Diffusion Generated Motion in Vision Applications

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Threshold Dynamics

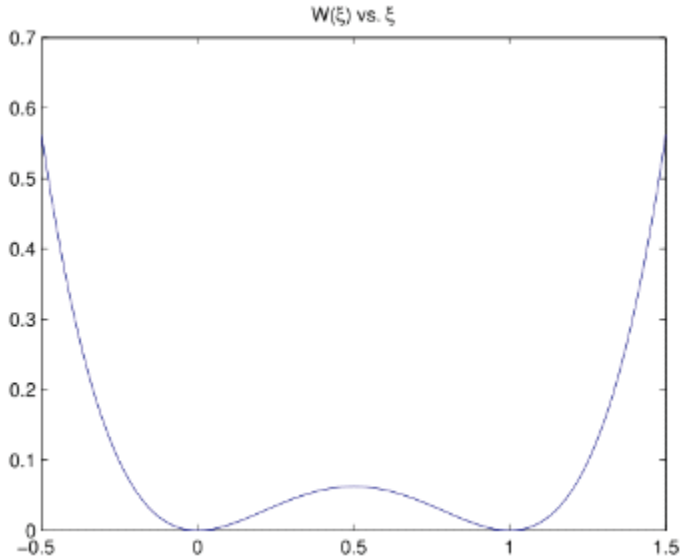
- Merriman-Bence-Osher (1992)
- To address stiffness of Allen-Cahn, time split the equation:
  - Step 1:

    $$u_t = \Delta u$$

  - Step 2:

    $$u_t = -\frac{1}{\varepsilon^2} W'(u)$$

- When $t \to \infty$, Step 2 turns into thresholding:



$$\lim_{t\to\infty} u(x,t) = \begin{cases} 0 & \text{if } u(x,0) < \frac{1}{2}, \\ 1 & \text{if } u(x,0) > \frac{1}{2}. \end{cases}$$

Threshold Dynamics

Fix a time step size $\delta t > 0$ and generate a discrete in time approximation $\{\Sigma^k\}_{k=0}^{\infty}$ to the flow as follows:

- MBO Algorithm:

  1. Convolution Step:

$$u(x,t) = \mathbf{1}_{\Sigma^k} * G_{\delta t} \quad \text{where} \quad G_t(x) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}}$$

