+\infty}\int_\Omega W\left(x,\left(\nabla_\alpha w_k\Big|\frac{1}{\varepsilon_{n_k}}\nabla_3 w_k\right)\right)dx = \int_\Omega\int_{\mathbb{M}^{3\times3}} W(A)d\mu_x(A)dx \\ &\le \lim_{n\to+\infty}\int_\Omega W\left(x,\left(\nabla_\alpha v_n\Big|\frac{1}{\varepsilon_n}\nabla_3 v_n\right)\right)dx = E(v,b), \end{aligned}$$

where we have used (1.4), (3.4)-(3.7) and (3.8). Thus,

$$\int_\Omega\int_{\mathbb{M}^{3\times3}} W(A)d\mu_x(A)dx = E(v,b),$$

and, in view of (1.6), we infer (1.7).

4 Characterization of bending Young measures

We devote this section to the proof of Theorem 1.4.

4.1 Necessity

Let $\nu \in Y(u,b)$ and consider sequences $\varepsilon_n \to 0^+$ and $\{v_n\} \subset W^{1,p}(\Omega;\mathbb{R}^3)$, such that the sequence $\left\{\left(\nabla_\alpha v_n\Big|\frac{1}{\varepsilon_n}\nabla_3 v_n\right)\right\}$ is bounded in $L^p(\Omega,\mathbb{M}^{3\times3})$, it generates ν, $v_n \rightharpoonup u$ weakly in $W^{1,p}(\Omega;\mathbb{R}^3)$, and $\frac{1}{\varepsilon_n}\int_I \nabla_3 v_n(\cdot,x_3)dx_3 \rightharpoonup b$ weakly in $L^p(\omega,\mathbb{R}^3)$. In view of Theorem 2.6 and Remark 2.8 we can assume, in addition and without loss of generality, that the sequence $\left\{\left(\nabla_\alpha v_n\Big|\frac{1}{\varepsilon_n}\nabla_3 v_n\right)\right\}$ is p-equi-integrable. Let A be any open subset of ω. By Theorem 2.3 (v) we obtain, in matrix notation,

$$\lim_{n\to+\infty}\int_{A\times I}\left(\nabla_\alpha v_n\Big|\frac{1}{\varepsilon_n}\nabla_3 v_n\right)dx = \int_A\int_I\int_{\mathbb{M}^{3\times3}} \Lambda d\nu_x(\Lambda)dx_3dx_\alpha.$$

On the other hand, the weak convergence gives

$$\lim_{n\to+\infty}\int_{A\times I}\left(\nabla_\alpha v_n\Big|\frac{1}{\varepsilon_n}\nabla_3 v_n\right)dx = \int_A \left(\nabla u(x_\alpha)\big|b(x_\alpha)\right)dx_\alpha,$$

and (1.9) may be obtained by a simple localization argument via Lebesgue's Differentiation Theorem. Note that (1.10) folows by direct application of Theorem 2.3 (iv) (take $f = |\cdot|^p$). It remains to prove (1.11).

The following lower semicontinuity result will be instrumental in our approach. Its proof uses a recent result of Babadjian and Francfort [3]; for the convenience of the reader, we have included a self-contained proof in Section 6 of the paper.

Lemma 4.1 *Let $f \in \mathcal{E}_p$ and let $\left\{\left(\nabla_\alpha u_n \middle| \frac{1}{\varepsilon_n}\nabla_3 u_n\right)\right\}$ be a p-equi-integrable sequence such that $\varepsilon_n \to 0^+$, $u_n \rightharpoonup 0$ weakly in $W^{1,p}(\Omega;\mathbb{R}^3)$, and $\displaystyle\frac{1}{\varepsilon_n}\int_I \nabla_3 u_n(\cdot, x_3)dx_3 \rightharpoonup 0$ weakly in $L^p(\omega,\mathbb{R}^3)$. Then*

$$\widehat{Q}f(0|0) \le \frac{1}{\mathcal{L}^2(A)}\liminf_{n\to+\infty}\int_{A\times I} f\left(x_3, \left(\nabla_\alpha u_n \middle| \frac{1}{\varepsilon_n}\nabla_3 u_n\right)\right)dx, \tag{4.1}$$

for all open subsets $A \subset \omega$.

Proof. We start by observing that since $f \in \mathcal{E}_p$, there exists a constant $C > 0$ such that

$$|f(x_3, X)| \le C(1 + |X|^p), \text{ for all } (x_3, X) \in I \times \mathbb{M}^{3\times3} \tag{4.2}$$

and

$$|f(x_3, X) - f(x_3, X')| \le C(1 + |X|^{p-1} + |X'|^{p-1})|X - X'|, \text{ for all } x_3 \in I \text{ and } X, X' \in \mathbb{M}^{3\times3}. \tag{4.3}$$

For $\delta,\ M > 0$ define $f_{\delta,M} := I \times \mathbb{M}^{3\times3} \to \mathbb{R}$ by $f_{\delta,M}(x_3, A) := f_M(x_3, A) + \delta|A|^p$, where $f_M := \max\{f, -M\}$. Then, $f_{\delta,M} \in p-\mathrm{Lip}(\mathrm{I}\times\mathbb{M}^{3\times3})$ and it satisfies

$$(1/C_{\delta,M})|A|^p - C_{\delta,M} \le f_{\delta,M}(x_3, A) \le C_{\delta,M}(1 + |A|^p), \tag{4.4}$$

for $\mathcal{L}^1$-a.e. $x_3 \in I$ and for all $A \in \mathbb{M}^{3\times3}$, where $C_{\delta,M} > 0$ is a real constant. Define $W_{\delta,M} : (-1,1)\times\mathbb{M}^{3\times3} \to \mathbb{R}$ by $W_{\delta,M}(x_3, A) := f_{\delta,M}(x_3/2, A)$.

Let A be an arbitrary open subset of ω. In view of (4.4), we may apply Lemma 3.6 of [3] (with $u \equiv 0,\ \bar{b} \equiv 0$, and $W = W_{\delta,M}$) to obtain, after a change of variables,

$$Q^*W_{\delta,M}(0|0) \le \frac{1}{\mathcal{L}^2(A)}\liminf_{n\to+\infty}\int_{A\times I} f_{\delta,M}\left(x_3, \left(\nabla_\alpha u_n \middle| \frac{1}{\varepsilon_n}\nabla_3 u_n\right)\right)dx, \tag{4.5}$$

where, following the definition (3.6) in [3], for $A \in \mathbb{M}^{3\times2},\ a \in \mathbb{R}^3$,

$$\begin{aligned} Q^*W_{\delta,M}(A|a) := \inf_{(\varphi,L)}\Bigg\{ &\frac{1}{2}\int_{Q'\times(-1,1)} QW_{\delta,M}\left(x_3, \left(A + \nabla_\alpha\varphi \middle| L\nabla_3\varphi\right)\right)dx : L > 0, \\ &\varphi \in W^{1,p}(Q'\times(-1,1);\mathbb{R}^3), \varphi(\cdot, x_3)\ Q' - \text{periodic for } \mathcal{L}^1 - \text{a.e. } x_3 \in (-1,1), \\ &\frac{L}{2}\int_{Q'\times(-1,1)} \nabla_3\varphi\, dx = a\Bigg\}, \end{aligned}$$

and where for $\mathcal{L}^1$-a.e $x_3 \in (-1,1)$, $QW_{\delta,M}(x_3,\cdot)$ denotes the quasiconvex envelope of $W_{\delta,M}(x_3,\cdot)$ defined by

$$QW_{\delta,M}(x_3, F) := \inf\left\{\int_{(0,1)^3} W_{\delta,M}(x_3, F + \nabla\Phi(y))\,dy\ :\ \Phi \in W_0^{1,p}\left((0,1)^3;\mathbb{R}^3\right)\right\} \text{ for } F \in \mathbb{M}^{3\times3}.$$

We claim that

$$\widehat{Q}f(0|0) \leq Q^*W_{\delta,M}(0|0), \text{ for all } \delta, M > 0. \tag{4.6}$$

To prove (4.6), for each $j \in \mathbb{N}$ let (φ_j, L_j) be admissible for the definition of $Q^*W_{\delta,M}(0|0)$ and such that

$$Q^*W_{\delta,M}(0|0) \geq \frac{1}{2} \int_{Q'\times(-1,1)} QW_{\delta,M}\left(x_3, \left(\nabla_\alpha \varphi_j \middle| L_j \nabla_3 \varphi_j\right)\right) dx - \frac{1}{j} \tag{4.7}$$

In view of (4.4), standard relaxation principles allow us to construct sequences $\{\varphi_{j,n}\}_n \subset W^{1,p}(Q' \times (-1,1); \mathbb{R}^{3\times 3})$ with $\varphi_{j,n} \rightharpoonup \varphi_j$ weakly in $W^{1,p}(Q' \times (-1,1); \mathbb{R}^{3\times 3})$ (as $n \to +\infty$), $\varphi(\cdot, x_3)$ Q'-periodic for $\mathcal{L}^1$-a.e. $x_3 \in (-1,1)$ and such that

$$\begin{aligned} &\int_{Q'\times(-1,1)} QW_{\delta,M}\left(x_3, \left(\nabla_\alpha \varphi_j \middle| L_j \nabla_3 \varphi_j\right)\right) dx \\ &= \lim_{n\to+\infty} \int_{Q'\times(-1,1)} W_{\delta,M}\left(x_3, \left(\nabla_\alpha \varphi_{j,n} \middle| L_j \nabla_3 \varphi_{j,n}\right)\right) dx, \text{ for every } j \in \mathbb{N}. \end{aligned} \tag{4.8}$$

In particular, and in view of the choice of the pair (φ_j, L_j), we have

$$\lim_{n\to+\infty} \frac{L_j}{2} \int_{Q'\times(-1,1)} \nabla_3 \varphi_{j,n} dx = \frac{L_j}{2} \int_{Q'\times(-1,1)} \nabla_3 \varphi_j dx = 0 \quad (j \in \mathbb{N}). \tag{4.9}$$

Define $\Psi_{j,n} \in W^{1,p}(Q' \times I; \mathbb{R}^{3\times 3})$ by

$$\Psi_{j,n}(x_\alpha, x_3) := \varphi_{j,n}(x_\alpha, 2x_3) - \left(\int_{Q'\times(-1,1)} \nabla_3 \varphi_{j,n} dy \right) x_3.$$

By (4.7), (4.8), (4.9), changing variables, taking into account the p-Lipschitz property of $f_{\delta,M}$ and using Hölder's inequality we obtain

$$\begin{aligned} Q^*W_{\delta,M}(0|0) &\geq \frac{1}{2} \lim_{n\to+\infty} \int_{Q'\times(-1,1)} W_{\delta,M}\left(x_3, \left(\nabla_\alpha \varphi_{j,n} \middle| L_j \nabla_3 \varphi_{j,n}\right)\right) dx - \frac{1}{j} \\ &= \lim_{n\to+\infty} \int_{Q'\times I} W_{\delta,M}\left(2x_3, \left(\nabla_\alpha \varphi_{j,n} \middle| L_j \nabla_3 \varphi_{j,n}\right)(x_\alpha, 2x_3)\right) dx - \frac{1}{j} \\ &= \lim_{n\to+\infty} \int_{Q'\times I} f_{\delta,M}\left(x_3, \left(\nabla_\alpha \varphi_{j,n} \middle| L_j \nabla_3 \varphi_{j,n}\right)(x_\alpha, 2x_3)\right) dx - \frac{1}{j} \\ &\geq \liminf_{n\to+\infty} \int_{Q'\times I} f_{\delta,M}\left(x_3, \left(\nabla_\alpha \Psi_{j,n} \middle| \frac{L_j}{2} \nabla_3 \Psi_{j,n}\right)(x_\alpha, x_3)\right) dx - \frac{1}{j} \\ &\geq \widehat{Q}f_{\delta,M}(0|0) - \frac{1}{j}, \end{aligned}$$

where the last inequality follows from the fact that for each $j \in \mathbb{N}$ the pair $(\Psi_{j,n}, L_j/2)$ is admissible for the definition of $\widehat{Q}f_{\delta,M}(0|0)$. Since $f \leq f_{\delta,M}$, (4.6) now follows by letting $j \to +\infty$.

