Optimal evolution of the free energy of interacting gazes and applications

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The Wasserstein manifold

Let Ω be open and convex subset of $\mathbb{R}^n$. Consider the space

$$M = \mathcal{P}_{ac}(\Omega) = \{\rho : \Omega \to \mathbb{R};\ \rho \geq 0 \text{ and } \int_\Omega \rho(x)dx = 1\}$$

equipped with the Wasserstein distance W_2 defined as:

$$W_2^2(\rho_0, \rho_1) := \inf_{\gamma \in \Gamma(\rho_0,\rho_1)} \int_{R^n \times R^n} |x-y|^2 d\gamma(x,y),$$

where $\Gamma(\rho_0, \rho_1)$ is the set of Borel probability measures on $\mathbb{R}^n \times \mathbb{R}^n$ with marginals ρ_0 and ρ_1, respectively.
(M, W_2) **is an "almost Riemannian" infinite dimensional manifold.** It is/will be a hugely important object for PDEs.

Basic model for interacting particles

(1) The **internal energy:** $\mathrm{H}^{\mathrm{F}}(\rho) := \int_\Omega F(\rho)dx$,
and the associated **Pressure**: $P_F(x) = xF'(x) - F(x)$,
where $F : [0,\infty) \to \mathbb{R}$.
(2) The **potential energy:** $\mathrm{H}_V(\rho) := \int_\Omega \rho V dx$
(3) the **interaction energy:** $\mathrm{H}^W(\rho) := \frac{1}{2}\int_\Omega \rho(W \star \rho)\,\mathrm{d}x$.
where V (confinement potential) and W(interaction potential) are C^2-real valued functions on $\mathbb{R}^n$.

The **Total Free Energy Functional** is then defined on $\mathcal{P}_{ac}(\Omega)$ as:

$$\mathrm{H}_V^{F,W}(\rho) := \int_\Omega \left[F(\rho) + \rho V + \frac{1}{2}(W \star \rho)\rho\right] \mathrm{d}x,$$

The **relative energy of ρ_0 with respect to ρ_1:**

$$\mathrm{H}_V^{F,W}(\rho_0|\rho_1) := \mathrm{H}_V^{F,W}(\rho_0) - \mathrm{H}_V^{F,W}(\rho_1).$$

Two basic general principles

In the most common models for energies

(1). (McCann and others) $\mathrm{H}_V^{F,W}$ is **geodesically convex** on the manifold (M, W_2). i.e., :
$t \rightarrow \mathrm{H}_V^{F,W}(\rho_t)$ is convex whenever $t \rightarrow \rho_t$ is a geodesic for the Wasserstein metric.

(2). (Otto and others) The evolution equations (Heat, Fokker-Planck, Porous media, McKean-Vlasov, etc...)

$$\frac{\partial \rho}{\partial t} = \mathrm{div}\left\{\rho \nabla \left(F'(\rho) + V + W \star \rho\right)\right\}$$

are the **gradient flows** of the energy $\mathrm{H}_V^{F,W}$ on the manifold (M, W_2), i.e.,

$$\dot{\rho}(t) = -\partial_{W_2} \mathrm{H}(\rho(t))$$

Relative Entropy Production of ρ with respect to ρ_V

Normally defined as

$$I_2(\rho|\rho_V) = \int_\Omega \rho \Big| \nabla \left(F'(\rho) + V + W \star \rho) \right) \Big|^2 \mathrm{d}x$$

in such a way that

$$\frac{d}{dt} \mathrm{H}_V^{F,W} \left(\rho(t)|\rho_V\right) = -I_2 \left(\rho(t)|\rho_V\right).$$

If ρ_V is (the stationnery state) a probability density that satisfies

$$\nabla \left(F'(\rho_V) + V + W \star \rho_V \right) = 0 \quad \text{a.e.}$$

then $I_2(\rho|\rho_V)$ is

$$\int_\Omega \rho |\nabla \left(F'(\rho) - F'(\rho_V) + W \star (\rho - \rho_V) \right|^2 \mathrm{d}x.$$

Notation ρ_V reflects our emphasis on its dependence on the confinement potential, though it obviously also depends on F, W.

