A HARRIS THEOREM FOR ENHANCED DISSIPATION, AND AN EXAMPLE OF PIERREHUMBERT

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Abstract. In many situations, the combined effect of advection and diffusion greatly increases the rate of convergence to equilibrium – a phenomenon known as *enhanced dissipation*. Here we study the situation where the advecting velocity field generates a random dynamical system satisfying certain *Harris conditions*. If κ denotes the strength of the diffusion, then we show that with probability at least $1 - o(\kappa^N)$ enhanced dissipation occurs on time scales of order $|\ln \kappa|$, a bound which is known to be optimal. Moreover, on long time scales, we show that the rate of convergence to equilibrium is almost surely *independent* of diffusivity. As a consequence we obtain enhanced dissipation for the randomly shifted alternating shears introduced by Pierrehumbert '94.

1. Introduction

1.1. **Main Results.** We begin by stating our results. Following this, we will survey the literature and place our work in the context of existing results. Let u be a (possibly time dependent) divergence free vector on the torus, $\kappa > 0$, and ρ solve the advection diffusion equation

$$\partial_t \rho + u \cdot \nabla \rho - \kappa \Delta \rho = 0\,, \tag{1.1}$$

on the d-dimensional torus $\mathbb{T}^d$. Multiplying (1.1) by ρ, using the fact that $\nabla \cdot u = 0$, integrating and using the Poincaré inequality implies

$$\|\rho(\cdot, t) - \bar\rho\|_{L^2} \leqslant e^{-\lambda_1 \kappa t} \|\rho_0 - \bar\rho\|_{L^2}\,, \tag{1.2}$$

where

$$\bar\rho = \int_{\mathbb{T}^d} \rho_0(x)\, dx\,,$$

is the (constant) equilibrium solution to (1.1) and $\lambda_1 > 0$ is the smallest non-zero eigenvalue of the negative Laplacian. *Enhanced dissipation* is the phenomenon where solutions to (1.1) converge to equilibrium faster than the upper bound (1.2). Our main result shows that if the flow of u generates a random dynamical system (RDS) that satisfies the Harris conditions (stated below), then enhanced dissipation occurs. The enhanced dissipation rate is optimal at short times with large probability, and is almost surely *independent* of the diffusivity for long times.

Theorem 1.1. *Suppose the flow of u generates a random dynamical system that satisfies the Harris conditions stated in Assumptions 2.1–2.4, below. For any $\alpha > 0$,*

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$q < \infty$ there exists $\gamma > 0$ and a κ-dependent random variable D_κ such that for every initial data $\rho_0 \in L^1(\mathbb{T}^d)$, the solution to (1.1) satisfies

$$\|\rho(\cdot,t) - \bar\rho\|_{L^\infty} \leqslant \frac{D_\kappa}{\kappa^{\frac{d}{2}+\alpha}} e^{-\gamma t} \|\rho_0 - \bar\rho\|_{L^1} . \tag{1.3}$$

Moreover, there exist a κ-independent, deterministic, constant $\bar D_q$ such that

$$\boldsymbol{E} D_\kappa^q \leqslant \bar D_q . \tag{1.4}$$

Remark 1.2. The bound (1.4) implies that for any $\beta > 0$, $\boldsymbol{P}(D_\kappa \geqslant \kappa^{-\beta}) \leqslant \kappa^{\beta q} \bar D_q$. Using this in (1.3) will show that with probability at least $1 - \bar D_q \kappa^{\beta q}$, we have

$$\|\rho(\cdot,t) - \bar\rho\|_{L^\infty} \leqslant \frac{1}{\kappa^{\frac{d}{2}+\alpha+\beta}} e^{-\gamma t} \|\rho_0 - \bar\rho\|_{L^1} , \tag{1.5}$$

for all $t \geqslant 0$, and all $\rho_0 \in L^1$.

An equivalent probabilistic formulation of this result is as follows. Consider the Markov process defined by the SDE

$$dX_t^\kappa(x) = -u(X_t^\kappa(x), t)\, dt + \sqrt{2\kappa}\, dW_t , \quad X_0^\kappa(x) = x , \tag{1.6}$$

on the torus $\mathbb{T}^d$. Since $\nabla \cdot u = 0$, the (unique) stationary distribution of X^κ is the Lebesgue measure. Let $p_t^\kappa(x,y)$ denote the transition density of X_t^κ, and recall the *uniform mixing time* [LP17, MT06] is defined by

$$t_{\text{mix}}^\infty(X^\kappa, \varepsilon) = \inf\Big\{ t \geqslant 0 \;\Big|\; \sup_{x \in \mathbb{T}^d} \|p_t^\kappa(x,\cdot) - 1\|_{L^\infty} < \varepsilon \Big\} .$$

It is easy to see that the *uniform mixing time* of X^κ satisfies

$$t_{\text{mix}}^\infty(X^\kappa, \varepsilon) \leqslant \frac{C |\ln \varepsilon|}{\kappa} ,$$

for some constant $C > 0$. Theorem 1.1 is equivalent to the following uniform mixing time estimate.

Theorem 1.3. *Suppose the flow of u generates a random dynamical system satisfying the Harris conditions (Assumptions 2.1–2.4, below). For any $\alpha > 0$, $q < \infty$ there exists $\gamma > 0$ and a κ-dependent random variable D_κ such that*

$$t_{\text{mix}}^\infty(X^\kappa, \varepsilon) \leqslant \frac{1}{\gamma} \ln\Big(\frac{D_\kappa}{\varepsilon \kappa^{\frac{d}{2}+\alpha}} \Big) \tag{1.7}$$

almost surely. Moreover, there exists a κ-independent (deterministic) constant $\bar D_q$ such that (1.4) holds.

Remark 1.4. We clarify that $t_{\text{mix}}^\infty(X^\kappa, \varepsilon)$ is random as it depends on u (it is, of course, independent of W). For any fixed $\varepsilon > 0$ (independent of κ), Chebychev's inequality (1.4) and (1.7) show that for any $\beta > 0$ we have

$$\boldsymbol{P}\Big(t_{\text{mix}}^\infty(X^\kappa, \varepsilon) \leqslant \frac{1}{\gamma} \ln\Big(\frac{1}{\varepsilon \kappa^{\frac{d}{2}+\alpha+\beta}} \Big) \Big) \geqslant 1 - \bar D_q \kappa^{\beta q} .$$

Moreover, as $\varepsilon \to 0$, we obtain the κ-independent uniform mixing time bound

$$\lim_{\varepsilon \to 0} \frac{t_{\text{mix}}(X^\kappa, \varepsilon)}{|\ln \varepsilon|} \leqslant \frac{1}{\gamma} , \quad \text{almost surely} .$$

Remark 1.5. While Assumptions 2.1–2.4 are easy to state (see Section 2.3, below), they aren't easy to verify in practice. Recent papers [BBPS22, BCZG22] instead assume certain conditions which are stronger than Harris conditions, but are easier to verify. For convenience of the reader, we state these conditions in Section 2.4, below.

Remark 1.6 (Pulsed diffusions). We can also obtain similar results for pulsed diffusions. Namely, define the Markov process Y^κ by

$$Y^\kappa_{n+1} = \varphi_{n+1}(Y^\kappa_n) + \zeta^\kappa_{n+1}\,,$$

where ζ^κ_n are i.i.d. periodized Gaussians with variance κ, and φ_n is a RDS on $\mathbb{T}^d$, independent of ζ^κ. If the RDS φ_n satisfies the Harris conditions (Assumption 2.1–2.3), then we obtain the same mixing time bound (1.7) for the process Y^κ. The proof is similar to the proof of Theorem 1.3, and in this setting several technical steps become much simpler.

As an immediate consequence, we can show enhanced dissipation if u is obtained by randomly shifting and alternating sinusoidal shears.

Corollary 1.7. *Let $A > 0$, $d = 2$ and ζ_n a sequence of i.i.d. random variables that are uniformly distributed on $[0,1]$. For $n \in \mathbb{N}$ and $t \in [2n, 2n+2)$ define*

$$u(x,t) = \begin{cases} A\sin(2\pi(x_2 - \zeta_{2n}))\,e_1 & t \in [2n, 2n+1)\,, \\ A\sin(2\pi(x_1 - \zeta_{2n+1}))\,e_2 & t \in [2n+1, 2n+2)\,, \end{cases} \tag{1.8}$$

where $x = (x_1, x_2) \in \mathbb{T}^2$, and e_i is the i-th standard basis vector. Then, almost surely, we have the enhanced dissipation bound (1.3) for a random variable D_κ that satisfies the κ-independent bounds (1.4). Consequently, with probability at least $1 - \bar{D}_q\kappa^{\beta q}$, the enhanced dissipation bound (1.5) holds.

This example was introduced by Pierrehumbert [Pie94] and [BCZG22] recently verified that it satisfies conditions that are stronger than our Harris assumptions. As a result the Corollary 1.7 follows immediately from Theorem 1.1 and [BCZG22] (see Proposition 2.9, below). We mention, however, that even though [BCZG22] abstractly check Assumption 2.2 (the existence of a Lyapunov function) for (1.8), the system is simple enough that a Lyapunov function can be constructed explicitly. We do this in Proposition 9.1 in Section 9, below.

Remark 1.8. Instead of using a sine shear profile in (1.8), we can use shears with a piecewise linear profile. In this case the results in [CFIN23] will show that Assumption 2.1–2.4 are satisfied, and so Theorems 1.1, 1.3 apply and will give the enhanced dissipation bound (1.3), and the equivalent uniform mixing time bound (1.7).

1.2. **Motivation and Literature Review.** We now survey the literature and place our results in the context of existing results. *Enhanced dissipation* is a phenomenon that can be observed in every day life: Pour some cream in your coffee. If left alone, it will take hours to mix. Stir it a little and it mixes right away. This effect arises due to the interaction between the advection (stirring) and diffusion, and plays an important role in many applications concerning hydrodynamic stability and turbulence and occurs on scales ranging from micro fluids to meteorological / cosmological [LTD11, Thi12, Are84, SSA04].

To describe this mathematically, let u be the velocity field of the ambient incompressible fluid, and ρ denote the concentration of a passively advected solute with molecular diffusivity $\kappa > 0$. The evolution of ρ is governed by the advection diffusion equation (1.1). For simplicity, in this paper we only consider (1.1) with periodic boundary conditions on the d-dimensional torus $\mathbb{T}^d$.

If the ambient fluid is incompressible, the velocity field u satisfies divergence free condition

$$\nabla \cdot u = 0\,.$$

In this case, an elementary energy estimate shows (1.2) and hence the L^2 distance of the concentration from the equilibrium distribution decreases at most exponentially with a rate proportional to κ.

Of course, (1.2) is only a crude upper bound. In many practical situations one expects the convergence to happen much faster than (1.2). Indeed, the advection term $u \cdot \nabla\rho$ typically causes filamentation and moves energy towards small scales. The diffusion term $\kappa\Delta\rho$ damps small scales faster, and the combination of these two effects leads to enhanced dissipation – faster convergence of $\rho(\cdot, t)$ to $\bar{\rho}$.

Several authors have proved enhanced dissipation by showing all solutions to (1.1) satisfy the decay estimate

$$\|\rho(\cdot,t) - \bar{\rho}\|_{L^2} \leqslant \exp\Big(-\Big(\frac{t}{T(\kappa)} - 1\Big)^+\Big) \|\rho_0 - \bar{\rho}\|_{L^2}\,, \tag{1.9a}$$

for every $t \geqslant 0$, and some time scale $T(\kappa)$ for which

$$\lim_{\kappa \to 0} \kappa T(\kappa) = 0\,. \tag{1.9b}$$

Seminal work of Constantin et al. [CKRZ08] (see also [Zla10, KSZ08]) shows that if u is time independent, then such a $T(\kappa)$ exists if and only if $u \cdot \nabla$ has no eigenfunctions in H^1. For shear flows classical work of Kelvin [Kel87] shows one can choose $T(\kappa) = \kappa^{-\alpha}$ for some $\alpha < 1$. There are now several results studying enhanced dissipation in more generality and for nonlinear equations (see [FNW04, Wei19, BCZ17, CZD21, FMN23, CZG23, ABN22, CH23, Sei23]).

The purpose of this paper is to further investigate the link between enhanced dissipation and mixing properties of u. Recall, a velocity field u is said to be *exponentially mixing* if, in the absence of diffusion, a dye that is initially localized to ball of size ε will get spread throughout the torus in time $O(|\ln \varepsilon|)$ (see for instance [SOW06]). In the presence of diffusion, a dye localized to a point gets spread to a ball of size $O(\sqrt{\kappa})$ in time $O(1)$. If u is exponentially mixing, then this dye is spread throughout the torus by the flow in time $O(|\ln \kappa|)$. As a result, in this case we expect (1.9a) should hold with $T(\kappa) = O(|\ln \kappa|)$.

Surprisingly, this is not easy to prove, and is an open question in this generality. Currently available results [FI19, Fen19, CZDE20] show that if u is exponentially mixing, then one can choose $T(\kappa) = O(|\ln \kappa|^2)$ in (1.9a). For a few specific exponentially mixing systems, available results [BBPS21, ELM23, ILN23] show that one can choose $T(\kappa) = O(|\ln \kappa|)$ in (1.9a). However, to the best of our knowledge, there is no general theorem (in discrete or continuous time) that shows that for any exponentially mixing flow one can choose $T(\kappa) = O(|\ln \kappa|)$ in (1.9a).