In view of (4.5) and (4.6) we have that

$$\widehat{Q}f(0|0) \leq \frac{1}{\mathcal{L}^2(A)} \liminf_{n\to+\infty} \int_{A\times I} f_{\delta,M}\left(x_3, \left(\nabla_\alpha u_n \Big| \frac{1}{\varepsilon_n}\nabla_3 u_n\right)\right) dx. \tag{4.10}$$

Recalling the definition of $f_{\delta,M}$, we obtain that

$$\begin{aligned}&\liminf_{n\to+\infty} \int_{A\times I} f_{\delta,M}\left(x_3, \left(\nabla_\alpha u_n \Big| \frac{1}{\varepsilon_n}\nabla_3 u_n\right)\right) dx \leq \\ &\liminf_{n\to+\infty} \int_{A\times I} f_M\left(x_3, \left(\nabla_\alpha u_n \Big| \frac{1}{\varepsilon_n}\nabla_3 u_n\right)\right) dx + \delta \limsup_{n\to+\infty} \int_{Q'\times I} \left|\nabla_\alpha u_n \Big| \frac{1}{\varepsilon_n}\nabla_3 u_n\right|^p dx.\end{aligned} \tag{4.11}$$

On the other hand,

$$\begin{aligned}&\liminf_{n\to+\infty} \int_{A\times I} f_M\left(x_3, \left(\nabla_\alpha u_n \Big| \frac{1}{\varepsilon_n}\nabla_3 u_n\right)\right) dx \\ &\leq \liminf_{n\to+\infty} \int_{A\times I} f\left(x_3, \left(\nabla_\alpha u_n \Big| \frac{1}{\varepsilon_n}\nabla_3 u_n\right)\right) dx - \liminf_{n\to+\infty} \int_{A(n,M)} f\left(x_3, \left(\nabla_\alpha u_n \Big| \frac{1}{\varepsilon_n}\nabla_3 u_n\right)\right) dx,\end{aligned} \tag{4.12}$$

where the subset $\left\{(x_\alpha, x_3) \in A\times I : f\left(x_3, \left(\nabla_\alpha u_n \Big| \frac{1}{\varepsilon_n}\nabla_3 u_n\right)(x_\alpha, x_3)\right) < -M\right\}$ of Ω has been denoted by $A(n, M)$. Combining (4.10)-(4.12) we may let $\delta \to 0^+$ to find

$$\begin{aligned}\widehat{Q}f(0|0) \leq &\frac{1}{\mathcal{L}^2(A)} \liminf_{n\to+\infty} \int_{A\times I} f\left(x_3, \left(\nabla_\alpha u_n \Big| \frac{1}{\varepsilon_n}\nabla_3 u_n\right)\right) dx \\ &- \frac{1}{\mathcal{L}^2(A)} \liminf_{n\to+\infty} \int_{A(n,M)} f\left(x_3, \left(\nabla_\alpha u_n \Big| \frac{1}{\varepsilon_n}\nabla_3 u_n\right)\right) dx.\end{aligned} \tag{4.13}$$

In view of (4.2), we have that

$$A(n,M) \subset \left\{x \in A\times I : \left|\left(\nabla_\alpha u_n \Big| \frac{1}{\varepsilon_n}\nabla_3 u_n\right)(x)\right| > \left(\frac{M}{C} - 1\right)^{1/p}\right\} =: B(n,M)$$

and

$$\begin{aligned}&\limsup_{M\to+\infty} \liminf_{n\to+\infty} \int_{A(n,M)} f\left(x_3, \left(\nabla_\alpha u_n \Big| \frac{1}{\varepsilon_n}\nabla_3 u_n\right)\right) dx \\ &\geq -C \liminf_{M\to+\infty} \limsup_{n\to+\infty} \int_{A(n,M)} \left(1 + \left|\left(\nabla_\alpha u_n \Big| \frac{1}{\varepsilon_n}\nabla_3 u_n\right)\right|^p\right) dx \\ &\geq -C \liminf_{M\to+\infty} \limsup_{n\to+\infty} \int_{B(n,M)} \left(1 + \left|\left(\nabla_\alpha u_n \Big| \frac{1}{\varepsilon_n}\nabla_3 u_n\right)\right|^p\right) dx = 0,\end{aligned} \tag{4.14}$$

where the last equality follows from the p-equi-integrability of $\left\{\left(\nabla_\alpha u_n \Big| \frac{1}{\varepsilon_n}\nabla_3 u_n\right)\right\}$ (via Theorem 2.1 (ii)). We may now deduce (4.1) by passing to the $\liminf$ as $M \to +\infty$ in (4.13). $\square$

We now return to the proof of (1.11). Let $\{\rho_k\}$ be a sequence of standard mollifiers in $\mathbb{R}^2$ and, for each $k \in \mathbb{N}$, let $b_k := \rho_k * b$ (the convolution of ρ_k and b), so that each b_k is smooth and $b_k \to b$ strongly in $L^p(\omega;\mathbb{R}^3)$. Define

$$U_{n,k}(x_\alpha, x_3) := v_n(x_\alpha, x_3) - u(x_\alpha) - \varepsilon_n x_3 b_k(x_\alpha).$$

For each $k \in \mathbb{N}$, $U_{n,k} \rightharpoonup 0$ weakly in $W^{1,p}(\Omega;\mathbb{R}^3)$ and $\displaystyle\frac{1}{\varepsilon_n}\int_I \nabla_3 U_{n,k}(\cdot, x_3)dx_3 \rightharpoonup b - b_k$ weakly in $L^p(\omega,\mathbb{R}^3)$, as $n \to \infty$. Since we also have

$$\lim_{k\to\infty}\lim_{n\to\infty}\int_\Omega |\varepsilon_n x_3 \nabla b_k(x_\alpha)|^p dx = 0 \text{ and } \limsup_{k\to\infty}\limsup_{n\to\infty}\int_\Omega \left|\left(\nabla_\alpha U_{n,k}\Big|\frac{1}{\varepsilon_n}\nabla_3 U_{n,k}\right)\right|^p dx < +\infty,$$

we may use a diagonalization process (e.g., see Lemma 7.2 in [12]) to extract an incresing subsequence $\{k(n)\}$ of $\{k\}$ such that, upon letting $w_n := U_{k(n),n}$ $(n \in \mathbb{N})$, $\left\{\left(\nabla_\alpha w_n\Big|\frac{1}{\varepsilon_n}\nabla_3 w_n\right)\right\}$ is bounded in $L^p(\Omega;\mathbb{R}^3)$, $w_n \rightharpoonup 0$ weakly in $W^{1,p}(\Omega;\mathbb{R}^3)$, $\displaystyle\frac{1}{\varepsilon_n}\int_I \nabla_3 w_n(\cdot, x_3)dx_3 \rightharpoonup 0$ weakly in $L^p(\omega,\mathbb{R}^3)$, and

$$\lim_{n\to\infty}\int_\Omega |\varepsilon_n x_3 \nabla b_{k(n)}(x_\alpha)|^p dx = 0. \tag{4.15}$$

In view of the p-equi-integrability of $\left\{\left(\nabla_\alpha v_n\Big|\frac{1}{\varepsilon_n}\nabla_3 v_n\right)\right\}$ and taking into account (4.15), it can be easily deduced that the sequence $\left\{\left(\nabla_\alpha w_n\Big|\frac{1}{\varepsilon_n}\nabla_3 w_n\right)\right\}$ is in fact p-equi-integrable.

Let $f \in \mathcal{E}_p$, let A be any open subset of ω, and note that we are now in the position to apply Lemma 4.1 with w_n in the role of u_n. It follows that

$$\widehat{Q}f(0|0) \le \frac{1}{\mathcal{L}^2(A)}\liminf_{n\to+\infty}\int_{A\times I} f\left(x_3, \left(\nabla_\alpha w_n\Big|\frac{1}{\varepsilon_n}\nabla_3 w_n\right)\right)dx. \tag{4.16}$$

In view of (4.3) we obtain

$$\begin{aligned}
&\int_{A\times I} f\left(x_3, \left(\nabla_\alpha v_n(x) - \nabla u(x_\alpha)\Big|\frac{1}{\varepsilon_n}\nabla_3 v_n(x) - b(x_\alpha)\right)\right)dx \ge \int_{A\times I} f\left(x_3, \left(\nabla_\alpha w_n\Big|\frac{1}{\varepsilon_n}\nabla_3 w_n\right)\right)dx \\
&- C\int_{A\times I}\left(1 + \left|\left(\nabla_\alpha v_n(x) - \nabla u(x_\alpha)\Big|\frac{1}{\varepsilon_n}\nabla_3 v_n(x) - b(x_\alpha)\right)\right|^{p-1}\right. \\
&\qquad\left. + \left|\left(\nabla_\alpha w_n(x)\Big|\frac{1}{\varepsilon_n}\nabla_3 w_n(x)\right)\right|^{p-1}\right)\left|\left(\varepsilon_n x_3 \nabla b_{k(n)}(x_\alpha)\big|b(x_\alpha) - b_{k(n)}(x_\alpha)\right)\right| dx.
\end{aligned}$$

By Hölder's inequality, together with (4.15) and the fact that $b_k \to b$ strongly in $L^p(A \times I;\mathbb{R}^3)$, we deduce that the last integral converges to zero as $n \to \infty$. Thus,

$$\begin{aligned}
&\liminf_{n\to\infty}\int_{A\times I} f\left(x_3, \left(\nabla_\alpha v_n(x) - \nabla u(x_\alpha)\Big|\frac{1}{\varepsilon_n}\nabla_3 v_n(x) - b(x_\alpha)\right)\right)dx \\
&\ge \liminf_{n\to\infty}\int_{A\times I} f\left(x_3, \left(\nabla_\alpha w_n\Big|\frac{1}{\varepsilon_n}\nabla_3 w_n\right)\right)dx.
\end{aligned} \tag{4.17}$$

Let $g : \Omega \times \mathbb{M}^{3\times 3} \to \mathbb{R}$ be the Carathéodory integrand defined by

$$g(x, (\Lambda|\lambda)) := f(x_3, (\Lambda - \nabla u(x_\alpha)|\lambda - b(x_\alpha))).$$

Since $\left\{g^-\left(\cdot, \left(\nabla_\alpha v_n\middle|\frac{1}{\varepsilon_n}\nabla_3 v_n\right)\right)\right\}$ and $\left\{g\left(\cdot, \left(\nabla_\alpha v_n\middle|\frac{1}{\varepsilon_n}\nabla_3 v_n\right)\right)\right\}$ are equi-integrable we have that

$$\begin{aligned}&\lim_{n\to\infty}\int_{A\times I} f\left(x_3, \left(\nabla_\alpha v_n(x) - \nabla u(x_\alpha)\middle|\frac{1}{\varepsilon_n}\nabla_3 v_n(x) - b(x_\alpha)\right)\right) dx\\ &= \int_{A\times I}\int_{\mathbb{M}^{3\times 2}\times\mathbb{R}^3} f(x_3, (\Lambda - \nabla u(x_\alpha)|\lambda - b(x_\alpha)))d\nu_x(\Lambda|\lambda)dx.\end{aligned}$$

Taking into account (4.16) and (4.17), we deduce that

$$\widehat{Q}f(0|0) \le \frac{1}{\mathcal{L}^2(A)}\int_A\left(\int_I\int_{\mathbb{M}^{3\times 2}\times\mathbb{R}^3} f\left(x_3, \left(\Lambda - \nabla u(x_\alpha)\middle|\lambda - b(x_\alpha)\right)\right) d\nu_x(\Lambda|\lambda)dx_3\right) dx_\alpha,$$

and we conclude that (1.11) holds (via Lebesgue's Differentiation Theorem).

4.2 Sufficiency

We now turn to the proof of the sufficiency part of Theorem 1.4.

Let $\nu = \{\nu_x\}_{x\in\Omega}$ be a weak* measurable family of probability measures and let $u \in W^{1,p}(\omega;\mathbb{R}^3)$ and $b \in L^p(\omega, \mathbb{R}^3)$ be such that (1.9), (1.10), and (1.11) hold. For $x := (x_\alpha, x_3) \in \Omega$, set $\mu_x := \Gamma_{-(\nabla u(x_\alpha)|b(x_\alpha))}\nu_x$ (see Proposition 2.4). We have by (1.9)

$$\begin{aligned}&\int_I\int_{\mathbb{M}^{3\times 3}} \Lambda d\mu_x(\Lambda)dx_3 = \int_I\int_{\mathbb{M}^{3\times 3}} (\Lambda - (\nabla u(x_\alpha)|b(x_\alpha)))d\nu_x(\Lambda)dx_3\\ &= \int_I\int_{\mathbb{M}^{3\times 3}} \Lambda d\nu_x(\Lambda)dx_3 - (\nabla u(x_\alpha)|b(x_\alpha)) = 0,\end{aligned}$$

for $\mathcal{L}^2$-a.e. $x_\alpha \in \omega$, and using (1.10)

$$\begin{aligned}&\int_\Omega\int_{\mathbb{M}^{3\times 3}} |\Lambda|^p d\mu_x(\Lambda)dx = \int_\Omega\int_{\mathbb{M}^{3\times 3}} |\Lambda - (\nabla u(x_\alpha)|b(x_\alpha))|^p d\nu_x(\Lambda)dx\\ &\le 2^{p-1}\left(\int_\Omega\int_{\mathbb{M}^{3\times 3}} |\Lambda|^p d\nu_x(\Lambda)dx + \int_\omega |(\nabla u(x_\alpha)|b(x_\alpha))|^p dx_\alpha\right) < +\infty.\end{aligned}$$

If $f \in \mathcal{E}_p$ then by (1.11)

$$\begin{aligned}&\int_I\int_{\mathbb{M}^{3\times 2}\times\mathbb{R}^3} f(x_3, (\Lambda|\lambda))d\mu_x(\Lambda|\lambda)dx_3\\ &= \int_I\int_{\mathbb{M}^{3\times 2}\times\mathbb{R}^3} f\left(x_3, (\Lambda - \nabla u(x_\alpha)|\lambda - b(x_\alpha))\right) d\nu_x(\Lambda|\lambda)dx_3 \ge \widehat{Q}f(0|0),\end{aligned}$$

for $\mathcal{L}^2$-a.e. $x_\alpha \in \omega$.

Define

$$\mathbb{X} := \Big\{ \nu : \Omega \to \mathcal{P}(\mathbb{M}^{3\times 3}) : \ \nu \text{ is weak} * \text{ measurable}, \int_\Omega \int_{\mathbb{M}^{3\times 3}} |\Lambda|^p d\nu_x(\Lambda) dx < +\infty,$$
$$\text{and} \int_I \int_{\mathbb{M}^{3\times 3}} \Lambda d\nu_x(\Lambda) dx_3 = 0 \text{ for } \mathcal{L}^2 - \text{a.e. } x_\alpha \in \omega \Big\},$$

$$\mathbb{Y} := \Big\{ \nu \in \mathbb{X} : \nu \text{ is a Young measure generated by a } p\text{–equi} - \text{integrable sequence}$$
$$\left\{ \left(\nabla_\alpha u_n \Big| \tfrac{1}{\varepsilon_n} \nabla_3 u_n \right) \right\}, \text{ with } \varepsilon_n \to 0^+, u_n \rightharpoonup 0 \text{ weakly in } W^{1,p}(\Omega;\mathbb{R}^3),$$
$$\text{and } \frac{1}{\varepsilon_n} \int_I \nabla_3 u_n(\cdot, x_3) dx_3 \rightharpoonup 0 \text{ weakly in } L^p(\omega, \mathbb{R}^3) \Big\},$$

and

$$\mathbb{T} := \left\{ \nu \in \mathbb{X} : \int_I \int_{\mathbb{M}^{3\times 3}} f(x_3, \Lambda) d\nu_x(\Lambda) dx_3 \geq \widehat{Q} f(0|0) \text{ for } \mathcal{L}^2 - \text{a.e. } x_\alpha \in \omega \ \text{ and all } f \in \mathcal{E}_p \right\}.$$

Proposition 4.2 *We have*

$$\mathbb{T} \subset \mathbb{Y}. \tag{4.18}$$

We postpone the proof of Proposition 4.2 for Section 5 of the paper.