  2. Thresholding Step:

$$\Sigma^{k+1} = \left\{ x \in \boldsymbol{R}^n : u(x) > \frac{1}{2} \right\}.$$

Threshold Dynamics

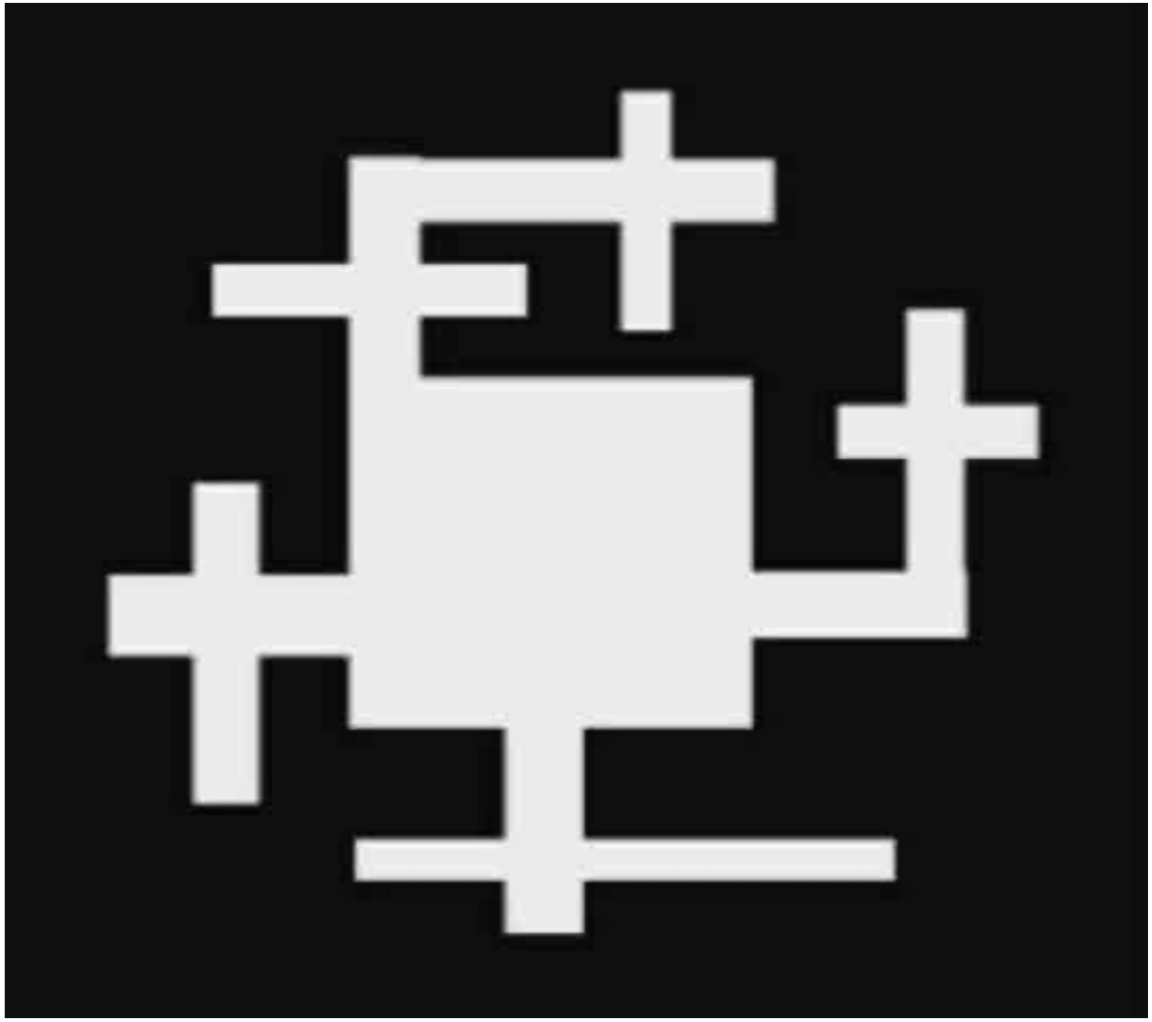
Benefits:

- Unconditionally stable:
  - Accuracy only concern in choosing time step size.
- Fast implementation:
  - Complexity only $O(N \log N)$, where N is the total number of grid points.
- Implicit representation of the front, allowing automatic topology changes.

