# A NOTE ON MEYERS' THEOREM IN $W^{k,1}$

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Abstract. Lower semicontinuity properties of multiple-integrals

$$u \in W^{k,1}(\Omega;\mathbb{R}^d) \mapsto \int_\Omega f(x, u(x), \cdots, \nabla^k u(x))\, dx$$

are studied when $f$ grows at most linearly with respect to the highest order derivative, $\nabla^k u$, and admissible $W^{k,1}(\Omega;\mathbb{R}^d)$ sequences converge strongly in $W^{k-1,1}(\Omega;\mathbb{R}^d)$. It is shown that under certain continuity assumptions on $f$, convexity, 1-quasiconvexity or $k$-polyconvexity of $\xi \longmapsto f(x_0, u(x_0), \cdots, \nabla^{k-1}u(x_0), \xi)$ ensure lower semicontinuity. The case where $f(x_0, u(x_0), \cdots, \nabla^{k-1}u(x_0), \cdot)$ is $k$-quasiconvex remains open except in some very particular cases, such as when $f(x, u(x), \cdots, \nabla^k u(x)) = h(x)g(\nabla^k u(x))$.

## 1. Introduction

In a classical paper Meyers [26] proved that $k$-quasiconvexity is a necessary and sufficient condition for (sequential) lower semicontinuity of a functional

$$u \mapsto \int_\Omega f(x, u(x), \cdots, \nabla^k u(x))\, dx,$$

with respect to weak convergence (resp. weak* convergence if $p = \infty$) in $W^{k,p}(\Omega;\mathbb{R}^d)$ and under appropriate growth and continuity conditions on the integrand $f$, thus extending to the case $k > 1$ the notion of quasiconvexity introduced by Morrey when $k = 1$. Here $\Omega$ is an open, bounded subset of $\mathbb{R}^N$, with $N \geq 1$, and $k, d \in \mathbb{N}$, $1 \leq p \leq \infty$. Meyers' theorem uses results of Agmon, Douglis and Nirenberg [1] concerning Poisson kernels for elliptic equations. Fusco [22] later gave a simpler proof using De Giorgi's Slicing Lemma. He also extended the result to Carathéodory integrands when $p = 1$, while the case $p > 1$ has been recently established by Guidorzi and Poggiolini [24] under the Lipschitz condition

$$|f(x, \mathbf{v}, \xi) - f(x, \mathbf{v}, \xi_1)| \leq C(1 + |\xi|^{p-1} + |\xi_1|^{p-1})|\xi - \xi_1|$$

(note that this condition is automatically satisfied for $k = 1$ and $k = 2$, see [25] and [24]), and by Braides, Fonseca and Leoni in [8], who obtained a general relaxation result in $W^{k,p}(\Omega;\mathbb{R}^d)$ with respect to weak convergence.

In most applications, the lower semicontinuity results mentioned above are completely satisfactory when $p > 1$ since bounded sequences in $W^{k,p}(\Omega;\mathbb{R}^d)$ admit weakly convergent subsequences. However, when $p = 1$ due to loss of reflexivity of the space $W^{k,1}(\Omega;\mathbb{R}^d)$ one can only conclude that an energy bounded sequence $\{u_n\} \subset W^{k,1}(\Omega;\mathbb{R}^d)$ with

$$\sup_n \|u_n\|_{W^{k,1}} < \infty$$

admits a subsequence (not relabelled) such that

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$$u_n \to u \quad \text{in } W^{k-1,1}(\Omega;\mathbb{R}^d), \tag{1.1}$$

where $u \in W^{k-1,1}(\Omega;\mathbb{R}^d)$ and $\nabla^{k-1}u$ is a vector-valued function of bounded variation. In this paper we seek to establish lower semicontinuity in the space $W^{k,1}(\Omega;\mathbb{R}^d)$ under this natural notion of convergence.

When $k=1$ the scalar case $d=1$ has been extensively treated, while the vectorial case $d>1$ was first studied by Fonseca and Müller in [19] where it was proven (sequential) lower semicontinuity in $W^{1,1}(\Omega;\mathbb{R}^d)$ of a functional

$$u \mapsto \int_\Omega f(x,u(x),\nabla u(x))\,dx,$$

with respect to strong convergence in $L^1(\Omega;\mathbb{R}^d)$ (see also [4], [20], [17], [18] and the references contained therein). The approach in [19] is based on blow–up and truncation methods.

Similar truncation techniques have been used quite successfully in the study of existence and qualitative properties of solutions of second order elliptic equations and systems (see e.g. the work of [7] and the references contained within). Their main drawback lies in the fact that they cannot be easily extended to truncate gradients or higher order derivatives. This may explain in part why several important results for second order elliptic equations have no analog for higher order equations.

The main result of this paper extends Meyers' Theorem to the case where weak convergence in $W^{k,1}(\Omega;\mathbb{R}^d)$ is replaced by (0.1) together with a weak form of coercivity of the convex, 1-quasiconvex or $k$-polyconvex density $f$ (see Theorems 2, 4 and 5 below). We start with the case where $f$ depends essentially only on $x$ and on the highest order derivatives, that is $\nabla^k u(x)$. This situation is significantly simpler than the general case, since it does not require to truncate the initial sequence $\{u_n\} \subset W^{k,1}(\Omega;\mathbb{R}^d)$. Using the notation and terminology introduced in Section 2, we state the following:

**Theorem 1.** *Let $f:\Omega\times E^d_{[k-1]}\times E^d_k \to [0,\infty)$ be a Borel integrand. Suppose that for all $(x_0,\mathbf{v}_0)\in\Omega\times E^d_{[k-1]}$ and $\varepsilon>0$ there exist $\delta_0>0$ and a modulus of continuity $\rho$, with $\rho(s)\le C_0(1+s)$ for $s>0$ and for some $C_0>0$, such that*

$$f(x_0,\mathbf{v}_0,\xi) - f(x,\mathbf{v},\xi) \le \varepsilon(1+f(x,\mathbf{v},\xi)) + \rho(|\mathbf{v}-\mathbf{v}_0|) \tag{1.2}$$

*for all $x\in\Omega$ with $|x-x_0|\le\delta_0$, and for all $(\mathbf{v},\xi)\in E^d_{[k-1]}\times E^d_k$. Assume also that one of the following three conditions is satisfied:*
*(a) $f(x_0,\mathbf{v}_0,\cdot)$ is $k$-quasiconvex in $E^d_k$ and*

$$\frac{1}{C_1}|\xi| - C_1 \le f(x_0,\mathbf{v}_0,\xi) \le C_1(1+|\xi|) \qquad \text{for all } \xi \in E^d_k, \tag{1.3}$$

*where $C_1>0$;*
*(b) $f(x_0,\mathbf{v}_0,\cdot)$ is 1-quasiconvex in $E^d_k$ and*

$$0 \le f(x_0,\mathbf{v}_0,\xi) \le C_1(1+|\xi|) \qquad \text{for all } \xi \in E^d_k, \tag{1.4}$$

*where $C_1>0$;*
*(c) $f(x_0,\mathbf{v}_0,\cdot)$ is convex in $E^d_k$.*
*Let $u\in BV^k(\Omega;\mathbb{R}^d)$ and let $\{u_n\}$ be a sequence of functions in $W^{k,1}(\Omega;\mathbb{R}^d)$ converging to $u$ in $W^{k-1,1}(\Omega;\mathbb{R}^d)$. Then*

$$\int_\Omega f(x,u,\dots,\nabla^k u)\,dx \le \liminf_{n\to\infty}\int_\Omega f(x,u_n,\dots,\nabla^k u_n)\,dx.$$

Here $\nabla^k u$ is the Radon–Nikodym derivative of the distributional derivative $D^k u$ of $\nabla^{k-1}u$, with respect to the $N$–dimensional Lebesgue measure $\mathcal{L}^N$. An important class of integrands which satisfy (0.2) of Theorem 1 is given by

$$f = f(x,\xi) := h(x)g(\xi),$$

where $h(x)$ is a nonnegative lower semicontinuous function and $g$ is a nonnegative function which satisfies either (a) or (b) or (c). The case where $h(x)\equiv 1$ and $g$ satisfies condition (a) was proved by Amar and De Cicco [2]. Theorem 1 extends a result of Fonseca and Leoni (Theorem 1.7 in [17]) to higher order derivatives, where the statement is exactly that of Theorem 1 setting $k=1$ and excluding part (a). Related results when

$k=1$ where obtained previously by Serrin [28] in the scalar case $d=1$ and by Ambrosio and Dal Maso [4] in the vectorial case $d>1$ (see also Fonseca and Müller [19], [20]). Even in the simple case where $f=f(\xi)$ it is not known if Theorem 1(a) still holds without the coercivity condition

$$f(\xi) \geq \frac{1}{C_1}|\xi| - C_1.$$

The main tool in the proof of Theorem 1, used also in an essential way in subsequent results, is the *blow–up method* introduced by Fonseca and Müller [19], [20], which reduces the domain $\Omega$ to a ball and the target function $u$ to a polynomial.

