Optimal evolution of the free energy of interacting gazes and applications-Proofs

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Contents

1 Main inequality between two configurations of interacting gases

Theorem 1.1 *Let Ω be open, bounded and convex subset of $I\!R^n$, let $F:[0,\infty)\to I\!R$ be differentiable function on $(0,\infty)$ with $F(0)=0$ and $x\mapsto x^nF(x^{-n})$ convex and non-increasing, and let $P_F(x):=xF'(x)-F(x)$ be its associated pressure function. Let $V:I\!R^n\to I\!R$ be a C^2-confinement potential with $D^2V\geq\lambda I$, and let W be an even C^2-interaction potential with $D^2W\geq\nu I$ where $\lambda,\nu\in I\!R$, and I denotes the identity map. Then, for any Young function $c:I\!R^n\to I\!R$, we have for all probability densities*

ρ_0 and ρ_1 on Ω, satisfying supp $\rho_0 \subset \Omega$ and $P_F(\rho_0) \in W^{1,\infty}(\Omega)$,

$$\begin{aligned} &\mathrm{H}_{V+c}^{F,W}(\rho_0|\rho_1) + \frac{\lambda+\nu}{2} W_2^2(\rho_0, \rho_1) - \frac{\nu}{2} |\mathrm{b}(\rho_0) - \mathrm{b}(\rho_1)|^2 \\ &\quad \le \mathrm{H}_{c+\nabla V\cdot x}^{-nP_F, 2x\cdot\nabla W}(\rho_0) + \int_\Omega \rho_0 c^* \left(-\nabla \left(F'(\rho_0) + V + W \star \rho_0\right)\right) \, dx. \end{aligned} \tag{1}$$

Furthermore, equality holds in (1) whenever $\rho_0 = \rho_1 = \rho_{V+c}$, where the latter satisfies

$$\nabla \left(F'(\rho_{V+c}) + V + c + W \star \rho_{V+c}\right) = 0 \quad a.e. \tag{2}$$

In particular, we have for any probability density ρ on Ω with supp $\rho \subset \Omega$ and $P_F(\rho) \in W^{1,\infty}(\Omega)$,

$$\begin{aligned} &\mathrm{H}_{V-x\cdot\nabla V}^{F+nP_F,\, W-2x\cdot\nabla W}(\rho) + \frac{\lambda+\nu}{2} W_2^2(\rho, \rho_{V+c}) - \frac{\nu}{2} |\mathrm{b}(\rho_0) - \mathrm{b}(\rho_{V+c})|^2 \\ &\quad \le \int_\Omega \rho c^\star \left(-\nabla \left(F'(\rho) + V + W \star \rho\right)\right) \, dx - \mathrm{H}^{P_F, W}(\rho_{V+c}) + K_{V+c}, \end{aligned} \tag{3}$$

where K_{V+c} is such that

$$F'(\rho_{V+c}) + V + c + W \star \rho_{V+c} = K_{V+c} \text{ while } \int_\Omega \rho_{V+c} = 1. \tag{4}$$

The proof is based on the recent advances in the theory of mass transport as developed by Brenier [7], Gangbo-McCann [15], [16], Caffarelli [8] and many others. For a survey, see Villani [27]. Here is a brief summary of the needed results.

Fix a non-negative C^1, strictly convex function $d : I\!R^n \to I\!R$ such that $d(0) = 0$. Given two probability measures μ and ν on $I\!R^n$, the minimum cost for transporting μ onto ν is given by

$$W_d(\mu, \nu) := \inf_{\gamma \in \Gamma(\mu,\nu)} \int_{R^n \times R^n} d(x - y) d\gamma(x, y), \tag{5}$$

where $\Gamma(\mu, \nu)$ is the set of Borel probability measures with marginals μ and ν, respectively. When $d(x) = |\,x\,|^2$, we have that $W_d = W_2^2$, where W_2 is the Wasserstein distance. We say that a Borel map $T : I\!R^n \to I\!R^n$ pushes μ forward to ν, if $\mu(T^{-1}(B)) = \nu(B)$ for any Borel set $B \subset I\!R^n$. The map T is then said to be d-optimal if

$$W_d(\mu, \nu) = \int_{R^n} d(x - Tx) d\mu(x) = \inf_S \int_{R^n} d(x - Sx) d\mu(x), \tag{6}$$

where the infimum is taken over all Borel maps $S : I\!R^n \to I\!R^n$ that push μ forward to ν.

For quadratic cost functions $d(z) = \frac{1}{2}|z|^2$, Brenier [7] characterized the optimal transport map T as the gradient of a convex function. An analogous result holds for general cost functions d, provided convexity is replaced by an appropriate notion of d-concavity. See [15], [8] for details.

Here is the lemma which leads to our main inequality (1). It is essentially a compendium of various observations by several authors. It describes the evolution of a generalized energy functional along optimal transport. The key idea lying behind it, is the concept of *displacement convexity* introduced by McCann [21]. For generalized cost functions, and when $V = 0$, it was first obtained by Otto [23] for the Tsallis entropy functionals and by Agueh [1] in general. The case of a nonzero confinement potential V and an interaction potential W was included in [13], [9]. Here, we state the results when the cost function is quadratic, $d(x) = |\, x\,|^2$.

Lemma 1.2 *Let $\Omega \subset I\!\!R^n$ be open, bounded and convex, and let ρ_0 and ρ_1 be probability densities on Ω, with $supp\, \rho_0 \subset \Omega$, and $P_F(\rho_0) \in W^{1,\infty}(\Omega)$. Let T be the optimal map that pushes $\rho_0 \in \mathcal{P}_a(\Omega)$ forward to $\rho_1 \in \mathcal{P}_a(\Omega)$ for the quadratic cost $d(x) = |\, x\,|^2$. Then*

- *Assume $F : [0, \infty) \to I\!\!R$ is differentiable on $(0, \infty)$, $F(0) = 0$ and $x \mapsto x^n F(x^{-n})$ is convex and non-increasing, then the following inequality holds for the internal energy:*
$$\mathrm{H}^F(\rho_1) - \mathrm{H}^F(\rho_0) \geq \int_\Omega \rho_0 (T - I) \cdot \nabla \left(F'(\rho_0)\right)\, dx. \tag{7}$$

- *Assume $V : I\!\!R^n \to I\!\!R$ is such that $D^2V \geq \lambda I$ for some $\lambda \in I\!\!R$, then the potential energy satisfies*
$$\mathrm{H}_V(\rho_1) - \mathrm{H}_V(\rho_0) \geq \int_\Omega \rho_0 (T - I) \cdot \nabla V\, dx + \frac{\lambda}{2} W_2^2(\rho_0, \rho_1). \tag{8}$$

- *Assume $W : I\!\!R^n \to I\!\!R$ is even, and $D^2W \geq \nu I$ for some $\nu \in I\!\!R$, then the interaction energy satisfies*
$$\begin{aligned}\mathrm{H}^W(\rho_1) - \mathrm{H}^W(\rho_0) \quad \geq \quad & \int_\Omega \rho_0 (T - I) \cdot \nabla (W \star \rho_0)\, dx \\ & + \frac{\nu}{2} \left(W_2^2(\rho_0, \rho_1) - |\mathrm{b}(\rho_0) - b(\rho_1)|^2 \right).\end{aligned} \tag{9}$$

Proof: If T ($T = \nabla\psi$, where ψ is convex) is the optimal map that pushes $\rho_0 \in \mathcal{P}_a(\Omega)$ forward to $\rho_1 \in \mathcal{P}_a(\Omega)$ for the quadratic cost $d(x) = |\, x\,|^2$, we have (see McCann [21]) that $\nabla T(x)$ is diagonalizable with positive eigenvalues for ρ_0 a.e., and the Monge-Ampère equation
$$0 \neq \rho_0(x) = \rho_1\left(T(x)\right) \det \nabla T(x) \tag{10}$$
holds for ρ_0 a.e. So, $\rho_1\left(T(x)\right) \neq 0$ for ρ_0 a.e. Here, $\nabla T(x) = \nabla^2\psi(x)$ denotes the derivative in the sense of Aleksandrov of ψ.

(1) The following proof of the internal energy inequality (7) is taken from Agueh [1]. Set
$$A(x) = x^n F(x^{-n}).$$
Since A is non-increasing by assumption, then P_F is non-negative and $x \mapsto \frac{F(x)}{x}$ is also non-increasing. We use that $F(0) = 0$, $T_\# \rho_0 = \rho_1$ and (10), to have that
$$\begin{aligned}\mathrm{H}^F(\rho_1) = \int_{[\rho_1 \neq 0]} \frac{F\left(\rho_1(y)\right)}{\rho_1(y)} \rho_1(y)\, \mathrm{d}y \quad &= \quad \int_\Omega \frac{F\left(\rho_1(Tx)\right)}{\rho_1(Tx)} \rho_0(x)\, \mathrm{d}x \\ &= \quad \int_\Omega F\left(\frac{\rho_0(x)}{\det \nabla T(x)}\right) \det \nabla T(x)\, \mathrm{d}x.\end{aligned} \tag{11}$$