Assuming that (4.18) holds, we deduce that $\mu = \{\mu_x\}_{x\in\Omega} \in \mathbb{Y}$ and, consequently, μ is generated by a p-equi-integrable sequence $\left\{ \left(\nabla_\alpha u_n \Big| \frac{1}{\varepsilon_n} \nabla_3 u_n \right) \right\}$, with $\varepsilon_n \to 0^+$, $u_n \rightharpoonup 0$ weakly in $W^{1,p}(\Omega;\mathbb{R}^3)$, and $\dfrac{1}{\varepsilon_n} \displaystyle\int_I \nabla_3 u_n(\cdot, x_3) dx_3 \rightharpoonup 0$ weakly in $L^p(\omega, \mathbb{R}^3)$. Note that for $x := (x_\alpha, x_3) \in \Omega$, we have that $\nu_x = \Gamma_{(\nabla u(x_\alpha)|b(x_\alpha))} \mu_x$ and, in view of Proposition 2.4, we deduce that ν is generated by $\left\{ \left(\nabla_\alpha u_n + \nabla u \Big| \frac{1}{\varepsilon_n} \nabla_3 u_n + b \right) \right\}$.

Assume, without loss of generality, that b is smooth (otherwise we may approximate b in $L^p(\omega;\mathbb{R}^3)$ by a sequence of smooth functions (using a sequence of standard mollifiers) and continue after first performing a diagonalization procedure similar to the one used to prove (1.11) in the necessity part of this section). Define $v_n \in W^{1,p}(\Omega;\mathbb{R}^3)$ by

$$v_n(x_\alpha, x_3) := u_n(x_\alpha, x_3) + u(x_\alpha) + \varepsilon_n x_3 b(x_\alpha).$$

We have that $v_n \rightharpoonup u$ weakly in $W^{1,p}(\Omega;\mathbb{R}^3)$ and $\dfrac{1}{\varepsilon_n} \displaystyle\int_I \nabla_3 v_n(\cdot, x_3) dx_3 \rightharpoonup b$ weakly in $L^p(\omega, \mathbb{R}^3)$.

Moreover, since

$$\left\| \left(\nabla_\alpha v_n \Big| \frac{1}{\varepsilon_n} \nabla_3 v_n \right) - \left(\nabla_\alpha u_n + \nabla u \Big| \frac{1}{\varepsilon_n} \nabla_3 u_n + b \right) \right\|^p_{L^p(\Omega, \mathbb{M}^{3\times 3})} = \int_\Omega |\varepsilon_n x_3 \nabla b(x_\alpha)|^p dx \to 0$$

as $n \to +\infty$, the sequences $\left\{\left(\nabla_\alpha v_n \middle| \frac{1}{\varepsilon_n}\nabla_3 v_n\right)\right\}$ and $\left\{\left(\nabla_\alpha u_n + \nabla u \middle| \frac{1}{\varepsilon_n}\nabla_3 u_n + b\right)\right\}$ must generate the same Young measure. Therefore, ν is generated by $\left\{\left(\nabla_\alpha v_n \middle| \frac{1}{\varepsilon_n}\nabla_3 v_n\right)\right\}$, and we conclude that $\nu \in Y(v, b)$.

5 Proof of Proposition 4.2

We will first restrict our attention to the case of x_α - homogeneous bending Young measures, i.e. bending Young measures $\nu : \Omega \to \mathcal{P}(\mathbb{M}^{3\times 3})$ such that ν_x is independent on x_α. We start with a lemma that will allow us to arbitrarily decide on the scale that appears in the last column of the sequence generating such a Young measure.

Lemma 5.1 *Let ν be an x_α - homogeneous bending Young measure generated by a sequence $\left\{\left(\nabla_\alpha u_n \middle| \frac{1}{\varepsilon_n}\nabla_3 u_n\right)\right\}$ bounded in $L^p(\Omega; \mathbb{M}^{3\times 3})$ and with $\varepsilon_n \to 0^+, u_n \rightharpoonup 0$ weakly in $W^{1,p}(\Omega; \mathbb{R}^3)$, and $\frac{1}{\varepsilon_n}\int_I \nabla_3 u_n(\cdot, x_3)dx_3 \rightharpoonup 0$ weakly in $L^p(\omega, \mathbb{R}^3)$. Let $\{\delta_k\}$ be any sequence of positive real numbers such that $\delta_k \to 0^+$. Then there exists a sequence $\{v_k\} \subset W^{1,p}(\Omega; \mathbb{R}^3)$ such that (up to a subsequence, not relabelled) $\left\{\left(\nabla_\alpha v_k \middle| \frac{1}{\delta_k}\nabla_3 v_k\right)\right\}$ is p-equi-integrable, generates ν and, in addition, $v_k \rightharpoonup 0$ weakly in $W^{1,p}(\Omega; \mathbb{R}^3)$, $v_k = 0$ on $\partial\omega \times I$, and $\frac{1}{\delta_k}\int_I \nabla_3 v_k(\cdot, x_3)dx_3 \rightharpoonup 0$ weakly in $L^p(\omega, \mathbb{R}^3)$.*

Proof. We will prove the lemma in two steps.
<u>*Step 1.*</u> We start by treating the particular case when ω is a cube (without loss of generality, we may assume that $\omega = Q' := (0, 1)^2$). In view of Remark 2.8 we assume, without loss of generality, that the sequence $\left\{\left(\nabla_\alpha u_n \middle| \frac{1}{\varepsilon_n}\nabla_3 u_n\right)\right\}$ is p-equi-integrable and that $u_n = 0$ on $\partial Q' \times I$. Extend $u_n(\cdot, x_3)$ by periodicity to the whole of $\mathbb{R}^2$ and define $u_{n,k} \in W^{1,p}(Q' \times I; \mathbb{R}^3)$ by

$$u_{n,k}(x_\alpha, x_3) := \frac{\delta_k}{\varepsilon_n} u_n\left(\frac{\varepsilon_n}{\delta_k}x_\alpha, x_3\right).$$

For $\theta \in C_0(Q'), \eta \in C_0(I)$ and $\Psi \in C_0(\mathbb{M}^{3\times 3})$, we have

$$\begin{aligned}
&\lim_{n\to\infty}\lim_{k\to\infty}\int_{Q'\times I}\theta(x_\alpha)\eta(x_3)\Psi\left(\left(\nabla_\alpha u_{n,k}\middle|\frac{1}{\delta_k}\nabla_3 u_{n,k}\right)\right)dx \\
&= \lim_{n\to\infty}\lim_{k\to\infty}\int_{Q'\times I}\theta(x_\alpha)\eta(x_3)\Psi\left(\left(\nabla_\alpha u_n\middle|\frac{1}{\varepsilon_n}\nabla_3 u_n\right)\left(\frac{\varepsilon_n}{\delta_k}x_\alpha, x_3\right)\right)dx \\
&= \left(\int_{Q'}\theta(x_\alpha)dx_\alpha\right)\lim_{n\to\infty}\int_{Q'\times I}\eta(x_3)\Psi\left(\left(\nabla_\alpha u_n\middle|\frac{1}{\varepsilon_n}\nabla_3 u_n\right)(y_\alpha, x_3)\right)dy_\alpha dx_3 \qquad (5.1)\\
&= \left(\int_{Q'}\theta(x_\alpha)dx_\alpha\right)\int_{Q'\times I}\eta(x_3)\int_{\mathbb{M}^{3\times 3}}\Psi(\Lambda)d\nu_{x_3}(\Lambda)dx \\
&= \int_{Q'\times I}\theta(x_\alpha)\eta(x_3)\int_{\mathbb{M}^{3\times 3}}\Psi(\Lambda)d\nu_{x_3}(\Lambda)dx,
\end{aligned}$$

where we have used Lebesgue's Dominated Convergence Theorem, Riemann-Lebesgue's lemma, and the fact that $\left\{\left(\nabla_\alpha u_n\Big|\frac{1}{\varepsilon_n}\nabla_3 u_n\right)\right\}$ is p-equi-integrable and generates $\nu = \{\nu_{x_3}\}_{x_3\in I}$. Similar arguments yield

$$\limsup_{n\to\infty}\limsup_{k\to\infty}\int_{Q'\times I}\left|\left(\nabla_\alpha u_{n,k}\Big|\frac{1}{\delta_k}\nabla_3 u_{n,k}\right)\right|^p dx =$$

$$\limsup_{n\to\infty}\int_{Q'\times I}\left|\left(\nabla_\alpha u_n\Big|\frac{1}{\varepsilon_n}\nabla_3 u_n\right)\right|^p dx < +\infty, \tag{5.2}$$

$$\lim_{k\to\infty}\int_{Q'\times I}|u_{n,k}|^p dx = 0, \text{ for all } n\in\mathbb{N}, \tag{5.3}$$

and, for every $\varphi \in L^{p'}(Q';\mathbb{R}^3)$,

$$\lim_{n\to\infty}\lim_{k\to\infty}\int_{Q'}\varphi(x_\alpha)\left(\frac{1}{\delta_k}\int_I \nabla_3 u_{n,k}(x_\alpha,x_3)dx_3\right)dx_\alpha = 0. \tag{5.4}$$

In view of (5.1)-(5.4), and using countable dense families $\{\theta\otimes\eta\}\subset C_0(Q'\times I)$, $\{\Psi\}\subset C_0(\mathbb{M}^{3\times 3})$ and $\{\varphi\}\subset L^{p'}(Q';\mathbb{R}^3)$, we can perform a diagonalization process to find an increasing subsequence $\{k(n)\}$ of $\{k\}$ such that, upon letting $\overline{u}_n := u_{n,k(n)}$,

$$\limsup_{n\to\infty}\int_{Q'\times I}\left|\left(\nabla_\alpha \overline{u}_n\Big|\frac{1}{\delta_{k(n)}}\nabla_3 \overline{u}_n\right)\right|^p dx = \limsup_{n\to\infty}\int_{Q'\times I}\left|\left(\nabla_\alpha u_n\Big|\frac{1}{\varepsilon_n}\nabla_3 u_n\right)\right|^p dx < +\infty, \tag{5.5}$$

the sequence $\left\{\left(\nabla_\alpha \overline{u}_n\Big|\frac{1}{\delta_{k(n)}}\nabla_3 \overline{u}_n\right)\right\}$ generates ν, $\overline{u}_n \rightharpoonup 0$ weakly in $W^{1,p}(Q'\times I;\mathbb{R}^3)$ and, in addition, $\displaystyle\frac{1}{\delta_{k(n)}}\int_I \nabla_3\overline{u}_n(\cdot,x_3)dx_3 \rightharpoonup 0$ weakly in $L^p(Q',\mathbb{R}^3)$. Taking into account Remark 2.8, we may find an increasing subsequence $\{n_j\}$ of $\{n\}$ and a sequence $\{w_j\}\subset W^{1,p}(Q'\times I;\mathbb{R}^3)$ such that $\left\{\left(\nabla_\alpha w_j\Big|\frac{1}{\delta_{k(n_j)}}\nabla_3 w_j\right)\right\}$ is p-equi-integrable and generates ν, $w_j \rightharpoonup 0$ weakly in $W^{1,p}(Q'\times I;\mathbb{R}^3)$, $w_j = 0$ on $\partial Q'\times I$, $\displaystyle\frac{1}{\delta_{k(n_j)}}\int_I \nabla_3 w_j(\cdot,x_3)dx_3 \rightharpoonup 0$ weakly in $L^p(Q',\mathbb{R}^3)$ and, in view of (2.3) and (5.5),

$$\limsup_{j\to\infty}\int_{Q'\times I}\left|\left(\nabla_\alpha w_j\Big|\frac{1}{\delta_{k(n_j)}}\nabla_3 w_j\right)\right|^p dx \le C\limsup_{n\to\infty}\int_{Q'\times I}\left|\left(\nabla_\alpha u_n\Big|\frac{1}{\varepsilon_n}\nabla_3 u_n\right)\right|^p dx < +\infty. \tag{5.6}$$

The sequence $\{v_k\}$ defined by setting $v_{k_{(n_j)}} := w_j$ ($j\in\mathbb{N}$) and $v_k := 0$ if $k\notin\{k(n_j) : j\in\mathbb{N}\}$ satisfies the desired properties.

<u>*Step 2.*</u> We now treat the general case. In view of Remark 2.8 we may assume that, in addition to the hypotheses in the statement of the lemma, the original sequence $\left\{\left(\nabla_\alpha u_n\Big|\frac{1}{\varepsilon_n}\nabla_3 u_n\right)\right\}$ is p-equi-integrable. In fact, passing eventually to a subsequence (not relabelled), we may further assume that

$$\left|\left(\nabla_\alpha u_n\Big|\frac{1}{\varepsilon_n}\nabla_3 u_n\right)\right|^p \rightharpoonup U \text{ weakly in } L^1(\Omega). \tag{5.7}$$

Let

$$\omega^{(m)} := \bigcup_{i=1}^{N(m)} (a_i + L_i Q') \subset\subset \omega, \ (m \in \mathbb{N}).$$

be a union of pairwise disjoint cubes such the $\mathcal{L}^2(\omega \setminus \omega^{(m)}) \leq \frac{1}{m}$, $a_i \in \omega$, and $L_i > 0$ ($i = 1, \cdots, N(m)$). We apply Step 1 on each cube $a_i + L_i Q'$ ($i = 1, \cdots, N(m)$), thus finding sequences $\{v_j^{(i)}\} \subset W^{1,p}((a_i + L_i Q') \times I; \mathbb{R}^3)$ such that $\left\{ \left(\nabla_\alpha v_j^{(i)} \Big| \frac{1}{\delta_j} \nabla_3 v_j^{(i)} \right) \right\}$ is p-equi-integrable and generates ν, $v_j^{(i)} \rightharpoonup 0$ weakly in $W^{1,p}((a_i + L_i Q') \times I; \mathbb{R}^3)$, $v_j^{(i)} = 0$ on $\partial(a_i + L_i Q') \times I$, and $\dfrac{1}{\delta_j} \displaystyle\int_I \nabla_3 v_j^{(i)}(\cdot, x_3) dx_3 \rightharpoonup 0$ weakly in $L^p((a_i + L_i Q'); \mathbb{R}^3)$, for all $i = 1, \cdots, N(m)$. Define $v_{j,m} \in W^{1,p}(\Omega; \mathbb{R}^3)$ by

$$v_{j,m}(x) := \sum_{i=1}^{N(m)} v_j^{(i)}(x) \chi_{(a_i + L_i Q') \times I}(x).$$

