A-B QUASICONVEXITY AND IMPLICIT PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. The study of existence of solutions of boundary-value problems for differential inclusions

$$\begin{cases} Bu(x) \in E & \text{a.e. } x \in \Omega, \\ u(x) = \varphi(x) & \text{for all } x \in \partial\Omega, \end{cases}$$

where $\varphi \in C^1_{\text{piec}}(\overline{\Omega}; \mathbb{R}^N)$, Ω is an open subset of $\mathbb{R}^n$, $E \subset \mathbb{R}^{m\times n}$ is a compact set, and B is a $m \times n$-valued first order differential operator, is undertaken. As an application, minima of the energy for large magnetic bodies

$$E(m) := \int_\Omega [\varphi(m) - \langle h_e; m\rangle]\, dx + \frac{1}{2}\int_{\mathbb{R}^3} |h_m|^2\, dx$$

where the magnetization $m : \Omega \to \mathbb{R}^3$ is taken with values on the unit sphere S^2, $h_m : \mathbb{R}^3 \to \mathbb{R}^3$ is the induced magnetic field satisfying $\operatorname{curl} h_m = 0$ and $\operatorname{div}(h_m + m\chi_\Omega) = 0$, φ is the anisotropic energy density, and the applied external magnetic field is given by $h_e \in \mathbb{R}^3$, are fully characterized. Setting $Z := \{\xi \in S^2 : \psi(\xi) = \min_{m\in S^2} \psi(m)\}$ with $\psi(m) := \varphi(m) - \langle h_e; m\rangle$, it is shown that E admits a minimizer $m \in L^\infty$ with $h_m \equiv 0$ if and only if either 0 is on a face of $\partial \operatorname{co} Z$ or $0 \in \operatorname{int} \operatorname{co} Z$, where $\operatorname{co} Z$ denotes the convex hull of Z.

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Contents

1. Introduction

In recent years considerable research activity has been dedicated to the study of Dirichlet problems for first order partial differential equations and systems of the form

$$\begin{cases} F_i\,(Du\,(x)) = 0 \;\; i = 1, ..., I, \text{ a.e. } x \in \Omega, \\ u = \varphi \text{ on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded, open set, $u : \Omega \to \mathbb{R}^m$, the functions $F_i : \mathbb{R}^{m\times n} \to \mathbb{R}$, $i = 1, \ldots, I$, are continuous, and the boundary datum φ is given.

The viscosity method was introduced to handle the case where the unknown function u is scalar (i.e. $m = 1$). In the vectorial case several methods have been developed, notably one using Baire Category Theorem and relaxation theorems, proposed by Cellina, Bressan-Flores and De Blasi-Pianigiani and extended by Dacorogna-Marcellini, and another one called convex integration and introduced by Gromov, further developed in many directions and in the present context by Müller-Šverák and others. We refer to [7] for a detailed bibliography on these matters.

The aim of the present article is to extend some of the results of [7], addressed to the gradient operator D, to the context of *compensated compactness* of Murat-Tartar, and in the framework of A-B quasiconvexity as introduced by Dacorogna [4] and studied in recent years by many authors following the work of Fonseca-Müller [12]. The property characterizing the operator D,

$$\operatorname{curl} Du = 0,$$

will be extended here to some more general first order differential operators A (replacing the operator curl) and B (replacing the operator gradient D) such that $AB \equiv 0$, namely A could be any combination of div or curl and therefore B would be composed of operators curl* and grad. Of course our analysis could and should be carried over to a more general class of operators d of the exterior differential calculus. However for the examples of applications we have in mind the present framework suffices.

Instead of summarizing the general results obtained in this paper, and gathered in Section 5, in this introduction we opt to discuss one significant example that has motivated our study. It concerns a problem in *micromagnetics*, and the findings described below were announced in [6].

Adopting Landau and Lifshitz [16] theory of micromagnetics (see also Brown [3]), we search for minimizers of the energy for a rigid ferromagnetic material occupying a configuration $\Omega \subset \mathbb{R}^3$, where Ω is an open, bounded, Lipschitz domain. The *magnetization* $m : \Omega \to \mathbb{R}^3$ represents a mass density of macroscopic magnetic moment and is subject to the constraint

$$|m(x)| = M_T \quad x \in \Omega, \tag{1.2}$$

where T is the temperature, and $M_T = 0$ above the Curie point, i.e. for $T \geq T_c$. Condition (1.2) ensures that the body is always saturated, and M_T is called *saturation magnetization.* We will assume that the temperature is held fixed, and, as it is usual, without loss of generality we will fix $M_T = 1$.

According to the theory of micromagnetics, observable states of a ferromagnetic body subject to a constant *external magnetic field* $h_e \in \mathbb{R}^3$ correspond to minimizers of the total energy

$$E_\alpha(m) := \frac{\alpha^2}{2}\int_\Omega |\nabla m|^2\,dx + \int_\Omega \varphi(m)\,dx - \int_\Omega \langle h_e; m\rangle\,dx + \frac{1}{2}\int_{\mathbb{R}^3} |h_m|^2\,dx$$

where $\varphi : S^2 \to \mathbb{R}$ is the nonnegative, even, continuous *anisotropic energy density,* $S^2 := \{\xi \in \mathbb{R}^3 : |\xi| = 1\}$, and $h_m : \mathbb{R}^3 \to \mathbb{R}^3$ is the *induced magnetic field* satisfying, in the sense of distributions,

$$\begin{cases} \operatorname{curl} h_m = 0 & \text{in } \mathbb{R}^3, \\ \operatorname{div}\,(h_m + \chi_\Omega m) = 0 & \text{in } \mathbb{R}^3, \end{cases} \tag{1.3}$$

where χ_Ω is the characteristic function of Ω. The four terms in E_α are denoted, respectively, *exchange energy*, *anisotropy energy*, *interaction energy* and *magnetostatic energy*. A scaling argument shows that for large bodies the exchange energy should become less important, and this leads us to the minimization of (see De Simone [8])

$$E(m) := E_0(m) = \int_\Omega [\varphi(m) - \langle h_e; m\rangle]\, dx + \frac{1}{2}\int_{\mathbb{R}^3} |h_m|^2\, dx.$$

Existence of solutions for E_α has been obtained by Visintin [24], and a thorough study of the limiting behavior of minimizers for E_α and how they relate to minimizers of E may be found in De Simone [8], [9].

Before proceeding further it is convenient to reformulate the problem. We therefore let $\psi(\eta) := \varphi(\eta) - \langle h_e; \eta\rangle$ and

$$Z := \left\{\xi \in S^2 \,:\, \psi(\xi) = \min_{|\eta|=1}\{\psi(\eta)\}\right\}.$$

It is easy to see that the question of finding minima of

$$(\mathcal{P}):\ \inf\left\{E(m)\,:\, m \in L^\infty\left(\mathbb{R}^3; \mathbb{R}^3\right) \text{ satisfies (1.3)}\right\}$$

is closely linked to the problem of finding $m \in L^\infty\left(\mathbb{R}^3; \mathbb{R}^3\right)$ ($m \equiv 0$ in $\mathbb{R}^3 \setminus \overline{\Omega}$) such that

$$\text{(1.4)} \qquad \begin{cases} m(x) \in Z & \text{a.e. } x \in \Omega, \\ \operatorname{div}(\chi_\Omega m) = 0 & \text{in } \mathbb{R}^3. \end{cases}$$

Precisely, if (1.4) has solutions then $(\mathcal{P})$ attains its minimum and minimizers must satisfy $h_m \equiv 0$ and $m \in Z$ for a.e. $x \in \Omega$.

When $h_e = 0$ James-Kinderlehrer [15] obtained certain characterizations of minimizers for E, by showing that in the *uniaxial case*, where $Z = \{\pm m_1\}$ for some $m_1 \in S^2$, the system (1.4) admits no solution, while if $\{\pm m_1, \pm m_2\} \subset Z$ for some orthogonal vectors $m_1, m_2 \in S^2$ then (1.4) does have a solution. The latter case falls within the so-called cubic-symmetry ferromagnetic crystals where $Z = \{\pm m_1, \pm m_2, \pm m_3\}$ with $\{m_1, m_2, m_3\}$ an orthonormal system in $\mathbb{R}^3$. Following the literature on magnetism, the argument is based on the construction of a prototype solution on a prism with cross-sectional shape dictated by the structure of the set Z, subsequently translated, scaled, and pieced together via Vitali Covering Theorem. In addition, James-Kinderlehrer [15] showed that E does not have a minimizer satisfying (1.4) in the presence of certain applied fields and for specific shapes of the domain. Precisely, they proved that if $h_e = \theta D m_1$ where $\varphi(m_1) = 0$, $\theta \in (0,1]$, and $D = D^T > 0$ is co-axial with the principal axis of the ellipsoid Ω, then a minimizer of E is given by the uniform magnetization $\overline{m} := D^{-1}h_e$, with corresponding nonzero induced magnetic field.

For further related work we refer to Anzellotti-Baldo-Visintin [1], Gioia-James [14], De Simone-Kohn-Müller-Otto [10], [11], Pedregal [19], and Tartar [22], [23].

In this paper, and using the abstract results developed below, we pursue further the analysis of James-Kinderlehrer [15] so as to give a complete characterization of the minimizers of E satisfying (1.4) and in the presence of a possibly non-vanishing external magnetic field h_e. Our analysis does not require that the function φ be even, although this is the natural framework in micromagnetics.

The main result of Section 6 (see Theorem 6.2) establishes that if $Z \subset S^2$ is compact, then problem (1.4) has or has no solution according to the following cases (in the sequel $\operatorname{co}Z$ will denote the convex hull of Z; for the notions of edges and faces of $\partial\operatorname{co}Z$ we refer to Section 6):

Case 1 : if $0 \notin \operatorname{co}Z$ then (1.4) has no solutions;
Case 2 : if $0 \in \partial\operatorname{co}Z$ and 0 is on an edge of $\partial\operatorname{co}Z$ then (1.4) has no solutions;
Case 3 : if $0 \in \partial\operatorname{co}Z$ and 0 is on a face de $\partial\operatorname{co}Z$ then (1.4) admits solutions;
Case 4 : if $0 \in \operatorname{int}\operatorname{co}Z$ (the interior of $\operatorname{co}Z$), then (1.4) admits solutions. Moreover, there exists $M \in C(\overline{\Omega};\mathbb{R}^3)$ satisfying $\operatorname{curl} M \in L^\infty(\Omega;\mathbb{R}^3)$ in the sense of distributions, and

$$\begin{cases} \operatorname{curl} M\,(x) \in Z & \text{a.e. in } \Omega, \\ M = 0 & \text{on } \partial\Omega. \end{cases}$$

We set $m := \operatorname{curl} M$.

The last case will be solved by the method presented in Sections 3, 4, and 5 of this article, while the other cases are handled by ad hoc methods similar to the ones used by James-Kinderlehrer [15] and strongly influenced by the magnetism literature.

2. Preliminaries and statement of the problem

We start by introducing the first order partial differential operators A and B. Let

$$U := (V, W) = \left(V^1, ..., V^k, W^{k+1}, ..., W^m\right) : \mathbb{R}^n \to \mathbb{R}^{k\times n} \times \mathbb{R}^{(m-k)\times n} = \mathbb{R}^{m\times n}$$

where $V^i : \mathbb{R}^n \to \mathbb{R}^n$, $V^i = \left(V^i_r\right)_{1\le r\le n}$, $i = 1, ..., k$, and $W^i : \mathbb{R}^n \to \mathbb{R}^n$, $W^i = \left(W^i_r\right)_{1\le r\le n}$, $i = k+1, ..., m$.

Consider the first order differential operator

$$AU := (\operatorname{div} V, \operatorname{curl} W) \in \mathbb{R}^k \times \mathbb{R}^{(m-k)\times\frac{n(n-1)}{2}}$$

where

$$\begin{aligned} \operatorname{div} V \quad &:= \quad \left(\operatorname{div} V^1, ..., \operatorname{div} V^k\right) \in \mathbb{R}^k, \\ \operatorname{div} V^i \quad &:= \quad \sum_{r=1}^n \frac{\partial V^i_r}{\partial x_r} \in \mathbb{R},\ i = 1, ..., k, \end{aligned}$$

and

$$\begin{aligned} \operatorname{curl} W \quad &:= \quad \left(\operatorname{curl} W^{k+1}, ..., \operatorname{curl} W^m\right) \in \mathbb{R}^{(m-k)\times\frac{n(n-1)}{2}}, \\ \operatorname{curl} W^i \quad &:= \quad \left(\frac{\partial W^i_r}{\partial x_s} - \frac{\partial W^i_s}{\partial x_r}\right)_{1\le r<s\le n} \in \mathbb{R}^{\frac{n(n-1)}{2}},\ i = k+1, ..., m. \end{aligned}$$

To each function

$$u = (v, w) \in \mathbb{R}^{k\times\frac{n(n-1)}{2}} \times \mathbb{R}^{(m-k)}$$

(in the sequel we will abbreviate $N := k\frac{n(n-1)}{2} + m - k$) we associate a first order differential operator

$$Bu := (\operatorname{curl}^* v, \operatorname{grad} w) \in \mathbb{R}^{k\times n} \times \mathbb{R}^{(m-k)\times n} = \mathbb{R}^{m\times n}$$

where

$$\begin{aligned}
\operatorname{curl}^* v &:= \left(\operatorname{curl}^* v^1, \ldots, \operatorname{curl}^* v^k\right) \in \mathbb{R}^{k\times n}, \\
\operatorname{curl}^* v^i &:= \left(\left(\operatorname{curl}^* v^i\right)_1, \ldots, \left(\operatorname{curl}^* v^i\right)_n\right) \in \mathbb{R}^n, \\
\left(\operatorname{curl}^* v^i\right)_\nu &:= \sum_{r=1}^{\nu-1} (-1)^{\nu+r} \frac{\partial v^i_{r\nu}}{\partial x_r} + \sum_{s=\nu+1}^{n} (-1)^{\nu+s+1} \frac{\partial v^i_{\nu s}}{\partial x_s} \in \mathbb{R},
\end{aligned}$$

with $v^i : \mathbb{R}^n \to \mathbb{R}^{\frac{n(n-1)}{2}}$, $v^i = \left(v^i_{rs}\right)_{1\le r<s\le n}$, $i = 1, \ldots, k$. It may be more convenient to write v^i as an $n \times n$ antisymmetric matrix (that is $v^i_{rs} = -\, v^i_{sr}$) and then we have, equivalently,

$$\left(\operatorname{curl}^* v^i\right)_\nu = \sum_{r=1}^{n} (-1)^{\nu+r} \frac{\partial v^i_{r\nu}}{\partial x_r}.$$

Similarly

$$\begin{aligned}
\operatorname{grad} w &:= \left(\operatorname{grad} w^{k+1}, \ldots, \operatorname{grad} w^m\right) \in \mathbb{R}^{(m-k)\times n}, \\
\operatorname{grad} w^i &:= \left(\frac{\partial w^i}{\partial x_1}, \ldots, \frac{\partial w^i}{\partial x_n}\right) \in \mathbb{R}^n,
\end{aligned}$$

where $w^i : \mathbb{R}^n \to \mathbb{R}$, $i = k+1, \ldots, m$.