Comparing the geometric mean $(\det \nabla T(x))^{1/n}$ to the arithmetic mean $\frac{\operatorname{tr} \nabla T(x)}{d}$, we have that

$$\frac{1}{\det \nabla T(x)} \geq \left(\frac{n}{\operatorname{tr} \nabla T(x)} \right)^n,$$

then, we use that $x \mapsto \frac{F(x)}{x}$ is non-decreasing, to get that

$$F\left(\frac{\rho_0(x)}{det\, \nabla T(x)}\right) \det \nabla T(x) \geq \Lambda^n F\left(\frac{\rho_0(x)}{\Lambda^n}\right) = \rho_0(x) A\left(\frac{\Lambda}{\rho_0(x)^{1/n}}\right), \tag{12}$$

where

$$\Lambda := \frac{\operatorname{tr} \nabla T(x)}{n}.$$

Next, we use that $A'(x) = -nx^{n-1}P_F(x^{-n})$, and that A is convex, to obtain that

$$\begin{aligned}
\rho_0(x) A\left(\frac{\Lambda}{\rho_0(x)^{1/n}}\right) &\geq \rho_0(x) \left[A\left(\frac{1}{\rho_0(x)^{1/n}}\right) + A'\left(\frac{1}{\rho_0(x)^{1/n}}\right)\left(\frac{\Lambda - 1}{\rho_0(x)^{1/n}}\right)\right] \\
&= \rho_0(x) \left[\frac{F(\rho_0(x))}{\rho_0(x)} - n(\Lambda - 1)\frac{P_F(\rho_0(x))}{\rho_0(x)} \right] \\
&= F(\rho_0(x)) - P_F(\rho_0(x)) \operatorname{tr}(\nabla T(x) - I).
\end{aligned} \tag{13}$$

We combine (11) - (13), to conclude that

$$\begin{aligned}
\mathrm{H}^F(\rho_1) - \mathrm{H}^F(\rho_0) &\geq -\int_\Omega P_F(\rho_0(x))\ tr\,(\nabla T(x) - I)\,\mathrm{d}x \\
&= -\int_\Omega P_F(\rho_0(x))\ \operatorname{div}(T(x) - I)\,\mathrm{d}x \\
&\geq \int_\Omega \rho_0\,(T - I) \cdot \nabla\left(F'(\rho_0)\right)\,\mathrm{d}x.
\end{aligned} \tag{14}$$

(2) To prove (8), use the fact that $D^2 V \geq \lambda I$, that is,

$$V(b) - V(a) \geq \nabla V(a) \cdot (b - a) + \frac{\lambda}{2} |\, a - b\,|^2 \tag{15}$$

for all $a, b \in \mathbb{R}^n$, and set $a = x$ and $b = T(x)$ in (15), where $T_{\#}\rho_0 = \rho_1$ is the optimal map in (6).

(3) The following proof of (9) is taken from Cordero-Gangbo-Houdré [13]. Indeed, following [13], we write the interaction energy as follows:

$$\begin{aligned}
\mathrm{H}^W(\rho_1) &= \frac{1}{2}\int_{\Omega\times\Omega} W(x - y)\rho_1(x)\rho_1(y)\,\mathrm{d}x\mathrm{d}y \\
&= \frac{1}{2}\int_{\Omega\times\Omega} W(T(x) - T(y))\rho_0(x)\rho_0(y)\,\mathrm{d}x\mathrm{d}y \\
&= \frac{1}{2}\int_{\Omega\times\Omega} W\left(x - y + (T - I)(x) - (T - I)(y)\right)\rho_0(x)\rho_0(y)\,\mathrm{d}x\mathrm{d}y
\end{aligned}$$

$$\begin{aligned}
&\geq \frac{1}{2}\int_{\Omega\times\Omega}\left[W(x-y)+\nabla W(x-y)\cdot((T-I)(x)-(T-I)(y))\,\rho_0(x)\rho_0(y)\right]\mathrm{d}x\mathrm{d}y\\
&\quad+\frac{\nu}{4}\int_{\Omega\times\Omega}|(T-I)(x)-(T-I)(y)|^2\rho_0(x)\rho_0(y)\,\mathrm{d}x\mathrm{d}y\\
&= \mathrm{H}^W(\rho_0)+\frac{1}{2}\int_{\Omega\times\Omega}\nabla W(x-y)\cdot((T-I)(x)-(T-I)(y))\,\rho_0(x)\rho_0(y)\,\mathrm{d}x\mathrm{d}y\\
&\quad+\frac{\nu}{4}\int_{\Omega\times\Omega}|(T-I)(x)-(T-I)(y)|^2\rho_0(x)\rho_0(y)\,\mathrm{d}x\mathrm{d}y, \qquad (16)
\end{aligned}$$

where we used above that $D^2W \geq \nu I$. The last term of the subsequent inequality can be written as:

$$\begin{aligned}
&\int_{\Omega\times\Omega}|(T-I)(x)-(T-I)(y)|^2\rho_0(x)\rho_0(y)\,\mathrm{d}x\mathrm{d}y\\
&\quad= 2\int_{\Omega}|(T-I)(x)|^2\rho_0(x)\,\mathrm{d}x - 2\Big|\int_{R^n}(T-I)(x)\rho_0(x)\,\mathrm{d}x\Big|^2\\
&\quad= 2\int_{\Omega}|(T-I)(x)|^2\rho_0(x)\,\mathrm{d}x - 2|\mathrm{b}(\rho_1)-\mathrm{b}(\rho_0)|^2. \qquad (17)
\end{aligned}$$

And since ∇W is odd (because W is even), we get for the second term of (16)

$$\begin{aligned}
&\int_{\Omega\times\Omega}\left[\nabla W(x-y)\cdot((T-I)(x)-(T-I)(y))\right]\rho_0(x)\rho_0(y)\,\mathrm{d}x\mathrm{d}y\\
&\quad= 2\int_{\Omega\times\Omega}\nabla W(x-y)\cdot(T-I)(x)\rho_0(x)\rho_0(y)\,\mathrm{d}x\mathrm{d}y\\
&\quad= 2\int_{\Omega\times\Omega}\rho_0(T-I)\cdot\nabla(W\star\rho_0)\,\mathrm{d}x. \qquad (18)
\end{aligned}$$

Combining (16) - (18), we obtain that

$$\begin{aligned}
&\mathrm{H}^W(\rho_1)-\mathrm{H}^W(\rho_0)\\
&\quad\geq \int_{\Omega\times\Omega}\rho_0(T-I)\cdot\nabla(W\star\rho_0)\,\mathrm{d}x+\frac{\nu}{2}\left(\int_{\Omega}|(T-I)(x)|^2\rho_0 dx-|\mathrm{b}(\rho_0)-\mathrm{b}(\rho_1)|^2\right).
\end{aligned}$$

This proves (9).

Proof of Theorem 1.1: Adding (7), (8) and (9), one gets

$$\begin{aligned}
&\mathrm{H}_V^{F,W}(\rho_0)-\mathrm{H}_V^{F,W}(\rho_1)+\frac{\lambda+\nu}{2}W_2^2(\rho_0,\rho_1)-\frac{\nu}{2}|\mathrm{b}(\rho_0)-\mathrm{b}(\rho_1)|^2 \qquad (19)\\
&\quad\leq \int_{\Omega}(x-Tx)\cdot\rho_0\nabla\left(F'(\rho_0)+V+W\star\rho_0\right)\,\mathrm{d}x.
\end{aligned}$$

Since $\rho_0\nabla(F'(\rho_0)) = \nabla\left(P_F(\rho_0)\right)$, we integrate by part $\int_\Omega \rho_0\nabla\left(F'(\rho_0)\right)\cdot x\,\mathrm{d}x$, and obtain that

$$\int_{\Omega} x\cdot\nabla(F'(\rho_0)+V+W\star\rho_0)\rho_0 = \mathrm{H}_{x\cdot\nabla V}^{-nP_F,\,2x\cdot\nabla W}(\rho_0).$$

This leads to

$$\begin{aligned}
&\mathrm{H}_V^{F,W}(\rho_0)-\mathrm{H}_V^{F,W}(\rho_1)+\frac{\lambda+\nu}{2}W_2^2(\rho_0,\rho_1)-\frac{\nu}{2}|\mathrm{b}(\rho_0)-\mathrm{b}(\rho_1)|^2 \qquad (20)\\
&\leq \mathrm{H}_{x\cdot\nabla V}^{-nP_F,\,2x\cdot\nabla W}(\rho_0)-\int_{\Omega}\rho_0\nabla\left(F'(\rho_0)+V+W\star\rho_0\right)\cdot T(x)\,\mathrm{d}x.
\end{aligned}$$

Now, use Young's inequality to get

$$
\begin{aligned}
&-\nabla\left(F'\left(\rho_0(x)\right)+V(x)+(W\star\rho_0)(x)\right)\cdot T(x) \\
&\quad\le c\left(T(x)\right)+c^\star\left(-\nabla\left(F'(\rho_0(x))+V(x)+(W\star\rho_0)(x)\right)\right),
\end{aligned}
\tag{21}
$$

and deduce that

$$
\begin{aligned}
&\mathrm{H}_V^{F,W}(\rho_0)-\mathrm{H}_V^{F,W}(\rho_1)+\frac{\lambda+\mu}{2}W_2^2(\rho_0,\rho_1)-\frac{\nu}{2}|\mathrm{b}(\rho_0)-\mathrm{b}(\rho_1)|^2 \\
&\quad\le \mathrm{H}_{x\cdot\nabla V}^{-nP_F,2x\cdot\nabla W}(\rho_0)+\int_\Omega \rho_0 c^\star\left(-\nabla\left(F'(\rho_0)+V+W\star\rho_0\right)\right)+\int_\Omega c(Tx)\rho_0\,\mathrm{d}x.
\end{aligned}
\tag{22}
$$

Finally, use again that T pushes ρ_0 forward to ρ_1, to rewrite the last integral on the right hand side of (22) as $\int_\Omega c(y)\rho_1(y)dy$ to obtain (1).
Now, set $\rho_0=\rho_1:=\rho_{V+c}$ in (20). We have that $T=I$, and equality then holds in (20). Therefore, equality holds in (1) whenever equality holds in (21), where $T(x)=x$. This occurs when (2) is satisfied.
(3) is straightforward when choosing $\rho_0:=\rho$ and $\rho_1:=\rho_{V+c}$ in (1).