When the integrand $f$ depends on the full set of variables in an essential way, the situation becomes significantly more complicated since one needs to truncate gradients and higher order derivatives in order to localize lower order terms.

The following theorem was proved for $k=1$ by Fonseca and Leoni in [17] Theorem 1.8. Here we extend the result to the higher order case.

**Theorem 2.** *Let $f : \Omega \times E^d_{[k-1]} \times E^d_k \to [0,\infty)$ be a Borel integrand, with $f(x,\mathbf{v},\cdot)$ 1-quasiconvex in $E^d_k$. Suppose that for all $(x_0,\mathbf{v}_0) \in \Omega \times E^d_{[k-1]}$ either $f(x_0,\mathbf{v}_0,\cdot) \equiv 0$, or for every $\varepsilon > 0$ there exist $C$, $\delta_0 > 0$ such that*

$$f(x_0,\mathbf{v}_0,\xi) - f(x,\mathbf{v},\xi) \leq \varepsilon(1+f(x,\mathbf{v},\xi)), \tag{1.5}$$

$$C|\xi| - \frac{1}{C} \leq f(x_0,\mathbf{v}_0,\xi) \leq C(1+|\xi|) \tag{1.6}$$

*for all $(x,\mathbf{v}) \in \Omega \times E^d_{[k-1]}$ with $|x-x_0|+|\mathbf{v}-\mathbf{v}_0| \leq \delta_0$ and for all $\xi \in E^d_k$. Let $u \in BV^k(\Omega;\mathbb{R}^d)$, and let $\{u_n\}$ be a sequence of functions in $W^{k,1}(\Omega;\mathbb{R}^d)$ converging to $u$ in $W^{k-1,1}(\Omega;\mathbb{R}^d)$. Then*

$$\int_\Omega f(x,u,\ldots,\nabla^k u)\,dx \leq \liminf_{n\to\infty} \int_\Omega f(x,u_n,\ldots,\nabla^k u_n)\,dx.$$

A standing open problem is to decide whether Theorem 2 continues to hold under the weaker assumption that $f(x,\mathbf{v},\cdot)$ is $k$-quasiconvex, which is the natural assumption in this context.

In the scalar case $d=1$, that is when $u$ is an $\mathbb{R}$-valued function, and for first order gradients, i.e. $k=1$, condition (0.6) can be eliminated, see Theorem 1.1 in [17]. In particular in [17] Fonseca and Leoni have shown the following result

**Proposition 1.** *(cf. [17], Corollary 1.2) Let $g : \mathbb{R}^N \to [0,\infty)$ be a convex function, and let $h : \Omega \times \mathbb{R} \to [0,\infty)$ be a lower semicontinuous function. If $u \in BV(\Omega;\mathbb{R})$ and $\{u_n\} \subset W^{1,1}(\Omega;\mathbb{R})$ converges to $u$ in $L^1(\Omega;\mathbb{R})$, then*

$$\int_\Omega h(x,u)g(\nabla u)\,dx \leq \liminf_{n\to\infty} \int_\Omega h(x,u_n)g(\nabla u_n)\,dx.$$

It is interesting to observe that an analog of this result is false when $k \geq 2$.

**Theorem 3.** *Let $\Omega := (0,1)^N$, $N \geq 3$, and let $h$ be a smooth cut-off function on $\mathbb{R}$ with $0 \leq h \leq 1$, $h(u)=1$ for $u \leq \frac{1}{2}$, $h(u)=0$ for $u \geq 1$. There exists a sequence of functions $\{u_n\}$ in $W^{2,1}(\Omega;\mathbb{R})$ converging to zero in $W^{1,1}(\Omega;\mathbb{R})$ such that $\{\|\Delta u_n\|_{L^1(\Omega;\mathbb{R})}\}$ is uniformly bounded and*

$$\limsup_{n\to\infty} \int_\Omega h(u_n)(1-\Delta u_n)^+\,dx < \int_\Omega h(0)\,dx.$$

As in Theorem 1, conditions (0.5) and (0.6) can be considerably weakened if we assume that $f(x,\mathbf{v},\cdot)$ is convex rather than 1-quasiconvex. Indeed we have the following result:

**Theorem 4.** *Let $f : \Omega \times E^d_{[k-1]} \times E^d_k \to [0,\infty]$ be a lower semicontinuous function, with $f(x,\mathbf{v},\cdot)$ convex in $E^d_k$. Suppose that for all $(x_0,\mathbf{v}_0) \in \Omega \times E^d_{[k-1]}$ either $f(x_0,\mathbf{v}_0,\cdot) \equiv 0$, or there exist $C_1$, $\delta_0 > 0$, and a continuous function $g : B(x_0,\delta_0) \times B(\mathbf{v}_0,\delta_0) \to E^d_k$ such that*

$$f(x,\mathbf{v},g(x,\mathbf{v})) \in L^\infty\left(B(x_0,\delta_0) \times B(\mathbf{v}_0,\delta_0);\mathbb{R}\right), \tag{1.7}$$

$$f(x,\mathbf{v},\xi) \geq C_1|\xi| - \frac{1}{C_1} \tag{1.8}$$

*for all* $(x,\mathbf{v}) \in \Omega \times E^d_{[k-1]}$ *with* $|x-x_0|+|\mathbf{v}-\mathbf{v}_0| \leq \delta_0$ *and for all* $\xi \in E^d_k$. *Let* $u \in BV^k(\Omega;\mathbb{R}^d)$, *and let* $\{u_n\}$ *be a sequence of functions in* $W^{k,1}(\Omega;\mathbb{R}^d)$ *converging to* $u$ *in* $W^{k-1,1}(\Omega;\mathbb{R}^d)$. *Then*

$$\int_\Omega f(x,u,\dots,\nabla^k u)\,dx \leq \liminf_{n\to\infty}\int_\Omega f(x,u_n,\dots,\nabla^k u_n)\,dx.$$

Theorem 4 was obtained by Fonseca and Leoni for the case $k=1$ in Theorem 1.1 [18]. It is interesting to observe that without a condition of the type (0.7) Theorem 4 is false in general. This has been recently proved by Černý and Malý in [12].

The proofs of Theorems 1(b) and (c), 2 and 4 can be deduced easily from the corresponding ones in [17], [18] where $k=1$. It suffices to write

$$\int_\Omega f(x,u(x),\dots,\nabla^k u(x))\,dx =: \int_\Omega F(x,\mathbf{v}(x),\nabla\mathbf{v}(x))\,dx$$

with $\mathbf{v} := \left(u,\dots,\nabla^{k-1}u\right)$, and then to perturb the new integrand $F$ in order to recover the full coercivity conditions necessary to apply the results in [17], [18]. This approach cannot be used for $k$-polyconvex integrands and a new proof is needed to treat this case. Thus Theorem 1(a) and Theorem 5 below are the only truly genuine higher order results, in that they cannot be reduced in a trivial way to a first order problem.

For each $\xi \in E^d_k$ let $\mathcal{M}(\xi) \in \mathbb{R}^\tau$ be the vector whose components are all the minors of $\xi$.

**Theorem 5.** *Let* $h : \Omega \times E^d_{[k-1]} \times \mathbb{R}^\tau \to [0,\infty]$ *be a lower semicontinuous function, with* $h(x,\mathbf{v},\cdot)$ *convex in* $\mathbb{R}^\tau$. *Suppose that for all* $(x_0,\mathbf{v}_0) \in \Omega \times E^d_{[k-1]}$ *either* $h(x_0,\mathbf{v}_0,\cdot) \equiv 0$, *or there exist* $C$, $\delta_0 > 0$, *and a continuous function* $g : B(x_0,\delta_0) \times B(\mathbf{v}_0,\delta_0) \to \mathbb{R}^\tau$ *such that*

$$h(x,\mathbf{v},g(x,\mathbf{v})) \in L^\infty\left(B(x_0,\delta_0)\times B(\mathbf{v}_0,\delta_0);\mathbb{R}\right), \tag{1.9}$$

$$h(x,\mathbf{v},v) \geq C|v| - \frac{1}{C} \tag{1.10}$$

*for all* $(x,\mathbf{v}) \in \Omega \times E^d_{[k-1]}$ *with* $|x-x_0|+|\mathbf{v}-\mathbf{v}_0| \leq \delta_0$ *and for all* $v \in \mathbb{R}^\tau$. *Let* $u \in BV^k(\Omega;\mathbb{R}^d)$, *and let* $\{u_n\}$ *be a sequence of functions in* $W^{k,p}(\Omega;\mathbb{R}^d)$ *which converges to* $u$ *in* $W^{k-1,1}(\Omega;\mathbb{R}^d)$, *where* $p$ *is the minimum between* $N$ *and the dimension of the vectorial space* $E^d_{k-1}$. *Then*

$$\int_\Omega h(x,u,\dots,\nabla^{k-1}u,\mathcal{M}(\nabla^k u))\,dx \leq \liminf_{n\to\infty}\int_\Omega h(x,u_n,\dots,\nabla^{k-1}u_n,\mathcal{M}(\nabla^k u_n))\,dx.$$

Theorem 5 is closely related to a result of Ball, Currie and Olver [6], where it was assumed that

$$h(x,\mathbf{v},v) \geq \gamma\left(|v|\right) - \frac{1}{C},$$

where

$$\frac{\gamma\left(s\right)}{s} \to \infty \text{ as } s \to \infty.$$

Also, as stated above and with $k=1$, Theorem 5 was proved by Fonseca and Leoni in [18], Theorem 1.4.