Remark 2.1. (i) The important fact linking the operators A and B is that

$$ABu \equiv 0 \text{ for all } u \in C^2\left(\mathbb{R}^n; \mathbb{R}^{k\times \frac{n(n-1)}{2}} \times \mathbb{R}^{(m-k)}\right). \tag{2.1}$$

(ii) The operators A and B are particular cases of the "d" operators of differential forms. For example $\operatorname{div} V^i$ (respectively $\operatorname{curl} W^i$, $\operatorname{curl}^* v^i$, $\operatorname{grad} w^i$) is an n (respectively 2, $(n-1)$, 1) form which, in turn, is the differential of an $(n-1)$ (respectively 1, $(n-2)$, 0) form. The identity (2.1) is just a rewriting of

$$dd\omega = 0.$$

Closely related to the operator A is the following set (so-called "characteristic cone" in the language of the theory of compensated compactness) introduced by Murat-Tartar,

$$\Lambda := \left\{ \begin{array}{c} \lambda = \begin{pmatrix} \lambda^1 \\ \vdots \\ \lambda^m \end{pmatrix} = \begin{pmatrix} \lambda^1_1 & \cdots & \lambda^1_n \\ \vdots & \ddots & \vdots \\ \lambda^m_1 & \cdots & \lambda^m_n \end{pmatrix} \in \mathbb{R}^{m\times n} : \\ \lambda^i \perp \lambda^j \text{ for all } i = 1, \ldots, k, \text{ and for all } j = k+1, \ldots, m, \\ \operatorname{rank} \begin{pmatrix} \lambda^1 \\ \vdots \\ \lambda^k \end{pmatrix} \le n-1,\ \operatorname{rank} \begin{pmatrix} \lambda^{k+1} \\ \vdots \\ \lambda^m \end{pmatrix} \le 1 \end{array} \right\}.$$

We now have the following definitions.

Definition 2.2. *(i) A Borel measurable function $f : \mathbb{R}^{m\times n} \to \mathbb{R}$ is said to be A-B quasiconvex if*

$$\int_U f\left(\xi + B\varphi(x)\right) dx \ge f(\xi) \operatorname{meas}(U)$$

for every bounded domain $U \subset \mathbb{R}^n$, $\xi \in \mathbb{R}^{m\times n}$, and $\varphi \in W_0^{1,\infty}\left(U; \mathbb{R}^N\right)$.

(ii) A function $f:\mathbb{R}^{m\times n}\to\overline{\mathbb{R}}=\mathbb{R}\cup\{+\infty\}$ *is said to be* Λ*-convex if*

$$f\left(t\xi_1+(1-t)\xi_2\right)\le t\,f\left(\xi_1\right)+(1-t)\;f\left(\xi_2\right)\textit{for all } \xi_1,\xi_2 \textit{ such that } \xi_1-\xi_2\in\Lambda.$$

(iii) Given a set $Z\subset\mathbb{R}^{m\times n}$ *we define* $\Lambda\mathrm{co}\,Z$*, the* Λ *convex hull of* Z*, by*

$$\Lambda\mathrm{co}\,Z:=\left\{\begin{array}{ll}\lambda\in\mathbb{R}^{m\times n} & : f\left(\lambda\right)\le 0 \text{ for all } \Lambda-\text{convex function}\\ & f:\mathbb{R}^{m\times n}\to\overline{\mathbb{R}}=\mathbb{R}\cup\{+\infty\}\text{ such that } f\mid_Z=0\end{array}\right\}.$$

Remark 2.3. It follows immediately from the above defintion and from the fact that every vector of the canonical orthonormal basis of $\mathbb{R}^{m\times n}$ may be identified with an element of Λ, that any Λ-convex function is separately convex, and thus locally Lipschitz.

Some basic facts about the preceding notions are (see [4], [12]):

Theorem 2.4. *(i) Let* $f:\mathbb{R}^{m\times n}\to\mathbb{R}$ *be a Borel measurable function.*
(i) if f *is convex then* f *is A-B quasiconvex;*
(ii) if f *is continuous then*

$$f\ \textit{A-B quasiconvex}\Rightarrow\ f\ \Lambda\textit{-convex;}$$

(iii) if f *is A-B quasiconvex and continuous, then for every sequence such that*

$$u_\nu\to u \textit{ in } L^\infty(\Omega) \textit{ and } Bu_\nu\overset{*}{\rightharpoonup} Bu \textit{ in } L^\infty(\Omega),$$

the lower semicontinuity property

$$\liminf_{\nu\to\infty}\int_\Omega f\left(Bu_\nu\left(x\right)\right)dx\ge\int_\Omega f\left(Bu\left(x\right)\right)dx$$

holds for every open, bounded set $\Omega\subset\mathbb{R}^n$*.*

The following are important examples of A-B quasiconvexity.

Example 2.5. (i) If $k=m\le n-1$ and $AU:=\left(\operatorname{div}V^1,...,\operatorname{div}V^k\right)$ then

$$\Lambda=\mathbb{R}^{m\times n},$$

and therefore Λ convexity reduces to the usual notion of convexity. In particular, the Λ convex hull of a given set is its ordinary convex hull.

If $k=m=n$ and $AU:=\left(\operatorname{div}V^1,...,\operatorname{div}V^n\right)$ then

$$\Lambda=\left\{\lambda\in\mathbb{R}^{n\times n}:\operatorname{rank}\left(\lambda\right)\le n-1\right\}.$$

An example of non convex function that is A-B quasiconvex is (see Tartar [21])

$$f\left(\xi\right)=(n-1)\operatorname{trace}\left(\xi^T\,\xi\right)-\left(\operatorname{trace}\xi\,\right)^2.$$

(ii) If $k=0$, $m=1$, and $AU:=\operatorname{curl}W\in\mathbb{R}^{\frac{n(n-1)}{2}}$ then

$$\Lambda=\mathbb{R}^n$$

and, as in (i) above, Λ convexity is ordinary convexity and the Λ convex hull is the usual convex hull. This case corresponds to scalar variational problems with underlying energy of the type

$$w\in W^{1,1}_{\mathrm{loc}}(\Omega;\mathbb{R})\mapsto\int_\Omega f(Dw)\,dx.$$

If $k=0$, $m>1$, and $AU := \operatorname{curl} W = \left(\operatorname{curl} W^1, ..., \operatorname{curl} W^m\right) \in \mathbb{R}^{m\times \frac{n(n-1)}{2}}$ then

$$\Lambda = \left\{\lambda \in \mathbb{R}^{n\times n} : \operatorname{rank}(\lambda) \le 1\right\}.$$

A-B quasiconvexity is then the usual quasiconvexity condition of Morrey [17] for energy densities of the type

$$w \in W^{1,1}_{\text{loc}}(\Omega;\mathbb{R}^m) \mapsto \int_\Omega f(Dw)\,dx.$$

In order to compute the Λ convex hull of a given set one can use the following iteration scheme.

Proposition 2.6. *Let $Z \subset \mathbb{R}^{m\times n}$, $\Lambda_0 \text{co}\, Z := Z$, and let for $i \in \mathbb{N}$*

$$\Lambda_i \text{co}\, Z := \{\xi \in \mathbb{R}^{m\times n} :\ \xi = t\xi_1 + (1-t)\xi_2,\quad \xi_1,\xi_2 \in \Lambda_{i-1}\text{co}\, Z,\\ \xi_1 - \xi_2 \in \Lambda,\ t\in[0,1]\}.$$

Then

$$\Lambda \text{co}\, Z = \cup_{i\in\mathbb{N}_0} \Lambda_i \text{co}\, Z.$$

Proof. It can be proved easily by induction (the case $i=0$ being trivial) that

$$\Lambda_i \text{co}\, Z \subset \Lambda \text{co}\, Z \quad \text{for all } i \in \mathbb{N}_0,$$

from what follows that

$$\cup_{i\in\mathbb{N}_0} \Lambda_i \text{co}\, Z \subset \Lambda \text{co}\, Z.$$

In order to establish the reverse inclusion, we first need to introduce the notion of Λ-convex envelope of a function $f : \mathbb{R}^{m\times n} \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$. We define recursively

$$\Lambda_0 f := f,$$

$$\Lambda_{i+1} f(\xi) := \inf\{t\Lambda_i f(\xi_1) + (1-t)\Lambda_i f(\xi_2) : t\in[0,1], \xi = t\xi_1 + (1-t)\xi_2, \xi_1 - \xi_2 \in \Lambda\},$$

and

$$\Lambda f(\xi) := \inf_{i\in\mathbb{N}_0} \Lambda_i f(\xi). \tag{2.2}$$

Since $\Lambda_{i+1} f \le \Lambda_i f$, we deduce that the infimum in (2.2) is actually a limit, and that Λf is the largest Λ-convex function smaller than or equal to f.

Consider the indicator function of the set Z,

$$I_Z(x) := \begin{cases} 0 & x \in Z, \\ +\infty & \text{otherwise.} \end{cases}$$

By induction it can be proved that for all $i \in \mathbb{N}_0$

$$\Lambda_i I_Z = I_{\Lambda_i \text{co}\, Z},$$

and thus

$$\Lambda I_Z = I_{\cup_{i\in\mathbb{N}_0} \Lambda_i \text{co}\, Z}. \tag{2.3}$$

Since ΛI_Z is non-negative and Λ-convex, and $\Lambda I_Z|Z = 0$, if $\xi \in \Lambda \text{co}\, Z$ then $\Lambda I_Z(\xi) = 0$, which, in view of (2.3), yields $\xi \in \cup_{i\in\mathbb{N}_0} \Lambda_i \text{co}\, Z$. ☐

We can now state the main problem we will address in the present article (see Section 5).

Problem 2.7. Let $\Omega \subset \mathbb{R}^n$ be an open set, let $E \subset \mathbb{R}^{m\times n}$ be compact, and consider the boundary datum $\varphi \in C^1\left(\overline{\Omega}; \mathbb{R}^N\right)$, where

$$N := k \times \frac{n(n-1)}{2} + (m-k).$$

Set

$$\varphi := (\psi, \chi)$$

where

$$\begin{aligned} \psi &:= \left(\psi^1, ..., \psi^k\right) \in \mathbb{R}^{k\times \frac{n(n-1)}{2}} \text{ and } \psi^i = \left(\psi^i_{rs}\right)_{1\le r<s\le n} \in \mathbb{R}^{\frac{n(n-1)}{2}}, \\ \chi &:= \left(\chi^{k+1}, ..., \chi^m\right) \in \mathbb{R}^{(m-k)}. \end{aligned}$$

Assume that

$$B\varphi(x) = (\operatorname{curl}^* \psi, \operatorname{grad} \chi) \in E \cup \operatorname{int} \Lambda \operatorname{co} E \text{ for every } x \in \Omega.$$

We seek to find a function $u \in C(\overline{\Omega}; \mathbb{R}^N)$ such that the distribution Bu belongs to $L^\infty(\Omega; \mathbb{R}^{m\times n})$ and

$$\begin{cases} Bu(x) \in E & \text{a.e. } x \in \Omega, \\ u(x) = \varphi(x) & \text{for all } x \in \partial\Omega. \end{cases}$$

3. The approximation lemma

The proof of our main existence result, Theorem 4.3, is hinged on a density argument together with the approximation lemma below. We adopt the notation introduced in Section 2.

Lemma 3.1 (Approximation lemma). *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, let $t \in (0,1)$, let $\xi, \eta \in \mathbb{R}^{m\times n}$ be such that*

$$\xi - \eta \in \Lambda,$$

and let $\varphi \in W^{1,\infty}\left(\Omega; \mathbb{R}^N\right)$ satisfy

$$B\varphi = t\xi + (1-t)\eta.$$

For every $\varepsilon > 0$ there exist $u \in W^{1,\infty}\left(\Omega; \mathbb{R}^N\right)$, and $\Omega_\xi, \Omega_\eta \subset \Omega$ disjoint open subsets, such that

$$\begin{cases} \left|\operatorname{meas}(\Omega_\xi) - t \operatorname{meas}(\Omega)\right|, \ \left|\operatorname{meas}(\Omega_\eta) - (1-t) \operatorname{meas}(\Omega)\right| \le \varepsilon, \\ u = \varphi \textit{ on a neighborhood of } \partial\Omega, \\ \|u-\varphi\|_{L^\infty} \le \varepsilon, \\ Bu(x) = \begin{cases} \xi & \textit{in } \Omega_\xi, \\ \eta & \textit{in } \Omega_\eta, \end{cases} \\ \operatorname{dist}\left(Bu(x), [\xi, \eta]\right) \le \varepsilon \textit{ for all } x \in \Omega. \end{cases} \tag{3.1}$$

Proof. We divide the proof into two steps.