Note that $v_{j,m|\partial\omega \times I} = 0$, that we have

$$\lim_{m \to \infty} \lim_{j \to \infty} \|v_{j,m}\|^p_{L^p(\Omega;\mathbb{R}^3)} = \lim_{m \to \infty} \lim_{j \to \infty} \sum_{i=1}^{N(m)} \|v_j^{(i)}\|^p_{L^p((a_i + L_i Q') \times I; \mathbb{R}^3)} = 0,$$

and

$$\lim_{m \to \infty} \lim_{j \to \infty} \int_\omega \varphi(x_\alpha) \left(\frac{1}{\delta_j} \int_I \nabla_3 v_{j,m}(x_\alpha, x_3) dx_3 \right) dx_\alpha =$$
$$\lim_{m \to \infty} \lim_{j \to \infty} \sum_{i=1}^{N(m)} \int_{a_i + L_i Q'} \varphi(x_\alpha) \left(\frac{1}{\delta_j} \int_I \nabla_3 v_j^{(i)}(x_\alpha, x_3) dx_3 \right) dx_\alpha = 0,$$

for all $\varphi \in L^{p'}(\omega; \mathbb{R}^3)$. Taking into account (5.6) and (5.7) we deduce that

$$\begin{aligned}
&\limsup_{m \to \infty} \limsup_{j \to \infty} \int_\Omega \left| \left(\nabla_\alpha v_{j,m} \Big| \frac{1}{\delta_j} \int_I \nabla_3 v_{j,m} \right) \right|^p dx \\
&\leq \limsup_{m \to \infty} \sum_{i=1}^{N(m)} \limsup_{j \to \infty} \int_{(a_i + L_i Q') \times I} \left| \left(\nabla_\alpha v_j^{(i)} \Big| \frac{1}{\delta_j} \nabla_3 v_j^{(i)} \right) \right|^p dx \\
&\leq \limsup_{m \to \infty} \sum_{i=1}^{N(m)} C \limsup_{n \to \infty} \int_{(a_i + L_i Q') \times I} \left| \left(\nabla_\alpha u_n \Big| \frac{1}{\varepsilon_n} \nabla_3 u_n \right) \right|^p dx \\
&= C \limsup_{m \to \infty} \sum_{i=1}^{N(m)} \int_{(a_i + L_i Q') \times I} U dx = C \limsup_{m \to \infty} \int_{\omega^{(m)} \times I} U dx \\
&= C \int_\Omega U dx < +\infty.
\end{aligned}$$

Finally, for $\theta\otimes\eta\in C_0(\Omega)$ and $\Psi\in C_0(\mathbb{M}^{3\times 3})$, we have

$$
\begin{aligned}
&\lim_{m\to\infty}\lim_{j\to\infty}\int_\Omega \theta(x_\alpha)\eta(x_3)\Psi\left(\left(\nabla_\alpha v_{j,m}\Big|\frac{1}{\delta_j}\nabla_3 v_{j,m}\right)\right)dx\\
&=\lim_{m\to\infty}\lim_{j\to\infty}\int_I \eta(x_3)\left(\sum_{i=1}^{N(m)}\int_{a_i+L_iQ'}\theta(x_\alpha)\Psi\left(\left(\nabla_\alpha v_j^{(i)}\Big|\frac{1}{\delta_j}\nabla_3 v_j^{(i)}\right)\right)dx_\alpha\right.\\
&\qquad\qquad\left.+\int_{(\omega\backslash\omega^{(m)})}\theta(x_\alpha)\Psi(0)dx_\alpha\right)dx_3\\
&=\lim_{m\to\infty}\left(\int_{\omega^{(m)}\times I}\theta(x_\alpha)\eta(x_3)\int_{\mathbb{M}^{3\times 3}}\Psi(\Lambda)d\nu_{x_3}(\Lambda)dx+\int_{(\omega\backslash\omega^{(m)})\times I}\theta(x_\alpha)\eta(x_3)\Psi(0)dx\right)\\
&=\int_\Omega\theta(x_\alpha)\eta(x_3)\int_{\mathbb{M}^{3\times 3}}\Psi(\Lambda)d\nu_{x_3}(\Lambda)dx.
\end{aligned}
$$

The proof may now be concluded by means of diagonalization and density arguments very similar to the ones used toward the end of Step 1. □

Let Y be the set of weak* measurable maps $\nu : I\to\mathcal{P}(\mathbb{M}^{3\times 3})$ satisfying

$$\int_I\int_{\mathbb{M}^{3\times 3}}|\Lambda|^p d\nu_{x_3}(\Lambda)dx_3<+\infty,$$

and define

$$
\begin{aligned}
\mathbb{H}:=\Bigg\{&\nu\in Y:\nu:=\{\nu_{x_3}\}_{x_3\in I}\text{ is a Young measure generated by a } p-\text{equi}-\text{integrable sequence}\\
&\left\{\left(\nabla_\alpha u_n\Big|\tfrac{1}{\varepsilon_n}\nabla_3 u_n\right)\right\},\ \varepsilon_n\to 0,\ u_n\rightharpoonup 0\text{ weakly in } W^{1,p}(Q'\times I;\mathbb{R}^3),\\
&\frac{1}{\varepsilon_n}\int_I\nabla_3 u_n(\cdot,x_3)dx_3\rightharpoonup 0\text{ weakly in } L^p(Q',\mathbb{R}^3)\Bigg\}.
\end{aligned}
$$

The following result is a "homogenized" version of Proposition 4.2.

Lemma 5.2 *Let $\nu\in Y$ be such that* $\displaystyle\int_I\int_{\mathbb{M}^{3\times 3}}\Lambda d\nu_{x_3}(\Lambda)dx_3=0$ *and*

$$\widehat{Q}f(0|0)\le\int_I\int_{\mathbb{M}^{3\times 2}\times\mathbb{R}^3}f\left(x_3,\Lambda\right)d\nu_{x_3}(\Lambda)dx_3\text{ for all } f\in\mathcal{E}_p. \tag{5.8}$$

Then $\nu\in\mathbb{H}$.

Proof. In what follows we will regard Y as a subset of the dual space $\mathcal{E}_p^{'}$ via the duality

$$\langle\mu;f\rangle:=\int_I\int_{\mathbb{M}^{3\times 3}}f\left(x_3,\Lambda\right)d\mu_{x_3}(\Lambda)dx_3$$

for $\mu \in Y,\ f \in \mathcal{E}_p$.

<u>*Claim 1*</u>. *If we denote by* $\overline{\mathbb{H}}^{\mathcal{E}_p'}$ *the weak* closure of* $\mathbb{H}$ *in* $\mathcal{E}_p'$, *then*

$$\overline{\mathbb{H}}^{\mathcal{E}_p'} \cap Y = \mathbb{H}.$$

Indeed, let $\nu \in \overline{\mathbb{H}}^{\mathcal{E}_p'} \cap Y$, and consider a sequence $\{\nu^{(k)}\} \subset \mathbb{H}$ such that $\nu^{(k)}$ converge to ν in the weak* topology of $\mathcal{E}_p'$. Thus, by Lemma 5.1 for each $k \in \mathbb{N}$ $\nu^{(k)}$ is generated by a p-equi-integrable sequence $\left\{\left(\nabla_\alpha u_n^{(k)} \middle|\, n\nabla_3 u_n^{(k)}\right)\right\}$, with $u_n^{(k)} \rightharpoonup 0$ weakly in $W^{1,p}(Q' \times I;\mathbb{R}^3)$, and such that $n\displaystyle\int_I \nabla_3 u_n^{(k)}(\cdot, x_3)dx_3 \rightharpoonup 0$ weakly in $L^p(Q',\mathbb{R}^3)$ as $n \to +\infty$. In view of Theorem 2.3 (v), and since the map $g : I \times \mathbb{M}^{3\times 3} \to \mathbb{R}$ defined by $g(x_3, X) := |X|^p$ is an element of $\mathcal{E}_p$, we have

$$\begin{aligned}\lim_{k\to\infty}\lim_{n\to\infty}\int_{Q'\times I}\left|\left(\nabla_\alpha u_n^{(k)} \middle|\, n\nabla_3 u_n^{(k)}\right)\right|^p dx &= \lim_{k\to\infty}\int_{Q'\times I}\int_{\mathbb{M}^{3\times 3}}|\Lambda|^p d\nu_{x_3}^{(k)}(\Lambda)dx \\ &= \int_I\int_{\mathbb{M}^{3\times 3}}|\Lambda|^p d\nu_{x_3}(\Lambda)dx_3 < +\infty, \qquad (5.9)\end{aligned}$$

where the last inequality follows from the fact that $\nu \in Y$.

Let $\theta \in W_0^{1,\infty}(Q' \times I)$ and $\Psi \in W^{1,\infty}(\mathbb{M}^{3\times 3})$. Using again Theorem 2.3, we have that

$$\lim_{n\to\infty}\int_{Q'\times I}\theta(x)\Psi\left(\nabla_\alpha u_n^{(k)} \middle|\, n\nabla_3 u_n^{(k)}\right)dx = \int_{Q'\times I}\theta(x)\int_{\mathbb{M}^{3\times 3}}\Psi(\Lambda)d\nu_{x_3}^{(k)}(\Lambda)dx \qquad (5.10)$$

for every $k \in \mathbb{N}$. On the other hand, for each fixed $x_\alpha \in \omega$, the map $I \times \mathbb{M}^{3\times 3} \ni (x_3, X) \mapsto \theta(x_\alpha, x_3)\Psi(X) \in \mathbb{R}$ belongs to $\mathcal{E}_p$, which implies that

$$\lim_{k\to\infty}\int_I\theta(x_\alpha, x_3)\int_{\mathbb{M}^{3\times 3}}\Psi(\Lambda)d\nu_{x_3}^{(k)}(\Lambda)dx_3 = \int_I\theta(x_\alpha, x_3)\int_{\mathbb{M}^{3\times 3}}\Psi(\Lambda)d\nu_{x_3}(\Lambda)dx_3.$$

Therefore, taking into account the last equality in (5.9) and the fact that $\nu \in Y$, we may invoke Lebesgue's Dominated Convergence Theorem to obtain

$$\lim_{k\to\infty}\int_{Q'\times I}\theta(x)\int_{\mathbb{M}^{3\times 3}}\Psi(\Lambda)d\nu_{x_3}^{(k)}(\Lambda)dx = \int_{Q'\times I}\theta(x)\int_{\mathbb{M}^{3\times 3}}\Psi(\Lambda)d\nu_{x_3}(\Lambda)dx.$$

In view of (5.10), we deduce that

$$\lim_{k\to\infty}\lim_{n\to\infty}\int_{Q'\times I}\theta(x)\Psi\left(\nabla_\alpha u_n^{(k)} \middle|\, n\nabla_3 u_n^{(k)}\right)dx = \int_{Q'\times I}\theta(x)\int_{\mathbb{M}^{3\times 3}}\Psi(\Lambda)d\nu_{x_3}(\Lambda)dx. \qquad (5.11)$$

Considering countable families $\{\theta\} \subset W_0^{1,\infty}(Q' \times I)$ and $\{\Psi\} \subset W^{1,\infty}(\mathbb{M}^{3\times 3})$ (that are dense in $C_0(Q' \times I)$ and $C_0(\mathbb{M}^{3\times 3})$ respectively) in (5.11), using the standard diagonalization procedure and invoking again Remark 2.8, we can construct a p-equi-integrable sequence of scaled gradients generating ν and having the right weak convergence properties as described in the definition of $\mathbb{H}$. We conclude that $\nu \in \mathbb{H}$.

Claim 2. The set $\mathbb{H}$ is convex.
Let $\nu, \mu \in \mathbb{H}$ and $\theta \in (0,1)$ be given. In view of Lemma 5.1, we may assume without loss of generality that ν is generated by $\{(\nabla_\alpha u_n | n\nabla_3 u_n)\}$ and that μ is generated by $\{(\nabla_\alpha v_n | n\nabla_3 v_n)\}$, with $u_n \rightharpoonup 0$, $v_n \rightharpoonup 0$ weakly in $W^{1,p}(Q' \times I;\mathbb{R}^3)$, $n\int_I \nabla_3 u_n(\cdot, x_3)dx_3 \rightharpoonup 0$, $n\int_I \nabla_3 v_n(\cdot, x_3)dx_3 \rightharpoonup 0$ weakly in $L^p(Q';\mathbb{R}^3)$, $u_n = v_n = 0$ on $\partial Q' \times I$ and, in addition, $\{(\nabla_\alpha u_n | n\nabla_3 u_n)\}$ and $\{(\nabla_\alpha v_n | n\nabla_3 v_n)\}$ are p-equi-integrable.