Non-quadratic versions of the entropy. Call *Young function*, any strictly convex C^1-function $c : \mathbb{R}^n \to \mathbb{R}$ such that $c(0) = 0$ and $\lim_{|x| \to \infty} \frac{c(x)}{|x|} = \infty$. c^* its Legendre conjugate defined by

$$c^*(y) = \sup_{z \in R^n} \{y \cdot z - c(z)\}.$$

Generalized relative entropy production-type function of ρ with respect to ρ_V measured against c^*:

$$\mathcal{I}_{c^*}(\rho|\rho_V) := \int_\Omega \rho c^\star \left(-\nabla \left(F'(\rho) + V + W \star \rho\right)\right) \mathrm{d}x,$$

However, writing

$$\frac{d}{dt} \mathrm{H}_V^{F,W} \left(\rho(t)|\rho_V\right) = -I_{c^*} \left(\rho(t)|\rho_V\right),$$

where

$$\frac{\partial \rho}{\partial t} = \operatorname{div} \{\rho \nabla c^*((F'(\rho) + V + W \star \rho))\}$$

The definition should then be defined as:

$$I_{c^*}(\rho|\rho_V := \int_\Omega \rho\nabla\left(F'(\rho) + V + W \star \rho\right)\cdot\nabla c^\star\left(\nabla\left(F'(\rho) + V + W \star \rho\right)\right) \mathrm{d}x.$$

But since $c^*(z) \leq z \cdot \nabla c^*(z)$ we have:

$$\mathcal{I}_{c^*}(\rho|\rho_V) \leq I_{c^*}(\rho|\rho_V).$$

When $c(x) = \frac{|\,x\,|^2}{2}$, we have

$$\begin{aligned} I_{c^*}(\rho|\rho_V) &= I_2(\rho|\rho_V) \\ &= \int_\Omega \rho\left|\,\nabla\left(F'(\rho) + V + W \star \rho\right)\,\right|^2 \mathrm{d}x \\ &= 2\mathcal{I}_{c^*}(\rho|\rho_V) \end{aligned}$$

Basic comparison principle (Agueh-Ghoussoub-Kang)

- Ω open, bounded convex subset of $\mathbb{R}^n$.
- $F : [0, \infty) \to \mathbb{R}$ is such that $F(0) = 0$ and $x \mapsto x^n F(x^{-n})$ convex and non-increasing. P_F is the associated pressure function.
- The confinement (resp., interaction) potentials are C^2-functions $V : \mathbb{R}^n \to \mathbb{R}$ with $D^2V \geq \lambda I$, $D^2W \geq \nu I$ where $\lambda, \nu \in \mathbb{R}$.

Then:
For any $\rho_0, \rho_1 \in \mathcal{P}_{ac}(\Omega)$ with $\operatorname{supp} \rho_0 \subset \Omega$ and $P_F(\rho_0) \in W^{1,\infty}(\Omega)$,
For any Young function $c : \mathbb{R}^n \to \mathbb{R}$:

$$\mathrm{H}_{V+c}^{F,W}(\rho_0|\rho_1) + \frac{\lambda+\nu}{2} W_2^2(\rho_0, \rho_1) - \frac{\nu}{2} |\mathrm{b}(\rho_0) - \mathrm{b}(\rho_1)|^2 \leq \mathrm{H}_{c+\nabla V \cdot x}^{-nP_F, 2x\cdot\nabla W}(\rho_0) + \mathcal{I}_{c^*}(\rho_0|\rho_V).$$

Equality holds whenever

$$\rho_0 = \rho_1 = \rho_{V+c}$$

$$\nabla \left(F'(\rho_{V+c}) + V + c + W \star \rho_{V+c}\right) = 0 \quad \text{a.e.}$$