## 2. Preliminaries

We start with some notation. Here $\Omega \subset \mathbb{R}^N$ is an open, bounded subset, $\mathcal{L}^N$ and $\mathcal{H}^{N-1}$ are, respectively, the $N$ dimensional Lebesgue measure and the $N-1$ dimensional Hausdorff measure in $\mathbb{R}^N$. Let $Q$ be the the unit cube $(-1/2,1/2)^N$ and set $Q(x_0,\varepsilon) := x_0 + \varepsilon Q$.

For any multi-index $\alpha = (\alpha_1,\dots,\alpha_N) \in \mathbb{N}^N$, we use the notation

$$\nabla^\alpha := \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\dots\partial x_N^{\alpha_N}}, \quad |\alpha| = \alpha_1 + \dots + \alpha_N,$$

and for each $j \in \mathbb{N}$ the symbol $\nabla^j u$ stands for the vector-valued function whose components are all the components of $\nabla^\alpha u$ for $|\alpha| = j$. If $u$ is $C^\infty$ then for $j \geq 2$ we have that $\nabla^j u(x) \in E^d_j$, where $E^d_j$ stands for the space of symmetric $j$-linear maps from $\mathbb{R}^{\mathbb{N}}$ into $\mathbb{R}^d$. We set $E^d_0 := \mathbb{R}^d$, $E^d_1 := \mathbb{R}^{d\times N}$ and

$$E^d_{[j-1]} := E^d_0 \times \cdots \times E^d_{j-1}, \quad E^d_{[0]} := E^d_0.$$

For any integer $k \geq 2$ we define

$$BV^k(\Omega;\mathbb{R}^d) := \left\{u \in W^{k-1,1}(\Omega;\mathbb{R}^d) : \nabla^{k-1}u \in BV(\Omega;E^d_{k-1})\right\},$$

where $\nabla^j u$ is the Radon–Nikodym derivative of the distributional derivative $D^j u$ of $\nabla^{j-1}u$, with respect to the $N$–dimensional Lebesgue measure $\mathcal{L}^N$.

We recall that a function $f : E^d_k \to \mathbb{R}$ is said to be *k-quasiconvex* if

$$f(\xi) \leq \int_Q f(\xi + \nabla^k w(y))\, dy$$

for all $\xi \in E^d_k$ and all $w \in C^\infty_0(Q;\mathbb{R}^d)$ .

The following theorem was proved in the case $k = 1$ by Ambrosio and Dal Maso [4], while Fonseca and Müller [19] treated general integrands of the form $f = f(x, u, \nabla u)$, but their argument requires coercivity. The case $k \geq 2$ is due to Amar and De Cicco [2]. For completeness we give a proof for all $k \geq 1$.

**Proposition 2.** *Let $f : E^d_k \to [0,\infty)$ be a k-quasiconvex function, such that*

$$0 \leq f(\xi) \leq C\,(1 + |\xi|)\,, \tag{2.1}$$

*for all $\xi \in E^d_k$. Moreover, when $k \geq 2$ assume that*

$$f(\xi) \geq C_1\,|\xi| \quad \text{for } |\xi| \text{ large.} \tag{2.2}$$

*Let $\{u_n\}$ be a sequence of functions in $W^{k,1}(Q;\mathbb{R}^d)$ converging to $0$ in $W^{k-1,1}(Q;\mathbb{R}^d)$. Then*

$$f(0) \leq \liminf_{n\to\infty} \int_Q f(\nabla^k u_n)\, dx.$$

*Proof.* We start with the case $k \geq 2$. Without loss of generality we may assume that

$$\liminf_{n\to\infty} \int_Q f(\nabla^k u_n)\, dx = \lim_{n\to\infty} \int_Q f(\nabla^k u_n)\, dx < \infty,$$

so that by condition (1.2)

$$K := \sup_n \int_Q |\nabla^k u_n|\, dx < \infty.$$

Let $\varepsilon > 0$, $M \in \mathbb{N}$, and decompose $L := Q \setminus (1-\varepsilon)\,Q$ into $M$ layers with mutually disjoint interiors, $L_i := \alpha_{i+1}Q \setminus \alpha_i Q$, so that $1 - \varepsilon = \alpha_1 < \alpha_2 < \ldots < \alpha_M < 1 =: \alpha_{M+1}$. Since

$$\sum_{i=1}^M \int_{L_i} \left(1 + |\nabla^k u_n|\right)\, dx \leq 1 + K$$

for all $n \in \mathbb{N}$, there exists $i_\varepsilon \in \{1, \ldots, M\}$ and a subsequence of $\{u_n\}$ (not relabelled) such that

$$\int_{L_{i_\varepsilon}} \left(1 + |\nabla^k u_n|\right) dx \leq \frac{1+K}{M} \qquad \text{for all } n \in \mathbb{N}. \tag{2.3}$$

Let $\varphi \in C^\infty_c(Q;[0,1])$ with $\varphi(x) = 1$ in $\alpha_{i_\varepsilon}Q$, $\varphi(x) = 0$ if $x \notin \alpha_{i_\varepsilon+1}Q$. Since $f$ is $k$-quasiconvex

$$\begin{aligned}
f(0) &\leq \liminf_{n\to\infty} \int_Q f(\nabla^k(\varphi u_n))\, dx \\
&\leq \liminf_{n\to\infty} \int_Q f(\nabla^k u_n)\, dx + \int_{Q \setminus \alpha_{i_\varepsilon+1}Q} f(0)\, dx \\
&\quad + C \limsup_{n\to\infty} \int_{L_{i_\varepsilon}} \left(1 + |\nabla^k(\varphi u_n)|\right)\, dx,
\end{aligned}$$

where we have used (1.1). As $u_n \to 0$ in $W^{k-1,1}(Q;\mathbb{R}^d)$ strongly, we have

$$\limsup_{n\to\infty}\int_{L_{i_\varepsilon}}\left(1+\left|\nabla^k\left(\varphi u_n\right)\right|\right)\,dx \le \limsup_{n\to\infty}\int_{L_{i_\varepsilon}}\left(1+\left|\nabla^k u_n\right|\right)\,dx \le \frac{1+K}{M}$$

by (1.3). We conclude that

$$(1-\varepsilon)^N f(0) \le \alpha_{i_\varepsilon+1}^N f(0) \le \liminf_{n\to\infty}\int_Q f(\nabla^k u_n)\,dx + \frac{1+K}{M},$$

and the result now follows by letting first $\varepsilon\to 0^+$ and then $M\to\infty$.