Let $\{\varphi_n\} \subset C_c^1(0,1)$ be such that $\varphi_n \to \chi_{(0,\theta)}$ strongly in $L^r(0,1)$ for any $1 \le r < +\infty$, and with $\mathcal{L}^1(\{x_1 \in (0,1) : 0 < \varphi_n(x_1) < 1\}) \approx \|u_n - v_n\|^{1/2}_{L^p(Q';\mathbb{R}^3)}$, $\|\varphi_n'\|_\infty \cdot \|u_n - v_n\|^{1/2}_{L^p(Q';\mathbb{R}^3)} \le C$, for some constant $C > 0$ and for all $n \in \mathbb{N}$. Extend by periodicity φ_n to $\mathbb{R}$ and, for $\mathcal{L}^1$-a.e. $x_3 \in I$, $u_n(\cdot, x_3)$, $v_n(\cdot, x_3)$ to the whole of $\mathbb{R}^2$. Define $w_{n,m} \in W^{1,p}(Q' \times I;\mathbb{R}^3)$ by

$$w_{n,m}(x_\alpha, x_3) := \frac{1}{m}\left[u_n(mx_\alpha, x_3) + \varphi_n(mx_1)(v_n(mx_\alpha, x_3) - u_n(mx_\alpha, x_3))\right].$$

We have

$$\lim_{n\to\infty}\lim_{m\to\infty}\int_{Q'\times I} |w_{n,m}|^p dx = 0, \tag{5.12}$$

and

$$\int_{Q'\times I} |(\nabla_\alpha w_{n,m} | nm\nabla_3 w_{n,m})|^p dx \le C\Bigg(\int_{Q'\times I} |(\nabla_\alpha u_n | n\nabla_3 u_n)(mx_\alpha, x_3)|^p dx +$$
$$\int_{Q'\times I} |(\nabla_\alpha v_n | n\nabla_3 v_n)(mx_\alpha, x_3)|^p dx + \int_{Q'\times I} |\varphi_n'(mx_1)|^p |(v_n - u_n)(mx_\alpha, x_3)|^p dx\Bigg).$$

Invoking the Riemann-Lebesgue Lemma and Lebesgue's Dominated Convergence Theorem we obtain

$$\lim_{m\to\infty}\int_{Q'\times I} |(\nabla_\alpha u_n | n\nabla_3 u_n)(mx_\alpha, x_3)|^p dx = \int_{Q'\times I} |(\nabla_\alpha u_n | n\nabla_3 u_n)|^p dx,$$
$$\lim_{m\to\infty}\int_{Q'\times I} |(\nabla_\alpha v_n | n\nabla_3 v_n)(mx_\alpha, x_3)|^p dx = \int_{Q'\times I} |(\nabla_\alpha v_n | n\nabla_3 v_n)|^p dx,$$

and

$$\lim_{m\to\infty}\int_{Q'\times I} |\varphi_n'(mx_1)|^p |(v_n - u_n)(mx_\alpha, x_3))|^p dx \le C\|u_n - v_n\|^{p/2}_{L^p(Q';\mathbb{R}^3)}.$$

Thus

$$\limsup_{n\to\infty}\lim_{m\to\infty}\int_{Q'\times I} |(\nabla_\alpha w_{n,m} | nm\nabla_3 w_{n,m})|^p dx$$
$$\le C\limsup_{n\to\infty}\Bigg(\int_{Q'\times I} |(\nabla_\alpha u_n | n\nabla_3 u_n)|^p dx + \int_{Q'\times I} |(\nabla_\alpha v_n | n\nabla_3 v_n)|^p dx\Bigg) < +\infty. \tag{5.13}$$

Let $\varphi \in L^{p'}(Q';\mathbb{R}^3)$. We have

$$\int_{Q'} \varphi(x_\alpha)\cdot\left(nm\int_I \nabla_3 w_{n,m}(x_\alpha,x_3)dx_3\right)dx_\alpha = \int_{Q'}\varphi(x_\alpha)\cdot\Bigg(\varphi_n(mx_1)\ n\int_I \nabla_3 v_n(mx_\alpha,x_3)dx_3$$

$$+\,(1-\varphi_n(mx_1))\ n\int_I \nabla_3 u_n(mx_\alpha,x_3)dx_3\Bigg)dx_\alpha.$$

By the Riemann-Lebesgue Lemma,

$$\lim_{m\to\infty}\int_{Q'}\varphi(x_\alpha)\cdot\left(\varphi_n(mx_1)\ n\int_I \nabla_3 v_n(mx_\alpha,x_3)dx_3+(1-\varphi_n(mx_1))\ n\int_I\nabla_3 u_n(mx_\alpha,x_3)dx_3\right)dx_\alpha$$

$$=\left(\int_{Q'}\varphi(x_\alpha)dx_\alpha\right)\cdot\left(\int_{Q'}\left(\varphi_n(x_1)\ n\int_I\nabla_3 v_n(x_\alpha,x_3)dx_3+(1-\varphi_n(x_1))\ n\int_I\nabla_3 u_n(x_\alpha,x_3)dx_3\right)dx_\alpha\right).$$

Since

$$\lim_{n\to\infty}\int_{Q'}\Psi(x_1)\ \left(n\int_I\nabla_3 u_n(x_\alpha,x_3)dx_3\right)dx_\alpha = 0 \text{ for } \Psi\in\{1,\chi_{(0,\theta)}\},$$

we have

$$\lim_{n\to\infty}\int_{Q'}(1-\varphi_n(x_1))\ \left(n\int_I\nabla_3 u_n(x_\alpha,x_3)dx_3\right)dx_\alpha$$

$$=\lim_{n\to\infty}\|\varphi_n-\chi_{(0,\theta)}\|_{L^{p'}((0,1))}\left\|n\int_I\nabla_3 u_n(\cdot,x_3)dx_3\right\|_{L^p(Q';\mathbb{R}^3)}=0.$$

Similarly,

$$\lim_{n\to\infty}\int_{Q'}\varphi_n(x_1)\ \left(n\int_I\nabla_3 v_n(x_\alpha,x_3)dx_3\right)dx_\alpha$$

$$=\lim_{n\to\infty}\int_{Q'}\left(\varphi_n(x_1)-\chi_{(0,\theta)}(x_1)\right)\ \left(n\int_I\nabla_3 v_n(x_\alpha,x_3)dx_3\right)dx_\alpha=0.$$

We deduce that

$$\lim_{n\to\infty}\lim_{m\to\infty}\int_{Q'}\varphi(x_\alpha)\cdot\left(nm\int_I\nabla_3 w_{n,m}(x_\alpha,x_3)dx_3\right)dx_\alpha=0. \tag{5.14}$$

Finally, we claim that for all $\xi\in C_0(Q'),\ \eta\in C_0(I)$ and $\Psi\in C_0(\mathbb{M}^{3\times 3})$,

$$\lim_{n\to\infty}\lim_{m\to\infty}\int_{Q'\times I}\xi(x_\alpha)\eta(x_3)\Psi(\nabla_\alpha w_{n,m}|nm\nabla_3 w_{n,m})dx$$

$$=\int_{Q'\times I}\xi(x_\alpha)\eta(x_3)\int_{\mathbb{M}^{3\times 3}}\Psi(\Lambda)d[\theta\mu_{x_3}+(1-\theta)\nu_{x_3}]dx. \tag{5.15}$$

The Riemann-Lebesgue Lemma yields

$$\lim_{m\to\infty}\int_{Q'\times I}\xi(x_\alpha)\eta(x_3)\Psi(\nabla_\alpha w_{n,m}|nm\nabla_3 w_{n,m})dx = (I_1^n+I_2^n+I_3^n)\int_{Q'}\xi(x_\alpha)dx_\alpha, \tag{5.16}$$

where

$$I_1^n := \int_I \eta(x_3)\int_{Q'\cap(\{\varphi_n=0\}\times(0,1))}\Psi(\nabla_\alpha u_n|n\ \nabla_3 u_n)dx_\alpha dx_3,$$

$$I_2^n := \int_I \eta(x_3)\int_{Q'\cap(\{\varphi_n=1\}\times(0,1))}\Psi(\nabla_\alpha v_n|n\ \nabla_3 v_n)dx_\alpha dx_3,$$

and

$$I_3^n := \int_I \eta(x_3)\int_{Q'\cap(\{0<\varphi_n<1\}\times(0,1))}\Psi(\varphi_n(x_1)(\nabla_\alpha v_n(x)\mid n\nabla_3 v_n(x))$$
$$+(1-\varphi_n(x_1))(\nabla_\alpha u_n(x)\mid n\nabla_3 u_n(x))+(\varphi_n'(x_1)(v_n-u_n)(x)|0|0))dx_\alpha dx_3.$$

In view of the particular choice of the cut-off functions φ_n $(n\in\mathbb{N})$ and taking into account that ν is generated by $\{(\nabla_\alpha u_n|n\nabla_3 u_n)\}$ and μ is generated by $\{(\nabla_\alpha v_n|n\nabla_3 v_n)\}$, we obtain

$$\begin{aligned}\lim_{n\to\infty} I_1^n &= \lim_{n\to\infty}\int_I \eta(x_3)\int_{(\theta,1)\times(0,1)}\Psi((\nabla_\alpha u_n|n\ \nabla_3 u_n))dx_\alpha dx_3\\ &= \int_{(\theta,1)\times(0,1)\times I}\eta(x_3)\int_{\mathbb{M}^{3\times 3}}\Psi(\Lambda)d\nu_{x_3}(\Lambda)dx\\ &= (1-\theta)\int_I\eta(x_3)\int_{\mathbb{M}^{3\times 3}}\Psi(\Lambda)d\nu_{x_3}(\Lambda)dx_3\end{aligned}$$

and

$$\begin{aligned}\lim_{n\to\infty} I_2^n &= \lim_{n\to\infty}\int_I \eta(x_3)\int_{(0,\theta)\times(0,1)}\Psi((\nabla_\alpha v_n|n\ \nabla_3 v_n))dx_\alpha dx_3\\ &= \theta\int_I\eta(x_3)\int_{\mathbb{M}^{3\times 3}}\Psi(\Lambda)d\mu_{x_3}(\Lambda)dx_3.\end{aligned}$$

Since $\mathcal{L}^1(\{x_1\in(0,1): 0<\varphi_n(x_1)<1\})\approx\|u_n-v_n\|^{1/2}_{L^p(Q';\mathbb{R}^3)}\to 0$ as $n\to+\infty$, we deduce that

$$\lim_{n\to\infty} I_3^n = 0.$$

Thus, by (5.16) we conclude that (5.15) holds. Considering countable dense families $\{\varphi\}\subset L^{p'}(Q';\mathbb{R}^3)$, $\{\xi\otimes\eta\}\subset C_0(Q'\times I)$, and $\{\Psi\}\subset C_0(\mathbb{M}^{3\times 3})$, and taking into account (5.12), (5.13), (5.14), and (5.15), we construct a diagonal subsequence $\{m(n)\}$ of $\{m\}$ such that, letting $\varepsilon_n := \frac{1}{m(n)n}$ and $w_n := w_{n,m(n)}$, the sequence $\left\{\left(\nabla_\alpha w_n\middle|\frac{1}{\varepsilon_n}\nabla_3 w_n\right)\right\}$ generates $\theta\mu+(1-\theta)\nu$ (and is, without loss of generality, p-equi-integrable), $\varepsilon_n \to 0$, $w_n \rightharpoonup 0$ weakly in $W^{1,p}(Q' \times I;\mathbb{R}^3)$, and $\frac{1}{\varepsilon_n}\int\limits_I \nabla_3 w_n(\cdot,x_3)dx_3 \rightharpoonup 0$ weakly in $L^p(Q',\mathbb{R}^3)$. We conclude that the Young measure $\theta\mu+(1-\theta)\nu$ belongs to $\mathbb{H}$, and the proof of Claim 2 is complete.

Having justified the claims, we may now focus on the proof of Lemma 5.2. Let

$$\overline{\mathrm{co}}^{\mathcal{E}_p'}\mathbb{H} := \bigcap\{H : H \text{ is a weakly} * \text{ closed half} - \text{space containing } \mathbb{H}\},$$

be the weakly* closed convex hull of $\mathbb{H}$ in $\mathcal{E}_p'$. In view of Claim 1 and Claim 2 above, we deduce that

$$\overline{\mathrm{co}}^{\mathcal{E}_p'}\mathbb{H} \cap Y = \mathbb{H}.$$

Therefore, given ν satisfying the hypotheses of Lemma 5.2, in order to prove that $\nu \in \mathbb{H}$ it suffices to show that $\nu \in H$ where H is an arbitrary weakly* closed half-space containing $\mathbb{H}$. To this aim, first note that there exist $t \in \mathbb{R}$ and a real weakly* continuous linear functional Ξ on $\mathcal{E}_p'$ such that $H = \{L \in \mathcal{E}_p' : \Xi(L) \geq t\}$. Let $f_0 \in \mathcal{E}_p$ be such that $\Xi(L) = \langle L; f_0\rangle$ for all $L \in \mathcal{E}_p'$. Since $\mathbb{H} \subset H$, we have that

$$\langle \mu; f_0\rangle \geq t \text{ for every } \mu \in \mathbb{H}. \tag{5.17}$$

We claim that we also have $\langle \nu; f_0\rangle \geq t$. Indeed, seeking a contradiction, assume that this is not true, i.e.,

$$\int\limits_I \int\limits_{\mathbb{M}^{3\times 2}\times\mathbb{R}^3} f_0\left(x_3,\Lambda\right) d\nu_{x_3}(\Lambda)dx_3 < t. \tag{5.18}$$

Consider an arbitrary pair (φ,λ) that is admissible for the definition of $\widehat{Q}f_0(0|0)$ (see (1.8)), define $V_n \in W^{1,p}(Q' \times I;\mathbb{R}^3)$ by the formula $V_n(x) := \frac{\lambda}{n}\varphi\left(\frac{n}{\lambda}x_\alpha, x_3\right)$, and let μ be the Young measure generated by (a subsequence, not relabelled, of) $\{(\nabla_\alpha V_n|n\nabla_3 V_n)\}$. Consider the weak* measurable map $\widetilde{\mu} : I \to \mathcal{P}(\mathbb{M}^{3\times 3})$ defined by

$$\langle \widetilde{\mu}_{x_3}, \Psi\rangle := \int\limits_{Q'} \Psi((\nabla_\alpha\varphi \mid \lambda\nabla_3\varphi)(x_\alpha,x_3))dx_\alpha, \quad \Psi \in C_c(\mathbb{M}^{3\times 3}).$$