One elementary observation is that if almost every realization of the stochastic flows X^κ is exponentially mixing, then one has enhanced dissipation as in (1.9a) with $T(\kappa) = C|\ln \kappa|$ for some C that can be explicitly computed in terms of the mixing rate. Thus, a natural question to ask is is whether or not the notion of exponentially mixing is stable with respect to κ.

Question 1.9. *If u is exponentially mixing, then for sufficiently small $\kappa > 0$ must almost every realization of X^κ be exponentially mixing (with a controlled rate)?*

Since the notion of exponentially mixing involves the long time behavior, it is not easy to determine the answer to Question 1.9. We instead look for stronger conditions on u which will will guarantee that for all sufficiently small $\kappa > 0$, almost every realization X^κ is exponentially mixing (with a controlled rate).

A general principle that is well known to the Sinai school is that for *random dynamical systems (RDS)*, geometric ergodicity of the two point process implies almost sure exponential mixing (see [DKK04, BBPS22], or the proof of Lemma 3.3, below). One could then ask whether or not this property is stable in κ.

Question 1.10. *If the two point process associated to the flow of a random dynamical system u is geometrically ergodic, then is the two point processes associated to the SDE* (1.6) *also geometrically ergodic for all small $\kappa > 0$?*

If Question 1.10 is answered affermatively, then for all sufficiently small $\kappa > 0$, must almost every realization X^κ be exponentially mixing. Not surprisingly, this question is also hard to answer. Geometric ergodicity involves questions about long time limits which are not stable as κ varies. There is, however, a classical result of Harris [Har55, MT09] that proves geometric ergodicity of a Markov process provided there is a Lyapunov function, and a small set. A version of this condition turns out to be stable in κ (see Lemma 3.1, below) which in turn leads to almost sure exponential mixing of the flows X^κ (see Lemma 3.3, below), which in turn yields Theorem 1.1.

Finally, we mention that if u is regular, and uniformly bounded in time, then we must have $T(\kappa) \geqslant O(\kappa)$ in (1.9a) (see for instance [MD18, Poo96, BBPS21, Sei22]). When u is irregular, one can even choose $T(\kappa) = O(1)$. This is known as *anomalous dissipation*, and examples of this were recently proved in [DEIJ22, CCS22, AV23]. In this paper we only consider regular velocity fields, and so are in a situation where anomalous dissipation can not ocur.

Plan of this paper. In Section 2.3 we define our notation and state Assumption 2.1–2.4 used in Theorems 1.1 and 1.3. In Section 3 we state the three lemmas (Lemmas 3.1–3.3) that will quickly yield Theorems 1.1 and 1.3, and use these lemmas to prove Theorems 1.1 and 1.3. The first of these lemmas (Lemma 3.1) is the main new contribution of this paper and which guarantees that our assumptions imply the Harris conditions hold for all sufficiently small $\kappa > 0$. The proof of Lemma 3.1 is split up into two steps – the existence of a Lyapunov function in Section 4, and the existence of a κ-independent small set in Section 5. Following this, we prove Lemma 3.1 in Section 6. The proof of Lemma 3.2 uses a quantitative version of Harris's theorem [HM11] and is presented in Section 7. The proof of Lemma 3.3 is based on the idea in [DKK04] and is presented in Section 8. Finally, we conclude this paper by explicitly finding a Lyapunov function for the randomly shifted alternating shear example in 1.7.

2. Notation and Preliminaries.

In this section we set up our notational convention and state the assumptions required for Theorem 1.1 and 1.3.

2.1. Notation and setup.

We will now define a setup that chooses the velocity field u randomly from a finite dimensional family of C^2, incompressible vector fields, and repeats this choice on after time intervals of length 1. Let $\mathscr{M}$ be a complete smooth Riemannian manifold, and $\mathscr{U} : \mathscr{M} \times \mathbb{T}^d \times [0,1] \to \mathbb{R}^d$ be a C^2-function such that

$$\nabla_x \cdot \mathscr{U}(\xi, x, t) = 0\,.$$

Now let $(\Omega_0, \mathcal{F}_0, \boldsymbol{P}_0)$ be a probability space and let $\omega = (\omega_1, \omega_2, \dots)$ be a sequence of $\mathscr{M}$-valued random variables whose distribution is absolutely continuous with respect to the volume measure on $\mathscr{M}$. Define the (random) velocity field u by

$$u(x,t) \stackrel{\text{def}}{=} \mathscr{U}(\omega_n, x, t-n) \quad \text{when } t \in [n, n+1)\,,$$

where, following standard convention, we suppress the dependence of u on ω. The flow of u is defined by the ODE

$$\partial_t X^0_t = u(X^0_t, t)\,, \qquad X^0_0 = \mathrm{Id}\,.$$

Restricting X^0 to integer times gives a Markov process, which we refer to as the RDS generated by u.

To define the processes X^κ, let $(\Omega_W, \mathcal{F}_W, \boldsymbol{P}_W)$ be a probability space, and W be a $\mathbb{T}^d$-valued Brownian motion on this space. We will now consider the product space $\Omega = \Omega_W \times \Omega_0$ with the product σ-algebra $\mathcal{F} = \mathcal{F}_W \otimes \mathcal{F}_0$ and product measure $\boldsymbol{P} \stackrel{\text{def}}{=} \boldsymbol{P}_W \otimes \boldsymbol{P}_0$. By a slight abuse of notation, we will sometimes treat random variables on each of the coordinate spaces Ω_0, Ω_W as random variables on the product space Ω by composing with the corresponding coordinate projection. In this sense, we may treat u, W as independent processes on Ω, and let X^κ_t be the solution of (1.6) on $\mathbb{T}^d$.

2.2. Two point and projective chains.

In order to state the Harris conditions, we need to define the *two point* and *projective chains*, and formalize the dependence on the noise history. Given $\xi = (\xi_1, \dots, \xi_n) \in \mathscr{M}^n$, define $\mathscr{U}_n : \mathscr{M}^n \times \mathbb{T}^d \times [0,n) \to \mathbb{R}^d$ by

$$\mathscr{U}_n(\xi, x, t) = \mathscr{U}(\xi_m, x, t-m) \quad \text{if } t \in [m, m+1)\,.$$

Now define $\mathscr{X}_n : \mathscr{M}^n \times \mathbb{T}^d \to \mathbb{T}^d$ to be the flow of $\mathscr{U}_n$ after time n. That is, define

$$\mathscr{X}_n(\xi, x) = \Phi_n(x)\,,$$

where Φ is defined by

$$\partial_t \Phi_t = \mathscr{U}_n(\xi, \Phi_t, t)\,, \quad \Phi_0 = \mathrm{Id}\,.$$

Given $\xi \in \mathscr{M}^n$, $(x,v) \in T\mathbb{T}^d$ (the tangent bundle of the torus), and $y \in \mathbb{T}^d$ with $\|v\| = 1$, $y \neq x$, we define the derivative, projective and two point maps by

$$\mathscr{A}_n(\xi, x) \stackrel{\text{def}}{=} D_x \mathscr{X}_n(\xi, x)\,, \quad \hat{\mathscr{X}}_n(\xi, x, v) = \Big(\mathscr{X}_n(\xi, x), \frac{\mathscr{A}_n(\xi, x) v}{\|\mathscr{A}_n(\xi, x) v\|} \Big)$$

and

$$\mathscr{X}^{(2)}_n(\xi, x, y) = (\mathscr{X}_n(\xi, x), \mathscr{X}_n(\xi, y))\,,$$

respectively. Note that $\hat{\mathscr{X}}_n$ takes values on $\mathscr{M}^n \times S\mathbb{T}^d$ where $S\mathbb{T}^d$ is the unit sphere bundle (see for instance [dC92]) defined by

$$S\mathbb{T}^d \stackrel{\text{def}}{=} \{(x,v) \in T\mathbb{T}^d \mid \|v\| = 1\},$$

The map $\mathscr{X}_n^{(2)}$ takes values on $\mathscr{M}^n \times \mathbb{T}^{d,(2)}$ where

$$\mathbb{T}^{d,(2)} \stackrel{\text{def}}{=} \mathbb{T}^d \times \mathbb{T}^d - \Delta,$$

and $\Delta \stackrel{\text{def}}{=} \{(x,y) \in \mathbb{T}^d \times \mathbb{T}^d \mid x = y\}$.

Notice that the RDS generated by u is precisely the Markov process

$$X_n^0(x) = \mathscr{X}_n(\underline{\omega_n}, x), \quad \text{where} \quad \underline{\omega_n} \stackrel{\text{def}}{=} (\omega_1, \dots, \omega_n).$$

We also define the derivative, projective and two point processes by

$$A_n^0(x) = \mathscr{A}_n(\underline{\omega_n}, x), \quad \hat{X}_n(x,v) = \hat{\mathscr{X}}_n(\underline{\omega_n}, x, v), \quad X_n^{0,(2)}(x,y) = \mathscr{X}_n^{(2)}(\underline{\omega_n}, x, y),$$

where as before $(x,v) \in S\mathbb{T}^d$ and $(x,y) \in \mathbb{T}^{d,(2)}$. We denote the n-step transition kernel of these processes by

$$P_0^n(x,A) = \boldsymbol{P}_0[X_n^0(x) \in A],$$

$$\hat{P}_0^n((x,v),A) \stackrel{\text{def}}{=} \boldsymbol{P}_0\Big[\hat{X}_n^0(x,v) \in A\Big],$$

$$P_0^{(2),n}((x,y),A) \stackrel{\text{def}}{=} \boldsymbol{P}_0\big[X_n^{0,(2)} \in A\big]. \tag{2.1}$$

For brevity, when $n = 1$ we will drop the superscript.

For $\kappa > 0$, we define the Markov process X^κ obtained by restricting the solution to (1.6) to integer times. Given $n \in \mathbb{N}$, let $P_\kappa^n(x,\cdot)$ denote the transition probability of X_n^κ. That is, for $n \in \mathbb{N}$ we define

$$P_\kappa^n(x,A) \stackrel{\text{def}}{=} \boldsymbol{P}(X_n^\kappa(x) \in A).$$

As before, when $n = 1$, we drop the superscript and simply write P_κ for P_κ^1. The two point process $X^{\kappa,(2)}$ is defined analogously by

$$X_n^{\kappa,(2)}(x,y) = (X_n^\kappa(x), X_n^\kappa(y)),$$

and its n-step transition kernel will be denoted by $P_\kappa^{(2),n}$.

2.3. **Harris Conditions.**

Harris theorems typically assume the existence of a Lyapunov function and a small set, and show that the associated system is uniformly geometrically ergodic [MT09, Har55, HM11]. We will now state these conditions for the two point chain $P_0^{(2)}$ defined in (2.1). We reiterate that our assumptions only concern the RDS generated by the flow of u with $\kappa = 0$.

Assumption 2.1. *The two point chain with kernel $P_0^{(2)}$ is Feller, topologically irreducible, strongly aperiodic.*

Our next assumption concerns the existence of a Lyapunov function for the two point chain near a small neighborhood of the diagonal. To state this, let $s > 0$ and define a punctured neighborhood of the diagonal $\Delta(s)$ by

$$\Delta(s) \stackrel{\text{def}}{=} \{(x,y) \in \mathbb{T}^d \times \mathbb{T}^d \mid 0 < d(x,y) < s\},$$

where $d(x,y)$ is the torus distance between x and y. On the flat torus it is easy to see that for $x, y \in \Delta(1/2)$, we can find a unique $v \in T_x\mathbb{T}^d$ such that $\exp_x(v) = y$, where $\exp_x$ denotes the exponential map (see for instance [dC92]).

Assumption 2.2. *There exists a Lyapunov function* $V\colon \mathbb{T}^{d,(2)} \to [1,\infty)$ *and constants* $\tilde\gamma \in (0,1)$, $s_* \in (0,1/2)$ *such that*

$$P_0^{(2)} V < \tilde\gamma V \quad \text{on } \Delta(s_*)\,. \tag{2.2}$$

Moreover, there exists $p \in (0,1)$ *such that the function* V *is of the form*

$$V(x,y) = d(x,y)^{-p} \psi\Big(x, \frac{\exp_x^{-1}(y)}{d(x,y)}\Big) \quad \text{on } \Delta(s_*)\,. \tag{2.3}$$

where $\psi : S\mathbb{T}^d \to \mathbb{R}^+$ *is a continuous, strictly positive function.*

The standard Harris Theorem [MT09] requires the assumptions 2.1, 2.2, and the existence of a small set for $P_0^{(2)}$, and shows ergodicity of the chain. However, for Lemma 3.3, we will need to show that for sufficiently small $\kappa > 0$ there exists a small set with κ-independent bound on the minorizing measure. We are presently unable to do this assuming only the existence of a small set for $P_0^{(2)}$. We can, however, prove this under a slightly stronger condition which is not hard to verify in practice.