2 Optimal Euclidean Sobolev inequalities

2.1 Euclidean Log-Sobolev inequalities

The following optimal Euclidean p-Log Sobolev inequality was established by Beckner [3] in the case where $p=1$, by Del Pino- Dolbeault [14] for $1<p<n$, and independently by Gentil for all $p>1$.

Corollary 2.1 *(*General Euclidean Log-Sobolev inequality*)*
Let $\Omega\subset I\!R^n$ be open bounded and convex, and let $c: I\!R^n\to I\!R$ be a Young functional such that its conjugate $c^\star$ is p-homogeneous for some $p>1$. Then,

$$
\int_{R^n}\rho\ln\rho\,dx\le\frac{n}{p}\ln\left(\frac{p}{ne^{p-1}\sigma_c^{p/n}}\int_{R^n}\rho c^\star\left(-\frac{\nabla\rho}{\rho}\right)\,dx\right),
\tag{23}
$$

for all probability densities ρ on $I\!R^n$, such that $\operatorname{supp}\rho\subset\Omega$ and $\rho\in W^{1,\infty}(I\!R^n)$. Here, $\sigma_c:=\int_{R^n}e^{-c}\,dx$. Moreover, equality holds in (23) if $\rho(x)=K_\lambda e^{-\lambda^q c(x)}$ for some $\lambda>0$, where $K_\lambda=\left(\int_{R^n}e^{-\lambda^q c(x)}\,dx\right)^{-1}$ and q is the conjugate of p ($\frac{1}{p}+\frac{1}{q}=1$).

Proof: Use $F(x)=x\ln(x)$ and $V=W=0$ in (3). Note that $P_F(x)=x$, and then, $H^{P_F}(\rho)=1$ for any $\rho\in\mathcal{P}_a(I\!R^n)$. So, $\rho_c(x)=\frac{e^{-c(x)}}{\sigma_c}$. We then have for $\rho\in\mathcal{P}_a(I\!R^n)\cap W^{1,\infty}(I\!R^n)$ such that $\operatorname{supp}\rho\subset\Omega$,

$$
\int_\Omega\rho\ln\rho\,\mathrm{d}x\le\int_{R^n}\rho c^\star\left(-\frac{\nabla\rho}{\rho}\right)\,\mathrm{d}x-n-\ln\left(\int_{R^n}e^{-c(x)}\,\mathrm{d}x\right),
\tag{24}
$$

with equality when $\rho = \rho_c$.

Now assume that $c^\star$ is p-homogeneous and set $\Gamma_\rho^c = \int_{R^n} \rho c^\star \left(-\frac{\nabla\rho}{\rho}\right) \mathrm{d}x$. Using $c_\lambda(x) := c(\lambda x)$ in (24), we get for $\lambda > 0$ that

$$\int_{R^n} \rho \ln\rho \,\mathrm{d}x \le \int_{R^n} \rho c^\star \left(-\frac{\nabla\rho}{\lambda\rho}\right) \mathrm{d}x + n\ln\lambda - n - \ln\sigma_c, \tag{25}$$

for all $\rho \in \mathcal{P}_a(I\!R^n)$ satisfying $\operatorname{supp}\rho \subset \Omega$ and $\rho \in W^{1,\infty}(\Omega)$. Equality holds in (25) if $\rho_\lambda(x) = \left(\int_{R^n} e^{-\lambda^q c(x)}\,\mathrm{d}x\right)^{-1} e^{-\lambda^q c(x)}$. Hence

$$\int_{R^n} \rho\ln\rho\,\mathrm{d}x \le -n - \ln\sigma_c + \inf_{\lambda>0}\left(G_\rho(\lambda)\right),$$

where

$$G_\rho(\lambda) = n\ln(\lambda) + \frac{1}{\lambda^p}\int_{R^n} \rho c^\star\left(-\frac{\nabla\rho}{\rho}\right) = n\ln(\lambda) + \frac{\Gamma_\rho^c}{\lambda^p}.$$

The infimum of $G_\rho(\lambda)$ over $\lambda > 0$ is attained at $\bar\lambda_\rho = \left(\frac{p}{n}\Gamma_\rho^c\right)^{1/p}$. Hence

$$\begin{aligned}
\int_{R^n} \rho\ln\rho\,\mathrm{d}x &\le G_\rho(\bar\lambda_\rho) - n - \ln(\sigma_c) \\
&= \frac{n}{p}\ln\left(\frac{p}{n}\Gamma_\rho^c\right) + \frac{n}{p} - n - \ln(\sigma_c) \\
&= \frac{n}{p}\ln\left(\frac{p}{ne^{p-1}\sigma_c^{p/n}}\Gamma_\rho^c\right),
\end{aligned}$$

for all probability densities ρ on $I\!R^n$, such that $\operatorname{supp}\rho \subset \Omega$, and $\rho \in W^{1,\infty}(I\!R^n)$.

Corollary 2.2 *(*Optimal Euclidean p-Log Sobolev inequality*)*

$$\int_{R^n} |\, f\,|^p \ln(|\, f\,|^p)\, dx \le \frac{n}{p}\ln\left(C_p\int_{R^n} |\,\nabla f\,|^p\, dx\right), \tag{26}$$

holds for all $p \ge 1$, and for all $f \in W^{1,p}(I\!R^n)$ such that $\|\, f\,\|_p = 1$, where

$$C_p := \begin{cases} \left(\frac{p}{n}\right)\left(\frac{p-1}{e}\right)^{p-1}\pi^{-\frac{p}{2}}\left[\frac{\Gamma(\frac{n}{2}+1)}{\Gamma(\frac{n}{q}+1)}\right]^{\frac{p}{n}} & \text{if } \ p>1, \\ \\ \frac{1}{n\sqrt{\pi}}\left[\Gamma(\frac{n}{2}+1)\right]^{\frac{1}{n}} & \text{if } \ p=1, \end{cases} \tag{27}$$

and q is the conjugate of p ($\frac{1}{p}+\frac{1}{q}=1$).

For $p > 1$, equality holds in (26) for $f(x) = Ke^{-\lambda^q\frac{|\,x-\bar x\,|^q}{q}}$ for some $\lambda > 0$ and $\bar x \in I\!R^n$, where $K = \left(\int_{R^n} e^{-(p-1)|\,\lambda x\,|^q}\, dx\right)^{-1/p}$.

Proof: First assume that $p > 1$, and set $c(x) = (p-1)|\, x\,|^q$ and $\rho = |\, f\,|^p$ in (23), where $f \in C_c^\infty(I\!R^n)$ and $\|\, f\,\|_p = 1$. We have that $c^\star(x) = \frac{|\, x\,|^p}{p^p}$, and then, $\int_{R^n} \rho c^* \left(-\frac{\nabla\rho}{\rho}\right) \mathrm{d}x = \int_{R^n} |\, \nabla f\,|^p \,\mathrm{d}x$. Therefore, (23) reads as

$$\int_{R^n} |\, f\,|^p \ln(|\, f\,|^p) \,\mathrm{d}x \le \frac{n}{p} \ln\left(\frac{p}{ne^{p-1}\sigma_c^{p/n}} \int_{R^n} |\, \nabla f\,|^p \,\mathrm{d}x\right). \tag{28}$$

Now, it suffices to note that

$$\sigma_c := \int_{R^n} e^{-(p-1)|\, x\,|^q} \,\mathrm{d}x = \frac{\pi^{\frac{n}{2}}\Gamma\left(\frac{n}{q}+1\right)}{(p-1)^{\frac{n}{q}}\Gamma\left(\frac{n}{2}+1\right)}. \tag{29}$$

To prove the case where $p = 1$, it is sufficient to apply the above to $p_\epsilon = 1+\epsilon$ for some arbitrary $\epsilon > 0$. Note that

$$C_{p\epsilon} = \left(\frac{1+\epsilon}{n}\right)\left(\frac{\epsilon}{e}\right)^\epsilon \pi^{-\frac{1+\epsilon}{2}} \left[\frac{\Gamma(\frac{n}{2}+1)}{\Gamma(\frac{n\epsilon}{1+\epsilon}+1)}\right]^{\frac{1+\epsilon}{n}},$$

so that when ϵ go to 0, we have

$$\lim_{\epsilon\to 0} C_{p_\epsilon} = \frac{1}{n\sqrt{\pi}}\left[\Gamma\left(\frac{n}{2}+1\right)\right]^{\frac{1}{n}} = C_1.$$

2.2 Sobolev and Gagliardo-Nirenberg inequalities

Corollary 2.3 *(*Gagliardo-Nirenberg inequalities*)*
Let $1 < p < n$ and $r \in \left(0, \frac{np}{n-p}\right)$ such that $r \ne p$. Set $\gamma := \frac{1}{r} + \frac{1}{q}$, where $\frac{1}{p} + \frac{1}{q} = 1$. Then, for any $f \in W^{1,p}(I\!R^n)$ we have

$$\|f\|_r \le C(p,r)\|\nabla f\|_p^\theta \,\|f\|_{r\gamma}^{1-\theta}, \tag{30}$$

where θ is given by

$$\frac{1}{r} = \frac{\theta}{p^*} + \frac{1-\theta}{r\gamma}, \tag{31}$$

$p^* = \frac{np}{n-p}$ and where the best constant $C(p,r) > 0$ can be obtained by scaling.