Step 1. We start by assuming that

$$\xi-\eta=:\lambda=\begin{pmatrix} 0 & \lambda_2^1 & \cdots & \lambda_n^1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \lambda_2^k & \cdots & \lambda_n^k \\ \lambda_1^{k+1} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^m & 0 & \cdots & 0 \end{pmatrix}. \tag{3.2}$$

In Step 2 we will treat the general case.

We first note that we can assume that Ω is the unit cube. Indeed we can express Ω as a disjoint union of cubes with faces parallel to the coordinate axes and a set of small measure. It then follows that a solution u for (3.1) with respect to Ω may be constructed from solutions of (3.1) when Ω is the cube Q by setting $u\equiv\varphi$ on the set of small measure and by using homothetics and translations in each of the small subcubes.

Let $\varepsilon>0$, let Ω_ε be a set compactly contained in Ω, and let $h\in C_0^\infty(\Omega)$ and $L=L(\Omega)>0$ be such that

$$\begin{cases} \operatorname{meas}(\Omega\setminus\Omega_\varepsilon)\leq\varepsilon/2, \\ 0\leq h(x)\leq 1 \ \text{ for all } x\in\Omega, \\ h(x)=1 \ \text{ for all } x\in\Omega_\varepsilon, \\ |Dh(x)|\leq\frac{L}{\varepsilon} \ \text{ for all } x\in\Omega\setminus\Omega_\varepsilon. \end{cases}$$

Fix $\delta\in(0,\varepsilon)$. Consider a C^∞ function $g:[0,1]\to\mathbb{R}$ and sets I_ξ, I_η which are unions of disjoint open subintervals of $[0,1]$ so that

$$\begin{cases} g'(x_1)=\begin{cases} 1-t & \text{if } x_1\in I_\xi, \\ -t & \text{if } x_1\in I_\eta, \end{cases} \\ g'(x_1)\in[-t,1-t] \ \text{ for all } x_1\in[0,1], \\ |\operatorname{meas}(I_\xi)-t|,\ |\operatorname{meas}(I_\eta)-(1-t)|\leq\varepsilon/2, \\ |g(x_1)|\leq\delta \text{ for all } x_1\in[0,1]. \end{cases}$$

The construction of such function is standard. Let

$$\Omega_\xi:=\{x\in\Omega_\varepsilon: x_1\in I_\xi\} \quad \text{and } \Omega_\eta:=\{x\in\Omega_\varepsilon: x_1\in I_\eta\}.$$

Define $\alpha:=(\beta,\gamma)\in C^\infty\left(\overline{\Omega};\mathbb{R}^N\right)$ by

$$\alpha:=(\beta,\gamma)=\left(\beta^1,...,\beta^k,\gamma^{k+1},...,\gamma^m\right)\in\mathbb{R}^{k\times\frac{n(n-1)}{2}}\times\mathbb{R}^{(m-k)}$$

where $\beta^i:\mathbb{R}^n\to\mathbb{R}^{\frac{n(n-1)}{2}}$, $\beta^i=\left(\beta^i_{rs}\right)_{1\leq r<s\leq n}$, $i=1,...,k$, and $\gamma^i:\mathbb{R}^n\to\mathbb{R}$, $i=k+1,...,m$, are given by

$$\beta^i_{rs}(x):=\begin{cases} (-1)^{s+1}g(x_1)\lambda^i_s & \text{if } r=1 \\ 0 & \text{if } r\neq 1 \end{cases},\ i=1,...,k,$$

and

$$\gamma^i(x):=g(x_1)\lambda^i_1,\ i=k+1,...,m.$$

We observe that

$$B\alpha=(\operatorname{curl}^*\beta,\operatorname{grad}\gamma)=g'(x_1)(\xi-\eta).$$

Indeed, for $i = 1, ..., k$, and $\nu = 1, ...n$, we have (recalling that $\lambda_1^i = 0$)

$$\begin{aligned}\left(\operatorname{curl}^* \beta^{\mathrm{i}}\right)_\nu &= \sum_{r=1}^{\nu-1} (-1)^{\nu+r} \frac{\partial \beta^i_{r\nu}}{\partial x_r} + \sum_{s=\nu+1}^{n} (-1)^{\nu+s+1} \frac{\partial \beta^i_{\nu s}}{\partial x_s} \\ &= (-1)^{\nu+1} \frac{\partial \beta^i_{1\nu}}{\partial x_1} = g'(x_1)\, \lambda^i_\nu\end{aligned}$$

which implies that

$$\operatorname{curl}^* \beta^i = g'(x_1)\, \lambda^i,\ i = 1, ..., k.$$

Similarly, for $i = k+1, ..., m$, we get

$$\operatorname{grad} \gamma^i = \left(g'(x_1)\, \lambda^i_1, 0, ..., 0\right) = g'(x_1)\, \lambda^i.$$

We claim that the function

$$u =: h\ (\alpha + \varphi) + (1-h)\ \varphi = h\, \alpha + \varphi$$

satisfies all properties listed in (3.1). The four three statements are immediate in light of the construction of u and since in Ω_ε the function $h \equiv 1$, and so we have

$$Bu = B\alpha + B\varphi = g'(x_1)\,(\xi - \eta) + t\xi + (1-t)\,\eta = \begin{cases} \xi & \text{in } \Omega_\xi, \\ \eta & \text{in } \Omega_\eta. \end{cases}$$

In order to prove the last property we observe that

$$\begin{aligned} Bu &= h\ (B\alpha + B\varphi) + (1-h)\ B\varphi + R\,(h, \alpha) \\ &= (t + hg')\xi + (1 - (t + hg'))\eta + R\,(h, \alpha) \end{aligned}$$

where

$$R\,(h, \alpha) = \left(\begin{array}{c} \left(\sum_{r=1}^{\nu-1} (-1)^{\nu+r}\, \frac{\partial h}{\partial x_r} \beta^i_{r\nu} + \sum_{s=\nu+1}^{n} (-1)^{\nu+s+1}\, \frac{\partial h}{\partial x_s} \beta^i_{\nu s}\right)^{1\le i\le k}_{1\le\nu\le n} \\ \left(\gamma^i \operatorname{grad} h\right)^{k+1\le i\le m} \end{array} \right).$$

It is clear that by choosing δ sufficiently small with respect to ε we find that

$$|R\,(h, \alpha)| \le \varepsilon, \tag{3.3}$$

and since $t + hg',\ 1 - t - hg' \ge 0$, by (3.3) we conclude that

$$\operatorname{dist}\left(Bu\,(x), [\xi, \eta]\right) \le \varepsilon,\ x \in \Omega.$$

This achieves the first step of the lemma.

Step 2. We first claim that since $\xi - \eta \in \Lambda$ we can find $Q \in \mathbb{R}^{n\times n}$ invertible (in fact $Q \in SO\,(n)$) such that

$$(\xi - \eta)\, Q = \begin{pmatrix} 0 & \lambda^1_2 & \cdots & \lambda^1_n \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \lambda^k_2 & \cdots & \lambda^k_n \\ \lambda^{k+1}_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \lambda^m_1 & 0 & \cdots & 0 \end{pmatrix}. \tag{3.4}$$

Write

$$\xi - \eta = \begin{pmatrix} \mu^1 \\ \vdots \\ \mu^k \\ \mu^{k+1} \\ \vdots \\ \mu^m \end{pmatrix}.$$

By definition of Λ we can find $q^1 \in \mathbb{R}^n$, $\left|q^1\right| = 1$, such that

$$q^1 \in \operatorname{span}\left\{\mu^{k+1}, ..., \mu^m\right\},\ q^1 \perp \mu^1, ..., \mu^k,$$

(if $\operatorname{span}\left\{\mu^{k+1}, ..., \mu^m\right\} = \{0\}$ then ignore the condition $q^1 \in \operatorname{span}\left\{\mu^{k+1}, ..., \mu^m\right\}$), and choose $q^2, ..., q^n \in \mathbb{R}^n$, orthonormal vectors, so that

$$q^2, ..., q^n \perp \mu^{k+1}, ..., \mu^m.$$

Let

$$Q^T = \begin{pmatrix} q^1 \\ \vdots \\ q^n \end{pmatrix} = \begin{pmatrix} q_1^1 & \cdots & q_n^1 \\ \vdots & \ddots & \vdots \\ q_1^n & \cdots & q_n^n \end{pmatrix}.$$

It is clear that

$$\begin{aligned} (\xi - \eta)\, Q &= \begin{pmatrix} \langle \mu^1; q^1 \rangle & \langle \mu^1; q^2 \rangle & \cdots & \langle \mu^1; q^n \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle \mu^k; q^1 \rangle & \langle \mu^k; q^2 \rangle & \cdots & \langle \mu^k; q^n \rangle \\ \langle \mu^{k+1}; q^1 \rangle & \langle \mu^{k+1}; q^2 \rangle & \cdots & \langle \mu^{k+1}; q^n \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle \mu^m; q^1 \rangle & \langle \mu^m; q^2 \rangle & \cdots & \langle \mu^m; q^n \rangle \end{pmatrix} \\ &= \begin{pmatrix} 0 & \lambda_2^1 & \cdots & \lambda_n^1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \lambda_2^k & \cdots & \lambda_n^k \\ \lambda_1^{k+1} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^m & 0 & \cdots & 0 \end{pmatrix} \end{aligned}$$

and we have established (3.4). The conclusion of the lemma now follows from Step 1 by a change of variables. Precisely, let

$$\widetilde{\Omega} := Q^T \Omega, \quad \widetilde{\xi} := \xi Q, \quad \widetilde{\eta} = \eta Q,$$

and for $i = 1, ..., k$, let

$$\widetilde{\varphi}_{rs}^i(y) = (-1)^{r+s} \sum_{a,b=1}^n (-1)^{a+b} \varphi_{ba}^i(Q\, y)\, q_b^r q_a^s, \tag{3.5}$$

(we consider here φ^i as an $n \times n$ antisymmetric matrix), while for $i = k+1, ..., m$, we set

$$\widetilde{\varphi}^i(y) := \varphi^i(Q\, y).$$

The above definition can be written in matrix form as follows

$$\widetilde{\varphi}^i(y) = \begin{cases} \varphi^i(Q\,y)\ \mathrm{adj}_2 Q & \text{if } i = 1, ..., k, \\ \varphi^i(Q\,y) & \text{if } i = k+1, ..., m. \end{cases} \tag{3.6}$$

(It is easy to see that when $n = 2$ then $\mathrm{adj}_2 Q = \det Q = 1$, while when $n = 3$ then $\mathrm{adj}_2 Q = Q$). An elementary computation shows that (see Step 3 below)

$$B\widetilde{\varphi}(y) = B\varphi(Q\,y)\ Q. \tag{3.7}$$

We therefore get that

$$B\widetilde{\varphi} = t\widetilde{\xi} + (1-t)\,\widetilde{\eta}$$

with $\widetilde{\xi} - \widetilde{\eta}$ as in (3.2). Apply Step 1 to these new data and find $\widetilde{u}$, $\widetilde{\Omega}_{\widetilde{\xi}}$, $\widetilde{\Omega}_{\widetilde{\eta}}$ satisfying (3.1) with respect to $\widetilde{\xi}$ and $\widetilde{\eta}$. It suffices now to set

$$\Omega_\xi := Q\,\widetilde{\Omega}_{\widetilde{\xi}}, \quad \Omega_\eta := Q\,\widetilde{\Omega}_{\widetilde{\eta}},$$

and to define for $i = 1, ..., k$,

$$u^i_{ba}(x) = (-1)^{a+b} \sum_{r,s=1}^{n} (-1)^{r+s}\, \widetilde{u}^i_{rs}\left(Q^T x\right) q^r_b q^s_a,$$

while, for $i = k+1, ..., m$,

$$u^i(x) := \widetilde{u}^i\left(Q^T x\right).$$

The result follows using the fact that $Bu(x) = B\widetilde{u}\left(Q^T x\right) Q^T$.

Step 3. It only remains to show the elementary result that if

$$\widetilde{\varphi}^i(y) = \begin{cases} \varphi^i(Q\,y)\ \mathrm{adj}_2 Q & \text{if } i = 1, ..., k, \\ \varphi^i(Q\,y) & \text{if } i = k+1, ..., m, \end{cases} \tag{3.8}$$

then

$$B\widetilde{\varphi}(y) = B\varphi(Q\,y)\ Q. \tag{3.9}$$

This is clear when $i = k+1, ..., m$, since

$$\mathrm{grad}\, \widetilde{\varphi}^i(y) = \mathrm{grad}\, \varphi^i(Q\,y)\ Q.$$

It remains to show that for $i = 1, ..., k$,

$$\mathrm{curl}^*\widetilde{\varphi}^i(y) = \mathrm{curl}^*\left[\varphi^i(Q\,y)\right] Q$$

i.e., that for $\nu = 1, .., n$,

$$\left(\mathrm{curl}^*\widetilde{\varphi}^i(y)\right)_\nu = \sum_{a=1}^{n} \left(\mathrm{curl}^*\left[\varphi^i(Q\,y)\right]\right)_a q^\nu_a.$$

Recall that

$$\left(\mathrm{curl}^*\widetilde{\varphi}^{\mathrm{i}}\right)_\nu = \sum_{r=1}^{n} (-1)^{\nu+r} \frac{\partial \widetilde{\varphi}^i_{r\nu}}{\partial y_r}.$$