Assumption 2.3. *There exist* $n \geqslant 1$ *and* $(\xi_*, x_*) \in \mathscr{M}^n \times \mathbb{T}^{d,(2)}$ *such that the following hold.*

1. *There exists* $c, \varepsilon > 0$ *such that for every* $\xi \in \mathscr{M}^n$ *with* $|\xi - \xi_*| < \varepsilon$, *we have* $\rho_n(\xi) \geqslant c > 0$. *Here* ρ_n *is the density of* $\underline{\omega_n}$ *on* $\mathscr{M}^n$.
2. *The map* $\mathscr{X}_n^{(2)}(\cdot, x_*)\colon \mathscr{M}^n \to \mathbb{T}^{d,(2)}$ *is a submersion at* $\xi = \xi_*$.

Finally, we need a uniform bound on the velocity field so that the gradients of the diffeomorphisms $\mathscr{X}_n$ are controlled uniformly in the noise.

Assumption 2.4. *The function* $\mathscr{U} : \mathscr{M} \times \mathbb{T}^d \times [0,1] \to \mathbb{R}^d$ *is such that*

$$\sup_{(\xi,t)\in\mathscr{M}\times[0,1]} \|\mathscr{U}(\xi,\cdot,t)\|_{C^2(\mathbb{T}^d)} < \infty \qquad \text{and} \qquad \sup_{(x,t)\in\mathbb{T}^d\times[0,1]} \|\nabla_\xi \mathscr{U}(\cdot,x,t)\|_{L^\infty(\mathscr{M})} < \infty\,.$$

We will prove (Section 4, below) that Assumptions 2.2 and 2.4 imply that V is a Lyapunov function for $P_\kappa^{(2)}$, and satisfies the drift condition (2.2) with slightly larger constants. Following this, we will show (Lemma 3.1, below) that these assumptions will also ensure $P_\kappa^{(2)}$ satisfies the assumptions of the Harris theorem with κ-independent constants. As a result, a quantitative Harris theorem [HM11] will show that $P_\kappa^{(2)}$ is uniformly geometrically ergodic with a κ-independent rate (Lemma 3.2 in Section 7, below). Once this has been established, an argument of [DKK04] will imply X^κ is exponentially mixing, and prove Lemma 3.3 (Section 8).

2.4. **Checkable Conditions that Guarantee the Harris Conditions.** In practice it is not easy to find a Lyapunov function (Assumption 2.2). As mentioned in Remark 1.5, recent papers [BBPS22, BCZG22] instead assume certain conditions which are stronger than Harris conditions, but are easier to verify. For convenience of the reader, we state these conditions here.

Assumption 2.5. *The transition kernels* P_0, $\hat P_0$ *and* $P_0^{(2)}$ *are all Feller, and topologically irreducible.*

Assumption 2.6. *Let $\mathscr{N}$ be one of the spaces $\mathbb{T}^d, S\mathbb{T}^d$, and $\mathbb{T}^{d,(2)}$ and $\mathscr{Y}_k$ be one of the corresponding maps $\mathscr{X}_k$, $\hat{\mathscr{X}}_k$, or $\mathscr{X}_k^{(2)}$. For each choice of $\mathscr{N}$ and $\mathscr{Y}$, there exist $n \geqslant 1$ and $(\xi_*, x_*) \in \mathscr{M}^n \times \mathscr{N}$ such that the following hold.*

(1) There exists $c, \varepsilon > 0$ such that for every $\xi \in \mathscr{M}^n$ with $|\xi - \xi_*| < \varepsilon$, we have $\rho_n(\xi) \geqslant c > 0$. Here ρ_n is the density of $\underline{\omega_n}$ on $\mathscr{M}^n$.

(2) The map $\mathscr{Y}_n(\cdot, x_*)\colon \mathscr{M}^n \to \mathscr{N}$ is a submersion at $\xi = \xi_*$.

Assumption 2.7. *For each choice of $\mathscr{N}$ and $\mathscr{Y}$ as in Assumption 2.6, there exist $\xi_{**} \in \operatorname{supp}(\operatorname{dist}(\zeta_1))$ and $y_* \in \mathscr{N}$ such that $\mathscr{Y}_1(\xi_{**}, y_*) = y_*$.*

Assumption 2.8. *Let $\mathscr{N} = \mathbb{T}^d, \mathscr{Y} = \mathscr{X}$, and let n, (ξ_*, x_*) be as in Assumption 2.6. Define a C^1-mapping $g\colon \mathscr{M}^n \to SL_d(\mathbb{R})$ by*

$$g(\xi) = \frac{1}{|\det \mathscr{A}_n(\xi, x_*)|^{\frac{1}{d}}} \mathscr{A}_n(\xi, x_*) \,.$$

Then, the restriction of the derivative $D_{\xi_*} g$ to $\ker D_\xi \mathscr{X}_n(\cdot, x_*) \subset T_\xi \mathscr{M}^n$ is surjective onto $T_{g(\xi_*)} SL_d(\mathbb{R})$.

These conditions are stronger than the Harris conditions in the following sense.

Proposition 2.9. *If Assumption 2.4 and 2.5–2.8 hold, then the Assumptions 2.1–2.3 also hold. Hence, in this case, the enhanced dissipation estimate (1.3) also holds for some random variable D_κ (depending on u and κ, but independent of W) satisfying (1.4).*

Proof of Proposition 2.9. The proof of Proposition 2.9 follows immediately from the results in [BCZG22]. Assumption 2.5 along with continuity of $\mathscr{U}$ (Assumption 2.4) and Assumption 2.7 implies the conditions in Assumption 2.1. Assumption 2.3 follows immediately from Assumption 2.6. To obtain the Lyapunov function (Assumption 2.2) we will apply Proposition 3.3 in [BCZG22]. In order to do this we note that a standard Gronwall argument (see for instance Lemma 4.2, below) and compactness imply that there exists a constant C_0' such that for all $\xi \in \mathscr{M}$, $x, y \in \mathbb{T}^d$, we have

$$\frac{1}{C_0'} d(x,y) \leqslant d(\mathscr{X}_1(\xi, x), \mathscr{X}_1(\xi, y)) \leqslant C_0' d(x,y) \,.$$

Along with Assumption 2.8, this allows us to apply Proposition 3.3 in [BCZG22], and gives positivity of the top Lyapunov exponent. Now Proposition 4.5 in [BCZG22] (and Assumptions 2.4–2.8) will imply the existence of a Lyapunov function as stated in Assumption 2.2. □

3. Proof of Theorem 1.1

As mentioned earlier, the main idea behind the proof of Theorems 1.1, 1.3 is to show that Assumption 2.1–2.4 imply that the Harris conditions hold for all sufficiently small $\kappa > 0$.

Lemma 3.1. *Suppose Assumptions 2.1–2.4 hold. Then there exist $l \in \mathbb{N}$, $\gamma_3 \in (0,1)$, $K > 0$, $R > \frac{2K}{1-\gamma_3}$, $\alpha \in (0,1)$, and a probability measure ν, such that for all sufficiently small $\kappa > 0$ we have*

$$P_\kappa^{(2),l} V \leqslant \gamma_3 V + K \,, \tag{3.1}$$

$$\inf_{x \in \{V \leqslant R\}} P_\kappa^{(2),l}(x, \cdot) \geqslant \alpha \nu(\cdot) \,. \tag{3.2}$$

Once Lemma 3.1 is proved, the remainder of the proof can be obtained using established methods (see for instance [BCZG22, BBPS21, DKK04]). First, a quantitative version of the Harris theorem [HM11, MT09] combined with Lemma 3.1 will show that the two point process $X^{\kappa,(2)}$ is V-geometrically ergodic.

Lemma 3.2 (V-geometric ergodicity)**.** *Suppose that* (3.1) *and* (3.2) *hold. There exist constants $C > 0$ and $\beta > 0$ such that for all sufficiently small $\kappa \geqslant 0$, all measurable $\varphi : \mathbb{T}^{d,(2)} \to \mathbb{R}$ such that $\|\varphi\|_V < \infty$, and any $n \in \mathbb{N}$, we have*

$$\Big\| P_\kappa^{(2),n}\varphi - \int \varphi \, d\pi^{(2)} \Big\|_V \leqslant C e^{-\beta n} \Big\| \varphi - \int \varphi \, d\pi^{(2)} \Big\|_V . \tag{3.3}$$

Combining this with Borel-Cantelli argument in [KDK05] (see also [BBPS21, BCZG22]) will show almost sure exponential mixing of X^κ.

Lemma 3.3. *Suppose that* (3.3) *holds. Then, for every $\alpha > 0$ and $0 < q < \infty$, there exists a random $D_\kappa \geqslant 1$ (which depends on u but is independent of W), and deterministic $\gamma > 0$ (independent of κ) such that for all sufficiently small $\kappa \geqslant 0$, every pair of mean-zero test functions $f, g \in \dot{H}^\alpha$, and every $n \in \mathbb{N}$ we have*

$$\langle f, g \circ X_n^\kappa \rangle \leqslant D_\kappa e^{-\gamma n} \|f\|_{H^\alpha} \|g\|_{H^\alpha} , \quad \textit{almost surely}. \tag{3.4}$$

Moreover, there exists a finite constant $\bar{D}_q$ (independent of κ) such that

$$\boldsymbol{E} D_\kappa^q \leqslant \bar{D}_q ,$$

for all sufficiently small $\kappa \geqslant 0$.

Remark 3.4. By duality, (3.4) is equivalent to

$$\|g \circ X_n^\kappa\|_{H^{-\alpha}} \leqslant D_\kappa e^{-\gamma t} \|g\|_{H^\alpha} , \tag{3.4$'$}$$

for every mean-zero test function $g \in H^{-\alpha}$.

Momentarily postponing the proof of Lemma 3.3, we prove Theorems 1.1 and 1.3.

Proof of Theorem 1.1. For notational convenience, we will use ρ_t to denote $\rho(\cdot, t)$, the slice of ρ at time t. Fix $\alpha > 0$, and let $n \in \mathbb{N}$ be a large time. Assumption 2.4 and parabolic regularity implies

$$\|\rho_n - \bar{\rho}\|_{L^\infty} \leqslant \frac{C}{\kappa^{\frac{2\alpha + d}{4}}} \|\rho_{n-1} - \bar{\rho}\|_{H^{-\alpha}} .$$

To bound the right hand side, we note that the Kolmogorov backward equation implies that for any $t \geqslant 1$ we have

$$\rho_{t+1}(x) = \boldsymbol{E}_W \rho_1 \circ X_{1,1+t}^\kappa(x) .$$

Here $\boldsymbol{E}_W$ denotes the expectation with respect to the $\boldsymbol{P}_W$ marginal of the product measure $\boldsymbol{P} = \boldsymbol{P}_W \otimes \boldsymbol{P}_0$, and $X_{1,\cdot}^\kappa$ is the solution of the SDE

$$dX_{1,t}^\kappa(x) = -u(X_{1,t}^\kappa(x), t)\, dt + \sqrt{2\kappa}\, dW_t , \quad X_{1,1}^\kappa(x) = x .$$

Since the distribution of u is time homogeneous, we may apply Lemma 3.3 to $X_{1,\cdot}^\kappa$. Thus, using (3.4$'$) yields

$$\begin{aligned} \|\rho_{n-1} - \bar{\rho}\|_{H^{-\alpha}} = \|\boldsymbol{E}_W \rho_1 \circ X_{1,n-1}^\kappa - \bar{\rho}\|_{H^{-\alpha}} &\leqslant \boldsymbol{E}_W \|\rho_1 \circ X_{1,n-1}^\kappa - \bar{\rho}\|_{H^{-\alpha}} \\ &\leqslant D_\kappa e^{-\gamma(n-1)} \|\rho_1 - \bar{\rho}\|_{H^\alpha} . \end{aligned}$$

Here we used the fact that since D_κ is independent of W which implies $D_\kappa = \boldsymbol{E}_W D_\kappa$. Finally, we note that parabolic regularity implies

$$\|\rho_1 - \bar\rho\|_{H^\alpha} \leqslant \frac{C}{\kappa^{\frac{2\alpha+d}{4}}}\|\rho_0 - \bar\rho\|_{L^1}\,.$$

Combining the above, we note

$$\|\rho_n - \bar\rho\|_{L^\infty} \leqslant \frac{Ce^{-\gamma(n-1)}D_\kappa}{\kappa^{\alpha+d/2}}\|\rho_0 - \bar\rho\|_{L^\infty}\,, \tag{3.5}$$

for all integer times n. Since the L^∞ norm is non-increasing, we can increase C by a factor of e^γ and ensure (3.5) holds for all $t \geqslant 0$, concluding the proof. $\square$

Theorem 1.3 is equivalent to Theorem 1.1 by a standard duality argument.

Proof of Theorem 1.3. Let ρ be a solution to (1.1) with initial data ρ_0. By the Kolmogorov backward equation

$$\rho_t(x) = \boldsymbol{E}_W \rho_0 \circ X_t^\kappa(x) = \int_{\mathbb{T}^d} p_t^\kappa(x,y)\rho_0(y)\,dy\,.$$

Thus

$$\sup_{x\in\mathbb{T}^d}\int_{\mathbb{T}^d}(p_t^\kappa(x,y)-1)\rho_0(y)\,dy\,dx \leqslant \|\rho_t - \bar\rho\|_{L^\infty} \leqslant \frac{D_\kappa}{\kappa^{\frac{d}{2}+\alpha}}e^{-\gamma t}\|\rho_0 - \bar\rho\|_{L^1}$$

Since $\rho_0 \in L^1$ is arbitrary, this implies

$$\sup_{x\in\mathbb{T}^d}\|p_t^\kappa(x,\cdot) - 1\|_{L^\infty} \leqslant \frac{D_\kappa}{\kappa^{\frac{d}{2}+\alpha}}e^{-\gamma t}\,,$$

which immediately yields (1.7) as desired. $\square$

The proofs of Lemmas 3.2 and 3.3 follow quickly from existing results, and the bulk of the remainder of the paper is devoted to proving Lemma 3.1. The proof can naturally be split into two parts – showing V is a Lyapunov function, and producing a κ-independent small set. We do each of these parts in Section 4 and 5 respectively, and then prove Lemma 3.1 in Section 6.