In view of the Riemann-Lebesgue's Lemma, Lebesgue's Dominated Convergence Theorem, and the choice of the pair (φ,λ), we have

$$\begin{aligned}\lim_{n\to+\infty}\int\limits_{A\times I} f(x_3,(\nabla_\alpha V_n|n\nabla_3 V_n))dx &= \mathcal{L}^2(A)\int\limits_I\int\limits_{Q'} f(x_3,(\nabla_\alpha\varphi|\lambda\nabla_3\varphi)(x_\alpha,x_3))dx_\alpha dx_3 \\ &= \int\limits_A \left(\int\limits_I \langle\widetilde{\mu}_{x_3}, f(x_3,\cdot)\rangle dx_3\right) dx_\alpha\end{aligned} \tag{5.19}$$

and, on the other hand,

$$\lim_{n\to+\infty}\int\limits_{A\times I} f(x_3,(\nabla_\alpha V_n|n\nabla_3 V_n))dx = \int\limits_{A\times I}\int\limits_{\mathbb{M}^{3\times 3}} f\left(x_3,\Lambda\right) d\mu_{(x_\alpha,x_3)}(\Lambda)dx \tag{5.20}$$

for any open subset A of Q' and every $f \in \mathcal{E}_p$. It follows that

$$\int_I \int_{\mathbb{M}^{3\times 3}} f\left(x_3, \Lambda\right) d\mu_{(x_\alpha, x_3)}(\Lambda) dx = \int_I \langle \widetilde{\mu}_{x_3}, f(x_3, \cdot)\rangle dx_3 \tag{5.21}$$

for $\mathcal{L}^2$-a.e. $x_\alpha \in \omega$. Testing with $f(x_3, \Lambda) = \theta(x_3)\Psi(\Lambda)$ in (5.21) for $\theta \in C_c(I)$, $\Psi \in C_c(\mathbb{M}^{3\times 3})$, we deduce that ($\mathcal{L}^2$-a.e. $x_\alpha \in \omega$ and $\mathcal{L}^1$-a.e. $x_3 \in I$) $\langle \mu_{(x_\alpha, x_3)}, \Psi\rangle = \langle \widetilde{\mu}_{x_3}, \Psi\rangle$ for all $\Psi \in C_c(\mathbb{M}^{3\times 3})$. Thus, we may assume without loss of generality that $\mu = \{\mu_{x_3}\}_{x_3 \in I}$ (independent on x_α). Invoking again the Riemann-Lebesgue's Lemma, Lebesgue's Dominated Convergence Theorem, and the choice of the pair (φ, λ), we obtain that $V_n \rightharpoonup 0$ weakly in $W^{1,p}(Q' \times I; \mathbb{R}^3)$ and $n \int_I \nabla_3 V_n(\cdot, x_3) dx_3 \rightharpoonup 0$ weakly in $L^p(Q', \mathbb{R}^3)$. Taking into account Remark 2.8, we obtain that $\mu \in \mathbb{H}$. Thus, by (5.17)

$$\int_I \int_{\mathbb{M}^{3\times 2}\times\mathbb{R}^3} f_0\left(x_3, \Lambda\right) d\mu_{x_3}(\Lambda) dx_3 \geq t. \tag{5.22}$$

By (5.19) and (5.20) (with $A = Q'$ and $f = f_0$), we find

$$\lim_{n\to+\infty} \int_{Q'\times I} f_0(x_3, (\nabla_\alpha V_n | n\nabla_3 V_n)) dx = \int_{Q'\times I} f_0(x_3, (\nabla_\alpha \varphi(x_\alpha, x_3) \mid \lambda \nabla_3 \varphi(x_\alpha, x_3))) dx.$$

and

$$\lim_{n\to+\infty} \int_{Q'\times I} f_0(x_3, (\nabla_\alpha V_n | n\nabla_3 V_n)) dx = \int_I \int_{\mathbb{M}^{3\times 3}} f_0\left(x_3, \Lambda\right) d\mu_{x_3}(\Lambda) dx_3.$$

Taking into account (5.22), we deduce that

$$\int_{Q'\times I} f_0(x_3, (\nabla_\alpha \varphi \mid \lambda \nabla_3 \varphi)(x)) dx \geq t.$$

By the arbitrariness of the pair (φ, λ), we may pass to the infimum in the above inequality to obtain that $\widehat{Q} f_0(0|0) \geq t$, which, in view of (5.8), contradicts (5.18). We conclude that $\nu \in H$. This finishes the proof of Lemma 5.2. □

Next, we will prove Proposition 4.2 for general inhomogeneous bending Young measures. We start with some notation needed in the proof. We say that a function $g : \Omega \times \mathbb{M}^{3\times 3} \to \mathbb{R}$ is p-Lipschitz if there exists a constant $C > 0$ such that

$$|g(x, X) - g(y, Y)| \leq C(1 + |X|^{p-1} + |Y|^{p-1})(|x - y| + |X - Y|) \text{ for all } (x, X), (y, Y) \in \Omega \times \mathbb{M}^{3\times 3}.$$

The space of p-Lipschitz functions $g : \Omega \times \mathbb{M}^{3\times 3} \to \mathbb{R}$ is a Banach space when endowed with the norm $\|g\| := |g(x_0, 0)| + \mathrm{Lip}(f)$, where x_0 is any fixed point of Ω, and

$$\mathrm{Lip}(g) := \sup \big\{C > 0 : |g(x, X) - g(y, Y)| \leq C(1 + |X|^{p-1} + |Y|^{p-1})(|x - y| + |X - Y|),$$
$$\text{for all } (x, X), (y, Y) \in \Omega \times \mathbb{M}^{3\times 3}\big\}.$$

Let $\mathcal{E}_{p,\Omega}$ be the closed subspace of the space of p-Lipschitz functions $g : \Omega \times \mathbb{M}^{3\times 3} \to \mathbb{R}$ having the additional property that the map

$$\Omega \times \mathbb{M}^{3\times 3} \ni (x, X) \longmapsto \frac{g(x, X)}{1 + |X|^p} \in \mathbb{R}$$

admits a continuous extension to $\overline{\Omega} \times (\mathbb{M}^{3\times 3} \cup \{+\infty\})$ (the closure of Ω times the one-point compactification of $\mathbb{M}^{3\times 3}$). Let Y_Ω be the set of weak* measurable maps $\nu : \Omega \to \mathcal{P}(\mathbb{M}^{3\times 3})$ satisfying

$$\int_\Omega \int_{\mathbb{M}^{3\times 3}} |\Lambda|^p d\nu_x(\Lambda) dx < +\infty,$$

which we regard as a subset of the dual space $\mathcal{E}'_{p,\Omega}$ via the duality

$$\langle \mu, g\rangle_{p,\Omega} := \int_\Omega \int_{\mathbb{M}^{3\times 3}} g(x,\Lambda)\, d\mu_x(\Lambda) dx$$

for $\mu \in Y_\Omega,\ g \in \mathcal{E}_{p,\Omega}$. Note that $\|\nu\| \leq C \int_\Omega \int_{\mathbb{M}^{3\times 3}} (1+|\Lambda|^p) d\nu_x(\Lambda) dx$, where C is a positive constant depending only on the diameter of Ω.

We will prove (4.18) in four steps.

Step 1. *If $\overline{\mathbb{Y}}^{\mathcal{E}'_{p,\Omega}}$ denotes the weak* closure of $\mathbb{Y}$ in $\mathcal{E}'_{p,\Omega}$, then*

$$\overline{\mathbb{Y}}^{\mathcal{E}'_{p,\Omega}} \cap \mathbb{X} = \mathbb{Y}. \tag{5.23}$$

It is clear that we have $\mathbb{Y} \subset \overline{\mathbb{Y}}^{\mathcal{E}'_{p,\Omega}} \cap \mathbb{X}$. Thus, it remains to show that the converse inclusion holds. Although this may be done using diagonalization and density arguments similar to those used earlier (see the proof of Claim 1 above), due to the slight technical differences and in order to make the diagonalization process explicit, we opt to give the details below.

Let $\nu \in \overline{\mathbb{Y}}^{\mathcal{E}'_{p,\Omega}} \cap \mathbb{X}$, and consider countable families $\{\varphi_l\}_{l\in\mathbb{N}} \subset L^{p'}(\omega;\mathbb{R}^3)$ dense in $L^{p'}(\omega;\mathbb{R}^3)$, $\{\eta_i\}_{i\in\mathbb{N}} \subset W^{1,\infty}_0(\Omega)$ dense in $C_0(\Omega)$, and $\{\Psi_j\}_{j\in\mathbb{N}} \subset W^{1,\infty}(\mathbb{M}^{3\times 3})$ dense in $C_0(\mathbb{M}^{3\times 3})$. Also, let $G : \Omega \times \mathbb{M}^{3\times 3} \to \mathbb{R}$ be defined by $G(x,X) := |X|^p$. Note that $G \in \mathcal{E}_{p,\Omega}$, and $\eta_i \otimes \Psi_j \in \mathcal{E}_{p,\Omega}$, for all $i,j \in \mathbb{N}$. For each $k \in \mathbb{N}$ there exists $\nu^{(k)} \in \mathbb{Y}$ such that, whenever $i+j \leq k$, we have

$$\left|\langle \nu^{(k)} - \nu, G\rangle_{p,\Omega}\right| + \left|\langle \nu^{(k)} - \nu, \eta_i \otimes \Psi_j\rangle_{p,\Omega}\right| < \frac{1}{k}. \tag{5.24}$$

By the definition of $\mathbb{Y}$, and in view of Remark 2.8, for each $k \in \mathbb{N}$ there exists a p-equi-integrable sequence $\left\{\left(\nabla_\alpha u_n^{(k)}\middle|\frac{1}{\varepsilon_n^{(k)}}\nabla_3 u_n^{(k)}\right)\right\}$ that generates $\nu^{(k)}$ and with $\varepsilon_n^{(k)} \to 0^+$, $u_n^{(k)} \rightharpoonup 0$ weakly in $W^{1,p}(\Omega;\mathbb{R}^3)$ and such that $\frac{1}{\varepsilon_n^{(k)}} \int_I \nabla_3 u_n^{(k)}(\cdot, x_3) dx_3 \rightharpoonup 0$ weakly in $W^{1,p}(\omega;\mathbb{R}^3)$ as $n \to +\infty$. Thus, we can find a sequence $\{u_k\} \subset W^{1,p}(\Omega;\mathbb{R}^3)$, and $\varepsilon_k \to 0^+$, such that

$$\begin{aligned}
&\left|\int_\omega \varphi_i(x_\alpha)\cdot\left(\frac{1}{\varepsilon_k}\int_I \nabla_3 u_k(x_\alpha,x_3)dx_3\right)dx_\alpha\right| + \int_\Omega |u_k|^p dx \\
&+\left|\int_\Omega \left|\left(\nabla_\alpha u_k\middle|\frac{1}{\varepsilon_k}\nabla_3 u_k\right)\right|^p dx - \int_\Omega\int_{\mathbb{M}^{3\times 3}} |\Lambda|^p d\nu_x^{(k)}(\Lambda) dx\right| \\
&+\left|\int_\Omega \eta_i(x)\Psi_j\left(\nabla_\alpha u_k\middle|\frac{1}{\varepsilon_k}\nabla_3 u_k\right)dx - \int_\Omega \eta_i(x)\int_{\mathbb{M}^{3\times 3}} \Psi_j(\Lambda) d\nu_x^{(k)}(\Lambda) dx\right| < \frac{1}{k},
\end{aligned}$$

for all $i, j \in \mathbb{N}$, with $i + j \leq k$. Taking (5.24) into account, we obtain that

$$\left|\int_{\omega} \varphi_i(x_\alpha) \cdot \left(\frac{1}{\varepsilon_k} \int_I \nabla_3 u_k(x_\alpha, x_3) dx_3\right) dx_\alpha\right| + \int_{\Omega} |u_k|^p dx$$
$$+ \left|\int_{\Omega} \left|\left(\nabla_\alpha u_k \Big| \frac{1}{\varepsilon_k} \nabla_3 u_k\right)\right|^p dx - \int_{\Omega} \int_{\mathbb{M}^{3\times 3}} |\Lambda|^p d\nu_x(\Lambda) dx\right|$$
$$+ \left|\int_{\Omega} \eta_i(x) \Psi_j\left(\nabla_\alpha u_k \Big| \frac{1}{\varepsilon_k} \nabla_3 u_k\right) dx - \int_{\Omega} \eta_i(x) \int_{\mathbb{M}^{3\times 3}} \Psi_j(\Lambda) d\nu_x(\Lambda) dx\right| < \frac{2}{k},$$

for all $i, j \in \mathbb{N}$, with $i + j \leq k$. We have

$$\lim_{k\to\infty} \int_{\omega} \varphi_i(x_\alpha) \cdot \left(\frac{1}{\varepsilon_k} \int_I \nabla_3 u_k(x_\alpha, x_3) dx_3\right) dx_\alpha = 0,$$

$$\lim_{k\to\infty} \int_{\Omega} \eta_i(x) \Psi_j\left(\nabla_\alpha u_k \Big| \frac{1}{\varepsilon_k} \nabla_3 u_k\right) dx = \int_{\Omega} \eta_i(x) \int_{\mathbb{M}^{3\times 3}} \Psi_j(\Lambda) d\nu_x(\Lambda) dx,$$

for all $i, j \in \mathbb{N}$,

$$\lim_{k\to\infty} \int_{\Omega} \left|\left(\nabla_\alpha u_k \Big| \frac{1}{\varepsilon_k} \nabla_3 u_k\right)\right|^p dx = \int_{\Omega} \int_{\mathbb{M}^{3\times 3}} |\Lambda|^p d\nu_x(\Lambda) dx,$$

and

$$\lim_{k\to\infty} \int_{\Omega} |u_k|^p dx = 0.$$

By the density properties of $\{\varphi_l\}_{l\in\mathbb{N}}$, $\{\eta_i\}_{i\in\mathbb{N}}$, $\{\Psi_j\}_{j\in\mathbb{N}}$, and in view of Theorem 2.3, we deduce that ν is generated by the p-equi-integrable sequence $\left\{\left(\nabla_\alpha u_k \Big| \frac{1}{\varepsilon_k} \nabla_3 u_k\right)\right\}$ with $\varepsilon_k \to 0^+$, $u_k \rightharpoonup 0$ weakly in $W^{1,p}(\Omega; \mathbb{R}^3)$ and $\frac{1}{\varepsilon_k} \int_I \nabla_3 u_k(\cdot, x_3) dx_3 \rightharpoonup 0$ weakly in $W^{1,p}(\omega; \mathbb{R}^3)$. Therefore $\nu \in \mathbb{Y}$, and we conclude that (5.23) holds.