Proof: Let $F(x) = \frac{x^\gamma}{\gamma-1}$, where $1 \ne \gamma > 1 - \frac{1}{n}$, which follows from the fact that $p \ne r \in \left(0, \frac{np}{n-p}\right)$. For this value of γ, the function F satisfies the conditions of Theorem 1.1. Let $c(x) = \frac{r\gamma}{q}|\, x\,|^q$ so that $c^*(x) = \frac{1}{p(r\gamma)^{p-1}}|\, x\,|^p$, and set $V = W = 0$. Inequality (3) then gives for all $f \in C_c^\infty(I\!R^n)$ such that $\|\, f\,\|_r = 1$,

$$\left(\frac{1}{\gamma-1}+n\right)\int_{R^n} |\, f\,|^{r\gamma} \le \frac{r\gamma}{p}\int_{R^n} |\, \nabla f\,|^p - H^{P_F}(\rho_\infty) + C_\infty. \tag{32}$$

where $\rho_\infty = h_\infty^r$ satisfies

$$-\nabla h_\infty(x) = x|\,x\,|^{q-2}h^{\frac{r}{p}}(x) \;\text{ a.e.}, \tag{33}$$

and where C_∞ insures that $\int h_\infty^r = 1$. The constants on the right hand side of (32) are not easy to calculate, so one can obtain θ and the best constant by a standard scaling procedure. Namely, write (32) as

$$\frac{r\gamma}{p}\frac{\|\nabla f\|_p^p}{\|f\|_r^p} - \left(\frac{1}{\gamma-1}+n\right)\frac{\|f\|_{r\gamma}^{r\gamma}}{\|f\|_r^{r\gamma}} \geq H^{P_F}(\rho_\infty) - C_\infty =: C, \tag{34}$$

for some constant C. Then apply (34) to $f_\lambda(x) = f(\lambda x)$ for $\lambda > 0$. A minimization over λ gives the required constant.

The limiting case where r is the critical Sobolev exponent $r = p^* = \frac{np}{n-p}$ (and then $\gamma = 1 - \frac{1}{n}$) leads to the Sobolev inequalities:

Corollary 2.4 *(*Sobolev inequalities*)*
If $1 < p < n$, then for any $f \in W^{1,p}(I\!\!R^n)$,

$$\|\,f\,\|_{p^*} \leq C(p,n)\|\,\nabla f\,\|_p \tag{35}$$

for some constant $C(p,n) > 0$.

Proof: It follows directly from (32), by using $\gamma = 1 - \frac{1}{n}$ and $r = p^*$.

Note that the scaling argument cannot be used here to compute the best constant $C(p,n)$ in (35), since $\|\,\nabla f_\lambda\,\|_p^p = \lambda^{p-n}\|\,\nabla f\,\|_p^p$ and $\|\,f_\lambda\,\|_r^p = \lambda^{p-n}\|\,f\,\|_r^p$ scale the same way in (34). Instead, one can proceed directly from (32) to have that

$$\|\,f\,\|_{p^*} = 1 \leq \left(\frac{r\gamma}{p\,[H^{P_F}(\rho_\infty) - C_\infty]}\right)^{1/p}\|\,\nabla f\,\|_p = \left(\frac{p^*(n-1)}{np\,[H^{P_F}(\rho_\infty) - C_\infty]}\right)^{1/p}\|\,\nabla f\,\|_p,$$

which shows that

$$C(p,n) = \left(\frac{p^*(n-1)}{np\,[H^{P_F}(\rho_\infty) - C_\infty]}\right)^{1/p}, \tag{36}$$

where $\rho_\infty = h_\infty^{p^*} = \left(\frac{p^*}{nq}|\,x\,|^q - \frac{C_\infty}{n-1}\right)^{-n}$ is obtained from (33), and C_∞ can be found using that ρ_∞ is a probability density,

$$C_\infty = (1-n)\left[\int_{R^n}\left(\frac{p^*}{nq}|\,x\,|^q + 1\right)^{-n}\,\mathrm{d}x\right]^{p/n}. \tag{37}$$

3 Optimal geometric inequalities

3.1 HWBI inequalities

We now establish HWBI inequalities relating the total energy of two arbitrary probability densities, their Wasserstein distance, their barycenters and their entropy production functional, and we deduce extensions of various powerful inequalities by Gross [18], Bakry-Emery[2], Talagrand [26], Otto-Villani [24], Cordero[12] and others.

Theorem 3.1 *(*HWBI inequality*)*
Let Ω be an open, bounded and convex subset of $I\!R^n$. Let $F:[0,\infty)\to I\!R$ be a differentiable function on $(0,\infty)$ with $F(0)=0$ and $x\mapsto x^nF(x^{-n})$ convex and non-increasing, and let $P_F(x):=xF'(x)-F(x)$ be its associated pressure function. Let $U:I\!R^n\to I\!R$ be a C^2-confinement potential with $D^2U\ge \mu I$, and let W be an even C^2-interaction potential with $D^2W\ge\nu I$ where $\mu,\nu\in I\!R$. Then we have for all probability densities ρ_0 and ρ_1 on Ω satisfying supp $\rho_0\subset\Omega$ and $P_F(\rho_0)\in W^{1,\infty}(\Omega)$,

$$\mathrm{H}_U^{F,W}(\rho_0|\rho_1)\le W_2(\rho_0,\rho_1)\sqrt{I_2(\rho_0|\rho_U)}-\frac{\mu+\nu}{2}W_2^2(\rho_0,\rho_1)+\frac{\nu}{2}|\mathrm{b}(\rho_0)-\mathrm{b}(\rho_1)|^2. \tag{38}$$

The proof of Theorem 3.1 relies on the following proposition.

Proposition 3.1 *Under the above hypothesis on Ω and F, let $U,W:I\!R^n\to I\!R$ be C^2-functions with $D^2U\ge\mu I$ and $D^2W\ge\nu I$, where $\mu,\nu\in I\!R$, and W is even. Then for any $\sigma>0$, we have for all probability densities ρ_0 and ρ_1 on Ω, satisfying supp $\rho_0\subset\Omega$, and $P_F(\rho_0)\in W^{1,\infty}(\Omega)$,*

$$\mathrm{H}_U^{F,W}(\rho_0|\rho_1)+\frac{1}{2}(\mu+\nu-\frac{1}{\sigma})W_2^2(\rho_0,\rho_1)-\frac{\nu}{2}|\mathrm{b}(\rho_0)-\mathrm{b}(\rho_1)|^2\le\frac{\sigma}{2}I_2(\rho_0|\rho_U), \tag{39}$$

Proof: Use (1) with $c(x)=\frac{1}{2\sigma}|\,x\,|^2$, $V=U-c$ and $\lambda=\mu-\frac{1}{\sigma}$ to obtain

$$\begin{aligned}\mathrm{H}_U^{F,W}(\rho_0|\rho_1)\quad&+\quad\frac{1}{2}(\mu+\nu-\frac{1}{\sigma})W_2^2(\rho_0,\rho_1)+\frac{\nu}{2}|\mathrm{b}(\rho_0)-\mathrm{b}(\rho_1)|^2\\&\le\quad\mathrm{H}_{c+\nabla(U-c)\cdot x}^{-nP_F,2x\cdot\nabla W}(\rho_0)+\int_\Omega\rho_0c^*\left(-\nabla\left(F'(\rho_0)+U-c+W\star\rho_0\right)\right)\mathrm{d}x.\end{aligned} \tag{40}$$

By elementary computations, we have

$$\begin{aligned}&\int_\Omega\rho_0c^*\left(-\nabla\left(F'\circ\rho_0+U-c+W\star\rho_0\right)\right)\mathrm{d}x\\&\quad=\frac{\sigma}{2}\int_\Omega\rho_0\Big|\,\nabla\left(F'(\rho_0)+U+W\star\rho_0\right)\,\Big|^2\mathrm{d}x+\frac{1}{2\sigma}\int_\Omega\rho_0|\,x\,|^2\,\mathrm{d}x-\int_\Omega\rho_0x\cdot\nabla\left(F'(\rho_0)\right)\mathrm{d}x\\&\qquad-\int_\Omega\rho_0x\cdot\nabla U\,\mathrm{d}x-\int_\Omega\rho_0x\cdot\nabla(W\star\rho_0)\,\mathrm{d}x,\end{aligned}$$

and

$$\mathrm{H}_{c+\nabla(U-c)\cdot x}^{-nP_F,2x\cdot\nabla W}(\rho_0)=-\mathrm{H}^{nP_F}(\rho_0)+\int_\Omega\rho_0x\cdot\nabla(W\star\rho_0)\,\mathrm{d}x+\int_\Omega\rho_0x\cdot\nabla U\,\mathrm{d}x-\frac{1}{2\sigma}\int_\Omega|\,x\,|^2\rho_0\,\mathrm{d}x.$$