Invoking (3.5), and using the fact that $Q \in SO(n)$, thus $\sum_r q_s^r q_b^r = \delta_{sb}$, we have

$$\begin{aligned}
\left(\operatorname{curl}^* \widetilde{\varphi}^i\right)_\nu &= \sum_{r=1}^{n} (-1)^{\nu+r} \left[(-1)^{\nu+r} \sum_{a,b=1}^{n} (-1)^{a+b} \frac{\partial}{\partial y_r} \left[\varphi_{ba}^i (Q\, y)\right] \, q_b^r q_a^\nu \right] \\
&= \sum_{a,b,r=1}^{n} (-1)^{a+b} \left[\sum_{s=1}^{n} \frac{\partial \varphi_{ba}^i}{\partial x_s} (Q\, y) \; q_s^r \right] q_b^r q_a^\nu \\
&= \sum_{a,b,s=1}^{n} (-1)^{a+b} \frac{\partial \varphi_{ba}^i}{\partial x_s} (Q\, y) \; \delta_{sb} \, q_a^\nu \\
&= \sum_{a,b=1}^{n} (-1)^{a+b} \frac{\partial \varphi_{ba}^i}{\partial x_b} (Q\, y) \; q_a^\nu \\
&= \sum_{a=1}^{n} \left(\operatorname{curl}^* \left[\varphi^i (Q\, y)\right]\right)_a q_a^\nu
\end{aligned}$$

as claimed. This concludes Step 3 and thus the lemma. □

Remark 3.2. (i) We note that, by choosing $\delta \in (0, \varepsilon^2)$ in the previous proof, in addition we may require in (3.1) that

$$\|Du\|_{L^\infty} \le \|D\varphi\|_{L^\infty} + |\xi - \eta| + \varepsilon.$$

(ii) It follows immediately from the construction that if $\varphi \in C^1(\overline{\Omega}; \mathbb{R}^N)$ (resp. $C^1_{\text{piec}}(\overline{\Omega}; \mathbb{R}^N)$) then so does u. Here, and in what follows, $C^1_{\text{piec}}(\overline{\Omega}; \mathbb{R}^N)$ stands for the space of continuous and piecewise C^1 functions on Ω.
(iii) Given $\xi \in \mathbb{R}^{m \times n}$ there exists an affine map u_ξ such that $Bu_\xi = \xi$ and $\|Du_\xi\|_{L^\infty(\Omega)} \le C|\xi|$, where $C = C(N)$. Write first

$$\xi = \begin{pmatrix} \xi^1 \\ \vdots \\ \xi^k \\ \xi^{k+1} \\ \vdots \\ \xi^m \end{pmatrix}.$$

Define

$$u_\xi := (v, w) = \left(v^1, ..., v^k, w^{k+1}, ..., w^m\right) \in \mathbb{R}^{k \times \frac{n(n-1)}{2}} \times \mathbb{R}^{(m-k)} = \mathbb{R}^N,$$

where $v^i : \mathbb{R}^n \to \mathbb{R}^{\frac{n(n-1)}{2}}$, $v^i = \left(v^i_{rs}\right)_{1 \le r < s \le n}$, $i = 1, ..., k$, and $w^i : \mathbb{R}^n \to \mathbb{R}$, $i = k+1, ..., m$, are given by

$$v^i_{rs}(x) := \begin{cases} (-1)^{s+1} \left[x_1 \xi^i_s - \frac{1}{n-1} x_s \xi^i_1\right] & \text{if } r = 1, \\ 0 & \text{if } r \ne 1, \end{cases} \quad , \; i = 1, ..., k,$$

$$w^i(x) := \left\langle \xi^i; x \right\rangle, \; i = k+1, ..., m.$$

We claim that $Bu_\xi = \xi$. Indeed, for $i = 1, ..., k$ and $\nu = 1$, we have

$$\left(\operatorname{curl}^* v^i\right)_1 = \sum_{s=2}^{n} (-1)^s \frac{\partial v^i_{1s}}{\partial x_s} = \xi^i_1$$

while for $\nu = 2, ...n$, we obtain

$$\begin{aligned}\left(\mathrm{curl}^* v^i\right)_\nu &= \sum_{r=1}^{\nu-1} (-1)^{\nu+r} \frac{\partial v^i_{r\nu}}{\partial x_r} + \sum_{s=\nu+1}^{n} (-1)^{\nu+s+1} \frac{\partial v^i_{\nu s}}{\partial x_s} \\ &= (-1)^{\nu+1} \frac{\partial v^i_{1\nu}}{\partial x_1} = \xi^i_\nu.\end{aligned}$$

The case $i = k+1, ..., m$, is trivial since

$$\mathrm{grad}\, w^i = \xi^i.$$

The L^∞ bound on Du_ξ is an immediate consequence of the explicit definition of u_ξ.

4. An abstract existence theorem

We introduce the notion of sets with the relaxation property with respect to a fixed, underlying set.

Definition 4.1 (Relaxation property). *Let $E, K \subset \mathbb{R}^{m\times n}$. We say that K has the relaxation property with respect to E if for every bounded domain $\Omega \subset \mathbb{R}^n$ and for every affine map u_ξ with*

$$Bu_\xi = \xi \in \mathrm{int} K$$

there exists a sequence $\{u_\nu\} \subset C^1_{\mathrm{piec}}\left(\overline{\Omega}; \mathbb{R}^N\right)$ such that

$$\begin{cases} u_\nu \in u_\xi + W_0^{1,\infty}\left(\Omega; \mathbb{R}^N\right), \\ u_\nu \to u_\xi \text{ in } L^\infty(\Omega; \mathbb{R}^N), Bu_\nu \overset{*}{\rightharpoonup} \xi \textit{ in } L^\infty(\Omega; \mathbb{R}^{m\times n}), \\ Bu_\nu(x) \in E \cup \mathrm{int}\, K \textit{ a.e. in } \Omega, \\ \int_\Omega \mathrm{dist}\left(Bu_\nu(x); E\right) dx \to 0 \textit{ as } \nu \to \infty. \end{cases}$$

Remark 4.2. Observe that if K has the relaxation property with respect to E and if $f : \mathbb{R}^{m\times n} \to \mathbb{R}$ is A-B quasiconvex and continuous satisfying $f\,|_E = 0$, then using Remark 3.2 (iii) and Theorem 2.4 (iii) we get

$$f(\xi) \leq 0 \text{ for all } \xi \in \mathrm{int}\, K.$$

Next we state and prove the main abstract existence theorem of this paper.

Theorem 4.3. *Let $\Omega \subset \mathbb{R}^n$ be open. Let $F_i : \mathbb{R}^{m\times n} \to \mathbb{R}$, $i = 1, ..., I$, be A-B quasiconvex, continuous, and let*

$$E := \left\{\xi \in \mathbb{R}^{m\times n} : F_i(\xi) = 0,\ i = 1, ..., I\right\}$$

be compact. Let $K \subset \mathbb{R}^{m\times n}$ be a compact set with the relaxation property with respect to E and let $\varphi \in C^1_{\mathrm{piec}}\left(\overline{\Omega}; \mathbb{R}^N\right)$ be such that

$$B\varphi(x) \in E \cup \mathrm{int}\, K \textit{ a.e. } x \in \Omega.$$

Then there exists $u \in C(\overline{\Omega}; \mathbb{R}^N)$ such that $Bu \in L^\infty(\Omega; \mathbb{R}^{m\times n})$ and

$$\begin{cases} Bu(x) \in E & \textit{a.e. } x \in \Omega, \\ u(x) = \varphi(x) & \textit{for all } x \in \partial\Omega. \end{cases} \tag{4.1}$$

Proof. We divide the proof into three steps and follow the framework of [7].

Step 1. We may assume that $\Omega \subset \mathbb{R}^n$ is bounded, since in the general case we decompose Ω as a countable union of open, bounded sets, on each one of which we solve (4.1).

Let V be the C^0 closure of the set

$$\left\{u \in \varphi + W_0^{1,\infty}(\Omega;\mathbb{R}^N) : u \in C^1_{\text{piec}}\left(\overline{\Omega};\mathbb{R}^N\right), Bu\left(x\right) \in E \cup \operatorname{int} K \text{ a.e in } \Omega\right\}.$$

Note that $\varphi \in V$ and V is a complete metric space when endowed with the C^0 norm. The compactness of E and K, the weak lower semicontinuity property of continuous A-B quasiconvex functions (see Theorem 2.4), and Remark 4.2 yield

$$V \subset \left\{ \begin{array}{ll} u \in C(\overline{\Omega};\mathbb{R}^N) & : Bu \in L^\infty(\Omega;\mathbb{R}^{m\times n}), u = \varphi \text{ on } \partial\Omega \\ & F_i\left(Bu\left(x\right)\right) \leq 0 \text{ for } i = 1, ..., I, \text{ a.e. } x \in \Omega \end{array} \right\}. \tag{4.2}$$

Step 2. For $u \in V$ set

$$L\left(u\right) := \sum_{i=1}^{I} \int_\Omega F_i\left(Bu\left(x\right)\right) dx.$$

Due to the continuity and A-B quasiconvexity of F_i (see Theorem 2.4) we have for every $u \in V$

$$\liminf_{u_s \to u, u_s \in V} L\left(u_s\right) \geq L\left(u\right), \tag{4.3}$$

where we have used the fact that if $u_s \in V$ are such that $u_s \to u$ uniformly then $Bu_s \overset{*}{\rightharpoonup} Bu$, and also, in view of (4.2),

$$L(u) \leq 0, \qquad L\left(u\right) = 0 \Leftrightarrow Bu\left(x\right) \in E \text{ a.e. in}\, \Omega. \tag{4.4}$$

Define

$$V^k := \left\{u \in V : L\left(u\right) > -\frac{1}{k}\right\}.$$

By (4.3) V^k is open in V, and in Step 3 below we prove that V^k is dense in V. We deduce from Baire Category Theorem that $\cap V^k$ is dense in V. In particular, we may find $u \in \cap V^k$ and the result now follows from the definition of V and by (4.4).

Step 3. It remains to prove that for fixed $k \in \mathbb{N}$, $u \in V$, and $\varepsilon \in (0, 1/k)$ sufficiently small, we can find $u_\varepsilon \in V^k$ so that

$$\left\|u_\varepsilon - u\right\|_\infty \leq \varepsilon.$$

We will prove this property under the further assumptions that $u \in C^1_{\text{piec}}(\overline{\Omega};\mathbb{R}^N)$ and that

$$Bu\left(x\right) \in E \cup \operatorname{int} K \text{ a.e in } \Omega.$$

The general case will follow by definition of V. Also, by working on each subdomain where u is C^1, we can assume, without loss of generality, that $u \in C^1\left(\overline{\Omega};\mathbb{R}^N\right)$. In particular, setting

$$\Omega_0 := \left\{x \in \Omega : Bu\left(x\right) \in E\right\}, \quad \Omega_1 := \Omega \setminus \Omega_0,$$

by continuity Ω_0 is closed and hence Ω_1 is open.

Before proceeding further we fix the constants. By compactness of E and K we have that

$$\xi \in E \cup K \;\Rightarrow\; \operatorname{dist}\left(\xi; E\right) \leq \beta$$

for some $\beta > 0$. From the continuity of F_i and from the definition of E we can find $\delta = \delta(\varepsilon) > 0$ such that for any measurable function $\xi : \mathbb{R}^n \to E \cup K$ it holds

$$\int_\Omega \operatorname{dist}(\xi(x); E)\, dx \le \delta \;\Rightarrow\; \sum_{i=1}^{I} \int_\Omega F_i(\xi(x))\, dx \ge -\varepsilon. \tag{4.5}$$

Using a density argument (see Theorem 10.16 in [7] modified accordingly), we can find $\tilde{u} \in C^1\left(\overline{\Omega}_1; \mathbb{R}^N\right)$, an integer $J = J(\varepsilon)$, and $\tilde{\Omega}_j \subset \Omega_1$, $1 \le j \le J$, disjoint open sets, such that

$$\begin{cases} \tilde{u} \equiv u \text{ on a neighborhood of } \partial\Omega_1, \\ \|\tilde{u} - u\|_{1,\infty} \le \frac{\varepsilon}{2}, \\ B\tilde{u}(x) \in \operatorname{int} K \text{ a.e. } x \in \Omega_1, \\ \operatorname{meas}\left(\Omega_1 \setminus \bigcup_{j=1}^{J} \tilde{\Omega}_j\right) \le \frac{\delta}{2\beta}, \\ B\tilde{u} \equiv \xi_j (\text{ constant}) \text{ on } \tilde{\Omega}_j. \end{cases}$$

Using the relaxation property we may find $u_j \in C^1_{\text{piec}}\left(\overline{\tilde{\Omega}_j}; \mathbb{R}^N\right)$ satisfying

$$\begin{cases} u_j \in \tilde{u} + W_0^{1,\infty}\left(\tilde{\Omega}_j; \mathbb{R}^N\right), \\ \|\tilde{u} - u_j\|_\infty \le \frac{\varepsilon}{2} \text{ in } \tilde{\Omega}_j, \\ Bu_j(x) \in E \cup \operatorname{int} K \text{ a.e. } x \in \tilde{\Omega}_j, \\ \int_{\tilde{\Omega}_j} \operatorname{dist}(Bu_{j,\nu}(x); E)\, dx \le \frac{\delta}{2} \frac{\operatorname{meas}(\tilde{\Omega}_j)}{\operatorname{meas}(\Omega_1)}. \end{cases}$$

Define

$$u_\varepsilon(x) := \begin{cases} u(x) & \text{if } x \in \Omega_0, \\ \tilde{u}(x) & \text{if } x \in \Omega_1 \setminus \bigcup_{j=1}^{J} \tilde{\Omega}_j, \\ u_j(x) & \text{if } x \in \tilde{\Omega}_j. \end{cases}$$

Observe that $u_\varepsilon \in C^1_{\text{piec}}\left(\overline{\Omega}; \mathbb{R}^N\right)$,

$$\begin{cases} u_\varepsilon \in u + W_0^{1,\infty}\left(\Omega; \mathbb{R}^N\right), \\ \|u_\varepsilon - u\|_\infty \le \varepsilon \text{ in } \Omega, \\ Bu_\varepsilon(x) \in E \cup \operatorname{int} K \text{ a.e. } x \in \Omega. \end{cases}$$

We also have that

$$\begin{aligned} \int_\Omega \operatorname{dist}(Bu_\varepsilon(x); E)\, dx &= \int_{\Omega_0} \operatorname{dist}(Bu_\varepsilon(x); E)\, dx + \int_{\Omega_1} \operatorname{dist}(Bu_\varepsilon(x); E)\, dx \\ &= \int_{\Omega_1} \operatorname{dist}(Bu_\varepsilon(x); E)\, dx \\ &= \int_{\Omega_1 \setminus \bigcup_{j=1}^{J} \tilde{\Omega}_j} \operatorname{dist}(Bu_\varepsilon(x); E)\, dx + \sum_{j=1}^{J} \int_{\tilde{\Omega}_j} \operatorname{dist}(Bu_\varepsilon(x); E)\, dx \\ &\le \beta \operatorname{meas}\left(\Omega_1 \setminus \bigcup_{j=1}^{J} \tilde{\Omega}_j\right) + \frac{\delta}{2} \\ &\le \frac{\delta}{2} + \frac{\delta}{2} = \delta. \end{aligned}$$

Hence, combining (4.5) with the above inequality, we get that

$$L(u_\varepsilon) \geq -\varepsilon > -\frac{1}{k}$$

which implies that $u_\varepsilon \in V^k$. The claimed density has therefore been established and the proof is complete. ☐

The verification of the relaxation property is, in general, a very delicate and subtle issue. The theorem below will provide a useful tool to establish the relaxation property for certain differential operators B (see Section 5).