Mother General Sobolev Inequality

For any $\rho \in \mathcal{P}_{ac}(\Omega)$ with $\operatorname{supp}\rho \subset \Omega$ and $P_F(\rho) \in W^{1,\infty}(\Omega)$

$$\mathrm{H}_{V-x\cdot\nabla V}^{F+nP_F,\, W-2x\cdot\nabla W}(\rho) + \frac{\lambda+\nu}{2} W_2^2(\rho, \rho_{V+c}) - \frac{\nu}{2}|\mathrm{b}(\rho_0) - \mathrm{b}(\rho_{V+c})|^2 \le \mathcal{I}_{c^*}(\rho|\rho_V) - \mathrm{H}^{P_F,W}(\rho_{V+c}) + K_{V+c},$$

where K_{V+c} is a constant such that:

$$F'(\rho_{V+c}) + V + c + W \star \rho_{V+c} = K_{V+c}$$

and $\int_\Omega \rho_{V+c} = 1$.

If V and W convex, then there is constant K, such that for any ρ,

$$\mathrm{H}_{V-x\cdot\nabla V}^{F+nP_F,\, W-2x\cdot\nabla W}(\rho) \le \mathcal{I}_{c^*}(\rho|\rho_V) + K$$

General Euclidean Sobolev Inequality

Assume $V = W = 0$, then for any Young function c and any $\rho \in \mathcal{P}_{ac}(\Omega)$ with $\operatorname{supp}\rho \subset \Omega$ and $P_F(\rho) \in W^{1,\infty}(\Omega)$

$$\mathrm{H}^{F+nP_F}(\rho) \leq \int_\Omega \rho c^\star \left(-\nabla(F' \circ \rho)\right) \mathrm{d}x + K_c,$$

where K_c is the constant determined by

$$F'(\rho_c) + c = K_c \text{ and } \int_\Omega \rho_c = 1.$$

Dirichlet Integrals are Entropy productions!

Euclidean Log-Sobolev inequality

Let $c : \mathbb{R}^n \to \mathbb{R}$ be Young such that $c^\star$ is p-homogeneous ($p > 1$). Then,

$$\int_{R^n} \rho \ln \rho \, dx \le \frac{n}{p} \ln \left(\frac{p}{n e^{p-1} \sigma_c^{p/n}} \int_{R^n} \rho c^\star \left(-\frac{\nabla \rho}{\rho} \right) dx \right)$$

for all probability densities $\rho \in W^{1,\infty}(\mathbb{R}^n)$. Here $\sigma_c := \int_{R^n} e^{-c} \, dx$. Equality holds if $\rho(x) = \left(\int_{R^n} e^{-\lambda^q c(x)} \, dx \right)^{-1} e^{-\lambda^q c(x)}$ for some λ.

Proof: Use $F(x) = x \ln(x)$ and $V = W = 0$. Note that $P_F(x) = x$ and $H^{P_F}(\rho) = 1$. So, $\rho_c(x) = \frac{e^{-c(x)}}{\sigma_c}$ and

$$\int_\Omega \rho \ln \rho \, dx \le \int_{R^n} \rho c^\star \left(-\frac{\nabla \rho}{\rho} \right) dx - n - \ln \left(\int_{R^n} e^{-c(x)} \, dx \right),$$

with equality when $\rho = \rho_c$.

When $c^\star$ is p-homogeneous, scale with $c_\lambda(x) := c(\lambda x)$ and minimize.

Optimal Euclidean p-Log Sobolev inequality

Beckner ($p = 1$), Dolbeault-DelPino ($p < n$), Gentil ($p > 1$)

$$\int_{R^n} |\, f \,|^p \ln(|\, f \,|^p)\, \mathrm{d}x \leq \frac{n}{p} \ln\left(C_p \int_{R^n} |\, \nabla f \,|^p \, \mathrm{d}x \right),$$

holds for all $p \geq 1$, and for all $f \in W^{1,p}(\mathbb{R}^n)$ such that $\|\, f \,\|_p = 1$, where

$$C_p := \begin{cases} \left(\frac{p}{n}\right) \left(\frac{p-1}{e}\right)^{p-1} \pi^{-\frac{p}{2}} \left[\frac{\Gamma(\frac{n}{2}+1)}{\Gamma(\frac{n}{q}+1)}\right]^{\frac{p}{n}} & \text{if} \quad p > 1, \\ \frac{1}{n\sqrt{\pi}} \left[\Gamma(\frac{n}{2}+1)\right]^{\frac{1}{n}} & \text{if} \quad p = 1, \end{cases}$$

and q is the conjugate of p ($\frac{1}{p} + \frac{1}{q} = 1$).