4. The existence of a κ-independent Lyapunov function.

The goal of this section is to produce the Lyapunov function V used in Lemma 3.1 in Section 6. For this we use the function V in Assumptions 2.2 and the fact that $u \in C^2$ (Assumption 2.4) to show that for sufficiently small κ, the function V is still a Lyapunov function for $P_\kappa^{(2)}$.

Lemma 4.1 (Existence of a κ-independent Lyapunov function)**.** *Suppose Assumptions 2.2 and 2.4 hold. Then there exists $\gamma_1 \in (0,1)$, $b > 0$ and $V : \mathbb{T}^{d,(2)} \to [1,\infty)$, all independent of κ, such that*

$$P_\kappa^{(2)} V \leqslant \gamma_1 V + b \tag{4.1}$$

holds for all sufficiently small $\kappa \geqslant 0$.

The main idea behind the proof of Lemma 4.1 is that the difference of four terms $X_1^\kappa(x), X_1^\kappa(y), X_1^0(x), X_1^0(y)$ can be estimated small when κ is close to 0 and x and y are close. To carry out the details, we need to first lift all the processes to $\mathbb{R}^d$. We identify the torus $\mathbb{T}^d$ with the set of equivalence classes $\{[\check x] \mid \check x \in \mathbb{R}^d\}$, where $[\check x]$

denotes the equivalence class of $\check{x}$ modulo $\mathbb{Z}^d$. Notice that for any $\check{x}, \check{y} \in \mathbb{R}^d$ such that $|\check{x} - \check{y}| < 1/2$, we have

$$d(x, y) = |\check{x} - \check{y}| \quad \text{and} \quad \exp_x^{-1} y = \check{y} - \check{x}\,, \qquad \text{where} \quad x = [\check{x}],\ y = [\check{y}]\,.$$

We will implicitly identify periodic functions on $\mathbb{R}^d$ with functions on the torus.

We also define the Brownian motion W on $\mathbb{T}^d$ by choosing a standard Brownian motion $\check{W}$ on $\mathbb{R}^d$, and setting

$$W_t = \check{W}_t \pmod{\mathbb{Z}^d}\,.$$

Now, let $\check{X}$ be the solution to (1.6) on $\mathbb{R}^d$ with the Brownian motion $\check{W}$, and notice $X_t = [\check{X}_t]$. To prove Lemma 4.1 we need two elementary estimates on the flows $\check{X}^\kappa$, which we state below.

Lemma 4.2. *There exists a constant $C_0 = C_0(\sup_{\xi,t}\|\mathscr{U}\|_{C^1}) \geqslant 1$ such that for every $\kappa \geqslant 0$ and $\check{x}, \check{y} \in \mathbb{R}^d$, we have*

$$\frac{|\check{x} - \check{y}|}{C_0} \leqslant |\check{X}_1^\kappa(\check{x}) - \check{X}_1^\kappa(\check{y})| \leqslant C_0|\check{x} - \check{y}|\,, \quad \textit{almost surely}. \tag{4.2}$$

Lemma 4.3. *For each $\check{x}, \check{y} \in \mathbb{R}^d$ and $t \in [0, 1]$, define $\varrho_t^\kappa(\check{x}, \check{y})$ by*

$$\varrho_t^\kappa(\check{x}, \check{y}) \stackrel{\text{def}}{=} \left|\check{X}_t^\kappa(\check{x}) - \check{X}_t^\kappa(\check{y}) - (\check{X}_t^0(\check{x}) - \check{X}_t^0(\check{y}))\right|.$$

There exists a constant $C_1 = C_1(\sup_{\xi,t}\|\mathscr{U}\|_{C^2})$ such that for any $\alpha > 0$ and $\kappa > 0$, we have

$$\mathbf{1}_{\{\check{W}_1^* \leqslant \alpha\}}\varrho_1^\kappa(\check{x}, \check{y}) \leqslant C_1(\alpha\sqrt{\kappa} + |\check{x} - \check{y}|)\left|\check{X}_1^0(\check{x}) - \check{X}_1^0(\check{y})\right|.$$

Here $\check{W}^*$ is the running maximum of $|\check{W}|$, defined by

$$\check{W}_t^* = \max_{s \leqslant t}|\check{W}_s|\,.$$

Momentarily postponing the proofs of Lemmas 4.2 and 4.3, we now prove Lemma 4.1.

Proof of Lemma 4.1. We prove this proposition in two following steps. First, we show that for any $c > 0$, there exists a constant $\varepsilon < s_*$ such that

$$\limsup_{\kappa \to 0^+} \sup_{(x,y) \in \Delta(\varepsilon)} \frac{|P_\kappa^{(2)} V - P_0^{(2)} V|}{V} < c\,. \tag{4.3}$$

Then, using (2.2) and choosing $c < 1 - \tilde{\gamma}$ imply that for some $\varepsilon > 0$ and all sufficiently small $\kappa \geqslant 0$,

$$P_\kappa^{(2)} V < (\tilde{\gamma} + c) V \quad \text{on } \Delta(\varepsilon)\,. \tag{4.4}$$

Outside $\Delta(\varepsilon)$, we will show that there exists a constant $b > 0$ such that

$$P_\kappa^{(2)} V \leqslant b \quad \text{on } \Delta(\varepsilon)^c\,, \tag{4.5}$$

for all sufficiently small $\kappa \geqslant 0$. Using (4.4) and (4.5) immediately implies (4.1) as desired.

In order to finish the proof we need to prove (4.3) and (4.5). To prove (4.3), let ε be a small κ-independent constant that will be chosen later. We let $x, y \in \Delta(\varepsilon)$ and note

$$P_\kappa^{(2)} V(x, y) - P_0^{(2)} V(x, y) = \boldsymbol{E}[V(X_1^\kappa(x), X_1^\kappa(y)) - V(X_1^0(x), X_1^0(y))]\,.$$

Choose $\varepsilon > 0$ such that for the constant C_0 in (4.2) and s_* from Assumption 2.2, we have $C_0\varepsilon < s_*$. Let $\check{x}, \check{y} \in \mathbb{R}^d$ such that $[\check{x}] = x$, $[\check{y}] = y$, and $d(x,y) = |\check{x} - \check{y}|$. Then by Lemma 4.2, we have

$$|\check{X}_1^\kappa(\check{x}) - \check{X}_1^\kappa(\check{y})| < C_0\varepsilon < s_* < \frac{1}{2},$$

for all $\kappa \geqslant 0$, and hence

$$\frac{1}{C_0} d(x,y) \leqslant d(X_1^\kappa(x), X_1^\kappa(y)) = |\check{X}_1^\kappa(\check{x}) - \check{X}_1^\kappa(\check{y})| \leqslant C_0 d(x,y) \leqslant s_* . \tag{4.6}$$

Then using (2.3), we get

$$P_\kappa^{(2)} V(x,y) - P_0^{(2)} V(x,y) = \boldsymbol{E}\Big[\frac{\psi^\kappa}{|\check{Z}_1 + \tilde{Z}_1^\kappa|^p} - \frac{\psi^0}{|\check{Z}_1|^p}\Big],$$

where

$$\begin{aligned}
\psi^\kappa &\stackrel{\text{def}}{=} \psi(X_1^\kappa(x), \hat{\omega}(X_1^\kappa(x), X_1^\kappa(y))),\\
\psi^0 &\stackrel{\text{def}}{=} \psi(X_1^0(x), \hat{\omega}(X_1^0(x), X_1^0(y))),\\
Z_1 &\stackrel{\text{def}}{=} \check{X}_1^0(\check{x}) - \check{X}_1^0(\check{y}),\\
\tilde{Z}_1^\kappa &\stackrel{\text{def}}{=} \check{X}_1^\kappa(\check{x}) - \check{X}_1^\kappa(\check{y}) - (\check{X}_1^0(\check{x}) - \check{X}_1^0(\check{y})),\\
\hat{\omega}(x', y') &= \frac{\exp_{x'}^{-1}(y')}{d(x',y')}.
\end{aligned}$$

In particular, we can rewrite the terms into

$$\frac{\psi^\kappa}{|\check{Z}_1 + \tilde{Z}_1^\kappa|^p} - \frac{\psi^0}{|\check{Z}_1|^p} = \psi^\kappa\Big(\frac{1}{|\check{Z}_1 + \tilde{Z}_1^\kappa|^p} - \frac{1}{|\check{Z}_1|^p}\Big) + \frac{1}{|\check{Z}_1|^p}(\psi^\kappa - \psi^0).$$

and deduce

$$\begin{aligned}
\frac{|P_\kappa^{(2)} V - P_0^{(2)} V|}{V} &\leqslant \frac{d(x,y)^p}{\inf_{SM} \psi_p}\Big(\boldsymbol{E}\Big[\psi^\kappa\Big|\frac{1}{|\check{Z}_1 + \tilde{Z}_1^\kappa|^p} - \frac{1}{|\check{Z}_1|^p}\Big| + \frac{1}{|\check{Z}_1|^p}|\psi^\kappa - \psi^0|\Big]\Big)\\
&\leqslant \frac{d(x,y)^p}{\inf_{SM} \psi}\Big(\boldsymbol{E}\mathbf{1}_{\{\check{W}_1^* \leqslant \alpha\}}(F_1 + F_2) + \boldsymbol{P}(\check{W}_1^* \geqslant \alpha)\sup_{\Omega}(F_1 + F_2)\Big),
\end{aligned} \tag{4.7}$$

where

$$F_1 \stackrel{\text{def}}{=} \psi^\kappa\Big|\frac{1}{|\check{Z}_1 + \tilde{Z}_1^\kappa|^p} - \frac{1}{|\check{Z}_1|^p}\Big|, \quad \text{and} \quad F_2 \stackrel{\text{def}}{=} \frac{1}{|\check{Z}_1|^p}|\psi^\kappa - \psi^0|,$$

and $\alpha > 0$ is a small κ-independent constant that will be chosen shortly.

We will now bound each term on the right of (4.7). On the event $\{\check{W}_1^* \leqslant \alpha\}$, we note that Lemma 4.3 implies

$$|\tilde{Z}_1^\kappa| \leqslant C_1(\alpha\sqrt{\kappa} + \varepsilon)|\check{Z}_1|, \tag{4.8}$$

so

$$|\check{Z}_1 + \tilde{Z}_1^\kappa| \geqslant |\check{Z}_1| - |\tilde{Z}_1^\kappa| \geqslant |\check{Z}_1|(1 - C_1(\alpha\sqrt{\kappa} + \varepsilon)) > 0, \tag{4.9}$$

for sufficiently small $\kappa, \varepsilon > 0$. Moreover,

$$|\check{Z}_1 + \tilde{Z}_1^\kappa| \leqslant |\check{Z}_1| + |\tilde{Z}_1^\kappa| \leqslant |\check{Z}_1|(1 + C_1(\alpha\sqrt{\kappa} + \varepsilon)).$$

These two inequalities imply that

$$\frac{1}{|\check{Z}_1|^p} - \frac{1}{|\check{Z}_1 + \tilde{Z}_1^\kappa|^p} \leqslant \frac{1}{|\check{Z}_1|^p}\Big(1 - \frac{1}{(1 + C_1(\alpha\sqrt{\kappa} + \varepsilon))^p}\Big)\,, \tag{4.10}$$

$$\frac{1}{|\check{Z}_1 + \tilde{Z}_1^\kappa|^p} - \frac{1}{|\check{Z}_1|^p} \leqslant \frac{1}{|\check{Z}_1|^p}\Big(\frac{1}{(1 - C_1(\alpha\sqrt{\kappa} + \varepsilon))^p} - 1\Big)\,. \tag{4.11}$$

By convexity of the function $\xi \mapsto \xi^{-p}$, we have

$$1 - \frac{1}{(1 + C_1(\alpha\sqrt{\kappa} + \varepsilon))^p} \leqslant \frac{1}{(1 - C_1(\alpha\sqrt{\kappa} + \varepsilon))^p} - 1\,.$$

Combined with (4.10) and (4.11) this implies

$$\Big|\frac{1}{|\check{Z}_1 + \tilde{Z}_1^\kappa|^p} - \frac{1}{|\check{Z}_1|^p}\Big| \leqslant \frac{1}{|\check{Z}_1|^p}\Big(\frac{1}{(1 - C_1(\alpha\sqrt{\kappa} + \varepsilon))^p} - 1\Big)\,. \tag{4.12}$$

Multiplying (4.12) by $\frac{d(x,y)^p}{\inf_{SM}\psi}\psi^\kappa$ and using (4.6) gives

$$\frac{d(x,y)^p}{\inf_{SM}\psi}\boldsymbol{E}\mathbf{1}_{\{\check{W}_1^* \leqslant \alpha\}}F_1 \leqslant C_0^p\frac{\|\psi\|_\infty}{\inf_{SM}\psi}\Big(\frac{1}{(1 - C_1(\alpha\sqrt{\kappa} + \varepsilon))^p} - 1\Big)\,. \tag{4.13}$$