Step 2. We construct a special subset $\mathbb{D}$ of $\mathbb{T}$ such that

$$\overline{\mathbb{D}}^{\mathcal{E}'_{p,\Omega}} \cap \mathbb{T} = \mathbb{T}. \tag{5.25}$$

For $k \in \mathbb{N}$, let $\mathcal{F}_k$ be the family of cubes of the form $\frac{1}{k}(x_\alpha + Q')$, $x_\alpha \in \mathbb{Z}^2$, $\frac{1}{k}(x_\alpha + Q') \subset \omega$, and define $\omega_k := \bigcup_{U\in\mathcal{F}_k} U \subset \omega$. Set

$$D_k := \left\{\nu \in \mathbb{T} : \nu_{|U\times I} \text{ depends only on } x_3 \in I \text{ if } U \in \mathcal{F}_k, \text{ and } \nu_{|(\omega\setminus\omega_k)\times I} = \delta_0\right\}.$$

We will show that the set

$$\mathbb{D} := \bigcup_{k\in\mathbb{N}} D_k$$

satisfies (5.25). To this aim, let ν be an arbitrary element of $\mathbb{T}$, and for each $k \in \mathbb{N}$ define $\nu^k = \{\nu_x^k\}_{x\in\Omega}$ by

$$\nu_x^k := \begin{cases} \mathrm{Av}[\nu, U]_{x_3} & \text{if } x_\alpha \in U,\ U \in \omega_k, \\ \delta_0 & \text{if } x \in (\omega \setminus \omega_k) \times I, \end{cases}$$

where $\mathrm{Av}[\nu, U] : I \to \mathcal{P}(\mathbb{M}^{3\times 3})$ is the weak* measurable map defined by

$$\langle \mathrm{Av}[\nu, U]_{x_3}, \varphi\rangle := \frac{1}{\mathcal{L}^2(U)} \int_U \int_{\mathbb{M}^{3\times 3}} \varphi(\Lambda) d\nu_{(x_\alpha,x_3)}(\Lambda) dx_\alpha,\ \varphi \in C_c(\mathbb{M}^{3\times 3}).$$

Fix $k \in \mathbb{N}$. Since $\nu \in \mathbb{T} \subset \mathbb{X}$, by Fubini's Theorem we have

$$\int_I \langle \mathrm{Av}[\nu, U]_{x_3}, \mathrm{id}\rangle dx_3 = \frac{1}{\mathcal{L}^2(U)} \int_U \left(\int_I \int_{\mathbb{M}^{3\times 3}} \Lambda d\nu_{(x_\alpha,x_3)}(\Lambda) dx_3 \right) dx_\alpha = 0.$$

Thus, we deduce that

$$\int_I \int_{\mathbb{M}^{3\times 3}} \Lambda d\nu^k_{(x_\alpha,x_3)}(\Lambda) dx_3 = 0, \text{ for } \mathcal{L}^2 - \text{a.e. } x_\alpha \in \omega.$$

We have

$$\begin{aligned} \int_\Omega \int_{\mathbb{M}^{3\times 3}} |\Lambda|^p d\nu_x^k(\Lambda) dx &= \sum_{U\in\mathcal{F}_k} \int_{U\times I} \int_{\mathbb{M}^{3\times 3}} |\Lambda|^p d\mathrm{Av}[\nu, U]_{x_3}(\Lambda) dx \\ &= \sum_{U\in\mathcal{F}_k} \int_{U\times I} \frac{1}{\mathcal{L}^2(U)} \int_U \int_{\mathbb{M}^{3\times 3}} |\Lambda|^p d\nu_{(y_\alpha,x_3)}(\Lambda) dy_\alpha dx = \sum_{U\in\mathcal{F}_k} \int_{U\times I} \int_{\mathbb{M}^{3\times 3}} |\Lambda|^p d\nu_x(\Lambda) dx \\ &= \int_{\omega_k\times I} \int_{\mathbb{M}^{3\times 3}} |\Lambda|^p d\nu_x(\Lambda) dx \le \int_\Omega \int_{\mathbb{M}^{3\times 3}} |\Lambda|^p d\nu_x(\Lambda) dx < +\infty, \end{aligned}$$

and thus $\nu^k \in \mathbb{X}$. Let $f \in \mathcal{E}_p$ and let $x_\alpha \in \omega$ be such that

$$\int_I \int_{\mathbb{M}^{3\times 3}} f(x_3, \Lambda) d\nu_{(x_\alpha,x_3)}(\Lambda) dx_3 \ge \widehat{Q}f(0|0).$$

Since $\nu \in \mathbb{T}$, the set of $x_\alpha \in \omega$ satisfying the above inequality is a subset of full $\mathcal{L}^2$-measure in ω. If $x_\alpha \in \omega \setminus \omega_k$ then

$$\int_I \int_{\mathbb{M}^{3\times 3}} f(x_3, \Lambda) d\nu^k_{(x_\alpha,x_3)}(\Lambda) dx_3 = \int_I f(x_3, 0) dx_3 \ge \widehat{Q}f(0|0),$$

where the last inequality follows by testing with $\varphi \equiv 0$ and $\lambda = 0$ in the definition of $\widehat{Q}f(0|0)$ (see (1.8)). On the other hand, if $x_\alpha \in U,\ U \in \mathcal{F}_k$, then

$$\begin{aligned} \int_I \int_{\mathbb{M}^{3\times 3}} f(x_3, \Lambda) d\nu^k_{(x_\alpha,x_3)}(\Lambda) dx_3 &= \int_I \frac{1}{\mathcal{L}^2(U)} \int_U \int_{\mathbb{M}^{3\times 3}} f(x_3, \Lambda) d\nu_{(x_\alpha,x_3)}(\Lambda) dx_\alpha dx_3 \\ &\ge \frac{1}{\mathcal{L}^2(U)} \int_U \widehat{Q}f(0|0) dx_\alpha = \widehat{Q}f(0|0). \end{aligned}$$

We have shown that $\nu^k \in \mathbb{T}$ and, consequently, $\nu^k \in D_k$ for every $k \in \mathbb{N}$. It remains to show that

$$\langle \nu^k, g\rangle_{p,\Omega} \to \langle \nu, g\rangle_{p,\Omega} \text{ as } k \to \infty \text{ for all } g \in \mathcal{E}_{p,\Omega}. \tag{5.26}$$

For each $U \in \mathcal{F}_k$, let $x_\alpha^U \in \left(\frac{1}{k}\mathbb{Z}\right)^2$ be the lower left corner of U, so that $U = x_\alpha^U + \frac{1}{k}Q'$, and fix $g \in \mathcal{E}_{p,\Omega}$. We have

$$\begin{aligned}
\left|\langle \nu^k, g\rangle_{p,\Omega} - \langle \nu, g\rangle_{p,\Omega}\right| \leq & \left| \int_{\omega_k \times I} \int_{\mathbb{M}^{3\times 3}} g\left(x,\Lambda\right) d\nu_x^k(\Lambda) dx - \int_{\omega_k \times I} \int_{\mathbb{M}^{3\times 3}} g\left(x,\Lambda\right) d\nu_x(\Lambda) dx \right| \\
& + \left| \int_{(\omega\setminus\omega_k) \times I} \int_{\mathbb{M}^{3\times 3}} g\left(x,\Lambda\right) d\nu_x^k(\Lambda) dx - \int_{(\omega\setminus\omega_k) \times I} \int_{\mathbb{M}^{3\times 3}} g\left(x,\Lambda\right) d\nu_x(\Lambda) dx \right| \\
\leq & \sum_{U \in \mathcal{F}_k} \left| \int_{U \times I} \int_{\mathbb{M}^{3\times 3}} g\left(x,\Lambda\right) d\nu_x^k(\Lambda) dx - \int_{U \times I} \int_{\mathbb{M}^{3\times 3}} g\left(x,\Lambda\right) d\nu_x(\Lambda) dx \right| \\
& + \left| \int_{(\omega\setminus\omega_k) \times I} \int_{\mathbb{M}^{3\times 3}} g\left(x,\Lambda\right) d\nu_x^k(\Lambda) dx - \int_{(\omega\setminus\omega_k) \times I} \int_{\mathbb{M}^{3\times 3}} g\left(x,\Lambda\right) d\nu_x(\Lambda) dx \right|.
\end{aligned} \tag{5.27}$$

Since $g \in \mathcal{E}_{p,\Omega},\ \nu \in \mathbb{X}$, and $\mathcal{L}^2(\omega \setminus \omega_k) \to 0$ as $k \to +\infty$, we obtain

$$\begin{aligned}
& \lim_{k\to+\infty} \left| \int_{(\omega\setminus\omega_k) \times I} \int_{\mathbb{M}^{3\times 3}} g\left(x,\Lambda\right) d\nu_x^k(\Lambda) dx - \int_{(\omega\setminus\omega_k) \times I} \int_{\mathbb{M}^{3\times 3}} g\left(x,\Lambda\right) d\nu_x(\Lambda) dx \right| \\
& = \lim_{k\to+\infty} \left\{ \left| \int_{(\omega\setminus\omega_k) \times I} g(x,0) dx \right| + \int_{(\omega\setminus\omega_k) \times I} \int_{\mathbb{M}^{3\times 3}} C(1+|\Lambda|^p) d\nu_x(\Lambda) dx \right\} = 0.
\end{aligned} \tag{5.28}$$

For $U \in \mathcal{F}_k$, we have

$$\begin{aligned}
& \left| \int_{U \times I} \int_{\mathbb{M}^{3\times 3}} g\left(x,\Lambda\right) d\nu_x^k(\Lambda) dx - \int_{U \times I} \int_{\mathbb{M}^{3\times 3}} g\left(x,\Lambda\right) d\nu_x(\Lambda) dx \right| \\
& \leq I_1^{k,U} + I_2^{k,U} + I_3^{k,U},
\end{aligned} \tag{5.29}$$

where

$$I_1^{k,U} := \left| \int_{U \times I} \int_{\mathbb{M}^{3\times 3}} g\left(x,\Lambda\right) d\nu_x^k(\Lambda) dx - \int_{U \times I} \int_{\mathbb{M}^{3\times 3}} g\left((x_\alpha^U, x_3),\Lambda\right) d\nu_x^k(\Lambda) dx \right|,$$

$$I_2^{k,U} := \left| \int_{U \times I} \int_{\mathbb{M}^{3\times 3}} g\left((x_\alpha^U, x_3),\Lambda\right) d\nu_x^k(\Lambda) dx - \int_{U \times I} \int_{\mathbb{M}^{3\times 3}} g\left((x_\alpha^U, x_3),\Lambda\right) d\nu_x(\Lambda) dx \right|,$$

and

$$I_3^{k,U} := \left| \int_{U \times I} \int_{\mathbb{M}^{3\times 3}} g\left((x_\alpha^U, x_3),\Lambda\right) d\nu_x(\Lambda) dx - \int_{U \times I} \int_{\mathbb{M}^{3\times 3}} g(x,\Lambda) d\nu_x(\Lambda) dx \right|.$$

We first note that $I_2^{k,U} = 0$. Also, we have

$$I_1^{k,U} \leq \int_{U\times I}\int_{\mathbb{M}^{3\times 3}} \left|g\left(x,\Lambda\right) - g\left(\left(x_\alpha^U, x_3\right),\Lambda\right)\right| d\nu_x^k(\Lambda)dx \leq \frac{C}{k}\int_{U\times I}\int_{\mathbb{M}^{3\times 3}} (1+2|\Lambda|^{p-1})d\nu_x^k(\Lambda)dx$$

$$= \frac{C}{k}\int_I\int_U\int_{\mathbb{M}^{3\times 3}} (1+2|\Lambda|^{p-1})d\nu_{(x_\alpha,x_3)}(\Lambda)dx_\alpha dx_3 \leq \frac{C}{k}\int_{U\times I}\left(3+2\int_{\mathbb{M}^{3\times 3}}|\Lambda|^p d\nu_x(\Lambda)\right)dx.$$

Similarly, we find

$$I_3^{k,U} \leq \frac{C}{k}\int_{U\times I}\int_{\mathbb{M}^{3\times 3}} (1+2|\Lambda|^{p-1})d\nu_x(\Lambda)dx \leq \frac{C}{k}\int_{U\times I}\left(3+2\int_{\mathbb{M}^{3\times 3}}|\Lambda|^p d\nu_x(\Lambda)\right)dx.$$

Thus, taking into account (5.29),

$$\sum_{U\in\mathcal{F}_k}\left|\int_{U\times I}\int_{\mathbb{M}^{3\times 3}} g\left(x,\Lambda\right)d\nu_x^k(\Lambda)dx - \int_{U\times I}\int_{\mathbb{M}^{3\times 3}} g\left(x,\Lambda\right)d\nu_x(\Lambda)dx\right|$$
$$\leq \frac{2C}{k}\left(3\mathcal{L}^2(\omega_k) + 2\int_\Omega\int_{\mathbb{M}^{3\times 3}}|\Lambda|^p d\nu_x(\Lambda)dx\right) \to 0 \text{ as } k\to+\infty,$$

where we have used again the fact that $\nu \in \mathbb{X}$. In view of (5.27) and (5.28) we deduce (5.26), thus completing Step 2.

Step 3. $\mathbb{D} \subset \mathbb{Y}$.

We show that

$$D_k \subset \mathbb{Y} \text{ for all } k\in\mathbb{N}. \tag{5.30}$$

Upon a rescaling argument, we may assume, without loss of generality, that $\omega \subset Q'$. Fix $k \in \mathbb{N}$, and denote the members of the family $\mathcal{F}_k$ by Q_i, $i\in\{1,\cdots,m\}$, for some positive integer m, so that $\omega_k = \bigcup_{i=1}^m Q_i$. Fix $\nu \in D_k$, and set $\nu^{(i)} = \{\nu_{x_3}^{(i)}\}_{x_3\in I} := \nu_{|Q_i\times I}$. Thus, $\nu^{(i)} \in Y$ (see the paragraph immediately following the proof of Lemma 5.1 for the definition of Y) and, since in particular $\nu \in \mathbb{T}$, we have

$$\int_I\int_{\mathbb{M}^{3\times 3}} \Lambda d\nu_{x_3}^{(i)}(\Lambda)dx_3 = 0, \text{ and } \int_I\int_{\mathbb{M}^{3\times 3}} f(x_3,\Lambda)d\nu_{x_3}^{(i)}(\Lambda)dx_3 \geq \widehat{Q}f(0|0) \text{ for all } f\in\mathcal{E}_p.$$

We are now in position to apply Lemma 5.2. We deduce that $\nu^{(i)} \in \mathbb{H}$ for every $i\in\{1,\cdots,m\}$. In view of Lemma 5.1, we may assume, without loss of generality, that each Young measure $\nu^{(i)}$ is generated by a p-equi-integrable sequence $\left\{\left(\nabla_\alpha v_n^{(i)}|n\nabla_3 v_n^{(i)}\right)\right\}$, with $v_n^{(i)} \rightharpoonup 0$ weakly in $W^{1,p}(Q'\times I;\mathbb{R}^3)$, $n\int_I \nabla_3 v_n^{(i)}(\cdot,x_3) \rightharpoonup 0$ weakly in $L^p(Q';\mathbb{R}^3)$ and, moreover, $v_n^{(i)} = 0$ on $\partial Q_i\times I$. Let $\{\varphi_n^{(i)}\} \subset C_c^\infty(Q_i;[0,1])$ be a sequence of smooth functions such that $\varphi_n^{(i)} \nearrow \chi_{Q_i}$, and define $u_n \in W^{1,p}(\omega;\mathbb{R}^3)$, by

$$u_n(x) := \sum_{i=1}^m \varphi_n^{(i)}(x_\alpha)v_n^{(i)}(x).$$

We have

$$\lim_{n\to+\infty}\int_\Omega |u_n|^p dx = 0,$$

$$n\int_I \nabla_3 u_n(\cdot,x_3) \rightharpoonup 0, \text{ weakly in } L^p(\omega;\mathbb{R}^3),$$

and, for $\theta\in C_0(\Omega), \Psi\in C_0(\mathbb{M}^{3\times 3})$,

$$\begin{aligned}
&\lim_{n\to+\infty}\int_\Omega \theta(x)\Psi(\nabla_\alpha u_n|n\nabla_3 u_n)dx\\
&=\lim_{n\to+\infty}\left(\sum_{i=1}^m\int_{Q_i\times I}\theta(x)\Psi(\nabla_\alpha v_n^{(i)}|n\nabla_3 v_n^{(i)})dx+\int_{(\omega\setminus\omega_k)\times I}\theta(x)\Psi(0)dx\right)\\
&=\sum_{i=1}^m\int_{Q_i\times I}\theta(x)\int_{\mathbb{M}^{3\times 3}}\Psi(\Lambda)d\nu_{x_3}^{(i)}(\Lambda)dx+\int_{(\omega\setminus\omega_k)\times I}\theta(x)\int_{\mathbb{M}^{3\times 3}}\Psi(\Lambda)d\delta_0(\Lambda)dx\\
&=\int_\Omega\theta(x)\int_{\mathbb{M}^{3\times 3}}\Psi(\Lambda)d\nu_x(\Lambda)dx,
\end{aligned}$$

and we deduce that ν is generated by $\{(\nabla_\alpha u_n|n\nabla_3 u_n)\}$. Therefore $\nu\in\mathbb{Y}$, asserting (5.30). This completes Step 3.