By combining the last 2 identities, we can rewrite the right hand side of (40) as

$$
\begin{aligned}
&\mathrm{H}_{c+\nabla(U-c)\cdot x}^{-nP_F,2x\cdot\nabla W}(\rho_0)+\int_\Omega \rho_0 c^*\left(-\nabla(F'\circ\rho_0+U-c+W\star\rho_0)\right)\mathrm{d}x\\
&\quad=\frac{\sigma}{2}\int_\Omega \rho_0|\,\nabla\left(F'(\rho_0)+U+W\star\rho_0\right)|^2\,\mathrm{d}x-\int_\Omega \rho_0 x\cdot\nabla\left(F'\circ\rho_0\right)\,\mathrm{d}x-\int_\Omega nP_F(\rho_0)\,\mathrm{d}x\\
&\quad=\frac{\sigma}{2}\int_\Omega \rho_0|\,\nabla\left(F'(\rho_0)+U+W\star\rho_0\right)|^2\mathrm{d}x+\int_\Omega \mathrm{div}\,(\rho_0 x)F'(\rho_0)\,\mathrm{d}x-\int_\Omega nP_F(\rho_0)\,\mathrm{d}x\\
&\quad=\frac{\sigma}{2}\int_\Omega \rho_0\Big|\,\nabla\left(F'(\rho_0)+U+W\star\rho_0\right)\Big|^2\,\mathrm{d}x+n\int_\Omega \rho_0F'(\rho_0)\,\mathrm{d}x+\int_\Omega x\cdot\nabla F(\rho_0)\,\mathrm{d}x\\
&\qquad-\int_\Omega nP_F(\rho_0)\,\mathrm{d}x\\
&\quad=\frac{\sigma}{2}\int_\Omega \rho_0\Big|\,\nabla\left(F'(\rho_0)+U+W\star\rho_0\right)\Big|^2\mathrm{d}x+\int_\Omega x\cdot\nabla F(\rho_0)\,\mathrm{d}x+n\int_\Omega F\circ\rho_0\,\mathrm{d}x\\
&\quad=\frac{\sigma}{2}\int_\Omega \rho_0\Big|\,\nabla\left(F'(\rho_0)+U+W\star\rho_0\right)\Big|^2\,\mathrm{d}x. \qquad (41)
\end{aligned}
$$

Inserting (41) into (40), we conclude (39).

Proof of Theorem 3.1: To establish the HWBI inequality (38), we rewrite (39) as

$$
\begin{aligned}
&\mathrm{H}_U^{F,W}(\rho_0|\rho_1)+\frac{\mu+\nu}{2}W_2^2(\rho_0,\rho_1)-\frac{\nu}{2}|\mathrm{b}(\rho_0)-\mathrm{b}(\rho_1)|^2\\
&\qquad\le\frac{1}{2\sigma}W_2^2(\rho_0,\rho_1)+\frac{\sigma}{2}I_2(\rho_0|\rho_U), \qquad (42)
\end{aligned}
$$

then minimize the right hand side of (42) over $\sigma>0$. The minimum is obviously achieved at $\bar\sigma=\frac{W_2(\rho_0,\rho_1)}{\sqrt{I_2(\rho_0|\rho_U)}}$. This yields (38).

Setting $W=0$ (and then $\nu=0$) in Theorem 3.1, we obtain in particular, the following HWI inequality first established by Otto-Villani [24] in the case of the classical entropy $F(x)=x\ln x$, and extended later on, for generalized entropy functions F by Carillo, McCann and Villani in [9].

Corollary 3.2 *(*HWI inequalities [9]*)*
Under the hypothesis on Ω and F in Theorem 3.1, let $U:\mathbb{R}^n\to\mathbb{R}$ be a C^2-function with $D^2U\ge\mu I$, where $\mu\in\mathbb{R}$. Then we have for all probability densities ρ_0 and ρ_1 on Ω, satisfying supp $\rho_0\subset\Omega$, and $P_F(\rho_0)\in W^{1,\infty}(\Omega)$,

$$
\mathrm{H}_U^F(\rho_0|\rho_1)\le W_2(\rho_0,\rho_1)\sqrt{I(\rho_0|\rho_U)}-\frac{\mu}{2}W_2^2(\rho_0,\rho_1). \qquad (43)
$$

If $U+W$ is uniformly convex (i.e., $\mu+\nu>0$) inequality (39) yields the following extensions of the Log-Sobolev inequality:

Corollary 3.3 *(*Log-Sobolev inequalities with interaction potentials*)*
In addition to the hypothesis on Ω, F, U and W in Theorem 3.1, assume $\mu+\nu>0$.

Then for all probability densities ρ_0 and ρ_1 on Ω, satisfying supp $\rho_0 \subset \Omega$*, and* $P_F(\rho_0) \in W^{1,\infty}(\Omega)$*, we have*

$$\mathrm{H}_U^{F,W}(\rho_0|\rho_1) - \frac{\nu}{2}|\mathrm{b}(\rho_0) - \mathrm{b}(\rho_1)|^2 \leq \frac{1}{2(\mu+\nu)} I_2(\rho_0|\rho_U). \tag{44}$$

In particular, if $b(\rho_0) = b(\rho_1)$, we have that

$$\mathrm{H}_U^{F,W}(\rho_0|\rho_1) \leq \frac{1}{2(\mu+\nu)} I_2(\rho_0|\rho_U). \tag{45}$$

Furthermore, if W is convex, then we have the following inequality, established in [9]

$$\mathrm{H}_U^{F,W}(\rho_0|\rho_1) \leq \frac{1}{2\mu} I_2(\rho_0|\rho_U). \tag{46}$$

Proof: (44) follows easily from (39) by choosing $\sigma = \frac{1}{\mu+\nu}$, and (46) follows from (44), using $\nu = 0$ because W is convex.

In particular, setting $W = 0$ in Corollary 3.3, one obtains the following generalized Log-Sobolev inequality obtained in [10], and in [13] for generalized cost functions.

Corollary 3.4 *(*Generalized Log-Sobolev inequalities *[10], [13])*
Assume that Ω and F satisfy the assumptions in Theorem 3.1, and that $U : I\!R^n \to I\!R$ is a C^2- uniformly convex function with $D^2U \geq \mu I$, where $\mu > 0$. Then for all probability densities ρ_0 and ρ_1 on Ω, satisfying supp $\rho_0 \subset \Omega$*, and* $P_F(\rho_0) \in W^{1,\infty}(\Omega)$*, we have*

$$\mathrm{H}_U^{F}(\rho_0|\rho_1) \leq \frac{1}{2\mu} I_2(\rho_0|\rho_U). \tag{47}$$

One can also deduce the following generalization of Talagrand's inequality. We note in particular that when $W = 0$, the result below is obtained previously by Blower [4], Otto-Villani [24] and Bobkov-Ledoux [5] for the Tsallis entropy $F(x) = x \ln x$, and by Carillo-McCann-Villani [9] for generalized entropy functions F.

Corollary 3.5 *(*Generalized Talagrand Inequality with interaction potentials*)*
In addition to the hypothesis on Ω, F, U and W in Theorem 3.1, assume $\mu + \nu > 0$. Then for all probability densities ρ on Ω, we have

$$\frac{\nu+\mu}{2} W_2^2(\rho, \rho_U) - \frac{\nu}{2}|\mathrm{b}(\rho) - \mathrm{b}(\rho_U)|^2 \leq \mathrm{H}_U^{F,W}(\rho|\rho_U). \tag{48}$$

In particular, if $b(\rho) = b(\rho_U)$, we have that

$$W_2(\rho, \rho_U) \leq \sqrt{\frac{2\mathrm{H}_U^{F,W}(\rho|\rho_U)}{\mu+\nu}}. \tag{49}$$

Furthermore, if W is convex, then the following inequality established in [9] holds:

$$W_2(\rho, \rho_U) \leq \sqrt{\frac{2\mathrm{H}_U^{F,W}(\rho|\rho_U)}{\mu}}. \tag{50}$$

Proof: (48) follows from (39) if we use $\rho_0 := \rho_U$, $\rho_1 := \rho$, notice that $I_2(\rho_U|\rho_U) = 0$, and then let σ go to ∞. (50) follows from (48), where we use $\nu = 0$ because W is convex.

3.2 Gaussian inequalities

Proposition 3.1 applied to $F(x) = x \ln x$ when $W = 0$, yields the following extension of Gross' Log-Sobolev inequality established by Bakry and Emery in [2]. First, we state the following HWI-type inequality from which we deduce Otto-Villani's HWI inequality [24], and the Log-Sobolev inequality of Gross [18] and Bakry-Emery [2].

Corollary 3.6 *Let $U : \mathbb{R}^n \to \mathbb{R}$ be a C^2-function with $D^2U \geq \mu I$ where $\mu \in \mathbb{R}$, and denote by ρ_U the normalized Gaussian $\frac{e^{-U}}{\sigma_U}$, where $\sigma_U = \int_{\mathbb{R}^n} e^{-U}\, dx$. Then for any $\sigma > 0$, the following holds for any nonnegative function f such that $f\rho_U \in W^{1,\infty}(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n} f\rho_U\, dx = 1$:*

$$\int_{R^n} f \ln(f)\, \rho_U\, dx + \frac{1}{2}(\mu - \frac{1}{\sigma}) W_2^2(f\rho_U, \rho_U) \leq \frac{\sigma}{2} \int_{R^n} \frac{|\nabla f|^2}{f}\, \rho_U\, dx. \tag{51}$$

Proof: First assume that f has compact support, and set $F(x) = x \ln x$, $\rho_0 = f\rho_U$, $\rho_1 = \rho_U$ and $W = 0$ in (39). We have that

$$H_U^F(f\rho_U|\rho_U) + \frac{1}{2}(\mu - \frac{1}{\sigma}) W_2^2(f\rho_U, \rho_U) \leq \frac{\sigma}{2} \int_{R^n} \Big| \frac{\nabla(f\rho_U)}{f\rho_U} + U \Big|^2 f\rho_U\, \mathrm{d}x. \tag{52}$$

By direct computations,

$$\frac{\nabla(f\rho_U)}{f\rho_U} = \frac{\nabla f}{f} - \nabla U, \tag{53}$$

and

$$\begin{aligned} \mathrm{H}_U^{F,W}(f\rho_U|\rho_U) &\leq \int_{R^n} [f\rho_U \ln(f\rho_U) + Uf\rho_U - \rho_U \ln \rho_U - U\rho_U]\, \mathrm{d}x \\ &= \int_{R^n} (f\rho_U \ln f)\, \mathrm{d}x + \ln \sigma_U \int_{R^n} (\rho_U - f\rho_U)\, \mathrm{d}x \\ &= \int_{R^n} f \ln(f) \rho_U\, \mathrm{d}x. \end{aligned} \tag{54}$$

Combining (52) - (54), we get (51). We finish the proof using a standard approximation argument.