Next, we bound $\boldsymbol{E}\mathbf{1}_{\{\check{W}_1^* \leqslant \alpha\}}F_2$. We note that by (4.25), (4.26), and (4.6), we can choose sufficiently small $\kappa, \alpha, \varepsilon$ to ensure

$$\operatorname{diam}\{\check{X}_1^\kappa(\check{x}), \check{X}_1^0(\check{x}), \check{X}_1^\kappa(\check{y}), \check{X}_1^0(\check{y})\} < \frac{1}{2}\,.$$

In this case we have

$$\begin{aligned}
d((X_1^\kappa(x),&\hat{\omega}(X_1^\kappa(x), X_1^\kappa(y))), (X_1^0(x), \hat{\omega}(X_1^0(x), X_1^0(y))))\\
&= |\check{X}_1^\kappa(\check{x}) - \check{X}_1^0(\check{x})| + \left|\frac{\check{X}_1^\kappa(\check{y}) - \check{X}_1^\kappa(\check{x})}{|\check{X}_1^\kappa(\check{y}) - \check{X}_1^\kappa(\check{x})|} - \frac{\check{X}_1^0(\check{y}) - \check{X}_1^0(\check{x})}{|\check{X}_1^0(\check{y}) - \check{X}_1^0(\check{x})|}\right|\\
&= |\check{X}_1^\kappa(\check{x}) - \check{X}_1^0(\check{x})| + \left|\frac{\check{Z}_1 + \tilde{Z}_1^\kappa}{|\check{Z}_1 + \tilde{Z}_1^\kappa|} - \frac{\check{Z}_1}{|\check{Z}_1|}\right|.
\end{aligned}$$

The inequalities (4.25) and the general bound

$$\Big|\frac{v}{|v|} - \frac{v'}{|v'|}\Big| \leqslant \frac{2|v - v'|}{\min\{|v|, |v'|\}}\,,$$

immediately imply

$$\begin{aligned}
|\check{X}_1^\kappa(\check{x}) - \check{X}_1^0(\check{x})| &+ \left|\frac{\check{Z}_1 + \tilde{Z}_1^\kappa}{|\check{Z}_1 + \tilde{Z}_1^\kappa|} - \frac{\check{Z}_1}{|\check{Z}_1|}\right|\\
&\leqslant C_0\alpha\sqrt{\kappa} + \frac{2|\tilde{Z}_1^\kappa|}{\min\{|\check{Z}_1 + \tilde{Z}_1^\kappa|, |\check{Z}_1|\}}\,.
\end{aligned} \tag{4.14}$$

Using (4.8) and (4.9) we see

$$C_0\alpha\sqrt{\kappa} + 2\frac{|\tilde{Z}_1^\kappa|}{\min\{|\check{Z}_1 + \tilde{Z}_1^\kappa|, |\check{Z}_1|\}} \leqslant C_0\alpha\sqrt{\kappa} + \frac{2C_1(\alpha\sqrt{\kappa} + \varepsilon)}{1 - C_1(\alpha\sqrt{\kappa} + \varepsilon)}\,. \tag{4.15}$$

Now fix $\eta > 0$ to be a κ-independent constant that will be chosen later. Using uniform continuity of ψ find $\delta > 0$ such that

$$\forall z, z' \in SM, \quad d(z, z') < \delta \implies |\psi(z) - \psi(z')| < \eta\,. \tag{4.16}$$

If κ, ε are sufficiently small, the right hand side of (4.15) can be made smaller than δ. Using (4.14) implies

$$d((X_1^\kappa(x), \hat{\omega}(X_1^\kappa(x), X_1^\kappa(y))), (X_1^0(x), \hat{\omega}(X_1^0(x), X_1^0(y)))) < \delta\,,$$

and using (4.16) implies

$$|\psi^\kappa - \psi^0| < \eta\,.$$

Thus by (4.6), we see

$$\frac{d(x,y)^p}{\inf_{SM}\psi}\boldsymbol{E}\mathbf{1}_{\{\check{W}_1^* \leqslant \alpha\}}F_2 \leqslant \frac{C_0^p}{\inf_{SM}\psi}\eta\,. \tag{4.17}$$

Finally, for the last term on the right of (4.7), we note

$$\psi^\kappa\Big|\frac{1}{|\check{Z}_1 + \tilde{Z}_1^\kappa|^p} - \frac{1}{|\check{Z}_1|^p}\Big| + \frac{1}{|\check{Z}_1|^p}|\psi^\kappa - \psi^0|$$
$$\leqslant \|\psi\|_\infty\Big(\frac{1}{|\check{Z}_1 + \tilde{Z}_1^\kappa|^p} + \frac{1}{|\check{Z}_1|^p}\Big) + \frac{2}{|\check{Z}_1|^p}\|\psi\|_\infty\,,$$

so using (4.6) shows

$$\frac{d(x,y)^p}{\inf_{SM}\psi}\boldsymbol{P}(\check{W}_1^* \geqslant \alpha)\sup_\Omega(F_1 + F_2) \leqslant \frac{4C_0^p\|\psi\|_\infty}{\inf_{SM}\psi}\boldsymbol{P}[\check{W}_1^* > \alpha]\,. \tag{4.18}$$

Using (4.13), (4.17) and (4.18) in (4.7) we obtain

$$\sup_{(x,y)\in\Delta(\varepsilon)}\frac{|P_\kappa^{(2)}V - P_0^{(2)}V|}{V} \leqslant C_0^p\frac{\|\psi\|_\infty}{\inf_{SM}\psi}\Big(\frac{1}{(1 - C_1(\alpha\sqrt{\kappa} + \varepsilon))^p} - 1\Big)$$
$$+ \frac{C_0^p}{\inf_{SM}\psi}\eta + \frac{4C_0^p\|\psi\|_\infty}{\inf_{SM}\psi}\boldsymbol{P}[W_1^* > \alpha]\,.$$

Thus,

$$\limsup_{\kappa\to 0+}\sup_{(x,y)\in\Delta(\varepsilon)}\frac{|P_\kappa^{(2)}V - P_0^{(2)}V|}{V} \leqslant C_0^p\frac{\|\psi\|_\infty}{\inf_{SM}\psi}\Big(\frac{1}{(1 - C_1\varepsilon)^p} - 1\Big) \tag{4.19}$$
$$+ \frac{C_0^p}{\inf_{SM}\psi}\eta + \frac{4C_0^p\|\psi\|_\infty}{\inf_{SM}\psi}\boldsymbol{P}[W_1^* > \alpha]\,.$$

Now, we choose α, η, and ε such that

$$\frac{4C_0^p\|\psi\|_\infty}{\inf_{SM}\psi}\boldsymbol{P}[W_1^* > \alpha] < \frac{1}{3}c\,,$$
$$\frac{C_0^p}{\inf_{SM}\psi}\eta < \frac{1}{3}c\,,$$
$$C_0^p\frac{\|\psi\|_\infty}{\inf_{SM}\psi}(\frac{1}{(1 - C_1\varepsilon)^p} - 1) < \frac{1}{3}c\,,$$

then using (4.19) will imply (4.3) as desired.

Finally, in order to prove (4.5), we see from the Assumption 2.2 that V is continuous on the compact set $K \stackrel{\text{def}}{=} \overline{\Delta(s_*)} - \Delta(\frac{s_*}{2})$ so it can be continuously extended to the compact set $K' \stackrel{\text{def}}{=} \mathbb{T}^{d,(2)} - \Delta(\frac{s_*}{2})$ such that

$$1 \leqslant \inf_K V = \inf_{K'} V \leqslant \sup_{K'} V = \sup_K V\,.$$

Now, let $(x, y) \in \Delta(\varepsilon)^c$. On the event $E_1 \stackrel{\text{def}}{=} \{(X_1^\kappa(x), X_1^\kappa(y)) \in \Delta(s_*)\}$, there exist $\check{x}, \check{y} \in \mathbb{R}^d$ such that $[\check{x}] = x$, $[\check{y}] = y$, and

$$d(X_1^\kappa(x), X_1^\kappa(y)) = |\check{X}_1^\kappa(\check{x}) - \check{X}_1^\kappa(\check{y})| < s_*\,.$$

By (4.2), $|\check{x} - \check{y}| < \frac{s_*}{C_0} < \frac{1}{2}$ so $|\check{x} - \check{y}| = d(x, y)$ and

$$d(X_1^\kappa(x), X_1^\kappa(y)) \geqslant C_0 d(x, y) \geqslant C_0 \varepsilon\,,$$

which implies

$$\boldsymbol{E}[V(X_1^\kappa(x), X_1^\kappa(y))\mathbf{1}_{E_1}] \leqslant C_0^{-p}\varepsilon^{-p}\|\psi\|_\infty\,. \tag{4.20}$$

On the event E_1^c, we have

$$\boldsymbol{E}[V(X_1^\kappa(x), X_1^\kappa(y))\mathbf{1}_{E_1^c}] \leqslant \sup_{K'} V\,. \tag{4.21}$$

Bounds (4.20) and (4.21) yield (4.5) with

$$b \stackrel{\text{def}}{=} C_0^{-p}\varepsilon^{-p}\|\psi\|_\infty + \sup_{K'} V\,.$$

This completes the proof. ∎

Gradient estimates on the stochastic flows. It remains to prove Lemmas 4.2 and 4.3. Lemma 4.2 is a direct result of applying Grönwall's inequality twice to the difference $|X_t^\kappa(\check{x}) - X_t^\kappa(\check{y})|$ near $t = 0$ and $t = 1$.

Proof of Lemma 4.2. For the upper bound, we see that for any $\kappa \geqslant 0$ and $0 \leqslant t \leqslant 1$,

$$\check{X}_t^\kappa(\check{x}) - \check{X}_t^\kappa(\check{y}) = \check{x} - \check{y} + \int_0^t u(X_s^\kappa(\check{x}), s) - u(\check{X}_s^\kappa(\check{y}), s)ds\,,$$

so

$$|\check{X}_t^\kappa(\check{x}) - \check{X}_t^\kappa(\check{y})| \leqslant |\check{x} - \check{y}| + A_1 \int_0^t |\check{X}_s^\kappa(\check{x}) - \check{X}_s^\kappa(\check{y})|ds\,.$$

By Grönwall's inequality, we have

$$|\check{X}_1^\kappa(x) - \check{X}_1^\kappa(y)| \leqslant e^{A_1}|\check{x} - \check{y}|\,,$$

where

$$A_1 \stackrel{\text{def}}{=} \|\nabla_x \mathscr{U}\|_{L^\infty(\mathscr{M} \times \mathbb{T}^d \times [0,1])} \tag{4.22}$$

Similarly, for the lower bound, we see that for any $\kappa \geqslant 0$ and $0 \leqslant t \leqslant 1$,

$$\begin{aligned}\check{X}_{1-t}^\kappa(\check{x}) - \check{X}_{1-t}^\kappa(\check{y}) &= \check{X}_1^\kappa(\check{x}) - \check{X}_1^\kappa(\check{y}) - \int_{1-t}^1 \Big(u(\check{X}_s^\kappa(\check{x}), s) - u(\check{X}_s^\kappa(\check{y}), s)\Big)ds \\ &= \check{X}_1^\kappa(\check{x}) - \check{X}_1^\kappa(\check{y}) - \int_0^t \Big(u(\check{X}_{1-s}^\kappa(\check{x}), s) - u(\check{X}_{1-s}^\kappa(\check{y}), s)\Big)ds\end{aligned}$$

so

$$|\check{X}_{1-t}^\kappa(\check{x}) - \check{X}_{1-t}^\kappa(\check{y})| \leqslant |\check{X}_1^\kappa(\check{x}) - \check{X}_1^\kappa(\check{y})| + A_1 \int_0^t |\check{X}_{1-s}^\kappa(\check{x}) - \check{X}_{1-s}^\kappa(\check{y})|ds\,.$$

By Grönwall's inequality, we have

$$|\check{x} - \check{y}| \leqslant e^{A_1}|\check{X}_1^\kappa(\check{x}) - \check{X}_1^\kappa(\check{y})|\,.$$

Thus, setting $C_0 \stackrel{\text{def}}{=} e^{A_1}$ concludes the proof. $\square$

To prove Lemma 4.3, we first explicitly write down the differences $\check{X}_t^\kappa(\check{x}) - \check{X}_t^\kappa(\check{y})$ and $\check{X}_t^0(\check{x}) - \check{X}_t^0(\check{y})$ by using the differential equations they satisfy. Then, we use mean value theorem and Grönwall's inequality multiple times to estimate their difference.