Theorem 4.4. *Let $E, E_\delta \subset \mathbb{R}^{m\times n}$, $\delta \in (0, \delta_0)$, be compact sets such that*
(i) $\Lambda\mathrm{co}\, E_\delta \subset \mathrm{int}\Lambda\mathrm{co}\, E$ for every $\delta \in (0, \delta_0)$;
(ii) for every $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that for any $\delta \in (0, \delta(\varepsilon))$

$$\eta \in E_\delta \ \Rightarrow \ \mathrm{dist}(\eta, E) \leq \varepsilon;$$

(iii) if $\eta \in \mathrm{int}\Lambda\mathrm{co}\, E$ then $\eta \in \Lambda\mathrm{co}\, E_\delta$ for every $\delta > 0$ sufficiently small.
Then $\Lambda\mathrm{co}\, E$ has the relaxation property with respect to E.

Remark 4.5. The above property between the sets E and E_δ is called the "approximation property" in [7] and resembles the in-approximation of convex integration (see Müller-Šverák [18]).

Proof. Let $\Omega \subset \mathbb{R}^n$ be a bounded, open set, and let u be an affine function with $Bu(x) = \xi$, $\xi \in \mathrm{int}\Lambda\mathrm{co}\, E$. We claim that there exists a sequence $u_\nu \subset C^1_{\mathrm{piec}}\left(\overline{\Omega}; \mathbb{R}^N\right)$ such that

$$\begin{cases} u_\nu \in u + W_0^{1,\infty}\left(\Omega; \mathbb{R}^N\right), \\ u_\nu \to u_\xi \text{ in } L^\infty(\Omega; \mathbb{R}^N), Bu_\nu \overset{*}{\rightharpoonup} \xi \text{ in } L^\infty(\Omega; \mathbb{R}^{m\times n}), \\ Bu_\nu(x) \in E \cup \mathrm{int}\,\Lambda\mathrm{co}\, E \text{ a.e. in } \Omega, \\ \int_\Omega \mathrm{dist}\left(Bu_\nu(x); E\right) dx \to 0 \text{ as } \nu \to \infty. \end{cases} \tag{4.6}$$

Fix $\varepsilon > 0$ and let $\delta = \delta(\varepsilon)$ be determined according to (ii). By (iii) we may find $\delta_1 < \delta$ such that

$$\xi \in \Lambda\mathrm{co} E_{\delta_1}.$$

In view of Proposition 2.6 we have $\xi \in \Lambda_J\mathrm{co}\, E_{\delta_1}$ for a certain J, and we now proceed by induction on J.

Step 1. Assume that $J = 1$. We can therefore write

$$Bu = \xi = t\xi_1 + (1-t)\xi_2, \ \xi_1 - \xi_2 \in \Lambda,$$

with

$$\xi_1, \xi_2 \in E_{\delta_1}.$$

In light of property (i), choose $\varepsilon' \in (0, \varepsilon)$ such that

$$\text{the } \varepsilon' - \text{neighborhood of } \Lambda\mathrm{co}E_{\delta_1} \text{ is contained in } \mathrm{int}\Lambda\mathrm{co}E. \tag{4.7}$$

Using the Approximation Lemma 3.1 and Remark 3.2 (ii), we may find $u_\varepsilon \in C^1\left(\overline{\Omega};\mathbb{R}^N\right)$ such that

$$\begin{cases} \operatorname{meas}(\Omega \setminus (\Omega_1 \cup \Omega_2)) = O(\varepsilon'), \\ u_\varepsilon \equiv u \text{ near } \partial\Omega, \\ \|u_\varepsilon - u\|_\infty \leq \varepsilon', \\ Bu_\varepsilon(x) = \begin{cases} \xi_1 & \text{in } \Omega_1, \\ \xi_2 & \text{in } \Omega_2, \end{cases} \\ \operatorname{dist}\left(Bu_\varepsilon(x); \Lambda\mathrm{co}\, E_{\delta_1}\right) \leq \varepsilon \quad \text{in } \Omega, \end{cases}$$

where we have the fact that (i) implies that

$$E_{\delta_1} \subset \mathrm{co}\, E \subset B(0, M)$$

for some $M > 0$ and all $\delta \in (0, \delta_0)$, and

$$\mathrm{co}\ \{\xi_1, \xi_2\} \subset \Lambda\mathrm{co}\, E_{\delta_1}.$$

Now (4.7) ensures that

$$Bu_\varepsilon \in \mathrm{int}\Lambda\mathrm{co}E,$$

and in view of (ii), taking into account that $\operatorname{dist}(Bu_\varepsilon; E)$ is a bounded function, we conclude that

$$\begin{aligned} \int_\Omega \operatorname{dist}(Bu_\varepsilon; E) &= \int_{\Omega_1} \operatorname{dist}(\xi_1; E) + \int_{\Omega_2} \operatorname{dist}(\xi_2; E) \\ &\quad + \int_{\Omega \setminus (\Omega_1 \cup \Omega_2)} \operatorname{dist}(Bu_\varepsilon; E) \\ &= O(\varepsilon). \end{aligned}$$

The claim (4.6) follows by letting $\varepsilon \to 0^+$.

Step 2. Let

$$\xi \in \Lambda_J \mathrm{co}\, E_\delta$$

for some $J > 1$. Then there exist $\xi_1, \xi_2 \in \mathbb{R}^{m\times n}$ such that

$$\xi = t\xi_1 + (1-t)\xi_2,\ \ \xi_1, \xi_2 \in \Lambda_{J-1}\mathrm{co}\, E_\delta,\ \ \xi_1 - \xi_2 \in \Lambda.$$

With ε' chosen as in Step 1, apply the Approximation Lemma 3.1 and Remark 3.2 (ii) to find a function $v_\varepsilon \in C^1(\overline{\Omega};\mathbb{R}^N)$ and Ω_1, Ω_2 disjoint open sets such that

$$\begin{cases} \operatorname{meas}(\Omega \setminus (\Omega_1 \cup \Omega_2)) \leq \varepsilon'/2, \\ v_\varepsilon \equiv u \text{ on a neighborhood of } \partial\Omega, \\ \|v_\varepsilon - u\|_\infty \leq \varepsilon'/2, \\ Bv_\varepsilon(x) = \begin{cases} \xi_1 \text{ in } \Omega_1, \\ \xi_2 \text{ in } \Omega_2, \end{cases} \\ \operatorname{dist}\left(Bv_\varepsilon(x); \Lambda\mathrm{co}\, E_\delta\right) \leq \varepsilon \text{ in } \Omega. \end{cases}$$

Note that again by (4.7) we have that

$$Bv_\varepsilon \in \mathrm{int}\Lambda\mathrm{co}E,$$

although now we are unable to guarantee that

$$\operatorname{dist}(\xi_i; E) \leq \varepsilon \quad i = 1, 2.$$

We use therefore the induction hypothesis on Ω_1, Ω_2 and ξ_1, ξ_2 to obtain $v_\varepsilon^1 \in C^1_{\text{piec}}$ in Ω_1, $v_\varepsilon^2 \in C^1_{\text{piec}}$ in Ω_2 and satisfying for $i = 1, 2$,

$$\begin{cases} v_\varepsilon^i \equiv v_\varepsilon \text{ near } \partial\Omega_i, \\ \left\| v_\varepsilon^i - v_\varepsilon \right\|_\infty \leq \varepsilon'/2 \text{ in } \overline{\Omega}_i, \\ Bv_\varepsilon^i \in E \cup \text{int}\Lambda\text{co}E \text{ a.e. } x \in \Omega_i, \\ \int_{\Omega_i} \text{dist}\left(Bv_\varepsilon^i; E\right) \leq \varepsilon'. \end{cases}$$

Setting

$$u_\varepsilon(x) := \begin{cases} v_\varepsilon(x) & \text{in } \Omega \setminus (\Omega_1 \cup \Omega_2), \\ v_\varepsilon^1(x) & \text{in } \Omega_1, \\ v_\varepsilon^2(x) & \text{in } \Omega_2, \end{cases}$$

we have indeed established (4.6) by choosing ε arbitrarily small. □

5. Existence theorems in the applications

In this section we solve Problem 2.7 in the case where $\Lambda = \mathbb{R}^{m\times n}$.

Theorem 5.1. *Let $\Omega \subset \mathbb{R}^n$ be open. Let $\Lambda = \mathbb{R}^{m\times n}$ and let $E \subset \mathbb{R}^{m\times n}$ be compact. Let $\varphi \in C^1_{\text{piec}}\left(\overline{\Omega}; \mathbb{R}^N\right)$ be such that*

$$B\varphi\left(x\right) \in E \cup \operatorname{intco} E \ \textit{a.e.}\ \ x \in \Omega.$$

Then there exists $u \in C(\overline{\Omega}; \mathbb{R}^N)$ with $Bu \in L^\infty(\Omega; \mathbb{R}^{m\times n})$ and satisfying

$$\begin{cases} Bu(x) \in E & \text{a.e. } x \in \Omega, \\ u(x) = \varphi(x) & \text{for all } x \in \partial\Omega, \end{cases}$$

Remark 5.2. (i) By imposing that $\Lambda = \mathbb{R}^{m\times n}$ we are, essentially, restricting to the scalar case.

(ii) Using a more refined version of Vitali Covering Theorem, as in [7] it is possible to handle the case where $\varphi \in W^{1,\infty}\left(\Omega; \mathbb{R}^N\right)$.

(iii) The result is in fact more precise in that if $\operatorname{intco} E$ is non empty then we will find solutions u such that $Bu\left(x\right) \in E_{ext}$ there where $B\varphi(x) \notin E$, and where E_{ext} is the set of extreme points of E in the convex sense.

In view of Example 2.5 we obtain the following direct corollary that will be used in the application to ferromagnetism (with $m = 1$ and $n = 3$).

Corollary 5.3. *Let $\Omega \subset \mathbb{R}^n$ be open. Let $m \leq n-1$ and let $E \subset \mathbb{R}^{m\times n}$ be compact. Let $\varphi \in C^1_{\text{piec}}\left(\overline{\Omega}; \mathbb{R}^N\right)$, $N = m \times n\left(n-1\right)/2$, be such that*

$$\left(\text{curl}^*\varphi^1\left(x\right), ..., \text{curl}^*\varphi^m\left(x\right)\right) \in E \cup \operatorname{intco} E \ \textit{a.e.}\ \ x \in \Omega.$$

Then there exists $u \in C(\overline{\Omega}; \mathbb{R}^N)$ with $\text{curl}^* u \in L^\infty(\Omega; \mathbb{R}^{m\times n})$ and satisfying

$$\begin{cases} \left(\text{curl}^*u^1\left(x\right), ..., \text{curl}^*u^m\left(x\right)\right) \in E & \textit{a.e. } \ x \in \Omega, \\ u(x) = \varphi(x) & \textit{for all } x \in \partial\Omega. \end{cases}$$

Before proceeding with the proof of the theorem we need this elementary result of convex analysis.

Proposition 5.4. *Let $E \subset \mathbb{R}^N$ be compact and such that $\operatorname{intco} E \neq \emptyset$. Then there exist convex functions $f_i : \mathbb{R}^N \to \mathbb{R}$, $i = 1, 2$, such that*

$$E_{\text{ext}} = \left\{\xi \in \mathbb{R}^N : f_i\left(\xi\right) = 0,\ i = 1, 2\right\},$$

$$\operatorname{co} E = \operatorname{co} E_{\text{ext}} = \left\{\xi \in \mathbb{R}^N : f_i\left(\xi\right) \leq 0,\ i = 1, 2\right\},$$

where E_{ext} *denotes the set of extreme points of* $\operatorname{co} E$.

Proof. We sketch the proof of Georgy [13] which follows ideas of Bressan-Flores [2] and De Blasi-Pianigiani [20].

The first function f_1 is the *gauge* associated to $\operatorname{co} E$, i.e. for any fixed $\xi_0 \in \operatorname{int} \operatorname{co} E$ we define

$$f_1(\xi) := -1 + \inf \left\{ t > 0 : \frac{\xi - \xi_0}{t} + \xi_0 \in \operatorname{co} E \right\},$$

which is a convex function that satisfies

$$f_1(\xi) \leq 0 \ \Leftrightarrow \ \xi \in \operatorname{co} E.$$

The second function f_2 is the *Choquet function*, and is defined as follows. Let

$$\varphi(\xi) := \begin{cases} -|\xi|^2 = -\sum_{i=1}^N \xi_i^2 & \text{if } \xi \in \operatorname{co} E, \\ +\infty & \text{otherwise,} \end{cases}$$

and, denoting by φ^{**} the convex envelope of φ, set

$$\psi(\xi) := \begin{cases} \varphi^{**}(\xi) - \varphi(\xi) & \text{if } \xi \in \operatorname{co} E, \\ +\infty & \text{otherwise.} \end{cases}.$$

The function ψ is exactly the Choquet function, which is convex and vanishes only on extreme points of E and otherwise is negative on $\operatorname{co} E$. It is easy to see that ψ is Lipschitz (since φ, and hence φ^{**}, is Lipschitz) all over the compact set $\operatorname{co} E$. It can therefore be extended in a finite and convex way to the whole of $\mathbb{R}^N$. It is this extension that we call f_2. The conclusion then follows. □

We may now return to the proof of Theorem 5.1.