For $p > 1$, equality holds for $f(x) = K e^{-\lambda^q \frac{|\, x - \bar{x} \,|^q}{q}}$ for some $\lambda > 0$ and $\bar{x} \in \mathbb{R}^n$, where $K = \left(\int_{R^n} e^{-(p-1)|\, \lambda x \,|^q}\, \mathrm{d}x\right)^{-1/p}$.

Gagliardo-Nirenberg inequalities

Let $1 < p < n$ and $r \in \left(0, \frac{np}{n-p}\right)$ such that $r \neq p$. Set $\gamma := \frac{1}{r} + \frac{1}{q}$, where $\frac{1}{p} + \frac{1}{q} = 1$. Then, for any $f \in W^{1,p}(\mathbb{R}^n)$ we have

$$\|f\|_r \leq C(p,r)\|\nabla f\|_p^{\theta}\,\|f\|_{r\gamma}^{1-\theta}, \tag{1}$$

where θ is given by

$$\frac{1}{r} = \frac{\theta}{p^*} + \frac{1-\theta}{r\gamma}, \tag{2}$$

$p^* = \frac{np}{n-p}$. Best constant $C(p,r) > 0$ can be obtained by scaling.

Proof: $V = W = 0$ and $F(x) = \frac{x^\gamma}{\gamma-1}$, where $1 \neq \gamma > 1 - \frac{1}{n}$ and $c(x) = \frac{r\gamma}{q}|\,x\,|^q$ so that for $\|\,f\,\|_r = 1$,

$$\left(\frac{1}{\gamma-1} + n\right)\int_{R^n} |\,f\,|^{r\gamma} \leq \frac{r\gamma}{p}\int_{R^n} |\,\nabla f\,|^p - H^{P_F}(\rho_\infty) + C_\infty.$$

where $\rho_\infty = h_\infty^r$ satisfies

$$-\nabla h_\infty(x) = x|\,x\,|^{q-2} h^{\frac{r}{p}}(x) \;\text{ a.e.},$$

and where C_∞ insures that $\int h_\infty^r = 1$. The constants on the right hand side are not easy to calculate, so one can obtain θ and the best constant by scaling. Rewrite as

$$\frac{r\gamma}{p} \frac{\|\nabla f\|_p^p}{\|f\|_r^p} - \left(\frac{1}{\gamma - 1} + n \right) \frac{\|f\|_{r\gamma}^{r\gamma}}{\|f\|_r} \geq H^{P_F}(\rho_\infty) - C_\infty$$

Scale with $f_\lambda(x) = f(\lambda x)$ for $\lambda > 0$. A minimization over λ gives the required constant.

Sobolev inequalities

If $1 < p < n$, then for any $f \in W^{1,p}(\mathbb{R}^n)$, $\| f \|_{p^*} \leq C(p,n) \| \nabla f \|_p$.

Proof: Use $\gamma = 1 - \frac{1}{n}$ and $r = p^*$.

$$1 = \| f \|_{p^*} \leq \left(\frac{r\gamma}{p\left[H^{P_F}(\rho_\infty) - C_\infty\right]} \right)^{1/p} \| \nabla f \|_p$$

which shows that

$$C(p,n) = \left(\frac{p^*(n-1)}{np\left[H^{P_F}(\rho_\infty) - C_\infty\right]} \right)^{1/p},$$

where $\rho_\infty = h_\infty^{p^*} = \left(\frac{p^*}{nq} | x |^q - \frac{C_\infty}{n-1} \right)^{-n}$ and C_∞ can be found using that ρ_∞ is a probability density,

$$C_\infty = (1-n) \left[\int_{R^n} \left(\frac{p^*}{nq} | x |^q + 1 \right)^{-n} \mathrm{d}x \right]^{p/n}.$$