Next we consider the case where $k=1$. Let $\varepsilon>0$, fix $n\in\mathbb{N}$, set

$$M_n := \left[n\int_Q \left(1+|\nabla u_n|\right)\,dx\right]+1,$$

where $[\cdot]$ denotes the integer part, and decompose $L:=Q\setminus(1-\varepsilon)\,Q$ into $M_n$ layers with mutually disjoint interiors, $L_i^{(n)} := \alpha_{i+1}^{(n)}Q\setminus\alpha_i^{(n)}Q$, so that $1-\varepsilon=\alpha_1^{(n)}<\alpha_2^{(n)}<\ldots<\alpha_{M_n}^{(n)}<1=:\alpha_{M+1}^{(n)}$ and, in addition $\alpha_{i+1}^{(n)}-\alpha_i^{(n)}=\frac{\varepsilon}{M_n}$, $i=1,\ldots,M_n$. Let $\varphi_i^{(n)}\in C_c^\infty(Q;[0,1])$ with $\varphi_i^{(n)}(x)=1$ in $\alpha_i^{(n)}Q$, $\varphi_i^{(n)}(x)=0$ if $x\notin\alpha_{i+1}^{(n)}Q$, $\|\nabla\varphi_i\|_\infty\le\frac{2M_n}{\varepsilon}$, $i=1,\ldots,M_n$. We have

$$\begin{aligned}\int_Q f\left(\nabla\left(\varphi_i^{(n)}u_n\right)\right)\,dx \le{}& \int_Q f(\nabla u_n)\,dx + \int_{Q\setminus\alpha_{i+1}^{(n)}Q} f(0)\,dx\\ &+C\int_{L_i^{(n)}}\left(1+|\nabla u_n|\right)\,dx + C\frac{2M_n}{\varepsilon}\int_{L_i^{(n)}}|u_n|\,dx.\end{aligned}$$

Thus

$$\begin{aligned}\frac{1}{M_n}\sum_{i=1}^{M_n}\int_Q f(\nabla\left(\varphi_i^{(n)}u_n\right))\,dx \le{}& \int_Q f(\nabla u_n)\,dx + \int_{Q\setminus\alpha_1^{(n)}Q} f(0)\,dx\\ &+\frac{C}{M_n}\int_{Q\setminus\alpha_1^{(n)}Q}\left(1+|\nabla u_n|\right)\,dx + \frac{C}{\varepsilon}\int_{Q\setminus\alpha_1^{(n)}Q}|u_n|\,dx\\ \le{}& \int_Q f(\nabla u_n)\,dx + \mathrm{O}(\varepsilon) + \frac{C}{n} + \frac{C}{\varepsilon}\int_{Q\setminus\alpha_1^{(n)}Q}|u_n|\,dx.\end{aligned}$$

We may, therefore, find $i=i(n,\varepsilon)\in\{1,\ldots,M\}$ such that, in view of the quasiconvexity of $f$,

$$f(0) \le \int_Q f\left(\nabla\left(\varphi_i^{(n)}u_n\right)\right)\,dx \le \int_Q f(\nabla u_n)\,dx + \mathrm{O}(\varepsilon) + \frac{C}{n} + \frac{C}{\varepsilon}\int_Q |u_n|\,dx,$$

and the conclusion follows by letting $n\to\infty$ and then $\varepsilon\to 0^+$. □

**Proposition 3.** *Let $h:\mathbb{R}^\tau\to[0,\infty)$ be a convex function such that*

$$h(v)\to\infty \qquad \text{as } |v|\to\infty.$$

*Let $u\in W^{1,p}(\Omega;\mathbb{R}^d)$, and let $\{u_n\}$ be a sequence of functions in $W^{1,p}(\Omega;\mathbb{R}^d)$ which converges to $u$ in $L^1(\Omega;\mathbb{R}^d)$, where $p=\min\{d,N\}$. Then*

$$\int_\Omega h(\mathcal{M}(\nabla u))\,dx \le \liminf_{n\to\infty}\int_\Omega h(\mathcal{M}(\nabla u_n))\,dx.$$

Proposition 3 has been proved by Dal Maso and Sbordone (cf. Theorem 2.2 in [14]) using cartesian currents and by Fusco and Hutchinson (cf. Theorem 2.6 in [23]).

Next we present an approximation result for convex functions.

**Proposition 4.** *Let $M$ be a closed set of $\mathbb{R}^p$, let $V$ be a reflexive and separable Banach space. Let $f : M \times V \to (0, +\infty]$ be a $M \times (w - V)$ sequentially lower semicontinuous function, convex in the last variable and such that there exists a continuous function $v_0 : M \to V$ with*

$$(f(\cdot, v_0(\cdot)))^+ \in L^\infty_{\mathrm{loc}}(M; \mathbb{R}). \tag{2.4}$$

*Then there exist two sequences of continuous functions*

$$a_j : M \to \mathbb{R}, \qquad b_j : M \to V^*,$$

*where $V^*$ is the dual space of $V$, such that*

$$f(t, v) = \sup_j \left(a_j(t) + \langle b_j(t), v\rangle\right)^+$$

*for all $t \in M$ and $v \in V$.*

Proposition 4 was proved by Fonseca and Leoni in [18], following closely the argument of Ambrosio in [3], who studied the case where (1.4) is replaced by the assumption that $f(\cdot, v_0(\cdot))$ is continuous.

## 3. Proof of Theorems 1 and 2

*Proof of Theorem 1.* Without loss of generality we may assume that

$$\liminf_{n\to\infty} \int_\Omega f(x, u_n(x), \dots, \nabla^k u_n(x))\, dx = \lim_{n\to\infty} \int_\Omega f(x, u_n(x), \dots, \nabla^k u_n(x))\, dx < \infty.$$

Passing to a subsequence, if necessary, there exists a nonnegative Radon measure $\mu$ such that

$$f(x, u_n(x), \dots, \nabla^k u_n(x))\, \mathcal{L}^N \lfloor \Omega \overset{*}{\rightharpoonup} \mu$$

as $n \to \infty$, weakly* in the sense of measures. We claim that

$$\frac{d\mu}{d\mathcal{L}^N}(x_0) = \lim_{\varepsilon\to 0^+} \frac{\mu(Q(x_0,\varepsilon))}{\varepsilon^N} \geq f(x_0, u(x_0), \dots, \nabla^k u(x_0)) \qquad \text{for } \mathcal{L}^N \text{ a.e. } x_0 \in \Omega. \tag{3.1}$$

If (2.1) holds, then the conclusion of the theorem follows immediately. Indeed, let $\varphi \in C_c(\Omega; \mathbb{R})$, $0 \leq \varphi \leq 1$. We have

$$\begin{aligned}\int_\Omega f(x, u_n, \dots, \nabla^k u_n)\, dx &\geq \liminf_{n\to\infty} \int_\Omega \varphi\, f(x, u_n, \dots, \nabla^k u_n)\, dx \\ &= \int_\Omega \varphi\, d\mu \geq \int_\Omega \varphi \frac{d\mu}{d\mathcal{L}^N}\, dx \geq \int_\Omega \varphi\, f(x, u, \dots, \nabla^k u)\, dx.\end{aligned}$$

By letting $\varphi \to 1$, and using Lebesgue Dominated Convergence Theorem, we obtain the desired result. Thus, to conclude the proof of the theorem it suffices to show (2.1).

Take $x_0 \in \Omega$ such that

$$\frac{d\mu}{d\mathcal{L}^N}(x_0) = \lim_{\varepsilon\to 0^+} \frac{\mu(Q(x_0,\varepsilon))}{\varepsilon^N} < \infty, \qquad \lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^N} \int_{Q(x_0,\varepsilon)} \frac{|u(x) - T_k(x)|}{|x - x_0|^k} dx = 0, \tag{3.2}$$

where

$$T_k(x) := \sum_{|\alpha|\leq k} \frac{1}{\alpha!} \nabla^\alpha u(x_0)(x - x_0)^\alpha,$$

and set

$$\mathbf{v}_0 := \left(u(x_0), \dots, \nabla^{k-1} u(x_0)\right).$$

Choosing $\varepsilon_h \searrow 0$ such that $\mu(\partial Q(x_0, \varepsilon_h)) = 0$, then

$$\lim_{h\to\infty} \frac{\mu(Q(x_0,\varepsilon_h))}{\varepsilon_h^N} = \lim_{h\to\infty}\lim_{n\to\infty} \frac{1}{\varepsilon_h^N} \int_{Q(x_0,\varepsilon_h)} f(x, u_n, \dots, \nabla^k u_n)\, dx$$
$$= \lim_{h\to\infty}\lim_{n\to\infty} \int_Q f(x_0 + \varepsilon_h y, T_{k-1}(x_0+\varepsilon_h y) + \varepsilon_h^k w_{n,h}(y), \nabla T_{k-1}(x_0 + \varepsilon_h y)$$
$$+ \varepsilon_h^{k-1}\nabla w_{n,h}(y), \nabla^2 T_{k-1}(x_0+\varepsilon_h y) + \varepsilon_h^{k-2}\nabla^2 w_{n,h}(y), \dots, \nabla^k w_{n,h}(y))\, dy,$$

where

$$w_{n,h}(y) := \frac{u_n(x_0+\varepsilon_h y) - T_{k-1}(x_0+\varepsilon_h y)}{\varepsilon_h^k}.$$