Corollary 3.7 *(*Otto-Villani's HWI inequality [24]*)*
Let $U : \mathbb{R}^n \to \mathbb{R}$ be a C^2-uniformly convex function with $D^2U \geq \mu I$, where $\mu > 0$, and denote by ρ_U the normalized Gaussian $\frac{e^{-U}}{\sigma_U}$, where $\sigma_U = \int_{\mathbb{R}^n} e^{-U}\, dx$. Then, for any nonnegative function f such that $f\rho_U \in W^{1,\infty}(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n} f\rho_U\, dx = 1$,

$$\int_{R^n} f \ln(f) \rho_U\, dx \leq W_2(\rho_U, f\rho_U) \sqrt{I(f\rho_U|\rho_U)} - \frac{\mu}{2} W_2^2(f\rho_U, \rho_U), \tag{55}$$

where

$$I(f\rho_U|\rho_U) = \int_{R^n} \frac{|\nabla f|^2}{f} \rho_U\, dx.$$

Proof: It is similar to the proof of Theorem 3.1. Rewrite (51) as

$$\int_{R^n} f\ln(f)\rho_U\,\mathrm{d}x + \frac{\mu}{2}W_2^2(f\rho_U,\rho_U) \le \frac{\mu}{2\sigma}W_2^2(f\rho_U,\rho_U) + \frac{\sigma}{2}I(f\rho_U|\rho_U),$$

and show that the minimum over $\sigma > 0$ of the right hand side is attained at $\bar{\sigma} = \frac{W_2(f\rho_U,\rho_U)}{\sqrt{I(f\rho_U|\rho_U)}}$.

Now, setting $f := g^2$ and $\sigma := \frac{1}{\mu}$ in (55), one obtains the following extension of Gross' [18] Log-Sobolev inequality first established by Bakry and Emery in [2].

Corollary 3.8 *(*Original Log Sobolev inequality [2], [18]*)*
Let $U : I\!R^n \to I\!R$ be a C^2-uniformly convex function with $D^2U \ge \mu I$ where $\mu > 0$, and denote by ρ_U the normalized Gaussian $\frac{e^{-U}}{\sigma_U}$, where $\sigma_U = \int_{R^n} e^{-U}\,dx$. Then, for any function g such that $g^2\rho_U \in W^{1,\infty}(I\!R^n)$ and $\int_{R^n} g^2\rho_U\,dx = 1$, we have

$$\int_{R^n} g^2\ln(g^2)\,\rho_U\,dx \le \frac{2}{\mu}\int_{R^n} |\nabla g|^2\rho_U\,dx. \tag{56}$$

As pointed out by Rothaus in [25], the above Log-Sobolev inequality implies the Poincaré's inequality.

Corollary 3.9 *(*Poincaré's inequality*)*
Let $U : I\!R^n \to I\!R$ be a C^2-uniformly convex function with $D^2U \ge \mu I$ where $\mu > 0$, and denote by ρ_U the normalized Gaussian $\frac{e^{-U}}{\sigma_U}$, where $\sigma_U = \int_{R^n} e^{-U}\,dx$. Then, for any function f such that $f\rho_U \in W^{1,\infty}(I\!R^n)$ and $\int_{R^n} f\rho_U\,dx = 0$, we have

$$\int_{R^n} f^2\rho_U\,dx \le \frac{1}{\mu}\int_{R^n} |\nabla f|^2\rho_U\,dx. \tag{57}$$

Proof: From (56), we have that

$$\int_{R^n} f_\epsilon\ln(f_\epsilon)\,\rho_U\,\mathrm{d}x \le \frac{1}{2\mu}\int_{R^n}\frac{|\nabla f_\epsilon|^2}{f_\epsilon}\rho_U\,\mathrm{d}x, \tag{58}$$

where $f_\epsilon = 1+\epsilon f$ for some $\epsilon > 0$. Using that $\int_{R^n} f\rho_U\,\mathrm{d}x = 0$, we have for small ϵ,

$$\int_{R^n} f_\epsilon\ln(f_\epsilon)\rho_U\,\mathrm{d}x = \frac{\epsilon^2}{2}\int_{R^n} f^2\rho_U\,\mathrm{d}x + o(\epsilon^3), \tag{59}$$

and

$$\int_{R^n}\frac{|\nabla f_\epsilon|^2}{f_\epsilon}\rho_U\,\mathrm{d}x = \epsilon^2\int_{R^n} |\nabla f|^2\rho_U\,\mathrm{d}x + o(\epsilon^3). \tag{60}$$

We combine (58) - (60) to have that

$$\int_{R^n} f^2\rho_U\,\mathrm{d}x \le \frac{1}{\mu}\int_{R^n} |\nabla f|^2\rho_U\,\mathrm{d}x + o(\epsilon). \tag{61}$$

We let ϵ go to 0 in (61) to conclude (57).

If we apply Corollary 3.5 to $F(x) = x\ln x$ when $W = 0$, we obtain the following extension of Talagrand's inequality established by Otto and Villani in [24].

Corollary 3.10 *(*Original Talagrand's inequality [26], [24]*)*
Let $U : I\!\!R^n \to I\!\!R$ be a C^2-uniformly convex function with $D^2U \geq \mu I$ where $\mu > 0$, and denote by ρ_U the normalized Gaussian $\frac{e^{-U}}{\sigma_U}$, where $\sigma_U = \int_{R^n} e^{-U}\, dx$. Then, for any nonnegative function f such that $\int_{R^n} f\rho_U\, dx = 1$, we have

$$W_2(f\rho_U, \rho_U) \leq \sqrt{\frac{2}{\mu}\int_{R^n} f\ln(f)\rho_U\, dx}. \tag{62}$$

In particular, if $f = \frac{I_B}{\gamma(B)}$ for some measurable subset B of $I\!\!R^n$, where $d\gamma(x) = \rho_U(x)\mathrm{d}x$ and $I\!\!I_B$ is the characteristic function of B, we obtain the following inequality in the concentration of measures in Gauss space, first proved by Bobkov and Götze in [6].

Corollary 3.11 *(*Concentration of measure inequality [6]*)*
Let $U : I\!\!R^n \to I\!\!R$ be a C^2-uniformly convex function with $D^2U \geq \mu I$ where $\mu > 0$, and denote by γ the normalized Gaussian measure with density $\rho_U = \frac{e^{-U}}{\sigma_U}$, where $\sigma_U = \int_{R^n} e^{-U}\, dx$. Then, for any ϵ-neighborhood B_ϵ of a measurable set B in $I\!\!R^n$, we have

$$\gamma(B_\epsilon) \geq 1 - e^{-\frac{\mu}{2}\left(\epsilon - \sqrt{\frac{2}{\mu}\ln\left(\frac{1}{\gamma(B)}\right)}\right)^2}, \tag{63}$$

where $\epsilon \geq \sqrt{\frac{2}{\mu}\ln\left(\frac{1}{\gamma(B)}\right)}$.

Proof: Using $f = f_B = \frac{I_B}{\gamma(B)}$ in (62), we have that

$$W_2(f_B\rho_U, \rho_U) \leq \sqrt{\frac{2}{\mu}\ln\left(\frac{1}{\gamma(B)}\right)},$$

and then, we obtain from the triangle inequality that

$$W_2(f_B\rho_U, f_{R^n\setminus B_\epsilon}\rho_U) \leq \sqrt{\frac{2}{\mu}\ln\left(\frac{1}{\gamma(B)}\right)} + \sqrt{\frac{2}{\mu}\ln\left(\frac{1}{1-\gamma(B_\epsilon)}\right)}. \tag{64}$$

But since $|\,x-y\,| \geq \epsilon$ for all $(x,y) \in B \times (I\!\!R^n \setminus B_\epsilon)$, we have that

$$W_2(f_B\rho_U, \rho_U) \geq \epsilon. \tag{65}$$

We combine (64) and (65) to deduce that

$$\ln\left(\frac{1}{1-\gamma(I\!\!R^n\setminus B_\epsilon)}\right) \geq \frac{\mu}{2}\left(\epsilon - \sqrt{\frac{2}{\mu}\ln\left(\frac{1}{\gamma(B)}\right)}\right)^2,$$

which leads to (63).

4 Trends to equilibrium

We use Corollary 3.4 and Corollary 3.5 to recover rates of convergence for solutions to equation

$$\begin{cases} \frac{\partial \rho}{\partial t} = \operatorname{div}\left\{\rho \nabla\left(F'(\rho) + V + W \star \rho\right)\right\} & \text{in} \quad (0,\infty) \times I\!R^n \\ \rho(t=0) = \rho_0 & \text{in} \quad \{0\} \times I\!R^n, \end{cases} \tag{66}$$

recently shown by Carillo, McCann and Villani in [9]. Here we consider the case where $V+W$ is uniformly convex and W convex, and the case when only $V+W$ is uniformly convex but the barycenter $b\left(\rho(t)\right)$ of any solution $\rho(t,x)$ of (66) is invariant in t. For a background and other cases of convergence to equilibrium for this equation, we refer to [9] and the references therein.