Generalized Gross Log Sobolev inequalities

Recall that

$$\mathrm{H}_{V+c}^{F,W}(\rho_0|\rho_1)+\frac{\lambda+\nu}{2}W_2^2(\rho_0,\rho_1)-\frac{\nu}{2}|\mathrm{b}(\rho_0)-\mathrm{b}(\rho_1)|^2 \le \mathrm{H}_{c+\nabla V\cdot x}^{-nP_F,2x\cdot\nabla W}(\rho_0)+\mathcal{I}_{c^*}(\rho_0|\rho_V).$$

Second term simplifies considerably in the case where c is a quadratic Young function

$$c(x) := c_\sigma(x) = \frac{1}{2\sigma}|\,x\,|^2 \quad \text{for } \sigma > 0.$$

In this case, we have the following identity:

$$\mathcal{I}_{c_\sigma^*}(\rho_0|\rho_V) + H_{c_\sigma+x\cdot\nabla V}^{-nP_F,2x\cdot\nabla W}(\rho_0) = \mathcal{I}_{c_\sigma^*}(\rho_0|\rho_{V+c_\sigma}) = \frac{\sigma}{2}I_2(\rho_0|\rho_{V+c_\sigma}).$$

so that

$$\mathrm{H}_{V+c_\sigma}^{F,W}(\rho_0|\rho_1)+\frac{\lambda+\nu}{2}W_2^2(\rho_0,\rho_1)-\frac{\nu}{2}|\mathrm{b}(\rho_0)-\mathrm{b}(\rho_1)|^2 \le \frac{\sigma}{2}I_2(\rho_0|\rho_{V+c_\sigma})$$

General Logarithmic Sobolev Inequality

For all probability densities ρ_0 and ρ_1 on Ω, satisfying $\operatorname{supp}\rho_0 \subset \Omega$, and $P_F(\rho_0) \in W^{1,\infty}(\Omega)$, we have for any $\sigma > 0$,

$$\mathrm{H}_V^{F,W}(\rho_0|\rho_1)+\frac{1}{2}(\lambda+\nu-\frac{1}{\sigma})W_2^2(\rho_0,\rho_1)-\frac{\nu}{2}|\mathrm{b}(\rho_0)-\mathrm{b}(\rho_1)|^2 \le \frac{\sigma}{2}I_2(\rho_0|\rho_V).$$

General Bakry-Emery Inequality

If $W = 0$, $\lambda > 0$, $\sigma = \frac{1}{\lambda}$ then

$$\mathrm{H}_V^F(\rho_0|\rho_1) \le \frac{1}{2\lambda}I_2(\rho_0|\rho_V).$$

HWBI inequality for interactive gases

Minimizing over $\sigma > 0$:

$$\mathrm{H}_V^{F,W}(\rho_0|\rho_1) \le W_2(\rho_0,\rho_1)\sqrt{I_2(\rho_0|\rho_V)}-\frac{\lambda+\nu}{2}W_2^2(\rho_0,\rho_1)+\frac{\nu}{2}|\mathrm{b}(\rho_0)-\mathrm{b}(\rho_1)|^2.$$

This extends the HWI inequality (Otto-Villani).

Generalized Talagrand Inequality with interaction potentials

Assume $\mu + \nu > 0$. Then for all probability densities ρ on Ω, we have

$$\frac{\nu + \mu}{2} W_2^2(\rho, \rho_U) - \frac{\nu}{2} |\mathrm{b}(\rho) - \mathrm{b}(\rho_U)|^2 \leq \mathrm{H}_U^{F,W}(\rho|\rho_U).$$

In particular, if $b(\rho) = b(\rho_U)$, we have that

$$W_2(\rho, \rho_U) \leq \sqrt{\frac{2\mathrm{H}_U^{F,W}(\rho|\rho_U)}{\mu + \nu}}.$$