Clearly $w_{n,h} \in W^{k,1}(Q;\mathbb{R}^d)$, and, by (2.2), $\lim_{h\to\infty}\lim_{n\to\infty} ||w_{n,h} - w_0||_{W^{k-1,1}(Q;\mathbb{R}^d)} = 0$, where

$$w_0(y) := \sum_{|\alpha|=k} \frac{1}{\alpha!} \nabla^\alpha u(x_0) y^\alpha.$$

By a standard diagonalization argument, we may extract a subsequence $w_h := w_{n_h,h}$ which converges to $w_0$ in $W^{k-1,1}(Q;\mathbb{R}^d)$, such that $\nabla^j w_h \to \nabla^j w$ pointwise a.e. for $j = 0, \dots, k$, and

$$\frac{d\mu}{d\mathcal{L}^N}(x_0) = \lim_{h\to\infty} \int_Q f(x_0+\varepsilon_h y, T_{k-1}(x_0+\varepsilon_h y) + \varepsilon_h^k w_h(y), \dots, \nabla^k w_h(y))\, dy. \tag{3.3}$$

By condition (0.2) for all $\varepsilon > 0$ and for $h$ large enough

$$(1+\varepsilon)\frac{d\mu}{d\mathcal{L}^N}(x_0) + \varepsilon \geq \lim_{h\to\infty} \left( \int_Q f(x_0, u(x_0), \dots, \nabla^{k-1}u(x_0), \nabla^k w_h(y))\ dy - \int_Q \rho(|z_h(y)|)\, dy \right),$$

where

$$z_h(y) := \big(T_{k-1}(x_0+\varepsilon_h y) + \varepsilon_h^k w_h(y), \dots, \nabla^{k-1}T_{k-1}(x_0+\varepsilon_h y) + \varepsilon_h \nabla^{k-1} w_h(y)\big) - \mathbf{v}_0.$$

By Fatou's Lemma, and since $\rho$ is continuous with $\rho(0) = 0$, we have

$$C_0 - \limsup_{h\to\infty} \int_Q \rho(\,|z_h(y)|)\, dy = \liminf_{h\to\infty} \int_Q [C_0(1 + \,|z_h(y)|) - \rho(|z_h(y)|)]\, dy$$
$$\geq \int_Q \liminf_{h\to\infty} [C_0(1 + |z_h(y)|) - \rho(\,|z_h(y)|)]\, dy = C_0,$$

and so

$$\int_Q \rho(|z_h(y)|)\, dy \to 0 \quad \text{as } h \to \infty.$$

Thus

$$(1+\varepsilon)\frac{d\mu}{d\mathcal{L}^N}(x_0) + \varepsilon \geq \lim_{h\to\infty} \int_Q f(x_0, \mathbf{v}_0, \nabla^k w_h(y))\ dy. \tag{3.4}$$

If $g(\xi) := f(x_0, \mathbf{v}_0, \xi)$ satisfies condition $(a)$ then use Proposition 2, and if either condition $(b)$ or $(c)$ hold then apply Theorem 1.7 in [17] to conclude that

$$(1+\varepsilon)\frac{d\mu}{d\mathcal{L}^N}(x_0) + \varepsilon \geq f(x_0, u(x_0), \dots, \nabla^k u(x_0)),$$

and it suffices to let $\varepsilon \to 0^+$. □

*Proof of Theorem 2.* Theorem 2 can be easily deduced from Theorem 1.8 in [17]. It suffices to write

$$\int_\Omega f(x, u(x), \dots, \nabla^k u(x))\, dx =: \int_\Omega F(x, \mathbf{v}(x), \nabla \mathbf{v}(x))\, dx$$

with $\mathbf{v} := \big(u, \dots, \nabla^{k-1}u\big)$. Note, however, that the coercivity condition (0.6) for $F$ now reads

$$F(x_0, \mathbf{v}_0, \eta) \geq C|\eta_k| - \frac{1}{C},$$

where

$$\eta = (\eta_1, \cdots, \eta_k) \in E_1^d \times \cdots \times E_k^d, \quad \text{and } F(x, \mathbf{v}, \eta) := f(x, \mathbf{v}, \eta_k).$$

In order to be in position to apply Theorem 1.8 we need to ensure full coercivity. Due to the strong convergence of admissible sequences $\{u_n\}$ in $W^{k-1,1}(\Omega; \mathbb{R}^d)$, and therefore of $\{v_n\}$ in $L^1\left(\Omega; E^d_{[k-1]}\right)$, it suffices to consider

$$F_\varepsilon(x, \mathbf{v}, \eta) := F(x, \mathbf{v}, \eta) + \varepsilon \chi_A(x, \mathbf{v}) \left|(\eta_1, \cdots, \eta_{k-1})\right|,$$

where $A := \left\{(x, \mathbf{v}) \in \Omega \times E^d_{[k-1]} : f(x, \mathbf{v}, \cdot) \not\equiv 0\right\}$. Theorem 1.8 in [17] now yields

$$\begin{aligned}
\liminf_{n\to\infty} \int_\Omega f(x, u_n, \dots, \nabla^k u_n)\, dx &= \liminf_{n\to\infty} \int_\Omega F(x, v_n, \nabla v_n)\, dx \\
&\geq \int_\Omega F(x, v, \nabla v)\, dx - \varepsilon \int_\Omega \left|\left(\nabla u, \dots, \nabla^{k-1} u\right)\right| \, dx \\
&= \int_\Omega f(x, u, \dots, \nabla^k u)\, dx - \varepsilon \int_\Omega \left|\left(\nabla u, \dots, \nabla^{k-1} u\right)\right| \, dx.
\end{aligned}$$

Let $\varepsilon \to 0^+$. □

## 4. Proof of Theorem 3

Throughout this section we assume that $N \geq 3$.

**Lemma 1.** *Let $D$ be a cube with $|D| \leq 1$. Then there exists constants $C > 0$, $\lambda \in (0,1)$ depending only on $N$, a function $u \in W^{2,\infty}(D; \mathbb{R})$ with compact support in $D$, and sets $A$, $E$, $G \subset D$, with $A \cup E \cup G = D$ and $|E| \leq \lambda |D|$, such that*

$$\|\Delta u\|_{L^1(D;\mathbb{R})} \leq C|D|, \quad \|u\|_{W^{1,1}(D;\mathbb{R})} \leq C|D|^{1+\frac{1}{N}}, \tag{4.1}$$

$$\Delta u = 1 \quad \text{on } A, \tag{4.2}$$

$$u = 0 \quad \text{on } E, \quad u \geq 1 \quad \text{on } G. \tag{4.3}$$

*Proof.* After a translation we may assume that there exists $B(0, R) \subset D$ such that

$$C^{-1} R^N \leq |D| \leq C R^N, \quad R \in (0, 1/2),$$

for some $C > 0$. We search for a radial function of type

$$u(x) := \varphi(|x|),$$

where $\varphi$ is a $C^2$-function on $(0, \infty)$ such that

$$\varphi(t) = 0 \quad \text{for } t \geq R, \tag{4.4}$$

$$\varphi'(0+) = 0. \tag{4.5}$$

Further we want that for some $a < 0$

$$\Delta u(x) = \begin{cases} -a & \text{if } |x| < r, \\ 1 & \text{if } r < |x| < R, \end{cases} \tag{4.6}$$

where $r$ is determined by the equation

$$r^{2-N} R^N = 2N(N-2). \tag{4.7}$$

Note that $r \in (0, R)$ because $R < 1$ and $N \geq 3$. In order to find $a$ and $\varphi$ satisfying (3.4), (3.5) and (3.6), we note that

$$\Delta u(x) = \varphi''(|x|) + |x|^{-1} (N-1)\, \varphi'(|x|), \quad \text{for } |x| \neq 0,$$

or, equivalently,

$$\Delta u(x) = t^{1-N} (t^{N-1} \varphi'(t))', \quad \text{where } t = |x|.$$

On the interval $(r,R)$ (3.6) now yields

$$(t^{N-1}\varphi'(t))' = t^{N-1},$$

and thus, by (3.4),

$$\varphi'(t) = \frac{t}{N}\left(1 - \frac{R^N}{t^N}\right). \tag{4.8}$$

On the interval $(0,r)$, and in view of (3.6), we have

$$(t^{N-1}\varphi'(t))' = -at^{N-1}$$

which, together with (3.5), implies that

$$\varphi'(t) = -\frac{at}{N}. \tag{4.9}$$

We have

$$-\frac{ar}{N} = \varphi'(r-) = \varphi'(r+) = \frac{r}{N}\left(1 - \frac{R^N}{r^N}\right)$$

and thus

$$a = \left(\frac{R^N}{r^N} - 1\right). \tag{4.10}$$

Now the function $u$ is uniquely determined by its properties. Obviously we have $(3.3)_1$ by setting

$$A := B(0,R) \setminus B(0,r), \quad E := D \setminus B(0,r), \quad G := B(0,r),$$

with $|E| \le \lambda |D|$ and $\lambda = \lambda(N)$. In light of (3.6) and (3.10) we have