Proof of Lemma 4.3. For $0 \leqslant t \leqslant 1$, we have

$$\begin{aligned}
\check{X}_t^\kappa(\check{x}) - \check{X}_t^\kappa(\check{y}) &= \check{x} - \check{y} + \int_0^t (u(\check{X}_s^\kappa(\check{x}), s) - u(\check{X}_s^\kappa(\check{y})), s)\, ds \\
&= \check{x} - \check{y} + \int_0^t \nabla_x u(\beta_1(s), s) \cdot (\check{X}_s^\kappa(\check{x}) - \check{X}_s^\kappa(\check{y}))\, ds
\end{aligned} \tag{4.23}$$

for some $\beta_1(s)$ and similarly

$$\check{X}_t^0(\check{x}) - \check{X}_t^0(\check{y}) = \check{x} - \check{y} + \int_0^t \nabla_x u(\beta_2(s), s) \cdot (\check{X}_s^0(\check{x}) - \check{X}_s^0(\check{y}))\, ds \tag{4.24}$$

for some $\beta_2(s)$. We define

$$S(t) \stackrel{\text{def}}{=} \check{X}_t^\kappa(\check{x}) - \check{X}_t^\kappa(\check{y}) - (\check{X}_t^0(\check{x}) - \check{X}_t^0(\check{y}))\,.$$

Then, by taking the difference between (4.23) and (4.24), we get

$$\begin{aligned}
S(t) = \int_0^t & \nabla_x u(\beta_1(s), s) \cdot S(s) \\
&+ (\nabla_x u(\beta_1(s), s) - \nabla_x u(\beta_2(s), s)) \cdot (\check{X}_s^0(\check{x}) - \check{X}_s^0(\check{y}))\, ds\,,
\end{aligned}$$

which implies

$$\varrho_t^\kappa(\check{x}, \check{y}) \leqslant A_1 \int_0^t \varrho_s^\kappa(\check{x}, \check{y}) ds + A_2 \int_0^t |\beta_1(s) - \beta_2(s)| \left|\check{X}_s^0(\check{x}) - \check{X}_s^0(\check{y})\right| ds\,,$$

where

$$A_2 \stackrel{\text{def}}{=} \|\nabla_x^2 \mathscr{U}\|_{L^\infty(\mathscr{M} \times \mathbb{T}^d \times [0,1])}\,.$$

We see that

$$\begin{aligned}
|\beta_1(s) - \beta_2(s)| \leqslant \max\{&|\check{X}_s^\kappa(\check{y}) - \check{X}_s^0(\check{y})|, |\check{X}_s^\kappa(\check{x}) - \check{X}_s^0(\check{x})|, \\
&|\check{X}_s^\kappa(\check{y}) - \check{X}_s^0(\check{x})|, |\check{X}_s^\kappa(\check{x}) - \check{X}_s^0(\check{y})|\}
\end{aligned}$$

and in particular,

$$\mathbf{1}_{\{\check{W}_1^* \leqslant \alpha\}} |\check{X}_s^\kappa(\check{x}) - \check{X}_s^0(\check{x})| \leqslant \alpha\sqrt{\kappa} e^{A_1 s} \tag{4.25}$$

$$|\check{X}_s^0(\check{x}) - \check{X}_s^0(\check{y})| \leqslant |\check{x} - \check{y}| e^{A_1 s} \tag{4.26}$$

by Grönwall's inequality.

As a result, we obtain

$$\begin{aligned}
\mathbf{1}_{\{\check{W}_1^* \leqslant \alpha\}} \varrho_t^\kappa(\check{x}, \check{y}) \leqslant A_1 \int_0^t & \mathbf{1}_{\{\check{W}_1^* \leqslant \alpha\}} \varrho_s^\kappa(\check{x}, \check{y}) ds \\
&+ A_2(\alpha\sqrt{\kappa} + |\check{x} - \check{y}|) e^{A_1 t} \int_0^t \left|\check{X}_s^0(\check{x}) - \check{X}_s^0(\check{y})\right| ds\,.
\end{aligned}$$

Using Grönwall's inequality and Lemma 4.2 this gives

$$\mathbf{1}_{\{\check{W}_1^* \leqslant \alpha\}} \varrho_1^{\kappa}(\check{x}, \check{y}) \leqslant C_1(\alpha\sqrt{\kappa} + |\check{x} - \check{y}|) \left| \check{X}_1^0(\check{x}) - \check{X}_1^0(\check{y}) \right| ,$$

for some constant $C_1 = C_1(A_1, A_2)$. □

5. A stable small set.

Next, in order to obtain the minorizing condition (3.2) stated in Lemma 3.1, we will show that Assumptions 2.3 and 2.4 imply the existence of a κ-independent small set.

Lemma 5.1. *Suppose Assumptions 2.3 and 2.4 hold. Then there exist nonempty open sets $A, B \subseteq \mathbb{T}^{d,(2)}$ and a constant $\beta > 0$ such that*

$$\inf_{x \in A} P_{\kappa}^{(2),n}(x, \cdot) \geqslant \beta \operatorname{Leb}|_B(\cdot) . \tag{5.1}$$

for all sufficiently small $\kappa \geqslant 0$. Here $\operatorname{Leb}|_B$ denotes the restriction of the Lebesgue measure to B.

To prove Lemma 5.1, we need the following two lemmas.

Lemma 5.2. *Let $f \colon B_r \subseteq \mathbb{R}^N \to \mathbb{R}^d$ be a C^1 function such that $f(0) = 0$ and $\operatorname{rank}(Df(0)) = d$. Then there exist $\delta, s > 0$ such that, for any $g \colon \mathbb{R}^N \to \mathbb{R}^d$ Lipschitz with $\|f - g\|_{L^\infty(B_r)} \leqslant \delta$, we have*

$$g(B_r) \supseteq B_s$$

Here B_r denotes the open ball centered at 0.

Proof. By the constant rank theorem, there are diffeomorphisms α and β which fix the origin such that $(\alpha \circ f \circ \beta)(x_1, \dots, x_N) = (x_1, \dots, x_d)$. Choose $\delta > 0$ so that $\|(\alpha \circ f \circ \beta) - (\alpha \circ g \circ \beta)\|_{L^\infty(B_1)} \leqslant \frac{1}{4}$. Let $x_{d+1}, \dots, x_N$ be arbitrary with $x_{d+1}^2 + \cdots + x_N^2 \leqslant \frac{1}{4}$ and define $h(x_1, \dots, x_d) := (\alpha \circ g \circ \beta)(x_1, \dots, x_d, x_{d+1}, \dots, x_N)$. Then $h \colon B_{3/4} \subseteq \mathbb{R}^d \to \mathbb{R}^d$ satisfies $|h(x) - x| \leqslant \frac{1}{4}$ at every point.

We claim that $h(B_{3/4}) \supseteq B_{1/2}$. Indeed, compute the degree $\deg(h, B_{3/4}, x) = 1$ for each $x \in B_{1/2}$ (the degree is a homotopy invariant, and h is homotopic to the identity).

Then $(\alpha \circ g \circ \beta)(B_1) \supseteq B_{1/2}$, and therefore $g(B_r) \supseteq B_s$ for some $s > 0$ depending only on α and β. □

Lemma 5.3. *For any $n \in \mathbb{N}$, $\kappa \geqslant 0$, $x \in \mathbb{T}^{d,(2)}$, $\xi, \xi' \in \mathscr{M}^n$, and almost any realization of the noise W, we have*

$$d(X_n^{\kappa,(2)}(\xi, x), X_n^{\kappa,(2)}(\xi', x)) \leqslant A_3 \exp(n(1 + A_1)) d_\infty(\xi, \xi') ,$$

where A_1 is defined as in (4.22), and

$$A_3 \stackrel{\text{def}}{=} \|\nabla_\xi \mathscr{U}\|_{L^\infty(\mathscr{M} \times \mathbb{T}^d \times [0,1])} ,$$
$$d_\infty(\xi, \xi') \stackrel{\text{def}}{=} \sup_{k \leqslant n} d_{\mathscr{M}}(\xi_k, \xi_k') .$$

Proof. The proof follows immediately from Grönwall's inequality and is very similar to the proof of Lemma 4.2. □

With these lemmas, we can prove the existence of a κ-independent small set.

Proof of Lemma 5.1. Let $n, x_*, \xi_*, \varepsilon, c, \rho_n$ be defined as in Assumption 2.3. Then, we see that $\mathscr{X}_n^{(2)}(\cdot, x_*)$ satisfies the assumption in Lemma 5.2 with $r = \varepsilon$. Let δ, s be the constants given in Lemma 5.2 for the map $\mathscr{X}_n^{(2)}(\cdot, x_*)$. Now, choose $\alpha > 0$ such that $\boldsymbol{P}[\check{W}_n^* \leqslant \alpha] \geqslant \frac{1}{2}$ and let η, κ be small enough so that

$$(\eta + n\sqrt{\kappa}\alpha)e^{nA_1} < \frac{1}{\sqrt{2}}\delta .$$

Then for each choice of $x \in B_\eta(x_*)$ and a realization of Brownian path such that $\{\check{W}_n^* \leqslant \alpha\}$, we have

$$d(X_n^{\kappa,(2)}(\xi, x), X_n^{0,(2)}(\xi, x_*)) < \delta ,$$

for any $\xi \in \mathscr{M}^n$. Thus, we can apply Lemma 5.2 to the map $g : \xi \mapsto X_n^{\kappa,(2)}(\xi, x)$ and see that

$$\begin{aligned}\boldsymbol{P}_\kappa^{(2),n}(x, A) &\geqslant \boldsymbol{E}_W\Big[\Big(\int_{g^{-1}(A)\cap B_\varepsilon(\xi_*)} \rho_n(\xi) d\xi\Big)\mathbf{1}_{\{W_n^* \leqslant \alpha\}}\Big] \\ &\geqslant c\boldsymbol{E}_W\Big[|g^{-1}(A) \cap B_\varepsilon(\xi_*)|\mathbf{1}_{\{W_n^* \leqslant \alpha\}}\Big] \\ &\geqslant \frac{c}{2}|A \cap B_s(\mathscr{X}(\xi_*, x_*))|\boldsymbol{E}_W \operatorname{Lip}(g)^{-\dim \mathscr{M}} . \end{aligned} \tag{5.2}$$

Using Lemma 5.3 we see

$$\boldsymbol{E}_W \operatorname{Lip}(g)^{-\dim \mathscr{M}} \geqslant (A_3 \exp(n(1 + A_1)))^{-\dim \mathscr{M}} . \tag{5.3}$$

Using (5.3) in (5.2) yields (5.1) as desired. ☐

6. Verification of the Harris Assumptions (Lemma 3.1).

Given the Lyapunov function V from Lemma 4.1 and the small set from Lemma 5.1, we will now prove Lemma 3.1 verifying Harris conditions for the two point chains, for all sufficiently small $\kappa \geqslant 0$.

Proof of Lemma 3.1. We first note that Lemma 4.1 and induction immediately imply that for every $l \in \mathbb{N}$,

$$P_\kappa^{(2),l} V \leqslant \gamma_1^l V + b_l ,$$

where

$$b_l \stackrel{\text{def}}{=} b \sum_{i=0}^{l-1} \gamma_1^i = b\frac{1 - \gamma_1^l}{1 - \gamma_1} . \tag{6.1}$$

We will now show that R and l can be chosen so that (3.2) is also satisfied.

We will first show that any compact set $A \subset \mathbb{T}^{d,(2)}$ is a small set, uniformly in κ. More precisely, we will prove that for any compact set $A \subset \mathbb{T}^{d,(2)}$, there exist $m \in \mathbb{N}$, $\beta \in (0, 1)$, and a probability measure μ, all independent of κ, such that

$$\inf_{x\in A} P_\kappa^{(2),m}(x, \cdot) \geqslant \beta\mu(\cdot) , \tag{6.2}$$

for all sufficiently small $\kappa \geqslant 0$. Next, we will show that for any $R > \frac{2b}{1-\gamma_1}$, there exists a compact set S such that

$$\{V \leqslant R\} \subset S . \tag{6.3}$$

To see why the above implies (3.2), we first fix any $R > \frac{2b}{1-\gamma_1}$ and then find a compact set S such that (6.3) holds. From (6.2), we can find κ-independent constants l, α and a probability measure ν such that for all sufficiently small $\kappa \geqslant 0$, we have

$$\inf_{x\in S} P_\kappa^{(2),l}(x,\cdot) \geqslant \alpha\nu(\cdot)\,.$$

For this particular $l \in \mathbb{N}$, we define

$$\gamma_3 \stackrel{\text{def}}{=} \gamma_1^l, \quad \text{and} \quad K \stackrel{\text{def}}{=} b_l\,.$$

Using (6.1) we observe

$$R > \frac{2b}{1-\gamma_1} = \frac{2b}{1-\gamma_1^l}\frac{1-\gamma_1^l}{1-\gamma_1} = \frac{2K}{1-\gamma_3}\,,$$

which implies (3.2) as claimed.