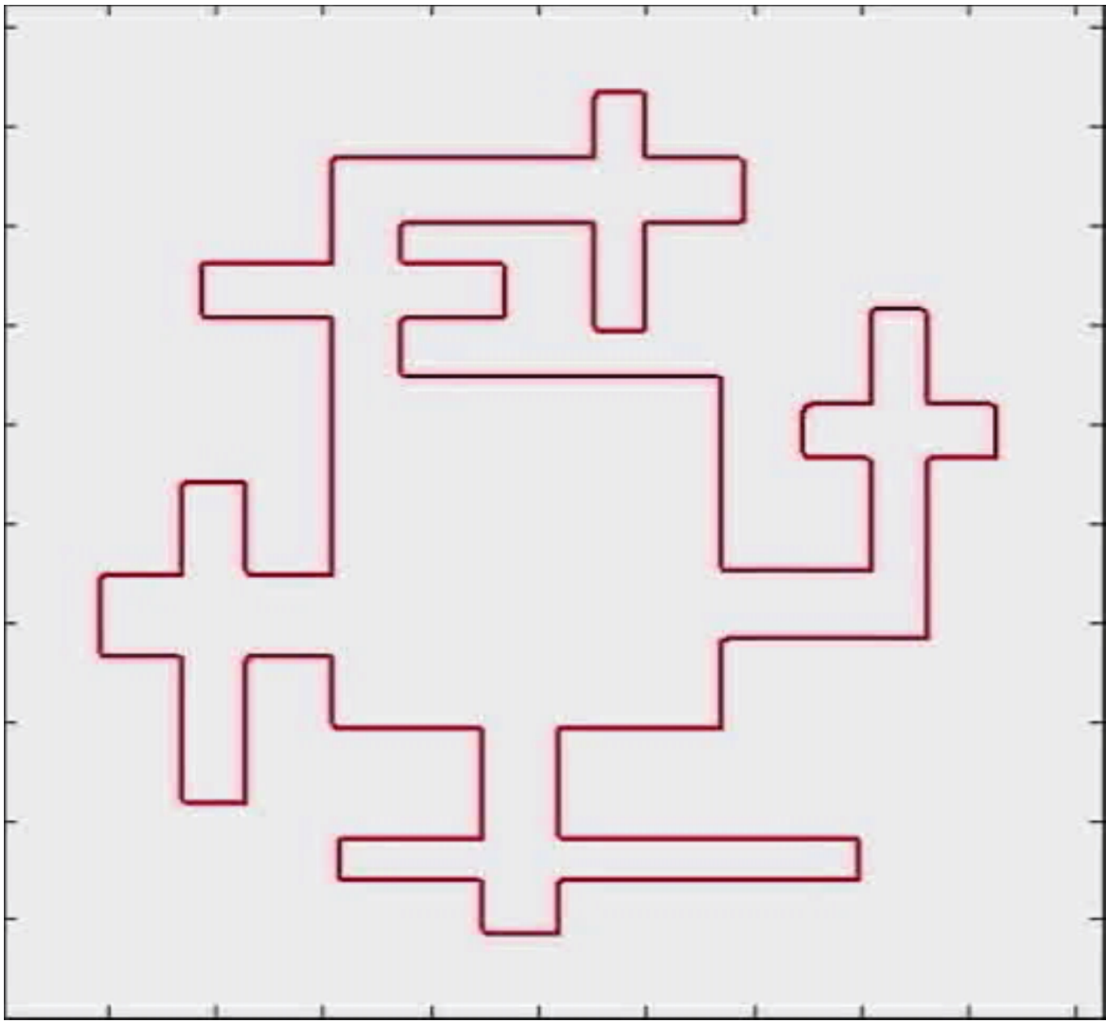
Caveats:

- No sub-grid accuracy.
- Dynamics can be very inaccurate on uniform grids.
- Can get "stuck" if δt too small.