$$\|\Delta u\|_{L^1(D;\mathbb{R})} \le |B(0,R) \setminus B(0,r)| + a\,|B(0,r)| = \omega_N\left(R^N - r^N + \left(\frac{R^N}{r^N} - 1\right)r^N\right) = \omega_N r^N,$$

where $\omega_N := |B(0,1)|$. If $x \in G$, we have by (3.9), (3.8) and (3.7)

$$\begin{aligned} u(x) \ge \varphi(r) &= -\int_r^R \varphi'(t)\,dt = \frac{1}{N}\int_r^R (R^N t^{1-N} - t)\,dt \\ &\ge \frac{1}{N(N-2)}(r^{2-N}R^N - R^2) - \frac{R^2}{2N} = 2 - \frac{R^2}{2\,(N-2)} \ge 1, \end{aligned}$$

as $R \le \frac{1}{2}$ as $B(0,R) \le D$ and the side length of $D$ does not exceed 1. This proves $(3.3)_2$. By (3.8) and (3.9),

$$\int_D |\nabla u|\,dx \le C\int_0^R t^{N-1}|\varphi'(t)|\,dt \le C\left(\int_0^r r^{-N}R^N t^N\,dt + \int_r^R R^N\,dt\right) \le CR^{N+1},$$

which, with the aid of the Poincaré inequality for zero boundary values, proves $(3.1)_2$. ☐

*Proof of Theorem 3.* We set $\Omega = (0,1)^N$, and we construct the $\frac{1}{n}$-periodic sequence $\{u_n\}$ as follows: divide $\Omega$ into small cubes $D_\alpha$ of measure $\frac{1}{n^N}$, $\alpha \in I_n$ where the set of indices $I_n$ has cardinality $n^N$. On each $D_\alpha$ we construct $u_n$ as indicated in Lemma 1, and denote by $A_\alpha$, $E_\alpha$, $G_\alpha$ the corresponding sets. Then $u_n \to 0$ in $W^{1,1}(\Omega;\mathbb{R})$ because

$$\|u_n\|_{W^{1,1}(\Omega;\mathbb{R})} = \sum_{\alpha \in I_n} \|u_n\|_{W^{1,1}(D_\alpha;\mathbb{R})} \le n^N C\left(\frac{1}{n^N}\right)^{1+\frac{1}{N}} \to 0 \quad \text{as } n \to \infty,$$

and $\{\|\Delta u_n\|_{L^1(\Omega;\mathbb{R})}\}$ is uniformly bounded since

$$\|\Delta u_n\|_{L^1(\Omega;\mathbb{R})} = \sum_{\alpha \in I_n} \|\Delta u_n\|_{L^1(D_\alpha;\mathbb{R})} \le n^N C\frac{1}{n^N} = C.$$

Consider the functional

$$F(v) := \int_\Omega h(v)(1 - \Delta v)^+\,dx.$$

For $\alpha \in I_n$ we have by (3.1)–(3.3)

$$\begin{aligned} h(u_n) = 1 \quad \text{and} \quad \Delta u_n &= 0 \quad \text{on } E_\alpha, \\ \Delta u_n &= 1 \quad \text{on } A_\alpha, \\ h(u_n) &= 0 \quad \text{on } G_\alpha, \end{aligned}$$

and thus

$$\int_{D_\alpha} h(u_n)(1-\Delta u_n)^+ \, dx = |E_\alpha| \leq \lambda |D_\alpha|.$$

Summing up over $\alpha \in I_n$, we conclude that

$$\int_\Omega h(u_n)(1-\Delta u_n)^+ \, dx \leq \lambda < 1 = F(0).$$

□

**Remark.** We cannot obtain an apriori bound on $\|u_n\|_{W^{2,1}(\Omega;\mathbb{R})}$ because the function

$$h(\mathbf{v})(1-\operatorname{trace}\xi)^+$$

is convex in the last variable, and, after adding a small multiple of $|\xi|$, the corresponding functional is lower semicontinuous on $W^{1,1}(\Omega;\mathbb{R})$ according to Theorem 4. A direct heuristic computation using the notation of Lemma 1 yields

$$\begin{aligned} \int_Q |\nabla^2 u| \, dx &\sim \int_0^R \left(t^{N-1}|\varphi''(t)| + t^{N-2}|\varphi'(t)|\right) dt \\ &\sim \left(\int_0^r t^{N-1} \frac{R^N}{r^N} \, dt + \int_r^R \frac{R^N}{t} \, dt\right) \\ &\sim R^N \log R, \end{aligned}$$

hence an inequality of the type

$$\|\nabla^2 u\|_{L^1\left(Q;E_2^1\right)} \leq C|Q|$$

will not hold.

## 5. Proof of Theorems 4 and 5

*Proof of Theorem 4.* As in the proof of Theorem 2, it is easy to obtain Theorem 4 from Theorem 1.1 in [18] by considering $\mathbf{v} := \left(u, \dots, \nabla^{k-1}u\right)$ and the reformulated functionals

$$\int_\Omega \left(F(x, \mathbf{v}(x), \nabla \mathbf{v}(x)) + \varepsilon \chi_A\left(x, \mathbf{v}\left(x\right)\right) \left|\left((\nabla \mathbf{v})_1, \cdots, (\nabla \mathbf{v})_{k-1}\right)\right|\right) dx,$$

with

$$\nabla \mathbf{v} := \left((\nabla \mathbf{v})_1, \cdots, (\nabla \mathbf{v})_k\right) \in E_1^d \times \cdots \times E_k^d.$$

□

Remark that, as opposed to Theorems 1(b) and (c), 2 and 4, Theorem 5 cannot be deduced easily from the analogous result already obtained in the case where $k=1$, precisely Theorem 1.4 in [18]. Indeed, there is no obvious way of perturbing the new integrand $H\left(x, \mathbf{v}, \mathcal{M}(\nabla \mathbf{v})\right) := h(x, u, \dots, \nabla^{k-1}u, \mathcal{M}(\nabla^k u))$, with $\mathbf{v} := \left(u, \dots, \nabla^{k-1}u\right)$, in such a way that (0.10) is satisfied for the perturbed integrand, i.e.

$$H_\varepsilon\left(x, \mathbf{v}, \mathcal{M}(\nabla \mathbf{v})\right) \geq C_\varepsilon \left|\mathcal{M}(\nabla \mathbf{v})\right| - \frac{1}{C_\varepsilon}$$

and

$$H_\varepsilon \geq H, \quad \liminf_{\varepsilon \to 0^+} \liminf_{n \to \infty} \int_\Omega H_\varepsilon\left(x, \mathbf{v}_n, \mathcal{M}(\nabla \mathbf{v}_n)\right) dx \leq \liminf_{n \to \infty} \int_\Omega H\left(x, \mathbf{v}_n, \mathcal{M}(\nabla \mathbf{v}_n)\right) dx.$$

This is due to the fact that $\mathcal{M}(\nabla \mathbf{v})$ involves terms of the form $\nabla u \nabla^k u$ for which we have no bounds.

*Proof of Theorem 5.* Let $f(x,\mathbf{v},\xi) := h(x,\mathbf{v},\mathcal{M}(\xi))$. We proceed as in the proof of Theorem 1 until we reach (2.3), precisely

$$\frac{d\mu}{d\mathcal{L}^N}(x_0) = \lim_{h\to\infty}\int_Q f(x_0+\varepsilon_h y, T_{k-1}(x_0+\varepsilon_h y)+\varepsilon_h^k w_h(y),\dots,\nabla^k w_h(y))\,dy, \tag{5.1}$$

where now $w_h \in W^{k,p}(Q;\mathbb{R}^d)$, and $||w_h - w_0||_{W^{k-1,1}(Q;\mathbb{R}^d)} \to 0$ as $h\to\infty$, where

$$w_0(y) := \sum_{|\alpha|=k}\frac{1}{\alpha!}\nabla^\alpha u(x_0)y^\alpha.$$

If $f(x_0,\mathbf{v}_0,\cdot)\equiv 0$ then there is nothing to prove. Otherwise, let $\delta_0>0$ be given by (0.9) and (0.10). Setting

$$Q_h := \left\{y\in Q:\ \left|(w_h(y),\dots,\nabla^{k-1}w_h(y))\right| \le \delta_0/(2\varepsilon_h)\right\},$$

by (4.1) and (0.10) we have

$$\begin{aligned}\frac{d\mu}{d\mathcal{L}^N}(x_0) &\ge \limsup_{h\to\infty}\int_{Q_h} f(x_0+\varepsilon_h y, T_{k-1}(x_0+\varepsilon_h y)+\varepsilon_h^k w_h(y),\dots,\nabla^k w_h(y))\,dy\\ &\ge C\limsup_{h\to\infty}\int_{Q_h}\left|\mathcal{M}(\nabla^k w_h(y))\right|dy - 1/C,\end{aligned}$$

and so there exists a constant $K>0$ such that

$$\int_{Q_h}|\mathcal{M}(\nabla^k w_h(y))|\,dy \le K \quad \text{for all } h\in\mathbb{N}. \tag{5.2}$$