It remains to prove (6.2) and (6.3). To prove (6.2), we first note that Assumptions 2.1 and 2.3 imply $P_0^{(2)}$ is an $\psi^{(2)}$-irreducible, aperiodic T-chain with the property that

$$\psi^{(2)}(\mathcal{V}) > 0, \quad \text{for all } \mathcal{V} \subset \mathbb{T}^{d,(2)} \text{ open}\,.$$

Assumption 2.3 and Lemma 5.1 imply that there exist open balls $B_r, B_s \subset \mathbb{T}^{d,(2)}$ and $n \in \mathbb{N}$ such that for all sufficiently small $\kappa \geqslant 0$,

$$\inf_{x\in B_r} P_\kappa^{(2),n}(x,\cdot) \geqslant \tilde\mu(\cdot)\,, \tag{6.4}$$

where B_r and B_s are open balls with radius $r, s > 0$, respectively, and

$$\tilde\mu(\cdot) \stackrel{\text{def}}{=} \operatorname{Leb}(\cdot \cap B_s)\,.$$

$B_{\frac12 r}$ is small for the chain $P_0^{(2)}$ and $\psi^{(2)}(B_{\frac12 r}) > 0$ so by Theorem 6.2.5 (ii) and Theorem 5.5.7 in [MT09], we see that there exist some $q \in \mathbb{N}$ and $c > 0$ such that

$$\inf_{x\in A} P_0^{(2),q}(x, B_{\frac12 r}) \geqslant c > 0\,. \tag{6.5}$$

Then, for each $x \in A$, the measure $P_\kappa^{(2),q}(x,\cdot)$ converges weak-* to the measure $P_0^{(2),q}(x,\cdot)$ as $\kappa \to 0$ so there exists $\kappa_0(x) > 0$ such that

$$\inf_{\kappa<\kappa_0(x)} P_\kappa^{(2),q}(x, B_{\frac12 r}) \geqslant \frac12 P_0^{(2),q}(x, B_{\frac12 r})\,. \tag{6.6}$$

Also, if we assume $r < 1$ without loss of generality and define A_1 as in (4.22), then for any $x \in \mathbb{T}^d$ and $y \in B(x, \frac12\exp(-A_1 q)r)$, we can find $\check x, \check y \in \mathbb{R}^d$ such that $[\check x] = x$, $[\check y] = y$, and $d(x,y) = |\check x - \check y|$, and use simple Gröwnwall bound to notice that

$$d(X_q^\kappa(x), X_q^\kappa(y)) = |\check X_q^\kappa(\check x) - \check X_q^\kappa(\check y)| \leqslant \exp(A_1 q)|\check x - \check y| \leqslant \frac12 r\,.$$

This immediately leads to the inequality

$$P_\kappa^{(2),q}(y, B_r) \geqslant P_\kappa^{(2),q}(x, B_{\frac12 r})\,. \tag{6.7}$$

Now, we cover the compact set A with open balls $\cup_{x\in A} B(x, \frac12\exp(-A_1 q)r)$ and find a finite cover $\bigcup_{i=1}^N B(x_i, \frac12\exp(-A_1 q)r)$. If we let $\kappa < \kappa_0 \stackrel{\text{def}}{=} \min_{i=1}^N \kappa_0(x_i)$ and $y \in A$, then by (6.7), (6.6), and (6.5), we see that

$$P_\kappa^{(2),q}(y, B_r) \geqslant \frac12 c\,.$$

This implies for any $0 \leqslant \kappa < \kappa_0$,

$$\inf_{x \in A} P_\kappa^{(2),q}(x, B_r) \geqslant \frac{1}{2} c\,. \tag{6.8}$$

Defining $m \stackrel{\text{def}}{=} n + q$ and using (6.4) with (6.8) yields

$$\inf_{x \in A} P_\kappa^{(2),m}(x, \cdot) \geqslant \frac{1}{2} c \tilde{\mu}(\cdot)$$

for all sufficiently small $\kappa \geqslant 0$. Then, normalizing $\frac{1}{2} c \tilde{\mu}(\cdot)$ immediately implies (6.2), as desired.

Finally, to prove (6.3), we notice that if $s'' < s_*$ and $(x, x') \in \Delta(s'')$, we have

$$V(x, x') = d(x, x')^{-p} \psi_p(x, \widehat{w}(x, x')) \geqslant (s'')^{-p} \Big(\inf_{\mathrm{SM}} \psi_p \Big) > 0\,.$$

Thus making $s'' > 0$ sufficiently small will ensure

$$\Delta(s'') \subset \{V > R\}\,,$$

which implies

$$\{V \leqslant R\} \subset \Delta(s'')^c\,.$$

Thus $S \stackrel{\text{def}}{=} \Delta(s'')^c$ is the desired compact set, proving (6.3). This concludes the proof. ☐

7. V-geometric ergodicity (Lemma 3.2).

Given Lemma 3.1, V-geometric ergodicity of the two point chains $P_\kappa^{(2)}$ follows directly from a theorem of Harris [Har55, MT09]. The usual Harris theorem, however, isn't quantitative enough to yield (3.3) with κ-independent constants C, β. We will instead use the version in [HM11] which is quantitative and can be used to prove Lemma 3.2.

For the proof, we define the metric ρ_β by

$$\rho_\beta(\mu_1, \mu_2) \stackrel{\text{def}}{=} \int_{\mathbb{T}^{d,(2)}} (1 + \beta V)\, d|\mu_1 - \mu_2|\,, \tag{7.1}$$

where $\beta \geqslant 0$, μ_1, μ_2 are probability measures, and $|\mu_1 - \mu_2|$ denotes the variation of the signed measure $\mu_1 - \mu_2$. The quantitative Harris theorem from [HM11] shows that $P_\kappa^{(2),l}$ is a contraction under ρ_β, and for readers convenience we now restate this result in our context.

Theorem 7.1 (Theorem 1.3 in [HM11])**.** *Make the same assumptions as in Lemma 3.1. For any $\alpha_0 \in (0, \alpha)$, and any $\gamma_0 \in (\gamma_3 + \frac{2K}{R}, 1)$, define*

$$\beta \stackrel{\text{def}}{=} \frac{\alpha_0}{K}\,, \qquad \bar{\alpha} \stackrel{\text{def}}{=} \max\Big\{ 1 - (\alpha - \alpha_0), \frac{2 + R\beta\gamma_0}{2 + R\beta} \Big\}\,.$$

Then, for any two probability measures μ_1, μ_2 we have

$$\rho_\beta(\mu_1 P_\kappa^{(2),l}, \mu_2 P_\kappa^{(2),l}) \leqslant \bar{\alpha} \rho_\beta(\mu_1, \mu_2)\,. \tag{7.2}$$

Referring to [HM11] for the proof of Theorem 7.1, we will now prove Lemma 3.2.

Proof of Lemma 3.2. We will first show there exist constants $C > 0$, $\gamma_2 \in (0,1)$, and $l \in \mathbb{N}$ such that for all sufficiently small $\kappa \geqslant 0$, any $\varphi : \mathbb{T}^{d,(2)} \to \mathbb{R}$ such that $\|\varphi\|_V < \infty$, and any $m \in \mathbb{N}$, we have

$$\Big\| P_\kappa^{(2),lm} \varphi - \int \varphi \, d\pi^{(2)} \Big\|_V \leqslant C \gamma_2^m \Big\| \varphi - \int \varphi \, d\pi^{(2)} \Big\|_V . \tag{7.3}$$

For this, we first define a weighted norm

$$\|\varphi\|_\beta \stackrel{\text{def}}{=} \sup_x \frac{|\varphi(x)|}{1 + \beta V(x)} .$$

Then, since $V \geqslant 1$ on $\mathbb{T}^{d,(2)}$, we see that the norms $\|\cdot\|_V$ and $\|\cdot\|_\beta$ are equivalent for any $\beta > 0$ with

$$\frac{1}{1+\beta} \|\varphi\|_V \leqslant \|\varphi\|_\beta \leqslant \frac{1}{\beta} \|\varphi\|_V . \tag{7.4}$$

We also note that given any two probability measures μ_1 and μ_2 on $\mathbb{T}^{d,(2)}$,

$$\rho_\beta(\mu_1, \mu_2) = \sup_{\|\varphi\|_\beta \leqslant 1} \langle \mu_1 - \mu_2, \varphi \rangle \tag{7.5}$$

always holds, where $\langle \mu, \varphi \rangle$ denotes the dual pairing

$$\langle \mu, \varphi \rangle \stackrel{\text{def}}{=} \int_{\mathbb{T}^{d,(2)}} \varphi \, d\mu .$$

Now, we're ready to prove (7.3). From here on, we set $P = P_\kappa^{(2),l}$ for notational simplicity. By Lemma 3.1 and Theorem 7.1 we obtain the contraction estimate (7.2). In particular, for any $x \in \mathbb{T}^{d,(2)}$ and $\mu_1 \stackrel{\text{def}}{=} \delta_x$, $\mu_2 \stackrel{\text{def}}{=} \pi^{(2)}$, we have

$$\rho_\beta(\delta_x P^n, \pi^{(2)}) \leqslant \bar{\alpha}^n \rho_\beta(\delta_x, \pi^{(2)}) .$$

By using (7.5) and then (7.1), we notice

$$\frac{|\langle \delta_x P^n - \pi^{(2)}, \varphi - \langle \pi^{(2)}, \varphi \rangle \rangle|}{\|\varphi - \langle \pi^{(2)}, \varphi \rangle\|_\beta} \leqslant \bar{\alpha}^n \rho_\beta(\delta_x, \pi^{(2)}) \leqslant \bar{\alpha}^n (1 + \beta V(x) + \langle \pi^{(2)}, 1 + \beta V \rangle) ,$$

and hence

$$\frac{\Big| (P^n \varphi)(x) - \langle \pi^{(2)}, \varphi \rangle \Big|}{1 + \beta V(x)} \leqslant \bar{\alpha}^n (1 + \langle \pi^{(2)}, 1 + \beta V \rangle) \|\varphi - \langle \pi^{(2)}, \varphi \rangle\|_\beta .$$

This holds for any $x \in \mathbb{T}^{d,(2)}$ so

$$\Big\| P^n \varphi - \langle \pi^{(2)}, \varphi \rangle \Big\|_\beta \leqslant C_\beta \bar{\alpha}^n \|\varphi - \langle \pi^{(2)}, \varphi \rangle\|_\beta .$$

Finally, using (7.4) yields (7.3) where C and γ_2 depend on α, γ_3, K, R but not on κ. This completes the proof for (7.3).

The proof that (7.3) implies (3.3) is a standard argument. For any $x \in \mathbb{T}^{d,(2)}$ and any mean 0 function g we note

$$\Big| \frac{(P_\kappa^{(2)} g)(x)}{V(x)} \Big| = \Big| \int_{\mathbb{T}^{d,(2)}} \frac{g(y)}{V(y)} \frac{V(y)}{V(x)} P_\kappa^{(2)}(x, dy) \Big| \leqslant \|g\|_V \frac{(P_\kappa^{(2)} V)(x)}{V(x)} \leqslant (\gamma_1 + b) \|g\|_V ,$$

where γ_1 and b are the constants defined in (4.1). Thus,

$$\big\| P_\kappa^{(2)} g \big\|_V \leqslant (\gamma_1 + b) \|g\|_V .$$

This and (7.3) imply that for any $m \in \mathbb{N}$, $0 \leqslant r < l$,

$$\left\| P_\kappa^{(2),lm+r} g \right\|_V \leqslant C \max(\gamma_1 + b, 1)^r \gamma_2^m \|g\|_V \leqslant C \gamma_2^m \|g\|_V ,$$

which proves

$$\left\| P_\kappa^{(2),n} g \right\|_V \leqslant C(\gamma_2^{\frac{1}{l}})^n \|g\|_V$$

for general $n \in \mathbb{N}$ with possibly different constants C in each line. This completes the proof of Lemma 3.2. □

8. **Exponential Mixing of the Stochastic Flows (Lemma 3.3).**

In general, the geometric ergodicity of the two-point chain implies almost sure exponential mixing of the random dynamical system. To the best of our knowledge this principle was introduced in [DKK04] and was also used in [BBPS22, BCZG22]. We reproduce it here keeping track of the constants introduced in the proof and their dependence on κ in order to prove that γ in (3.4) is κ-independent.

Proof of Lemma 3.3. Let $\mathbb{Z}_0^d \stackrel{\text{def}}{=} \mathbb{Z}^d - \{0\}$ and denote $\{e_m : m \in \mathbb{Z}^d\}$ as the orthogonal basis $e_m(x) = e^{im\cdot x}$ for $L^2(\mathbb{T}^d)$ and denote

$$f = \sum_{m \in \mathbb{Z}_0^d} \hat{f}_m e_m , \quad g = \sum_{m \in \mathbb{Z}_0^d} \hat{g}_m e_m ,$$

as the fourier expansions of f and g. We note that and $\hat{f}_0 = \hat{g}_0 = 0$ as f and g are mean-zero.