Proof. Since $\Lambda = \mathbb{R}^{m \times n}$ we have $\Lambda \mathrm{co} E = \mathrm{coE}$. If $\operatorname{intco} E = \emptyset$ then the result is trivial, so we may assume that $\operatorname{intco} E \neq \emptyset$. In addition, without loss of generality we suppose that $\varphi \in C^1(\overline{\Omega}; \mathbb{R}^N)$.

Let

$$\Omega_0 := \{x \in \Omega : B\varphi(x) \in E\}, \quad \Omega_1 := \Omega \setminus \Omega_0.$$

Then Ω_1 is open and $B\varphi(x) \in \mathrm{intco} E$ for all $x \in \Omega_1$. By Proposition 5.4 we may assume that

$$E = E_{\text{ext}} = \left\{ \xi \in \mathbb{R}^N : f_i(\xi) = 0,\ i = 1, 2 \right\}$$

and

$$\operatorname{co} E = \operatorname{co} E_{\text{ext}} = \left\{ \xi \in \mathbb{R}^N : f_i(\xi) \leq 0,\ i = 1, 2 \right\},$$

with f_i convex functions, hence A-B quasiconvex and continuous, for $i = 1, 2$. In view of Theorem 4.3 it suffices to prove that $\operatorname{co} E$ has the relaxation property with respect to E to find a function $u_1 \in C(\overline{\Omega_1}; \mathbb{R}^N)$ with $Bu_1 \in L^\infty(\Omega_1; \mathbb{R}^N)$ and satisfying

$$\begin{cases} Bu_1(x) \in E_{\text{ext}} & \text{a.e. } x \in \Omega_1, \\ u_1(x) = \varphi(x) & \text{for all } x \in \partial\Omega, \end{cases}$$

and we then set

$$u(x) := \begin{cases} \varphi(x) & \text{if } x \in \Omega_0, \\ u_1(x) & \text{if } x \in \Omega_1. \end{cases}$$

In order to establish the relaxation property we use Theorem 4.4. Choose $\xi_0 \in \operatorname{intco} E$ and define for $\delta \in (0, 1)$ the sets

$$E_\delta := \delta \xi_0 + (1 - \delta) E.$$

Observe that

$$\operatorname{co} E_\delta = \delta\xi_0 + (1-\delta)\operatorname{co} E \subset \operatorname{intco} E$$

because if $B(\xi_0, r) \subset \operatorname{co}E$ then it follows that for all $\eta \in \operatorname{co}E$

$$B(\delta\xi_0 + (1-\delta)\eta; \delta r) \subset \operatorname{co}E.$$

Hypothesis (ii) of Theorem 4.4 can be easily verified, and as for (iii) we remark that if $\eta \in \operatorname{intco}E$ then, since

$$\left|\left(\frac{\eta}{1-\delta} - \frac{\delta}{1-\delta}\xi_0\right) - \eta\right| \to 0 \text{ as } \delta \to 0,$$

we have

$$\frac{\eta}{1-\delta} - \frac{\delta}{1-\delta}\xi_0 \in \operatorname{co}E,$$

i.e. $\eta \in \delta\xi_0 + (1-\delta)\operatorname{co}E = \operatorname{co}E_\delta$. □

Suppose now that the set E is the set of zeroes of just one A-B quasiconvex function.

Theorem 5.5. *Let $\Omega \subset \mathbb{R}^n$ be open. Let $F : \mathbb{R}^{m\times n} \to \mathbb{R}$, be A-B quasiconvex, continuous, and coercive (i.e. $F(\xi) \to +\infty$ as $|\xi| \to +\infty$). Let $\varphi \in C^1_{\text{piec}}\left(\overline{\Omega}; \mathbb{R}^N\right)$ be such that*

$$F(B\varphi(x)) \leq 0 \ \ a.e.\ \ x \in \Omega.$$

Then there exists $u \in C(\overline{\Omega}; \mathbb{R}^N)$ with $Bu \in L^\infty(\Omega; \mathbb{R}^{m\times n})$ and satisfying

$$\begin{cases} F(Bu(x)) = 0 & a.e.\ \ x \in \Omega, \\ u(x) = \varphi(x) & \text{for all } x \in \partial\Omega. \end{cases}$$

Remark 5.6. In fact, as it will be clear from the proof, one can weaken the coercivity condition and replace it by coercivity in, at least, one direction $\lambda \in \Lambda$, i.e. for every bounded set $\Xi \subset \mathbb{R}^{m\times n}$ there exists a continuous function $\rho : \mathbb{R} \to \mathbb{R}$, with $\lim_{t\to\pm\infty} \rho(t) = +\infty$, such that

$$F(\xi + t\lambda) \geq \rho(t) \text{ for all } t \in \mathbb{R} \text{ and for all } \xi \in \Xi.$$

Proof. By working on each subdomain where $u \in C^1$, we may assume thar $u \in C^1(\overline{\Omega}; \mathbb{R}^N)$.

We claim that

$$\Lambda\operatorname{co} E = \{F \leq 0\}.$$

Since F is Λ-convex (see Theorem 2.4 (ii)) we have that

$$\Lambda\operatorname{co} E \subset \{F \leq 0\}.$$

Conversely, if $F(\xi) = 0$ then $\xi \in E \subset \Lambda\operatorname{co} E$, and if $F(\xi) < 0$ then choose $\lambda \in \Lambda\backslash\{0\}$. By the coercivity hypothesis there must exist $t_1 < 0 < t_2$ such that $F(\xi_i) = 0$, where $\xi_i := \xi + t_i\lambda$. Now $\xi_1 - \xi_2 \in \Lambda$ and $\xi = \theta\xi_1 + (1-\theta)\xi_2$ where $\theta \in [0,1]$ is such that $0 = \theta t_1 + (1-\theta)t_2$. In view of Proposition 2.6 we conclude that $\xi \in \Lambda\operatorname{co} E$.

As it is usual we may assume that $F(B\varphi) < 0$, by setting $u \equiv \varphi$ on the closed set where $F(B\varphi) = 0$. By continuity of F we have

$$B\varphi \in \operatorname{int} \Lambda\operatorname{co} E.$$

Due to the continuity and coercivity of F, the set $E := \{F = 0\}$ is compact, and in particular $\Lambda\operatorname{co} E$ is also compact. Therefore, in order to apply Theorem 4.3 we only have to check that $\Lambda\operatorname{co} E$ has the relaxation property with respect to E. Let

$\xi \in \operatorname{int} \Lambda \mathrm{co}\, E$. If $F(\xi) = 0$ then, and in view of Remark 3.2, the relaxation property is trivial. Assume now that $F(\xi) < 0$ and fix $\lambda \in \Lambda \backslash \{0\}$. The coercivity assumption leads to the existence of $t_1 < 0 < t_2$ such that

$$F(\xi + t_1 \lambda) = 0 = F(\xi + t_2 \lambda),\ F(\xi + t\lambda) < 0 \text{ for all } t \in (t_1, t_2).$$

Let now u_ξ be an affine map with $Bu_\xi = \xi$, and choose $\varepsilon > 0$ small enough so that $\theta := \frac{t_2 - \varepsilon}{t_2 - t_1 - 2\varepsilon} > 0$. We have

$$Bu_\xi = \theta \xi_1 + (1 - \theta) \xi_2$$

where

$$\xi_1 := \xi + (t_1 + \varepsilon)\lambda,\ \xi_2 := \xi + (t_2 - \varepsilon)\lambda,\ \xi_1, \xi_2 \in \operatorname{int} \Lambda \mathrm{co}\, E,\ \xi_1 - \xi_2 \in \Lambda.$$

Choose $0 < \varepsilon' < \varepsilon$ such that

$$\{\operatorname{dist}(\cdot; [\xi_1, \xi_2]) < \varepsilon'\} \subset \operatorname{int} \Lambda \mathrm{co}\, E.$$

By the Approximation Lemma 3.1 we may find u_ε which agrees with u_ξ on a neighborhood of $\partial\Omega$, with

$$||u_\varepsilon - u_\xi||_{L^\infty} < \varepsilon', \operatorname{dist}(Bu_\varepsilon(x), [\xi_1, \xi_2]) \leq \varepsilon'.$$

We conclude that $Bu_{\varepsilon'} \in \operatorname{int} \Lambda \mathrm{co}\, E$, and it suffices to let $\varepsilon \to 0$. $\square$

6. Exact equilibrium solutions in ferromagnetism

In the sequel we will adopt the notations of the Introduction, and we will use the notions of *edge* and *face* of a convex set. Precisely

Definition 6.1. *Given $z_1, ..., z_N \in \mathbb{R}^3$ we denote by span $\{z_1, ..., z_N\}$ the subspace generated by these vectors and by dim span its dimension. If $Z \subset \mathbb{R}^3$ is compact and if $\xi \in \partial coZ \setminus Z$ then we say that ξ is on an edge of $\partial \mathrm{co}\, Z$ if*

$$\left.\begin{array}{c} \xi = \sum_{i=1}^{l} t_i z_i \\ z_i \in Z,\ t_i > 0,\ \sum_{i=1}^{l} t_i = 1 \end{array}\right\} \Rightarrow \dim \operatorname{span} \{z_1, ..., z_l\} \leq 1.$$

If this is not the case then we say that ξ is on a face of $\partial \mathrm{co}\, Z$.

The main result of this section is the theorem below.

Theorem 6.2. *Let $\Omega \subset \mathbb{R}^3$ be a bounded, open set with Lipschitz boundary, let $Z \subset S^2$ be compact and consider the problem*

$$\left\{\begin{array}{ll} m(x) \in Z & a.e.\ x \in \Omega, \\ \operatorname{div}(\chi_\Omega m) = 0 & in\, \mathbb{R}^3, \end{array}\right. \tag{6.1}$$

where $m \in L^\infty\left(\mathbb{R}^3; \mathbb{R}^3\right)$ ($m \equiv 0$ in $\mathbb{R}^3 \diagdown \overline{\Omega}$).

Case 1 : if $0 \notin \mathrm{co}\, Z$ then (6.1) has no solutions;

Case 2 : if $0 \in \partial \mathrm{co}\, Z$ and 0 is on an edge of $\partial \mathrm{co}\, Z$ then (6.1) has no solutions;

Case 3 : if $0 \in \partial \mathrm{co}\, Z$ and 0 is on a face de $\partial \mathrm{co}\, Z$ then (6.1) admits solutions;

Case 4 : if $0 \in \operatorname{intco} Z$ then (6.1) admits solutions. Moreover, there exists $M \in C(\overline{\Omega}; \mathbb{R}^3)$, with $m := \operatorname{curl} M \in L^\infty(\Omega; \mathbb{R}^3)$, satisfying

$$\left\{\begin{array}{ll} \operatorname{curl} M \in Z & a.e.\ in\ \Omega, \\ M = 0 & on\ \partial\Omega. \end{array}\right.$$

Remark 6.3. (i) Note that the condition div $(m\chi_\Omega) = 0$ implies $\int_\Omega m = 0$.

(ii) In the particular case where the set Z is symmetric with respect to the origin, i.e.

$$\xi \in Z \;\Rightarrow\; -\xi \in Z,$$

and under some further hypotheses, Cases 1 to 3 may be found in the work of James-Kinderlehrer [15] (see also De Simone [8]). Their results were inspired by the magnetism literature and we will also follow these ideas in our proofs for these two cases. However, Case 4 is new and follows from the theory developed earlier in this article. It could happen that in Case 4 problem (6.1) has geometrically constructed solutions of the same type as those we exhibit in Case 3, although we have not followed this avenue in our approach to the proof.

(iii) To contrast with Case 4 we will find in Case 3 a vector valued function $M \in W^{1,\infty}\left(\Omega;\mathbb{R}^3\right)$ satisfying

$$\begin{cases} \operatorname{curl} M\left(x\right) \in Z & \text{a.e. in } \Omega, \\ M \parallel \nu & \text{on } \partial\Omega, \end{cases}$$

where ν is a normal to $\partial\Omega$, and we set $m := \operatorname{curl} M$.

Proof. We will study each case separately

Case 1. If $0 \notin \operatorname{co} Z$ then (6.1) has no solutions. Indeed let $\rho : \mathbb{R}^3 \to \mathbb{R}$ be the gauge associated to $\operatorname{co} Z$, i.e.

$$\rho\left(\xi\right) \leq 1 \text{ if } \xi \in \operatorname{co} Z \quad \text{and } \rho\left(\xi\right) > 1 \text{ if } \xi \notin \operatorname{co} Z.$$

Since ρ is convex we have by Jensen inequality, and in view of Remark 6.3 (i),

$$1 < \rho\left(0\right) = \rho\left(\frac{1}{\operatorname{meas}(\Omega)}\int_\Omega m\right) \leq \frac{1}{\operatorname{meas}(\Omega)}\int_\Omega \rho\left(m\right).$$

This implies that $m \notin \operatorname{co} Z$ on a set of positive measure; hence the result.

Case 2. Assume now that $0 \in \partial\operatorname{co} Z$ and 0 is on an edge of $\partial\operatorname{co} Z$. As in Case 1 (6.1) has no solutions. To prove this fact we proceed in three steps.