If W is convex, then

$$W_2(\rho, \rho_U) \leq \sqrt{\frac{2\mathrm{H}_U^{F,W}(\rho|\rho_U)}{\mu}}.$$

Trend to equilibrium

Let $\rho(t)$ be a solution of $\frac{\partial\rho}{\partial t} = \operatorname{div}\left\{\rho\nabla\left(F'(\rho) + V + W \star \rho\right)\right\}$ starting at ρ_0 which has finite total energy. Then

(1). If $V+W$ is unif. convex (i.e., $\lambda+\nu > 0$) and W is convex, then:

$$\mathrm{H}_V^{F,W}\left(\rho(t)|\rho_V\right) \leq e^{-2\lambda t}\mathrm{H}_V^{F,W}(\rho_0|\rho_V),$$

$$W_2\left(\rho(t),\rho_V\right) \leq e^{-\lambda t}\sqrt{\frac{2\mathrm{H}_V^{F,W}(\rho_0|\rho_V)}{\lambda}}.$$

(2). If $V+W$ is unif. convex $(\lambda+\nu > 0)$ and if we assume the barycentre of the solution $\rho(t,x)$ is invariant in t, then,

$$\mathrm{H}_V^{F,W}\left(\rho(t)|\rho_V\right) \leq e^{-2(\lambda+\nu)t}\mathrm{H}_V^{F,W}(\rho_0|\rho_V),$$

and

$$W_2\left(\rho(t),\rho_V\right) \leq e^{-2(\lambda+\nu)t}\sqrt{\frac{2\mathrm{H}_V^{F,W}(\rho_0|\rho_V)}{\lambda+\nu}}.$$

Energy-Entropy production Duality

For any probability density $\rho_0 \in W^{1,\infty}(\Omega)$, and any $\rho_1 \in \mathcal{P}_{ac}(\Omega)$,

$$-\mathrm{H}_c^F(\rho_1) \le -\mathrm{H}^{F+nP_F}(\rho_0) + \int_\Omega \rho_0 c^\star \left(-\nabla(F' \circ \rho_0)\right) \mathrm{d}x.$$

Equality holds whenever $\rho_0 = \rho_1 = \rho_c$ where ρ_c is a probability density on Ω such that $\nabla(F'(\rho_c) + c) = 0$ a.e.

$$\sup\{J(\rho);\ \int_\Omega \rho(x)dx = 1\} \le \inf\{I(f);\ \int_\Omega \psi(f(x))dx = 1\}$$

where

$$I(f) = \int_\Omega \left[c^*(-\nabla f(x)) - G\left(\psi \circ f(x)\right)\right] dx$$

and

$$J(\rho) = -\int_\Omega [F(\rho(y)) + c(y)\rho(y)]dy$$

with $G(x) = (1-n)F(x) + nxF'(x)$ and where ψ is computable from F and c.

Equality holds whenever there exists $\bar{f}$ (and $\bar{\rho} = \psi(\bar{f})$) that satisfies:

$$-(F' \circ \psi)'(\bar{f})\nabla \bar{f}(x) = \nabla c(x) \text{ a.e.}$$

The extrema are achieved at $\bar{f}$ (resp. $\bar{\rho} = \psi(\bar{f})$). Hence a solution for the quasilinear (or semi-linear) equation

$$\text{div}\{\nabla c^*(-\nabla f)\} - (G \circ \psi)'(f) = \psi'(f)$$

since it is the L^2-Euler-Lagrange equation of I on

$$\{f \in C_0^\infty(\Omega); \int_\Omega \psi(f(x))dx = 1\}.$$

Interesting: To the latter quasi-linear (or semi-linear) eqaution, we can associate J, a Free Energy functional on $\mathcal{P}_{ac}(\Omega)$, whose gradient flow with respect to the Wasserstein distance is the evolution equation

$$\frac{\partial u}{\partial t} = \text{div}\{u\nabla(F'(u) + c)\}$$

A remarkable correspondence

(1) General (nonlinear) Fokker-Planck evolution equations
(2) Quasilinear or semi-linear equations
The latter appear as Euler-Lagrange equations of entropy production functionals associated to the free energies.
(Cordero-Nazareth-Villani) For $p = 2^* = \frac{2n}{n-2}$,