By Proposition 4, with $M=(x_0+\varepsilon_1\overline{Q})\times\overline{B(\mathbf{v}_0,\delta_0/2)}$ and $V=\mathbb{R}^\tau$, in view of (0.9) there exist two sequences of continuous functions

$$a_j : M\to\mathbb{R}, \qquad b_j : M\to\mathbb{R}^\tau,$$

such that

$$h(x,\mathbf{v},\eta) = \sup_j\,(a_j(x,\mathbf{v})+b_j(x,\mathbf{v})\cdot\eta)^+ \tag{5.3}$$

for all $(x,\mathbf{v})\in M$ and $\eta\in\mathbb{R}^\tau$. Define

$$h_j(x,\mathbf{v},\eta) := (a_j(x,\mathbf{v})+b_j(x,\mathbf{v})\cdot\eta)^+, \quad f_j(x,\mathbf{v},\xi) := h_j(x,\mathbf{v},\mathcal{M}(\xi)).$$

Clearly $h_j$ is continuous, convex in $\eta$ and

$$0\le h_j(x,\mathbf{v},\eta)\le C_j(|\eta|+1), \tag{5.4}$$

for all $(x,\mathbf{v})\in M$ and $\eta\in\mathbb{R}^\tau$, where

$$C_j := \max\{|a_j(x,\mathbf{v})|+|b_j(x,\mathbf{v})| :\ (x,\mathbf{v})\in M\}.$$

Fix $\varepsilon>0$ and find $0<\delta_j\le\delta_0/2$ such that

$$|a_j(x,\mathbf{v})-a_j(x_0,\mathbf{v}_0)|+|b_j(x,\mathbf{v})-b_j(x_0,\mathbf{v}_0)|\le\varepsilon$$

for all $(x,\mathbf{v})\in(x_0+\delta_j\overline{Q})\times\overline{B(\mathbf{v}_0,\delta_j)}$. Since the function $s\mapsto s^+$ is Lipschitz continuous with Lipschitz constant 1, we have

$$|f_j(x,\mathbf{v},\xi)-f_j(x_0,\mathbf{v}_0,\xi)|\le|a_j(x,\mathbf{v})-a_j(x_0,\mathbf{v}_0)|+|b_j(x,\mathbf{v})-b_j(x_0,\mathbf{v}_0)|\,|\mathcal{M}(\xi)|\le\varepsilon(1+|\mathcal{M}(\xi)|) \tag{5.5}$$

for all $(x,\mathbf{v})\in(x_0+\delta_j\overline{Q})\times\overline{B(\mathbf{v}(x_0),\delta_j)}$ and all $\xi\in E_k^d$. By (4.1) and for any $j\in\mathbb{N}$ we obtain

$$
\begin{aligned}
\frac{d\mu}{d\mathcal{L}^N}(x_0) &\geq \liminf_{h\to\infty} \int_{Q_h} f(x_0+\varepsilon_h y, T_{k-1}(x_0+\varepsilon_h y)+\varepsilon_h^k w_h(y),\dots,\nabla^k w_h(y))\,dy \\
&\geq \liminf_{h\to\infty} \int_{Q_h} f_j(x_0+\varepsilon_h y, T_{k-1}(x_0+\varepsilon_h y)+\varepsilon_h^k w_h(y),\dots,\nabla^k w_h(y))\,dy \\
&\geq \liminf_{h\to\infty} \left( \int_{Q_h} f_j(x_0,\mathbf{v}_0,\nabla^k v_h(y))\,dy - \varepsilon - \varepsilon \int_{Q_h} \left|\mathcal{M}\left(\nabla^k w_h(y)\right)\right|\,dy \right) \\
&\geq \liminf_{h\to\infty} \int_{Q_h} f_j(x_0,\mathbf{v}_0,\nabla^k v_h(y))\,dy - \varepsilon - \varepsilon K,
\end{aligned}
\tag{5.6}
$$

where we have used (4.5) and (4.2). Define

$$
\mathbf{z}_h(y) := \left(\varepsilon_h^{k-1} w_h(y),\dots,\varepsilon_h \nabla^{k-2} w_h(y)\right), \quad \mathbf{u}_h(y) := \nabla^{k-1} w_h(y).
$$

Fix an integer $P \in \mathbb{N}$ such that $e^P > 1 + ||\nabla^{k-1} w_0||_\infty$. For $h$ sufficiently large, say $h \geq h_P$, we have $e^{2P+1} \leq \delta_0/(2\varepsilon_h)$, so in view of (4.2) we may find $i_h \in \{P+1,\cdots,2P\}$ such that

$$
\left\{y \in Q : \, e^{i_h} \leq |(\mathbf{z}_h(y),\mathbf{u}_h(y))| \leq e^{i_h+1}\right\} \subset Q_h,
$$

and

$$
\int_{\left\{y\in Q:\, e^{i_h} \leq |(\mathbf{z}_h(y),\mathbf{u}_h(y))| \leq e^{i_h+1}\right\}} \left(1+|\mathcal{M}(\nabla^k w_h(x))|\right)\,dx \leq \frac{1+K}{P}.
$$

Since $\{P+1,\cdots,2P\}$ is a finite set, we may find $i_P \in \{P+1,\cdots,2P\}$ such that

$$
\int_{\left\{y\in Q:\, e^{i_P} \leq |(\mathbf{z}_h(y),\mathbf{u}_h(y))| \leq e^{i_P+1}\right\}} \left(1+|\mathcal{M}(\nabla^k w_h(x))|\right)\,dx \leq \frac{1+K}{P}.
\tag{5.7}
$$

for infinitely many indices $h \in \mathbb{N}$. From now until the end of the proof we assume without loss of generality that the whole sequence satisfy (4.7).

Set

$$
\mathbf{v}_h(y) := g\left(|(\mathbf{z}_h(y),\mathbf{u}_h(y))|\right)\mathbf{u}_h(y)
$$

and

$$
\begin{aligned}
D_h &:= \left\{y \in Q : |(\mathbf{z}_h(y),\mathbf{u}_h(y))| < e^{i_P}\right\}, \\
D_h^- &:= \left\{y \in Q : \, e^{i_P} \leq |(\mathbf{z}_h(y),\mathbf{u}_h(y))| \leq e^{i_P+1}\right\}, \\
D_h^+ &:= \left\{y \in Q : \, |(\mathbf{z}_h(y),\mathbf{u}_h(y))| > e^{i_P+1}\right\},
\end{aligned}
$$

where

$$
g(s) := \begin{cases} 1 & \text{if } s < e^{i_P}, \\ \dfrac{e^{i_P+1}-s}{e^{i_P+1}-e^{i_P}} & \text{if } e^{i_P} \leq s \leq e^{i_P+1}, \\ 0 & \text{if } s > e^{i_P+1}. \end{cases}
$$

Note that

$$
\begin{aligned}
\left|D_h^- \cup D_h^+\right| &= \left|\left\{y \in Q : \, |(\mathbf{z}_h(y),\mathbf{u}_h(y))| \geq e^{i_P}\right\}\right| \\
&\leq \left|\left\{y \in Q : \, \left|(\mathbf{z}_h(y),\mathbf{u}_h(y)) - \left(\mathbf{0},\nabla^{k-1} w_0(y)\right)\right| \geq 1\right\}\right| \\
&\leq ||\mathbf{z}_h||_{L^1(Q)} + ||\mathbf{u}_h - \nabla^{k-1} w_0||_{L^1(Q)} \to 0 \qquad \text{as } h \to \infty,
\end{aligned}
\tag{5.8}
$$

where we have used the fact that $e^{i_P} > 1 + ||\nabla^{k-1} w_0||_\infty$. Also,

$$
|\nabla \mathbf{z}_h| = \left|\left(\varepsilon_h^{k-1}\nabla w_h,\dots,\varepsilon_h \nabla^{k-1} w_h\right)\right| \leq \varepsilon_h \left|\left(\varepsilon_h^{k-1} w_h, \varepsilon_h^{k-2}\nabla w_h,\dots,\varepsilon_h \nabla^{k-1} w_h\right)\right| = \varepsilon_h \left|(\mathbf{z}_h,\mathbf{u}_h)\right|.
\tag{5.9}
$$