Step 1. Since 0 is on an edge of $\partial\operatorname{co} Z$, we find that there is a unique $z_0 \in Z$ such that $-z_0 \in Z$ and

$$0 = \frac{1}{2}z_0 + \frac{1}{2}\left(-z_0\right). \tag{6.2}$$

Indeed assume for the sake of contradiction that there exist $z_0, z_1, z_2, z_3 \in Z$, all distinct, and $s, t \in (0,1)$ so that

$$0 = tz_0 + \left(1-t\right)z_1 = sz_2 + \left(1-s\right)z_3.$$

Since the $z_i \in Z \subset S^2$, we deduce that $t = s = 1/2$, $z_1 = -z_0$ and $z_2 = -z_3$. From this we immediately get

$$0 = \frac{1}{4}z_0 + \frac{1}{4}z_1 + \frac{1}{4}z_2 + \frac{1}{4}z_3$$

with

$$\dim\operatorname{span}\left\{z_0, z_1, z_2, z_3\right\} = \dim\operatorname{span}\left\{z_0, z_2\right\} = 2,$$

contradicting the fact that 0 is on an edge of $\partial\operatorname{co} Z$. The equation (6.2) thus holds.

Step 2. We will now prove that if (6.1) has a solution $m \in L^\infty\left(\mathbb{R}^3;\mathbb{R}^3\right)$ $(m \equiv 0$ in $\mathbb{R}^3 \setminus \overline{\Omega})$ then necessarily

$$m\left(x\right) \in \left\{\pm z_0\right\} \text{ a.e. in } \Omega. \tag{6.3}$$

Let $\varepsilon > 0$ be arbitrary and let

$$Z_{0,\varepsilon} := \{z \in Z : \min\{|z - z_0|, |z + z_0|\} < \varepsilon\}.$$

Define next

$$\begin{aligned}\Omega_0 \quad &: \quad = \{x \in \Omega : m(x) \in Z_{0,\varepsilon} \text{ a.e.}\} \\ \Omega_1 \quad &: \quad = \{x \in \Omega : m(x) \in Z \setminus Z_{0,\varepsilon} \text{ a.e.}\}.\end{aligned}$$

Suppose, for the sake of contradiction, that $\text{meas}(\Omega_1) > 0$. It follows from Jensen inequality, the definition of Ω_1, and from the fact that $Z \setminus Z_{0,\varepsilon}$ is closed, that

$$\frac{1}{\text{meas}(\Omega_1)} \int_{\Omega_1} m \in \text{co}\,(Z \setminus Z_{0,\varepsilon}).$$

By Carathéodory Theorem we can find $z_1, z_2, z_3, z_4 \in Z \setminus Z_{0,\varepsilon}$ and $t_1, t_2, t_3, t_4 > 0$ with $\sum_{i=1}^4 t_i = 1$, such that

$$\frac{1}{\text{meas}(\Omega_1)} \int_{\Omega_1} m = \sum_{i=1}^4 t_i z_i.$$

Similarly, there exist $z_1', z_2', z_3', z_4' \in \overline{Z}_{0,\varepsilon}$ and $t_1', t_2', t_3', t_4' > 0$ with $\sum_{i=1}^4 t_i' = 1$, such that

$$\frac{1}{\text{meas}(\Omega_0)} \int_{\Omega_0} m = \sum_{i=1}^4 t_i' z_i'.$$

Combining these two facts we obtain

$$\begin{aligned} 0 \quad &= \quad \frac{1}{\text{meas}(\Omega)} \int_\Omega m = \frac{1}{\text{meas}(\Omega)} \int_{\Omega_1} m + \frac{1}{\text{meas}(\Omega)} \int_{\Omega_0} m \\ &= \quad \sum_{i=1}^4 \frac{\text{meas}(\Omega_1)}{\text{meas}(\Omega)} t_i z_i + \sum_{i=1}^4 \frac{\text{meas}(\Omega_0)}{\text{meas}(\Omega)} t_i' z_i', \end{aligned}$$

which contradicts the fact that 0 is on an edge of $\partial \text{co}\, Z$. Hence $\text{meas}(\Omega_1) = 0$ and thus

$$m(x) \in Z_{0,\varepsilon}, \text{ a.e. in } \Omega.$$

The arbitrariness of ε leads to the desired conclusion (6.3).

Step 3. We are now in a position to conclude. Assume for the sake of contradiction that (6.1) has a solution $m \in L^\infty(\mathbb{R}^3; \mathbb{R}^3)$ $(m \equiv 0 \text{ in } \mathbb{R}^3 \setminus \overline{\Omega})$. Let

$$\Omega_\pm := \{x \in \Omega : m(x) = \pm z_0 \text{ a.e.}\}.$$

Since div $(m\chi_\Omega) = 0$ in the sense of distributions, and in view of (6.3), we obtain for every $\zeta \in C_0^\infty(\mathbb{R}^3)$,

$$0 = \int_{\mathbb{R}^3} \langle m; \nabla\zeta \rangle \chi_\Omega dx = \int_{\mathbb{R}^3} (\chi_{\Omega_+} - \chi_{\Omega_-}) \langle z_0; \nabla\zeta \rangle dx = \int_{\mathbb{R}^3} (\chi_{\Omega_+} - \chi_{\Omega_-}) \frac{\partial\zeta}{\partial z_0} dx$$

which implies that the function $g := \chi_{\Omega_+} - \chi_{\Omega_-}$ depends only on the variables that are orthogonal to z_0, more precisely g is constant on all rays with direction z_0, and this is not possible.

Case 3. If $0 \in \partial \text{co}\, Z$ and if 0 is on a face of $\partial \text{co}\, Z$, then problem (6.1) admits a solution. To assert this fact we address separately two subcases which correspond to the alternatives of Lemma 6.4 below.

Case 3a. There exist $z_1, z_2, z_3 \in Z$, all distinct, and $t_1, t_2, t_3 > 0$ with $\sum_{i=1}^{3} t_i = 1$, such that

$$0 = \sum_{i=1}^{3} t_i z_i .$$

We will show that for a given Ω_0 we can find $M \in W^{1,\infty}\left(\Omega_0; \mathbb{R}^3\right)$ such that

$$\begin{cases} \operatorname{curl} M(x) \in \{z_1, z_2, z_3\} & \text{a.e. in } \Omega_0, \\ M \parallel \nu & \text{on } \partial\Omega_0, \end{cases} \tag{6.4}$$

where ν denotes the exterior normal to $\partial\Omega_0$. Since the domain for which we will make such a construction is a sort of a prism, and hence its boundary is not C^1, the boundary condition is to be interpreted in the almost everywhere sense with respect to the boundary measure.

The general case follows then by using Vitali Theorem (for more details see James-Kinderlehrer [15]).

Since $0 = \sum_{i=1}^{3} t_i z_i$ we have that, for example, $\{z_1, z_2\}$ are linearly independent. We will then let $\alpha := \frac{t_1}{t_3}$ and $\beta := \frac{t_2}{t_3}$ (hence $z_3 = -\alpha z_1 - \beta z_2$). We will also define

$$T^3 := z_1 \wedge z_2,\ T^2 := \frac{z_2 - \langle z_1; z_2\rangle z_1}{1 - \left(\langle z_1; z_2\rangle\right)^2},\ T^1 := \frac{\langle z_1; z_2\rangle z_2 - z_1}{1 - \left(\langle z_1; z_2\rangle\right)^2}$$

and note that (recalling that $Z \subset S^2$)

$$T^2 \wedge T^3 = z_1,\ T^1 \wedge T^3 = z_2.$$

Setting

$$T := \begin{pmatrix} T^1 \\ T^2 \\ T^3 \end{pmatrix} = \begin{pmatrix} T_1^1 & T_2^1 & T_3^1 \\ T_1^2 & T_2^2 & T_3^2 \\ T_1^3 & T_2^3 & T_3^3 \end{pmatrix},$$

we have $\det T = T^1 \cdot (T^2 \wedge T^3) = -1$, and so the matrix T is invertible. Consider the triangle

$$\Delta := \left\{(X_1, X_2) \in \mathbb{R}^2 : 0 < X_1 < 1/\beta, 0 < X_2 < 1/\alpha, \beta X_1 + \alpha X_2 < 1\right\}$$

that we subdivide into a disjoint union of three triangles

$$\begin{aligned} \Delta_1 &:= \left\{(X_1, X_2) \in \mathbb{R}^2 : 0 < X_2 < X_1 < \frac{1}{\beta},\ \beta X_1 + (1+\alpha) X_2 < 1\right\}, \\ \Delta_2 &:= \left\{(X_1, X_2) \in \mathbb{R}^2 : 0 < X_1 < X_2 < \frac{1}{\alpha},\ (1+\beta) X_1 + \alpha X_2 < 1\right\}, \\ \Delta_3 &:= \left\{ \begin{array}{ll} (X_1, X_2) \in \mathbb{R}^2 : & 1 < (1+\beta) X_1 + \alpha X_2, \\ & \beta X_1 + \alpha X_2 < 1, 1 < \beta X_1 + (1+\alpha) X_2 \end{array} \right\}. \end{aligned}$$

We next define $f : \Delta \to \mathbb{R}$ as

$$f(X_1, X_2) := \begin{cases} X_2 & \text{if } (X_1, X_2) \in \Delta_1, \\ X_1 & \text{if } (X_1, X_2) \in \Delta_2, \\ 1 - (\beta X_1 + \alpha X_2) & \text{if } (X_1, X_2) \in \Delta_3, \end{cases}$$

and observe that $f\,|_{\partial\Delta} = 0$.

Define Ω_0 and $M : \mathbb{R}^3 \to \mathbb{R}^3$ by

$$\Omega_0 := \left\{x \in \mathbb{R}^3 : Tx = \left(\langle T^1; x\rangle, \langle T^2; x\rangle, \langle T^3; x\rangle\right) \in \Delta \times (0,1)\right\},$$

$$M(x) := f\left(\langle T^1; x\rangle, \langle T^2; x\rangle\right) T^3.$$

It remains to show that M has all the claimed properties. The fact that $\operatorname{curl} M \in \{z_1, z_2, z_3\}$ follows from the definitions of T and M, and from the elementary observation that

$$\operatorname{curl} M = \frac{\partial f}{\partial X_1}\left(T^1 \wedge T^3\right) + \frac{\partial f}{\partial X_2}\left(T^2 \wedge T^3\right).$$

Note that in Δ_3 the field $\operatorname{curl} M$ reduces to $-\beta z_2 - \alpha z_1$ which is equal to z_3. Since

$$\begin{aligned}\partial\Omega_0 &= T^{-1}\left(\Delta \times \{0\}\right) \cup T^{-1}\left(\Delta \times \{1\}\right) \cup T^{-1}\left(\partial\Delta \times [0,1]\right) \\ &=: A_0 \cup A_1 \cup A_\partial,\end{aligned}$$

we deduce that the boundary condition is satisfied on A_∂ because $f\,|_{\partial\Delta} = 0$, hence $M = 0$ on A_∂, and also on $A_0 \cup A_1$ where $\nu \parallel T^3$ and $M \parallel T^3$.

Case 3b. There exist $z_1, z_2 \in Z$, $z_1 \neq \pm z_2$, with $-z_1, -z_2 \in Z$. We therefore get

$$0 = \frac{1}{4}z_1 + \frac{1}{4}z_2 + \frac{1}{4}(-z_1) + \frac{1}{4}(-z_2).$$

The proof is almost identical to the preceding one except that we need to change somewhat the definitions of Δ and f. Now Δ is the unit square $(0,1)^2$, that we subdivide into four triangles:

$$\begin{aligned}\Delta_1 &:= \left\{(X_1, X_2) \in \mathbb{R}^2 : 0 < X_2 < X_1 < 1 - X_2\right\}, \\ \Delta_2 &:= \left\{(X_1, X_2) \in \mathbb{R}^2 : 0 < 1 - X_1 < X_2 < X_1\right\}, \\ \Delta_3 &:= \left\{(X_1, X_2) \in \mathbb{R}^2 : 0 < 1 - X_2 < X_1 < X_2\right\}, \\ \Delta_4 &:= \left\{(X_1, X_2) \in \mathbb{R}^2 : 0 < X_1 < X_2 < 1 - X_1\right\}.\end{aligned}$$

The function $f : \Delta \to \mathbb{R}$ is then given by

$$f(X_1, X_2) := \begin{cases} X_2 & \text{if } (X_1, X_2) \in \Delta_1, \\ 1 - X_1 & \text{if } (X_1, X_2) \in \Delta_2, \\ 1 - X_2 & \text{if } (X_1, X_2) \in \Delta_3, \\ X_1 & \text{if } (X_1, X_2) \in \Delta_4. \end{cases}$$

Case 4. $0 \in \operatorname{int}\operatorname{co} Z$. This is a particular case of Corollary 5.3 with $m = 1$, $E := Z$, $\varphi \equiv 0$, recalling that, in $\mathbb{R}^3$, $\operatorname{curl} M = \operatorname{curl}^* M$. □

We have used in Step 3 of the proof of Theorem 6.2 the following elementary result of convex analysis.

Lemma 6.4. *Let $Z \subset S^2$ be compact and such that 0 is on a face of $\partial\operatorname{co} Z$. Then one of the following two properties holds:*

(i) There exist $z_1, z_2, z_3 \in Z$, all distinct, and $t_1, t_2, t_3 > 0$ with $\sum_{i=1}^3 t_i = 1$, such that

$$0 = \sum_{i=1}^{3} t_i z_i.$$

(ii) There exist $z_1, z_2 \in Z$, $z_1 \neq \pm z_2$, with $-z_1, -z_2 \in Z$. In particular

$$0 = \frac{1}{4}z_1 + \frac{1}{4}z_2 + \frac{1}{4}(-z_1) + \frac{1}{4}(-z_2).$$

Remark 6.5. The two properties are not exclusive. We know by Carathéodory theorem that 0 is always a convex combination of four elements of Z. The lemma asserts that if 0 is not a convex combination of only three elements then necessarily Z contains the four distinct elements $\pm z_1$, $\pm z_2$.

Proof. We will proceed in two steps.