$$\inf\left\{ \left(\frac{n-1}{n-2}\right)^2 \int_{R^n} |\nabla f|^2 dx;\ \int_{R^n} |f|^p dx = 1 \right\}$$

is equal to:

$$\sup\left\{ n \int_{R^n} |u(x)|^{\frac{p(n-1)}{n}} dx - \frac{1}{2} \int_{R^n} |x|^2 |u(x)|^p dx; \int_{R^n} |u|^p dx = 1 \right\}.$$

The latter is really:

$$\sup\{ n \int_{R^n} \rho(x)^{\frac{n-1}{n}} dx - \frac{1}{2} \int_{R^n} |x|^2 \rho(x) dx; \int_{R^n} \rho dx = 1 \}$$

Equality is achieved at U that satisfies:

$$\frac{2(n-1)}{n-2} U^{-\frac{p}{2}} \nabla U(x) = x$$

Looks like a correspondence between the "Yamabe" equation

$$-\Delta f = |f|^{2^*-2} f \text{ on } \mathbb{R}^n,$$

where $2^* = \frac{2n}{n-2}$ is the critical Sobolev exponent, and the non-linear Fokker-Planck equation

$$\frac{\partial u}{\partial t} = \Delta u^{1-\frac{1}{n}} + \operatorname{div}(x.u),$$

which –after appropriate scaling– reduces to the fast diffusion equation:

$$\frac{\partial u}{\partial t} = \Delta u^{1-\frac{1}{n}}.$$

Evolution of energies along Wasserstein geodesics

Let $\Omega \subset \mathbb{R}^n$ be open, bounded and convex, ρ_0, ρ_1 two probability densities on Ω, with $\operatorname{supp} \rho_0 \subset \Omega$, and $P_F(\rho_0) \in W^{1,\infty}(\Omega)$. Let T be the optimal map that pushes ρ_0 forward to ρ_1. Then

1) (Otto-Agueh) If $F(0) = 0$ and $x \mapsto x^n F(x^{-n})$ is convex and non-increasing, then

$$\mathrm{H}^F(\rho_1) - \mathrm{H}^F(\rho_0) \geq \int_\Omega \rho_0 (T - I) \cdot \nabla \left(F'(\rho_0)\right) \mathrm{d}x$$

2) Let $V : \mathbb{R}^n \to \mathbb{R}$ with $D^2 V \geq \lambda I$, then

$$\mathrm{H}_V(\rho_1) - \mathrm{H}_V(\rho_0) \geq \int_\Omega \rho_0 (T - I) \cdot \nabla V \mathrm{d}x + \frac{\lambda}{2} W_2^2(\rho_0, \rho_1).$$

3) (Cordero-Gangbo-Houdre) $W : \mathbb{R}^n \to \mathbb{R}$ is even, and

$D^2W \geq \nu I$, then

$$\begin{aligned} \mathrm{H}^W(\rho_1) - \mathrm{H}^W(\rho_0) &\geq \int_\Omega \rho_0 (T - I) \cdot \nabla (W \star \rho_0) \mathrm{d}x \\ &+ \frac{\nu}{2} W_2^2(\rho_0, \rho_1) \\ &- \frac{\nu}{2} |\mathrm{b}(\rho_0) - b(\rho_1)|^2. \end{aligned}$$

Use Young's inequality to get

$$\begin{aligned} &-\nabla \left(F'\left(\rho_0(x)\right) + V(x) + (W \star \rho_0)(x) \right) \cdot T(x) \\ &\qquad \leq c\left(T(x)\right) + c^\star \left(-\nabla \left(F'(\rho_0(x)) + V(x) + (W \star \rho_0)(x) \right) \right), \end{aligned}$$

and again by transport that

$$\int_\Omega c(Tx) \rho_0 \,\mathrm{d}x = \int_\Omega c(y) \rho_1 \,\mathrm{d}y.$$