Now, fix $\zeta > 0$ and for $m, m' \in \mathbb{Z}_0^d$ and $\kappa > 0$, define random variables

$$N_{m,m'}^\kappa \stackrel{\text{def}}{=} \max\Big\{n \in \mathbb{N}, \Big| \int e_m(x) e_{m'} \circ X_n^\kappa(x) \pi(dx) \Big| > e^{-\zeta n} \Big\} ,$$

$$K_\kappa \stackrel{\text{def}}{=} \max\Big\{ |m| \vee |m'| : e^{\zeta N_{m,m'}^\kappa} > |m||m'| \Big\} ,$$

$$\widehat{D}_\kappa \stackrel{\text{def}}{=} \max_{|m|,|m'| \leqslant K_\kappa} e^{\zeta N_{m,m'}^\kappa} .$$

Then by the definition of $N_{m,m'}^\kappa$ and Chebyshev, we get

$$\begin{aligned} \boldsymbol{P}[N_{m,m'}^\kappa > l] &\leqslant \sum_{n > l} \boldsymbol{P}\Big[\Big| \int e_m(x) e_{m'} \circ X_n^\kappa(x) \pi(dx) \Big| > e^{-\zeta n} \Big] \\ &\leqslant \sum_{n>l} e^{2\zeta n} \boldsymbol{E} \Big| \int e_m(x) e_{m'} \circ X_n^\kappa(x) \pi(dx) \Big|^2 . \end{aligned}$$

Observe

$$\boldsymbol{E} \Big| \int e_m(x) e_{m'} \circ X_n^\kappa(x) \pi(dx) \Big|^2 = \int e_{m'}^{(2)} P_\kappa^{(2),n} e_m^{(2)} ,$$

where

$$e_m^{(2)}(x,y) \stackrel{\text{def}}{=} e_m(x) \overline{e_m}(y) ,$$

$$\pi^{(2)}(dx,dy) \stackrel{\text{def}}{=} \pi(dx)\pi(dy) .$$

From (3.3), we see

$$\Big| \int e_{m'}^{(2)} P_\kappa^{(2),n} e_m^{(2)} d\pi^{(2)} \Big| = \Big| \int \Big(e_{m'}^{(2)} P_\kappa^{(2),n} - \Big(\int e_{m'}^{(2)} d\pi^{(2)} \Big) \Big) e_m^{(2)} d\pi^{(2)} \Big|$$

$$\leqslant \int \Big| e^{(2)}_{m'} P^{(2),n}_{\kappa} - \int e^{(2)}_{m'} d\pi^{(2)} \Big| \, d\pi^{(2)}$$
$$\leqslant C e^{-\beta n} \|e^{(2)}_{m'}\|_V \int V d\pi^{(2)} = C_V e^{-\beta n} .$$

This implies

$$\boldsymbol{P}[N^{\kappa}_{m,m'} > l] \leqslant C_V e^{(2\zeta - \beta) l} , \tag{8.1}$$

provided

$$2\zeta - \beta < 0 . \tag{8.2}$$

From now on, we make additional assumptions that ζ is small enough to satisfy

$$d + \frac{2\zeta - \beta}{\zeta} < 0 , \tag{8.3}$$

$$\frac{1}{\zeta q}(2\zeta - \beta) + 1 < -1 , \tag{8.4}$$

$$\frac{5d}{2} + \frac{2\zeta - \beta}{\zeta} < -1 . \tag{8.5}$$

Equations (8.1) and (8.2) imply that $\boldsymbol{P}(N^{\kappa}_{m,m'} < \infty) = 1$ and we have the estimate

$$\Big| \int e_m(x) e_{m'} \circ X^{\kappa}_n(x) \pi(dx) \Big| \leqslant e^{\zeta N^{\kappa}_{m,m'} - \zeta n} ,$$

$$\text{hence } \Big| \int f(x) g \circ X^{\kappa}_n(x) \pi(dx) \Big| \leqslant e^{-\zeta n} \sum_{m,m'} |\hat{f}_m| |\hat{g}_{m'}| e^{\zeta N^{\kappa}_{m,m'}} . \tag{8.6}$$

Moreover, using (8.1) and (8.3), we observe that

$$\begin{aligned} \boldsymbol{P}[K_{\kappa} > l] &\leqslant 2 \sum_{m,m' \in \mathbb{Z}^d_0, |m| > l} \boldsymbol{P}\Big[e^{\zeta N^{\kappa}_{m,m'}} > |m||m'| \Big] \\ &\leqslant 2 \sum_{m' \in \mathbb{Z}^2_0} |m'|^{\frac{2\zeta - \beta}{\zeta}} \sum_{m \in \mathbb{Z}^2_0, |m| > l} |m|^{\frac{2\zeta - \beta}{\zeta}} \\ &\lesssim \sum_{n=1}^{\infty} n^{\frac{2\zeta - \beta}{\zeta} + d - 1} \sum_{n > l}^{\infty} n^{\frac{2\zeta - \beta}{\zeta} + d - 1} \lesssim l^{d + \frac{2\zeta - \beta}{\zeta}} , \end{aligned} \tag{8.7}$$

where the constants in the inequalities are independent of κ. Hence $\boldsymbol{P}(K_{\kappa} < \infty) = 1$.

Noting that

$$e^{\zeta N^{\kappa}_{m,m'}} \leqslant \widehat{D}_{\kappa} |m||m'|$$

always holds, we conclude from (8.6) that

$$\Big| \int f(x) g \circ X^{\kappa}_n(x) \pi(dx) \Big| \leqslant \widehat{D}_{\kappa}(\underline{\omega}) e^{-\zeta n} \|f\|_{H^{\frac{d}{2}+2}} \|g\|_{H^{\frac{d}{2}+2}} .$$

Finally, the same arguments in Lemma 7.1 and Section 7.3 of [BBPS22] show that for any $s, q > 0$,

$$\Big| \int f(x) g \circ X^{\kappa}_n(x) \pi(dx) \Big| \leqslant \widehat{D}_{\kappa}(\underline{\omega}) e^{-(\frac{2s\zeta}{d+4}) n} \|f\|_{H^s} \|g\|_{H^s} .$$

Moreover, using (8.1) and (8.7), they show that

$$\boldsymbol{E}[\widehat{D}^q_{\kappa}] = \sum_{l=1}^{\infty} \boldsymbol{E}\big[1_{\{K_{\kappa} = l\}} \max_{|m|, |m'| \leqslant l} e^{\zeta q N^{\kappa}_{m,m'}} \big]$$

$$
\begin{aligned}
&\leqslant \sum_{l=1}^{\infty} \boldsymbol{P}[K_\kappa = l]^{\frac{1}{2}} \Big\| \max_{|m|,|m'|\leqslant l} e^{\zeta q N^\kappa_{m,m'}} \Big\|_{L^2} \\
&\lesssim \sum_{l=1}^{\infty} l^{d+\frac{2\zeta-\beta}{\zeta}} \Big(\sum_{|m|,|m'|\leqslant l} \|e^{\zeta q N^\kappa_{m,m'}}\|_{L^2} \Big) \\
&\leqslant \Big(1 + \frac{\zeta q}{\beta - 2\zeta(1+q)}\Big)^{\frac{1}{2}} \sum_{l=1}^{\infty} l^{\frac{5d}{2}+\frac{2\zeta-\beta}{\zeta}} < \infty\,,
\end{aligned}
$$

provided (8.4) and (8.5). This completes the proof of the theorem with the choice of $\gamma \stackrel{\text{def}}{=} \frac{2s\zeta}{d+4}$ which can be made independent of κ since the conditions for ζ, (8.2)–(8.5), are independent of κ. ☐

9. An explicit Lyapunov function for Pierrehumbert flows.

It was proved in Section 5 of [BCZG22] that the Pierrehumbert flows defined in Corollary 1.7 satisfy the assumptions 2.5–2.8. Then by Proposition 2.9, we see that Assumption 2.1–2.3 (and hence Corollary 1.7) must hold. However, in the case of Pierrehumbert flows, we can explicitly construct a simple Lyapunov function and verify Assumption 2.2 directly.

Proposition 9.1. *Consider the RDS of alternating shears defined in Corollary 1.7. If the flow amplitude A (in (1.8)) is sufficiently large, then there exists $s_*, p > 0$ such that the function V defined by*

$$
V(x,y) \stackrel{\text{def}}{=} |x-y|_\infty^{-p} \text{ on } \Delta(s_*)\,,
$$

and extended continuously to $\mathbb{T}^{d,(2)}$ is a Lyapunov function that satisfies Assumption 2.2. (Here $|z|_\infty = \max_i |z_i|$.)

Proof. In this proof, we use C as a generic constant that doesn't depend on A or κ. For a given $x \in \mathbb{T}^2$, we write x_1, x_2 as the first and second coordinates of x, respectively. Let $u(x,t)$ be defined as in Corollary 1.7.

First, we note that for some small $s_* \in (0, \frac{1}{2})$ and any $x, y \in \Delta(s_*)$, we can find $\check{x}, \check{y} \in \mathbb{R}^2$ such that $[\check{x}] = x$, $[\check{y}] = y$, and

$$
d(X_2^0(x), X_2^0(y)) = |\check{X}_2^0(\check{x}) - \check{X}_2^0(\check{y})| \leqslant CA^2 |\check{x} - \check{y}| = CA^2 d(x,y) < \frac{1}{2}\,,
$$

so with slight abuse of notation, we still write $x, y, X_2^0(x)$, and $X_2^0(y)$ for $\check{x}, \check{y}, \check{X}_2^0(\check{x})$, and $\check{X}_2^0(\check{y})$, respectively.

We aim to show that there is a constant $0 < \beta < 1$ such that, for any $(x,y) \in \Delta(s_*)$, we have

$$
\boldsymbol{E}[V(\Phi_2(x), \Phi_2(y))] \leqslant \beta V(x,y),
$$

where Φ_t denotes the flow map induced by u at time t.

Fix $(x,y) \in \Delta(s_*)$. First, note that we have $|x-y|_\infty \leqslant CA^2 |\Phi_2(x) - \Phi_2(y)|_\infty$ for some large A and therefore $V(\Phi_2(x), \Phi_2(y)) \leqslant (CA^2)^p V(x,y)$ almost surely. We break into two cases.

Case I: $|x_2 - y_2| \geqslant |x_1 - y_1|$. Let E_0 be the event

$$
E_0 = \{ |\Phi_1(x)_1 - \Phi_1(y)_1| < 2|x_2 - y_2| \}\,.
$$

Since $|\Phi_2(x)-\Phi_2(y)|_\infty \geqslant |\Phi_1(x)_1-\Phi_1(y)_1|_\infty$, we have

$$\begin{aligned}\boldsymbol{E}[V(\Phi_2(x),\Phi_2(y))] &\leqslant \boldsymbol{E}[|\Phi_1(x)_1-\Phi_1(y)_1|^{-p}]\\ &\leqslant \boldsymbol{E}[|\Phi_1(x)_1-\Phi_1(y)_1|^{-p}\mathbf{1}_{E_0}] + \boldsymbol{E}[|\Phi_1(x)_1-\Phi_1(y)_1|^{-p}\mathbf{1}_{E_0^c}]\\ &\leqslant (CA^2)^p V(x,y)P(E_0) + 2^{-p}V(x,y)\,. \end{aligned}\tag{9.1}$$

We will now estimate $P(E_0)$. For this, we use the explicit form of the vector field u to write

$$\Phi_1(x)_1-\Phi_1(y)_1 = x_1-y_1+A\sin(2\pi(x_2-\zeta_0)) - A\sin(2\pi(y_2-\zeta_0)).$$

Thus E_0 is contained in the event that

$$|x_1-y_1| + |A\sin(2\pi(x_2-\zeta_0)) - A\sin(2\pi(y_2-\zeta_0))| < 2|x_2-y_2|.$$

In view of the assumption $|x_2-y_2|\geqslant|x_1-y_1|$, the above inequality is implied by

$$|A\sin(2\pi(x_2-\zeta_0)) - A\sin(2\pi(y_2-\zeta_0))| < |x_2-y_2|.\tag{9.2}$$

Using the fundamental theorem of calculus, the left-hand side above can be written as the convolution $|\big(2\pi A\cos(2\pi\cdot) * \mathbf{1}_{[x_2,y_2]}\big)(\zeta_0)|$. Here $[x_2,y_2]$ denotes the smallest interval (mod $\mathbb{Z}$) with x_2 and y_2 as endpoints. Since the derivative of $\cos(2\pi\cdot)$ is bounded away from zero near the zeros of $\cos(2\pi\cdot)$, the same is true for the convolution (rescaling by $|x_2-y_2|$). This implies that the set of ζ_0 which satisfy (9.2) has measure at most CA^{-1}. This in turn implies $\boldsymbol{P}(E_0 \leqslant C/A)$.

Using this in (9.1) implies

$$\boldsymbol{E}[V(\Phi_2(x),\Phi_2(y))] \leqslant CA^{2p-1}V(x,y) + 2^{-p}V(x,y)\,.$$

Choosing $p\in(0,\frac{1}{2})$ and $A>0$ sufficiently large, we can ensure

$$\boldsymbol{E}[V(\Phi_2(x),\Phi_2(y))] \leqslant 2^{-p/2}V(x,y)\,,$$

as desired.

Case II: $|x_2-y_2|<|x_1-y_1|$. Let E_1 be the event that $|\Phi_1(x)_1-\Phi_1(y)_1|\leqslant A^{-1/2}|x_1-y_1|$. Then $\boldsymbol{P}[E_1]\leqslant CA^{-1/2}$ by the same argument as in the previous case. On the other hand, if E_2 is the event that $|\Phi_2(x)_2-\Phi_2(y)_2|\leqslant 2A^{1/2}|\Phi_1(x)_1-\Phi_1(y)_1|$, then we similarly have $\boldsymbol{P}[E_2]\leqslant CA^{-1/2}$. Putting these together, we conclude

$$\begin{aligned}\boldsymbol{E}[V(\Phi_2(x),\Phi_2(y))] &= \boldsymbol{E}[V(\Phi_2(x),\Phi_2(y))\mathbf{1}_{E_1\cup E_2}] + \boldsymbol{E}[V(\Phi_2(x),\Phi_2(y))\mathbf{1}_{(E_1\cup E_2)^c}]\\ &\leqslant (CA^2)^p V(x,y)(CA^{-1/2}) + 2^{-p}V(x,y),\end{aligned}$$

so for $p\in(0,\frac{1}{4})$ we can choose $A>0$ large enough to conclude as in the previous case. ∎

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