Step 1. We will start with a preliminary step. Assume that there exist four points $z_1, z_2, z_3, z_4 \in Z$ and $t_1, t_2, t_3, t_4 > 0$ with $\sum_{i=1}^4 t_i = 1$ such that

$$0 = \sum_{i=1}^{4} t_i z_i \text{ and } \dim \operatorname{span}\{z_1, z_2, z_3, z_4\} = 3.$$

We will show that necessarily $0 \in \operatorname{int} \operatorname{co} Z$ (in fact the converse is also true, see [7] Lemma 2.11). Assume, without loss of generality, that $\{z_1, z_2, z_3\}$ are independent; we therefore have

$$z_4 = -(\alpha_1 z_1 + \alpha_2 z_2 + \alpha_3 z_3)$$

where $\alpha_i := t_i/t_4 > 0$. For $\varepsilon > 0$ define next

$$C_\varepsilon := \left\{\xi \in \mathbb{R}^3 : \xi = \varepsilon_1 z_1 + \varepsilon_2 z_2 + \varepsilon_3 z_3 \text{ with } |\varepsilon_i| < \varepsilon, i = 1,2,3\right\}.$$

Clearly C_ε is an open set and $0 \in C_\varepsilon$. We will show that, for $\varepsilon > 0$ sufficiently small, $C_\varepsilon \subset \operatorname{co} Z$, and this will establish the result. Let $\xi \in C_\varepsilon$, i.e.

$$\xi = \varepsilon_1 z_1 + \varepsilon_2 z_2 + \varepsilon_3 z_3.$$

Set

$$s_4 := \frac{1 - (\varepsilon_1 + \varepsilon_2 + \varepsilon_3)}{1 + (\alpha_1 + \alpha_2 + \alpha_3)},\ s_i := \alpha_i s_4 + \varepsilon_i,\ i = 1,2,3.$$

To ensure that all s_i are positive it suffices to choose

$$\varepsilon < \min_{i=1,2,3} \left[\frac{1}{3}, \frac{\alpha_i}{1 + (\alpha_1 + \alpha_2 + \alpha_3) + 3\alpha_i}\right].$$

Since

$$\begin{cases} s_1 + s_2 + s_3 + s_4 = 1, \\ \xi = s_1 z_1 + s_2 z_2 + s_3 z_3 + s_4 z_4, \end{cases}$$

we conclude that $\xi \in \operatorname{co} Z$.

Step 2. In view of Step 1, since $0 \in \partial \operatorname{co} Z$ we deduce that

$$\left.\begin{array}{c} 0 = \sum_{i=1}^4 t_i z_i \\ z_i \in Z,\ t_i > 0,\ \sum_{i=1}^4 t_i = 1 \end{array}\right\} \Rightarrow \dim \operatorname{span}\{z_1, z_2, z_3, z_4\} \leq 2.$$

This, together with the fact that 0 is not on an edge of $\partial \operatorname{co} Z$ yields the existence of $z_i \in Z$, $t_i > 0, i = 1, \dots, 4$ such that

$$0 = \sum_{i=1}^{4} t_i z_i, \sum_{i=1}^{4} t_i = 1, \text{ and } \dim \operatorname{span}\{z_1, z_2, z_3, z_4\} = 2. \tag{6.5}$$

Assume therefore, without loss of generality, that $\{z_1, z_2\}$ in (6.5) are linearly independent and that

$$z_3 = -(\alpha z_1 + \beta z_2),\ z_4 = -(\gamma z_1 + \delta z_2),$$

for some $\alpha, \beta, \gamma, \delta \in \mathbb{R}$. We thus get

$$\begin{cases} t_1 + t_2 + t_3 + t_4 = 1, \\ t_1 = \alpha t_3 + \gamma t_4, \\ t_2 = \beta t_3 + \delta t_4. \end{cases} \tag{6.6}$$

We will now discuss all the possibilities according to the signs of α, β, γ and δ. We will see that unless $\alpha=\delta=0$ and $\beta=\gamma=1$ or $\alpha=\delta=1$ and $\beta=\gamma=0$ (i.e. under the conditions of (ii)), the statement (i) of the lemma always holds.

Case 1. $\alpha,\beta>0$. Set

$$s_1 := \frac{\alpha}{1+\alpha+\beta},\ s_2 := \frac{\beta}{1+\alpha+\beta},\ s_3 := \frac{1}{1+\alpha+\beta},\ (s_4 := 0)\,.$$

We get

$$0 = s_1 z_1 + s_2 z_2 + s_3 z_3$$

and (i) holds.

Case 2. $\alpha \le 0$ and $\beta \le 0$. From (6.6) we deduce that γ, $\delta > 0$ and thus choosing

$$s_1 := \frac{\gamma}{1+\gamma+\delta},\ s_2 := \frac{\delta}{1+\gamma+\delta},\ s_4 := \frac{1}{1+\gamma+\delta},\ (s_3 := 0)\,.$$

we have

$$0 = s_1 z_1 + s_2 z_2 + s_4 z_4,$$

which asserts (i).

Case 3. $\alpha \ge 0$ and $\beta < 0$. From (6.6) we obtain that $\delta > 0$. Two possibilities can then happen: either $\gamma > 0$, and we are in a position to apply Case 2, or $\gamma \le 0$. Using again (6.6) we find that

$$(\alpha\delta - \beta\gamma)\, t_3 = \delta t_1 - \gamma t_2 > 0$$

and thus $\alpha\delta - \beta\gamma > 0$. Letting

$$s_1 := \frac{\alpha\delta-\beta\gamma}{\delta-\beta+\alpha\delta-\beta\gamma},\ s_3 := \frac{\delta}{\delta-\beta+\alpha\delta-\beta\gamma},\ s_4 := \frac{-\beta}{\delta-\beta+\alpha\delta-\beta\gamma},$$

and $s_2 = 0$, the lemma follows from the observation

$$0 = s_1 z_1 + s_3 z_3 + s_4 z_4.$$

Case 4. $\alpha < 0$ and $\beta \ge 0$. This may be treated as Case 3 with the roles of z_1 and z_2 interchanged.

Case 5. $\alpha = 0$ and $\beta > 0$. Since $z_3 = -\beta z_2$, $\beta > 0$ and $z_2, z_3 \in Z \subset S^2$ we deduce that $\beta = 1$. From (6.6) we infer that $\gamma > 0$. Three possibilities can then happen: either $\delta > 0$, which was handled in Case 2, or $\delta < 0$, or $\delta = 0$. If $\delta < 0$ then with the choice

$$s_1 := \frac{\gamma}{1+\gamma-\delta},\ s_3 := \frac{-\delta}{1+\gamma-\delta},\ s_4 := \frac{1}{1+\gamma-\delta},\ (s_2 := 0)$$

and in light of the observation

$$0 = s_1 z_1 + s_3 z_3 + s_4 z_4$$

we are led to the assertion (i) of the lemma. Therefore it remains only to discuss the case $\delta = 0$. Since $\gamma > 0$, $z_1, z_4 \in Z \subset S^2$ and $z_4 = -\gamma z_1$ we get $\gamma = 1$. We find that $\pm z_1, \pm z_2 \in Z$ and hence

$$0 = \frac{1}{4} z_1 + \frac{1}{4} z_2 + \frac{1}{4}(-z_1) + \frac{1}{4}(-z_2)\,,$$

which is the conclusion (ii) of the lemma.

Case 6. $\alpha > 0$ and $\beta = 0$. This case is almost identical to the above one and we leave out the details.

The proof of the lemma is therefore complete. $\square$

We finish this section with some comments on the structure of the set Z in the case of ferromagnetics. We recall that

$$Z := \left\{ \xi \in S^2 \ : \ \psi(\xi) = \min_{|\eta|=1} \{\psi(\eta)\} \right\},$$

where $\psi(\eta) := \varphi(\eta) - \langle h_e; \eta \rangle$. Here φ is a non-negative, continuous, even function (i.e. $\varphi(m) = \varphi(-m)$ for all $m \in S^2$), and h_e is a given external magnetic field.

If $h_e = 0$ then Z reduces to the set of minimizers of φ on S^2, and since φ is even it follows immediately that $0 \in \operatorname{co} Z$. Next we discuss the case where $h_e \neq 0$.

Proposition 6.6. *If $h_e \neq 0$ then $0 \notin \operatorname{int} \operatorname{co} Z$.*

Assume, in addition, that φ is an even function separately with respect to each component, i.e.

$$\varphi(\xi_1, \xi_2, \xi_3) = \varphi(-\xi_1, \xi_2, \xi_3) = \varphi(\xi_1, -\xi_2, \xi_3) = \varphi(\xi_1, \xi_2, -\xi_3) \tag{6.7}$$

for all $\xi \in S^2$. If either $\varphi \in C^1$ or $h_e = (h_e^1, h_e^2, h_e^3)$ is such that $h_e^i \neq 0$ for all $i \in \{1, 2, 3\}$, then $0 \notin \operatorname{co} Z$.

Remark 6.7. (i) The conclusions of the above proposition are sharp. Indeed, we may have $h_e \neq 0$ and still $0 \in \operatorname{co} Z$ for some even, continuous function φ. Consider, as an example,

$$\varphi(\xi_1, \xi_2, \xi_3) := |\xi_1 + \xi_2 + \xi_3|, \quad h_e := (1, 1, 1).$$

It can be seen easily that

$$Z = \left\{ \xi \in S^2 : \xi_1 + \xi_2 + \xi_3 \geq 0 \right\},$$

thus, in spite of the fact that $0 \notin \operatorname{int} \operatorname{co} Z$, we still have

$$0 \in \operatorname{co} Z = \left\{ \xi \in \overline{B(0,1)} : \xi_1 + \xi_2 + \xi_3 \geq 0 \right\}.$$

Also, the function

$$\varphi(\xi_1, \xi_2, \xi_3) := 1 - \xi_3^2 + |\xi_1|$$

satisfies (6.7), and taking $h_e := (1, 0, 0)$ then $(0, 0, 1), (0, 0, -1) \in Z$, and we conclude that $0 \in \operatorname{co} Z$. Note, however, that $\varphi \notin C^1$.

(ii) The anisotropic energy densities φ considered by Landau and Lifschitz in [16] satisfy property (6.7) with $\varphi \in C^1$. Note that although $0 \notin \operatorname{co} Z$, we may still have existence of minima for the energy E, as illustrated by the example treated by James-Kinderlehrer [15] and addressed in the Introduction where $h_e = \theta D m_1$ and $\varphi(m_1) = 0$. Indeed, it is easy to prove that here $Z = \{m_1\}$, and James-Kinderlehrer [15] provide a solution with induced magnetic field $h_m \neq 0$.

Proof. If $\xi \in Z$ then

$$\varphi(\xi) - \langle h_e; \xi \rangle = \psi(\xi) \leq \psi(-\xi) = \varphi(-\xi) + \langle h_e; \xi \rangle = \varphi(\xi) + \langle h_e; \xi \rangle,$$

and so

$$Z \subset \{\xi \in S^2 : \langle h_e; \xi \rangle \geq 0\}, \quad \operatorname{co} Z \subset \{\xi \in \overline{B(0,1)} : \langle h_e; \xi \rangle \geq 0\}. \tag{6.8}$$

If $h_e \neq 0$ it follows immediately that $0 \notin \operatorname{int} \operatorname{co} Z$.

Assume now that φ satisfies (6.7) and that $h_e^i \neq 0$ for all $i \in \{1, 2, 3\}$. If $0 \in \operatorname{co} Z$ then

$$0 = \sum_{i=1}^{4} t_i \xi^i, \ \xi^i \in Z, t_i \geq 0, \sum_{i=1}^{4} t_i = 1.$$

By (6.7) we have that for $i \in \{1,2,3,4\}$

$$\begin{aligned}\varphi(\xi_1^i,\xi_2^i,\xi_3^i) - \left\langle h_e;(\xi_1^i,\xi_2^i,\xi_3^i)\right\rangle &\le \varphi(-\xi_1^i,\xi_2^i,\xi_3^i) - \left\langle h_e;(-\xi_1^i,\xi_2^i,\xi_3^i)\right\rangle \\ &= \varphi(\xi_1^i,\xi_2^i,\xi_3^i) - \left\langle h_e;(-\xi_1^i,\xi_2^i,\xi_3^i)\right\rangle,\end{aligned}$$

and thus $h_e^1\,\xi_1^i \ge 0$; more generally

$$h_e^j\,\xi_j^i \ge 0 \quad \text{for all}\, j \in \{1,2,3\}, i \in \{1,2,3,4\}. \tag{6.9}$$

We have

$$0 = \langle h_e; 0\rangle = \sum_{i=1}^{4} t_i \left\langle h_e;\xi^i\right\rangle$$

which, together with (6.8), yields $\left\langle h_e;\xi^i\right\rangle = 0$ for all $i \in \{1,2,3,4\}$. In view of (6.9) we now have that $h_e^j\,\xi_j^i = 0$ for all $j \in \{1,2,3\}, i \in \{1,2,3,4\}$, hence $\xi^i = 0$ for all $i \in \{1,2,3,4\}$, contradicting the fact that $\xi \in S^2$. We conclude that $0 \notin \operatorname{co} Z$.

Finally, consider the case where $h_e \neq 0$, and $\varphi \in C^1$ satisfies (6.7). Without loss of generality we assume that $h_e^1 > 0$. We claim that $Z \subset \{x > 0\}$, from what it will follow that $\operatorname{co} Z \subset \{x > 0\}$, and hence $0 \notin \operatorname{co} Z$. Let $\xi = (x,y,z) \in Z$. Since

$$\varphi(x,y,z) - \langle h_e;(x,y,z)\rangle \le \varphi(-x,y,z) - \langle h_e;(-x,y,z)\rangle,$$

by (6.7) we deduce that $2h_e^1 x \ge 0$, i.e. $x \ge 0$. If $x = 0$ then there exists a Lagrange multiplier $\lambda/2 \in \mathbb{R}$ such that

$$\nabla\psi(0,y,z) + \frac{\lambda}{2}\nabla|\mathrm{id}|^2(0,y,z) = 0,$$

and so

$$\frac{\partial\varphi}{\partial x}(0,y,z) - h_e^1 = 0.$$

But (6.7) implies that $\frac{\partial\varphi}{\partial x}(0,y,z) = 0$, and we conclude that $h_e^1 = 0$, what is clearly in contradiction with our assumptions. $\square$

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