Corollary 4.1 *(*Trend to equilibrium*)*
Let $F:[0,\infty) \to I\!R$ be strictly convex, differentiable on $(0,\infty)$ and satisfies $F(0)=0$, $\lim_{x\to\infty}\frac{F(x)}{x} = \infty$, and $x \mapsto x^n F(x^{-n})$ is convex and non-increasing. Let V, $W : I\!R^n \to [0,\infty)$ be respectively C^2-confinement and interaction potentials with $D^2 V \geq \lambda I$ and $D^2 W \geq \nu I$, where $\lambda, \nu \in I\!R$. Assume that the initial probability density ρ_0 has finite total energy. Then

(i). If $V+W$ is uniformly convex (i.e., $\lambda+\nu>0$) and W is convex (i.e. $\nu \geq 0$), then, for any solution ρ of (66), such that $\mathrm{H}_V^{F,W}\left(\rho(t)\right) < \infty$, we have:

$$\mathrm{H}_V^{F,W}\left(\rho(t)|\rho_V\right) \leq e^{-2\lambda t}\mathrm{H}_V^{F,W}(\rho_0|\rho_V), \tag{67}$$

and

$$W_2\left(\rho(t), \rho_V\right) \leq e^{-\lambda t}\sqrt{\frac{2\mathrm{H}_V^{F,W}(\rho_0|\rho_V)}{\lambda}}. \tag{68}$$

(ii). If $V+W$ is uniformly convex (i.e., $\lambda+\nu>0$) and if we assume that the barycenter $b\left(\rho(t)\right)$ of any solution $\rho(t,x)$ of (66) is invariant in t, then, for any solution ρ of (66) such that $\mathrm{H}_V^{F,W}\left(\rho(t)\right) < \infty$, we have:

$$\mathrm{H}_V^{F,W}\left(\rho(t)|\rho_V\right) \leq e^{-2(\lambda+\nu)t}\mathrm{H}_V^{F,W}(\rho_0|\rho_V), \tag{69}$$

and

$$W_2\left(\rho(t), \rho_V\right) \leq e^{-2(\lambda+\nu)t}\sqrt{\frac{2\mathrm{H}_V^{F,W}(\rho_0|\rho_V)}{\lambda+\nu}}. \tag{70}$$

Proof: Under the assumptions on F, V and W in Corollary 4.1, it is known (see [9], and references therein) that the total energy $\mathrm{H}_V^{F,W}$ – which is a Lyapunov functional associated with (66) – has a unique minimizer ρ_V defined by

$$\rho_V \nabla\left(F'(\rho_{_V}) + V + W \star \rho_{_V}\right) = 0 \quad \text{a.e.}$$

Now, let ρ be a – smooth – solution of (66). We have the following energy dissipation equation

$$\frac{d}{dt}\mathrm{H}_V^{F,W}\left(\rho(t)|\rho_V\right) = -I_2\left(\rho(t)|\rho_V\right). \tag{71}$$

Combining (71) with (46), we have that

$$\frac{d}{dt}\mathrm{H}_V^{F,W}\left(\rho(t)|\rho_V\right) \leq -2\lambda\mathrm{H}_V^{F,W}\left(\rho(t)|\rho_V\right). \tag{72}$$

We integrate (72) over $[0,t]$ to conclude (67). (68) follows directly from (50) and (67). To prove (69), we use (71) and (45) to have that

$$\frac{d}{dt}\mathrm{H}_V^{F,W}\left(\rho(t)|\rho_V\right) \leq -2(\lambda+\nu)\mathrm{H}_V^{F,W}\left(\rho(t)|\rho_V\right). \tag{73}$$

We integrate (73) over $[0,t]$ to conclude (69). As before, (70) is a consequence of (69) and (49).

Below, we apply Corollary 4.1 to obtain rates of convergence to equilibrium for some equations of the form (66) studied in the literature by many authors.

Examples:

- If $W=0$ and $F(x)=x\ln x$ in which case (66) is the linear Fokker-Planck equation $\frac{\partial\rho}{\partial t}=\Delta\rho+\mathrm{div}(\rho\nabla V)$, Corollary 4.1 gives an exponential decay in relative entropy of solutions of this equation to the Gaussian density $\rho_V=\frac{e^{-V}}{\sigma_V}$, $\sigma_V=\int_{R^n}e^{-V}\,\mathrm{d}x$, at the rate 2λ when $D^2V\geq\lambda I$ for some $\lambda>0$, and an exponential decay in the Wasserstein distance, at the rate λ.

- If $W=0$, $F(x)=\frac{x^m}{m-1}$ where $1\neq m\geq 1-\frac{1}{n}$, and $V(x)=\lambda\frac{|x|^2}{2}$ for some $\lambda>0$, in which case (66) is the rescaled porous medium equation ($m>1$), or fast diffusion equation ($1-\frac{1}{n}\leq m<1$), that is $\frac{\partial\rho}{\partial t}=\Delta\rho^m+\mathrm{div}(\lambda x\rho)$, Corollary 4.1 gives an exponential decay in relative entropy of solutions of this equation to the Barenblatt-Prattle profile $\rho_V(x)=\left[\left(C+\frac{\lambda(1-m)}{2m}|x|^2\right)^{\frac{1}{m-1}}\right]^+$ (where $C>0$ is such that $\int_{R^n}\rho(x)\,\mathrm{d}x=1$) at the rate 2λ, and an exponential decay in the Wasserstein distance at the rate λ.

5 A remarkable duality

In this section, we apply Theorem 1.1 when $V=W=0$, to obtain an intriguing duality between ground state solutions of some quasilinear PDEs and stationary solutions of Fokker-Planck type equations.

Corollary 5.1 *Let $\Omega \subset I\!R^n$ be open, bounded and convex, let $F : [0, \infty) \to I\!R$ be differentiable on $(0, \infty)$ such that $F(0) = 0$ and $x \mapsto x^n F(x^{-n})$ be convex and non-increasing. Let $\psi : I\!R \to [0, \infty)$ differentiable be chosen in such a way that $\psi(0) = 0$ and $|\,\psi^{\frac{1}{p}}(F' \circ \psi)'\,| = K$ where $p > 1$, and K is chosen to be 1 for simplicity. Then, for any Young function c with p-homogeneous Legendre transform c^*, we have the following inequality:*

$$\sup\{-\int_\Omega F(\rho)+c\rho; \rho \in \mathcal{P}_a(\Omega)\} \leq \inf\{\int_\Omega c^*(-\nabla f) - G_F \circ \psi(f); f \in C_0^\infty(\Omega), \int_\Omega \psi(f) = 1\} \tag{74}$$

where $G_F(x) := (1-n)F(x) + nxF'(x)$.
Furthermore, equality holds in (74) if there exists $\bar{f}$ (and $\bar{\rho} = \psi(\bar{f})$) that satisfies

$$-(F' \circ \psi)'(\bar{f})\nabla \bar{f}(x) = \nabla c(x) \ \text{ a.e.} \tag{75}$$

Moreover, $\bar{f}$ solves

$$\begin{array}{ll} \operatorname{div}\{\nabla c^*(-\nabla f)\} - (G_F \circ \psi)'(f) = \lambda \psi'(f) & \text{in } \Omega \\ \nabla c^*(-\nabla f) \cdot \nu = 0 & \text{on } \partial\Omega, \end{array} \tag{76}$$

for some $\lambda \in I\!R$, while $\bar{\rho}$ is a stationary solution of

$$\begin{array}{ll} \frac{\partial \rho}{\partial t} = \operatorname{div}\{\rho \nabla\left(F'(\rho) + c\right)\} & \text{in } (0, \infty) \times \Omega \\ \rho \nabla\left(F'(\rho) + c\right) \cdot \nu = 0 & \text{on } (0, \infty) \times \partial\Omega. \end{array} \tag{77}$$

Proof: Assume that c^* is p-homogeneous, and let $Q''(x) = x^{\frac{1}{q}} F''(x)$. Let

$$J(\rho) := -\int_\Omega [F(\rho(y)) + c(y)\rho(y)]dy$$

and

$$\tilde{J}(\rho) := -\int_\Omega (F + nP_F)(\rho(x))dx + \int_\Omega c^*(-\nabla(Q'(\rho(x)))dx.$$

Equation (1) (where we use $V = W = 0$, and then $\lambda = \nu = 0$) then becomes

$$J(\rho_1) \leq \tilde{J}(\rho_0) \tag{78}$$

for all probability densities ρ_0, ρ_1 on Ω such that $\operatorname{supp} \rho_0 \subset \Omega$ and $P_F(\rho_0) \in W^{1,\infty}(\Omega)$. If $\bar{\rho}$ satisfies

$$-\nabla(F'(\bar{\rho}(x))) = \nabla c(x) \text{ a.e.},$$

then equality holds in (78), and $\bar{\rho}$ is an extremal of the variational problems

$$\sup\{J(\rho);\ \rho \in \mathcal{P}_a(\Omega)\} = \inf\{\tilde{J}(\rho); \rho \in \mathcal{P}_a(\Omega), \operatorname{supp} \rho \subset \Omega, P_F(\rho) \in W^{1,\infty}(\Omega)\}.$$

In particular, $\bar{\rho}$ is a solution of

$$\begin{array}{ll} \operatorname{div}\{\rho \nabla(F'(\rho) + c)\} = 0 & \text{in } \Omega \\ \rho \nabla(F'(\rho) + c) \cdot \nu = 0 & \text{on } \partial\Omega. \end{array} \tag{79}$$