We claim that

$$
|\mathcal{M}(\nabla \mathbf{v}_h(y))| \leq C\left(1+\varepsilon_h\, e^{i_P+1}\right)|\mathcal{M}(\nabla^k w_h(y))|.
\tag{5.10}
$$

In view of the definition of $g$ this is immediate for $x \in D_h \cup D_h^+$. Thus it remains to assert (4.10) in $D_h^-$. We have

$$\nabla \mathbf{v}_h = \left( g\left(|(\mathbf{z}_h, \mathbf{u}_h)|\right) \mathcal{I} + g'\left(|(\mathbf{z}_h, \mathbf{u}_h)|\right) \frac{\mathbf{u}_h \otimes \mathbf{u}_h}{|(\mathbf{z}_h, \mathbf{u}_h)|} \right) \nabla \mathbf{u}_h + g'\left(|(\mathbf{z}_h, \mathbf{u}_h)|\right) \frac{\mathbf{u}_h \otimes \mathbf{z}_h}{|(\mathbf{z}_h, \mathbf{u}_h)|} \nabla \mathbf{z}_h,$$

where $\mathcal{I}$ is the identity matrix. Since in $D_h^-$ it holds

$$\left| g\mathcal{I} + g' \frac{\mathbf{u}_h \otimes \mathbf{u}_h}{|(\mathbf{z}_h, \mathbf{u}_h)|} \right| + \left| g' \frac{\mathbf{u}_h \otimes \mathbf{z}_h}{|(\mathbf{z}_h, \mathbf{u}_h)|} \right| \le C,$$

we have

$$\left| \mathcal{M}_l \left( \left( g\mathcal{I} + g' \frac{\mathbf{u}_h \otimes \mathbf{u}_h}{|(\mathbf{z}_h, \mathbf{u}_h)|} \right) \nabla \mathbf{u}_h \right) \right| \le \left| \mathcal{M}_l \left( g\mathcal{I} + g' \frac{\mathbf{u}_h \otimes \mathbf{u}_h}{|(\mathbf{z}_h, \mathbf{u}_h)|} \right) \right| \left| \mathcal{M}_l\left(\nabla \mathbf{u}_h\right) \right| \le C \left| \mathcal{M}_l\left(\nabla \mathbf{u}_h\right) \right|,$$

and

$$\left| \mathcal{M}_l \left( g' \frac{\mathbf{u}_h \otimes \mathbf{z}_h}{|(\mathbf{z}_h, \mathbf{u}_h)|} \nabla \mathbf{z}_h \right) \right| \le \left| \mathcal{M}_l \left( g' \frac{\mathbf{u}_h \otimes \mathbf{z}_h}{|(\mathbf{z}_h, \mathbf{u}_h)|} \right) \right| \left| \mathcal{M}_l\left(\nabla \mathbf{z}_h\right) \right| \le \begin{cases} 0 & l > 1, \\ C \left| \mathcal{M}_1\left(\nabla \mathbf{z}_h\right) \right| & l = 1, \end{cases}$$

where $\mathcal{M}_l(X)$ is the vector whose components are all the minors of $X$ of order $l$. Here we have used the facts that

$$|\mathcal{M}_l(X+Y)| \le C \sum_{i=0}^{l} |\mathcal{M}_i(X)|\, |\mathcal{M}_{l-i}(Y)|,$$

that

$$|\mathcal{M}_l(XY)| \le |\mathcal{M}_l(Y)|\, |\mathcal{M}_l(Y)|,$$

and that $\mathbf{u}_h \otimes \mathbf{z}_h$ is a rank one matrix. Then, in view of (4.9),

$$\begin{aligned} |\mathcal{M}_l(\nabla \mathbf{v}_h(y))| &\le C \sum_{i=0}^{l} \left| \mathcal{M}_i \left( \left( g\mathcal{I} + g' \frac{\mathbf{u}_h \otimes \mathbf{u}_h}{|(\mathbf{z}_h, \mathbf{u}_h)|} \right) \nabla \mathbf{u}_h \right) \right| \left| \mathcal{M}_{l-i} \left( g' \frac{\mathbf{u}_h \otimes \mathbf{z}_h}{|(\mathbf{z}_h, \mathbf{u}_h)|} \nabla \mathbf{z}_h \right) \right| \\ &\le C \left( \left| \mathcal{M}_l\left(\nabla \mathbf{u}_h\right) \right| + \left| \mathcal{M}_{l-1}\left(\nabla \mathbf{u}_h\right) \right| \left| \mathcal{M}_1\left(\nabla \mathbf{z}_h\right) \right| \right) \\ &\le C \left( \left| \mathcal{M}_l\left(\nabla \mathbf{u}_h\right) \right| + \varepsilon_h \left|(\mathbf{z}_h, \mathbf{u}_h)\right| \left| \mathcal{M}_{l-1}\left(\nabla \mathbf{u}_h\right) \right| \right). \end{aligned}$$

As $|(\mathbf{z}_h, \mathbf{u}_h)| \le e^{i_P+1}$ for $x \in D_h^-$, we conclude that (4.10) holds.

Since $D_h \subset Q_h$, it follows from (4.6)

$$\frac{d\mu}{d\mathcal{L}^N}(x_0) \ge \liminf_{h\to\infty} \int_{D_h} f_j(x_0, \mathbf{v}_0, \nabla \mathbf{v}_h)\, dy - \varepsilon - \varepsilon K. \tag{5.11}$$

By (4.4) and (4.8)

$$\int_{D_h^+} f_j(x_0, \mathbf{v}_0, \nabla \mathbf{v}_h)\, dy \le C_j \int_{D_h^+} \left(1 + |\mathcal{M}(\nabla \mathbf{v}_h)|\right) dy = C_j \left| D_h^+ \right| \to 0, \tag{5.12}$$

while from (4.4), (4.10) and (4.7), and taking $h > h_P$ so that $\varepsilon_h e^{P+1} < 1$,

(5.13)
$$\int_{D_h^-} f_j(x_0, \mathbf{v}_0, \nabla \mathbf{v}_h)\, dy \le C_j \int_{D_h^-} \left(1 + |\mathcal{M}(\nabla \mathbf{v}_h)|\right) dy \le CC_j \int_{D_h^-} \left(1 + \left|\mathcal{M}(\nabla^k w_h(y))\right|\right) dy \le CC_j \frac{1+K}{P}.$$

Consequently, in view of (4.11), (4.12) and (4.13)

$$\frac{d\mu}{d\mathcal{L}^N}(x_0) \ge \liminf_{h\to\infty} \int_{Q} f_j(x_0, \mathbf{v}_0, \nabla \mathbf{v}_h)\, dy - \varepsilon - \varepsilon K - CC_j \frac{1+K}{P}, \tag{5.14}$$

and by (4.2), (4.7), and in view of the fact that $\mathbf{v}_h \equiv 0$ in $D_h^+$,

$$\sup_h \int_Q |\mathcal{M}(\nabla \mathbf{v}_h)|\, dy \le K_1 < \infty, \tag{5.15}$$

where $K_1$ is independent of $h$ and $j$. Define

$$h_{j,\varepsilon}(v) := h_j(x_0, \mathbf{v}_0, v) + \varepsilon |v|, \quad f_{j,\varepsilon}(\xi) := h_{j,\varepsilon}(\mathcal{M}(\xi)).$$

Then by (4.14), (4.15), and Proposition 3

$$\begin{aligned}\frac{d\mu}{d\mathcal{L}^N}(x_0) &\geq \liminf_{h\to\infty}\int_Q h_{j,\varepsilon}(\mathcal{M}(\nabla \mathbf{v}_h(y)))\ dy - \varepsilon - \varepsilon\,(K+K_1) - CC_j\frac{1+K}{P}\\ &\geq h_{j,\varepsilon}(\mathcal{M}(\nabla^k w_0(x_0)))\ \ - \varepsilon - \varepsilon\,(K+K_1) - CC_j\frac{1+K}{P}\\ &= f_j(x_0,u(x_0),\cdots,\nabla^k u(x_0)) + \varepsilon\left|\mathcal{M}(\nabla^k w_0(x_0))\right| - \varepsilon - \varepsilon\,(K+K_1) - CC_j\frac{1+K}{P}.\end{aligned}$$

Letting first $P\to\infty$, then taking the supremum in $j$ yields

$$\frac{d\mu}{d\mathcal{L}^N}(x_0) \geq f(x_0,u(x_0),\cdots,\nabla^k u(x_0)) + \varepsilon\left|\mathcal{M}(\nabla^k w_0(x_0))\right| - \varepsilon - \varepsilon\,(K+K_1)\,,$$

by (4.3). To complete the proof it suffices to let $\varepsilon\to 0^+$. □

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