Step 4. Let ν be an arbitrary element of $\mathbb{T}$. By Step 2, $\nu\in\overline{\mathbb{D}}^{\mathcal{E}'_{p,\Omega}}$. In view of Step 3, and taking into account the fact that $\mathbb{T}\subset\mathbb{X}$, we obtain that $\nu\in\overline{\mathbb{Y}}^{\mathcal{E}'_{p,\Omega}}\cap\mathbb{X}$. Thus, $\nu\in\mathbb{Y}$ (by Step 1). We conclude that (4.18) holds, which completes the proof of Proposition 4.2.

6 Appendix

In this appendix we provide an alternative, self-contained proof of Lemma 4.1. Apart from slight technical differences, the proof follows essentially that of Lemma 3.1 in [11].

Step 1. We first prove the lemma under the additional assumption that $A = Q' := (0,1)^2\subset\omega$. Since $\left\{\left(\nabla_\alpha u_n\middle|\frac{1}{\varepsilon_n}\nabla_3 u_n\right)\right\}$ is p-equi-integrable, and taking into account (4.2), we may assume, after eventually passing to a subsequence (not relabelled), that

$$L:=\liminf_{n\to+\infty}\int_{Q'\times I} f\left(x_3,\left(\nabla_\alpha u_n\middle|\frac{1}{\varepsilon_n}\nabla_3 u_n\right)\right)dx=\lim_{n\to+\infty}\int_{Q'\times I} f\left(x_3,\left(\nabla_\alpha u_n\middle|\frac{1}{\varepsilon_n}\nabla_3 u_n\right)\right)dx<+\infty.$$

We may construct a subsequence $\{n_k\}$ of $\{n\}$ and a sequence $\{v_k\}\subset W^{1,p}(Q'\times I;\mathbb{R}^3)$ such that $\left\{\left(\nabla_\alpha v_k\middle|\frac{1}{\varepsilon_{n_k}}\nabla_3 v_k\right)\right\}$ is bounded in $L^p(\Omega,\mathbb{M}^{3\times 3})$, $\displaystyle\frac{1}{\varepsilon_{n_k}}\int_I\nabla_3 v_k(\cdot,x_3)dx_3\rightharpoonup 0$ weakly in $L^p(Q',\mathbb{R}^3)$, $v_k\rightharpoonup 0$ weakly in $W^{1,p}(Q'\times I;\mathbb{R}^3)$, $v_k = 0$ on $\partial Q'\times I$, and

$$\limsup_{k\to+\infty}\int_{Q'\times I} f\left(x_3,\left(\nabla_\alpha v_k\middle|\frac{1}{\varepsilon_{n_k}}\nabla_3 v_k\right)\right)dx\le L. \tag{6.1}$$

This can be done in the usual way (see the proofs of Lemma 2.2 in [11], and Lemma 2.7 in [12]). Note that since our original sequence $\left\{\left(\nabla_\alpha u_n\middle|\frac{1}{\varepsilon_n}\nabla_3 u_n\right)\right\}$ is p-equi-integrable, a coercivity condition on f (which is usually needed in such an analysis; see hypotheses (H1) in [11] and (2.1) in [12]) is not necessary in this case.

We may now extend $v_k(\cdot, x_3)$ by periodicity to the whole $\mathbb{R}^2$ and we define $w_k \in W^{1,p}(\mathbb{R}^2 \times I)$ by

$$w_k(x_\alpha, x_3) := v_k(x_\alpha, x_3) - x_3 \int_{Q'\times I} \nabla_3 v_k dy.$$

Note that $\left\{\left(\nabla_\alpha w_k\middle|\frac{1}{\varepsilon_{n_k}}\nabla_3 w_k\right)\right\}$ is bounded in $L^p(\Omega, \mathbb{M}^{3\times 3})$, $w_k(\cdot, x_3)$ is Q' -periodic, for $\mathcal{L}^1$ - a.e. $x_3 \in I$, $\dfrac{1}{\varepsilon_{n_k}} \displaystyle\int_{Q'\times I} \nabla_3 w_k dx = 0$. In particular, the pair $\left(w_k, \frac{1}{\varepsilon_{n_k}}\right)$ is admissible for the definition of $\widehat{Q}f(0|0)$. It follows that

$$\widehat{Q}f(0|0) \leq \int_{Q'\times I} f\left(x_3, \left(\nabla_\alpha w_k\middle|\frac{1}{\varepsilon_{n_k}}\nabla_3 w_k\right)\right) dx \text{ for all } k \in \mathbb{N}. \tag{6.2}$$

In view of (4.3), the fact that $\dfrac{1}{\varepsilon_{n_k}} \displaystyle\int_I \nabla_3 v_k(\cdot, x_3) dx_3 \rightharpoonup 0$ weakly in $L^p(Q', \mathbb{R}^3)$, the uniform bounds on $\left\{\left(\nabla_\alpha v_k\middle|\frac{1}{\varepsilon_{n_k}}\nabla_3 v_k\right)\right\}$ and $\left\{\left(\nabla_\alpha w_k\middle|\frac{1}{\varepsilon_{n_k}}\nabla_3 w_k\right)\right\}$ in $L^p(\Omega, \mathbb{M}^{3\times 3})$, and by means of Hölder's inequality, we obtain that

$$\lim_{k\to+\infty} \left| \int_{Q'\times I} f\left(x_3, \left(\nabla_\alpha w_k\middle|\frac{1}{\varepsilon_{n_k}}\nabla_3 w_k\right)\right) dx - \int_{Q'\times I} f\left(x_3, \left(\nabla_\alpha v_k\middle|\frac{1}{\varepsilon_{n_k}}\nabla_3 v_k\right)\right) dx \right| = 0.$$

Thus,

$$\liminf_{k\to+\infty} \int_{Q'\times I} f\left(x_3, \left(\nabla_\alpha v_k\middle|\frac{1}{\varepsilon_{n_k}}\nabla_3 v_k\right)\right) dx \geq \liminf_{k\to+\infty} \int_{Q'\times I} f\left(x_3, \left(\nabla_\alpha w_k\middle|\frac{1}{\varepsilon_{n_k}}\nabla_3 w_k\right)\right) dx.$$

Taking into account (6.1) and (6.2), we deduce that

$$L \geq \widehat{Q}f(0|0),$$

and this completes Step 1.

Step 2. We now turn to the general case. As in Step 1, and eventually passing to a subsequence (not relabelled), we may assume that

$$L := \liminf_{n\to+\infty} \int_{A\times I} f\left(x_3, \left(\nabla_\alpha u_n\middle|\frac{1}{\varepsilon_n}\nabla_3 u_n\right)\right) dx = \lim_{n\to+\infty} \int_{A\times I} f\left(x_3, \left(\nabla_\alpha u_n\middle|\frac{1}{\varepsilon_n}\nabla_3 u_n\right)\right) dx < +\infty.$$

For each $n \in \mathbb{N}$, define the measure $\mu_n := \left(\displaystyle\int_I f\left(x_3, \left(\nabla_\alpha u_n\middle|\frac{1}{\varepsilon_n}\nabla_3 u_n\right)\right) dx_3\right) \mathcal{L}^2 \lfloor A$. Since the sequence $\{\mu_n\}$ is bounded, there exists μ such that, upon extracting a further subsequence (not relabelled) μ_n convergence weakly* to μ. Let $\mu^{(a)}$ be the absolute continuous part of μ with respect to $\mathcal{L}^2$. The set $A^0 := \left\{ z \in A : \mu^{(a)}(z) = \lim_{\delta \to 0} \frac{\mu(z+\delta Q')}{\delta^2} \right\}$ is a set of full $\mathcal{L}^2$ measure in A. Fix x^0 to be an arbitrary element of A^0. Let $\delta_k \to 0^+$ be such that $\mu(\partial(x^0 + \delta_k Q')) = 0$ for all $k \in \mathbb{N}$, and define $u_{n,k}(y) := \frac{1}{\delta_k} u_n(x^0 + \delta_k y_\alpha, y_3)$. We have, changing variables,

$$\mu^a(x^0) = \lim_{k\to\infty} \frac{\mu(x^0 + \delta_k Q')}{\delta_k^2} = \lim_{k\to\infty} \lim_{n\to\infty} \frac{1}{\delta_k^2} \int_{(x^0+\delta_k Q')\times I} f\left(y_3, \left(\nabla_\alpha u_n \Big| \frac{1}{\varepsilon_n} \nabla_3 u_n\right)\right) dy$$

$$= \lim_{k\to\infty} \lim_{n\to\infty} \int_{Q'\times I} f\left(y_3, \left(\nabla_\alpha u_{n,k} \Big| \frac{\delta_k}{\varepsilon_n} \nabla_3 u_{n,k}\right)\right) dy \tag{6.3}$$

Let $\varphi \in L^{p'}(Q'; \mathbb{R}^3)$. Since $\frac{1}{\varepsilon_n} \int_I \nabla_3 u_n(\cdot, x_3) dx_3 \rightharpoonup 0$ weakly in $L^p(A, \mathbb{R}^3)$, we have

$$\lim_{k\to\infty} \lim_{n\to\infty} \int_{Q'} \left(\frac{\delta_k}{\varepsilon_n} \int_I \nabla_3 u_{n,k}(x_\alpha, x_3) dx_3 \right) \varphi(x_\alpha) dx_\alpha$$

$$= \lim_{k\to\infty} \lim_{n\to\infty} \int_{Q'} \left(\frac{1}{\varepsilon_n} \int_I \nabla_3 u_n(x^0 + \delta_k x_\alpha, x_3) dx_3 \right) \varphi(x_\alpha) dx_\alpha$$

$$= \lim_{k\to\infty} \lim_{n\to\infty} \frac{1}{\delta_k^2} \int_{x^0+\delta_k Q'} \left(\frac{1}{\varepsilon_n} \int_I \nabla_3 u_n(y_\alpha, y_3) dy_3 \right) \varphi\left(\frac{y_\alpha - x^0}{\delta_k}\right) dy_\alpha = 0. \tag{6.4}$$

Also, note that $\lim_{k\to\infty} \lim_{n\to\infty} \frac{\varepsilon_n}{\delta_k} = 0$, $\lim_{k\to\infty} \lim_{n\to\infty} \|u_{n,k}\|_{L^p((Q'\times I;\mathbb{R}^3)} = 0$,

$$\limsup_{k\to\infty} \limsup_{n\to\infty} \int_{Q'\times I} \left| \left(\nabla_\alpha u_{n,k} \Big| \frac{\delta_k}{\varepsilon_n} \nabla_3 u_{n,k} \right) \right|^p dx < +\infty,$$

and, recalling (6.3) and (6.4), a standard diagonalization argument allows us to construct a subsequence $\{n(k)\}$ of $\{n\}$ such that $\overline{\varepsilon}_k := \frac{\varepsilon_{n(k)}}{\delta_k} \to 0^+$ and, after defining $\overline{u}_k := u_{n(k),k}$, we have $\overline{u}_k \rightharpoonup 0$ weakly in $W^{1,p}(Q' \times I; \mathbb{R}^3)$, $\left\{ \left(\nabla_\alpha \overline{u}_k \Big| \frac{1}{\overline{\varepsilon}_k} \nabla_3 \overline{u}_k \right) \right\}$ is bounded in $L^p(Q' \times I; \mathbb{M}^{3\times 3})$,

$$\frac{1}{\overline{\varepsilon}_k} \int_I \nabla_3 \overline{u}_k(\cdot, x_3) dx_3 \rightharpoonup 0 \text{ weakly in } L^p(Q', \mathbb{R}^3),$$

and

$$\mu^a(x^0) = \lim_{k\to\infty} \int_{Q'\times I} f\left(x_3, \left(\nabla_\alpha \overline{u}_k \Big| \frac{1}{\overline{\varepsilon}_k} \nabla_3 \overline{u}_k\right)\right) dx.$$

It follows that the sequences $\{\overline{\varepsilon}_k\}$ and $\{\overline{u}_k\}$ satisfy the conditions of Step 1 and we have that

$$\mu^a(x^0) \geq \widehat{Q} f(0|0). \tag{6.5}$$

In view of the weak* convergence of μ_n to μ, and since (6.5) holds for $\mathcal{L}^2$-a.e. $x^0 \in A$, we deduce (4.1).

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