Suppose now $\psi : I\!\!R \to [0,\infty)$ differentiable, $\psi(0)=0$ and that $\bar{f} \in C_0^\infty(\Omega)$ satisfies $-(F'\circ\psi)'(\bar{f})\nabla\bar{f}(x) = \nabla c(x)$ a.e. Then equality holds in (78), and $\bar{f}$ and $\bar{\rho} = \psi(\bar{f})$ are extremals of the following variational problems

$$\inf\{I(f);\ f \in C_0^\infty(\Omega), \int_\Omega \psi(f) = 1\} = \sup\{J(\rho); \rho \in \mathcal{P}_a(\Omega)\}$$

where

$$I(f) = \tilde{J}(\psi(f)) = -\int_\Omega [F\circ\psi + nP_F\circ\psi](f) + \int_\Omega c^*(-\nabla(Q'\circ\psi(f))).$$

If now ψ is such that $|\,\psi^{\frac{1}{p}}(F'\circ\psi)'\,| = 1$, then $|\,(Q'\circ\psi)'\,| = 1$ and

$$I(f) = -\int_\Omega [F\circ\psi + nP_F\circ\psi](f) + \int_\Omega c^*(-\nabla f),$$

because c^* is p-homogeneous. This proves (74). The Euler-Lagrange equation of the variational problem

$$\inf\Big\{\int_\Omega c^*(-\nabla(f)) - [F\circ\psi + nP_F\circ\psi](f);\ \int_\Omega \psi(f) = 1\Big\}$$

reads as

$$\begin{array}{ll} \operatorname{div}\{\nabla c^*(-\nabla f)\} - (G_F\circ\psi)'(f) = \lambda\psi'(f) & \text{in } \Omega \\ \nabla c^*(-\nabla f)\cdot\nu = 0 & \text{on } \partial\Omega \end{array} \tag{80}$$

where $\lambda \in I\!\!R$ is a Lagrange multiplier, and $G(x) = (1-n)F(x) + nxF'(x)$. This proves (76). To prove that the maximizer $\bar{\rho}$ of

$$\sup\{-\int_\Omega (F(\rho) + c\rho)\ \mathrm{d}x;\ \rho \in \mathcal{P}_a(\Omega)\}$$

is a stationary solution of (77), we refer to [19] and [22].

Now, we apply Corollary 5.1 to the functions $F(x) = x\ln x, \psi(x) = |\,x\,|^p$ and $c(x) = (p-1)|\,\mu x\,|^q$, with $\mu > 0$ and $c^*(x) = \frac{1}{p}\Big|\,\frac{x}{\mu}\,\Big|^p$ and $\frac{1}{p} + \frac{1}{q} = 1$, to derive a duality between stationary solutions of Fokker-Planck equations, and ground state solutions of some semi-linear equations. We note here that the condition $|\,\psi^{\frac{1}{p}}(F'\circ\psi)\,| = K$ holds for $K = p$. We obtain the following:

Corollary 5.2 *Let $p > 1$ and let q be its conjugate ($\frac{1}{p} + \frac{1}{q} = 1$). For all $f \in W^{1,p}(I\!\!R^n)$, such that $\|\,f\,\|_p = 1$, any probability density ρ such that $\int_{R^n} \rho(x)|x|^q dx < \infty$, and any $\mu > 0$, we have*

$$J_\mu(\rho) \le I_\mu(f), \tag{81}$$

where

$$J_\mu(\rho) := -\int_{R^n} \rho\ln(\rho)\ dy - (p-1)\int_{R^n} |\,\mu y\,|^q \rho(y)\ dy,$$

and

$$I_\mu(f) := -\int_{\mathbb{R}^n} |\, f\,|^p \ln(|\, f\,|^p) + \int_{\mathbb{R}^n} \Big|\, \frac{\nabla f}{\mu} \Big|^p - n.$$

Furthermore, if $h \in W^{1,p}(\mathbb{R}^n)$ *is such that* $h \geq 0$, $\|\, h\,\|_p = 1$, *and*

$$\nabla h(x) = -\mu^q x|\, x\,|^{q-2} h(x) \quad a.e.,$$

then

$$J_\mu(h^p) = I_\mu(h).$$

Therefore, h *(resp.,* $\rho = h^p$*) is an extremum of the variational problem:*

$$\sup\{\, J_\mu(\rho) : \rho \in W^{1,1}(\mathbb{R}^n),\ \|\, \rho\,\|_1 = 1\} = \inf\{\, I_\mu(f) : f \in W^{1,p}(\mathbb{R}^n), \|\, f\,\|_p = 1\}.$$

It follows that h satisfies the Euler-Lagrange equation corresponding to the constraint minimization problem, i.e., h is a solution of

$$\mu^{-p}\Delta_p f + pf|\, f\,|^{p-2} \ln(|\, f\,|) = \lambda f|\, f\,|^{p-2}, \tag{82}$$

where λ is a Lagrange multiplier. On the other hand, $\rho = h^p$ is a stationary solution of the Fokker-Planck equation:

$$\frac{\partial u}{\partial t} = \Delta u + \mathrm{div}(p\mu^q |x|^{q-2} x u). \tag{83}$$

We can also apply Corollary 5.1 to recover the duality associated to the Gagliardo-Nirenberg inequalities obtained recently in [11].

Corollary 5.3 *Let* $1 < p < n$, *and* $r \in \left(0, \frac{np}{n-p}\right]$ *such that* $r \neq p$. *Set* $\gamma := \frac{1}{r} + \frac{1}{q}$, *where* $\frac{1}{p} + \frac{1}{q} = 1$. *Then, for* $f \in W^{1,p}(\mathbb{R}^n)$ *such that* $\|\, f\,\|_r = 1$, *for any probability density* ρ *and for all* $\mu > 0$, *we have*

$$J_\mu(\rho) \leq I_\mu(f) \tag{84}$$

where

$$J_\mu(\rho) := -\frac{1}{\gamma - 1} \int_{\mathbb{R}^n} \rho^\gamma - \frac{r\gamma\mu^q}{q} \int_{\mathbb{R}^n} |\, y\,|^q \rho(y)(y)\, dy,$$

and

$$I_\mu(f) := -\left(\frac{1}{\gamma - 1} + n\right) \int_{\mathbb{R}^n} |\, f\,|^{r\gamma} + \frac{r\gamma}{p\mu^p} \int_{\mathbb{R}^n} |\, \nabla f\,|^p.$$

Furthermore, if $h \in W^{1,p}(\mathbb{R}^n)$ *is such that* $h \geq 0$, $\|\, h\,\|_r = 1$, *and*

$$\nabla h(x) = -\mu^q x|\, x\,|^{q-2} h^{\frac{r}{p}}(x) \quad a.e.,$$

then

$$J_\mu(h^r) = I_\mu(h).$$

Therefore, h *(resp.,* $\rho = h^r$*) is an extremum of the variational problems*

$$\sup\{\, J_\mu(\rho) : \rho \in W^{1,1}(\mathbb{R}^n),\ \|\, \rho\,\|_1 = 1\} = \inf\{\, I_\mu(f) : f \in W^{1,p}(\mathbb{R}^n), \|\, f\,\|_r = 1\}.$$

Proof: Again, the proof follows from Corollary 5.1, by using now $\psi(x) = |\, x\,|^r$ and $F(x) = \frac{x^\gamma}{\gamma-1}$, where $1 \neq \gamma \geq 1 - \frac{1}{n}$, which follows from the fact that $p \neq r \in \left(0, \frac{np}{n-p}\right]$. Indeed, for this value of γ, the function F satisfies the conditions of Corollary 5.1. The Young function is now $c(x) = \frac{r\gamma}{q}|\,\mu x\,|^q$, that is, $c^*(x) = \frac{1}{p(r\gamma)^{p-1}}\left|\,\frac{x}{\mu}\,\right|^p$, and the condition $|\,\psi^{\frac{1}{p}}(F' \circ \psi)'\,| = K$ holds with $K = r\gamma$.
Moreover, if $h \geq 0$ satisfies (75), which is here,

$$-\nabla h(x) = \mu^q x|\,x\,|^{q-2} h^{\frac{r}{p}}(x) \ \text{ a.e.},$$

then h is extremal in the minimization problem defined in Corollary 5.3.

As above, we also note that h satisfies the Euler-Lagrange equation corresponding to the constraint minimization problem, that is, h is a solution of

$$\mu^{-p}\Delta_p f + \left(\frac{1}{\gamma - 1} + n\right) f|\,f\,|^{r\gamma-2} = \lambda f|\,f\,|^{r-2}, \tag{85}$$

where λ is a Lagrange multiplier. On the other hand, $\rho = h^r$ is a stationary solution of the evolution equation:

$$\frac{\partial u}{\partial t} = \Delta u^\gamma + \mathrm{div}(r\gamma\mu^q|x|^{q-2}xu). \tag{86}$$

Example: In particular, when $\mu = 1, p = 2, \gamma = 1 - \frac{1}{n}$ and then $r = 2^* = \frac{2n}{n-2}$ is the critical Sobolev exponent, then Corollary 5.3 yields a duality between solutions of (85), which here the Yamabe equation:

$$-\Delta f = \lambda f|\,f\,|^{2^*-2},$$

(where λ is the Lagrange multiplier due to the constraint $\|\,f\,\|_{2^*} = 1$), and stationary solutions of (86), which is here the rescaled fast diffusion equation:

$$\frac{\partial u}{\partial t} = \Delta u^{1-\frac{1}{n}} + \mathrm{div}\left(\frac{2n-2}{n-2}xu\right).$$

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