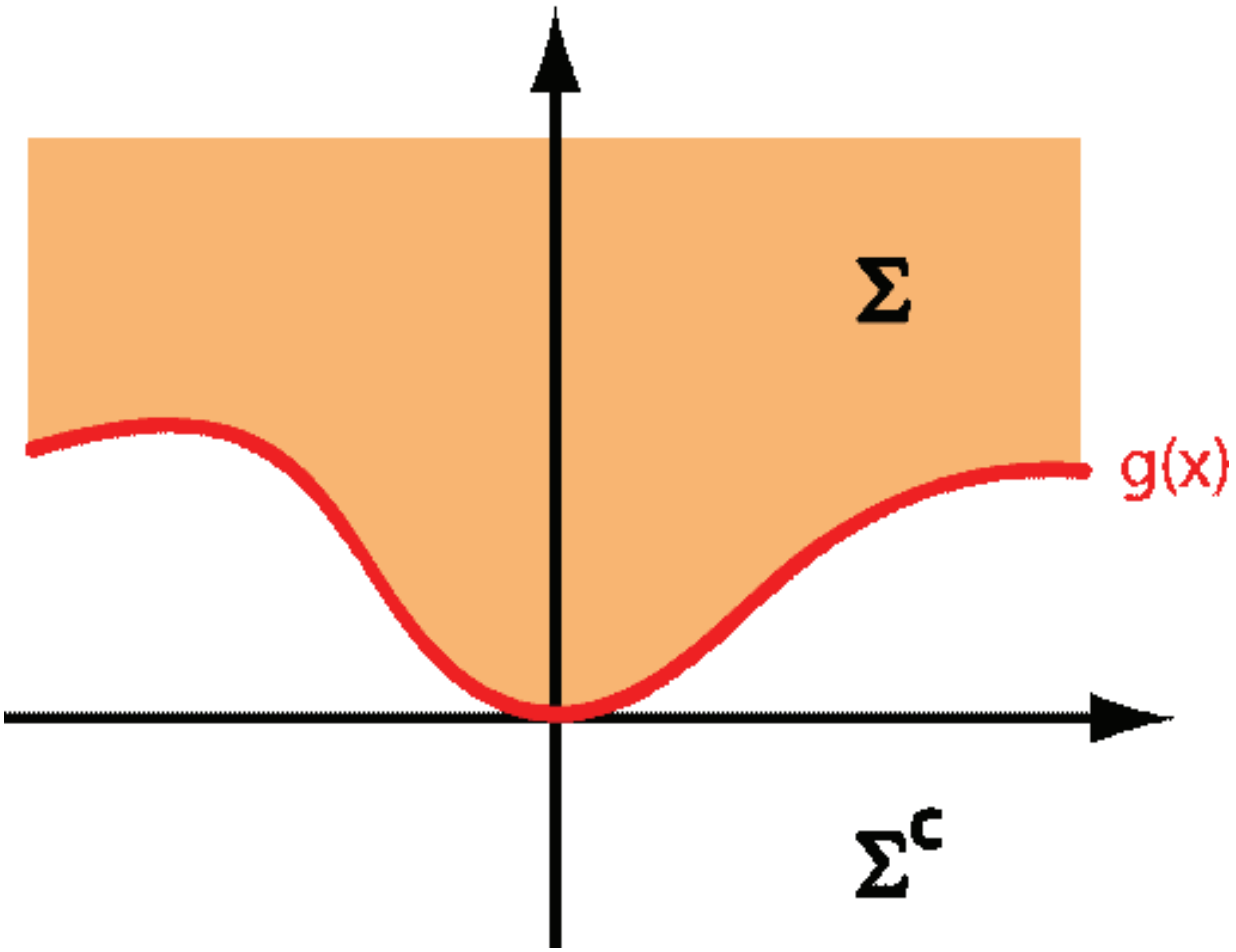
Threshold Dynamics



Threshold Dynamics



Threshold Dynamics



Threshold Dynamics

- Mascarenhas calculated the following: Let

$$F(x, y, t) = (\mathbf{1}_{\Sigma} * G_t)(x, y).$$

- Express the convolution in terms of $g(\xi)$.
- Taylor expand $F(0, y, t)$ at $y = 0$:

$$F(0, y, t) = \frac{1}{2} - \frac{1}{2\sqrt{\pi}} y t^{-\frac{1}{2}} + \frac{1}{2\sqrt{\pi}} g''(0) t^{\frac{1}{2}} + O\left(t^{\frac{3}{2}}\right)$$

  provided that $y = O(t)$ as $t \to 0^+$.

- At the origin, we have

$$g_{ss} = g'' = \kappa$$

Threshold Dynamics

- Thresholding is equivalent to setting

$$F(0, y, t) = \frac{1}{2}$$

which gives

$$\frac{1}{2} = F(0, y, t) \approx \frac{1}{2} - \frac{1}{2\sqrt{\pi}} y\, t^{-\frac{1}{2}} + \frac{1}{2\sqrt{\pi}} \kappa t^{\frac{1}{2}}$$

- Solving for y:

$$y \approx \kappa t.$$

- UPSHOT: Interface moves with normal speed κ for duration t in the normal direction.

Threshold Dynamics

- Mascarenhas also proposed the generalization:

$$v_n = \kappa + \alpha$$

1. Step 1: Convolution.

$$u_{k+1} = (\mathbf{1}_{\Sigma_k} * G_t)$$

2. Step 2: Dilation.

$$u_{k+1} \to u_{k+1} + \alpha\sqrt{\delta t}$$

3. Step 3: Thresholding.

$$\Sigma_{k+1} = \left\{x : u_{k+1}(x) \geq \frac{1}{2}\right\}.$$

Threshold Dynamics for Mumford-Shah

- Recall normal speed for P.C. Mumford-Shah:

$$v_n = \kappa + \lambda\big((f - c_2)^2 - (f - c_1)^2\big)$$

- One option for threshold dynamics:

1. Step 1: Convolution.

$$u_{k+1} = \mathbf{1}_{\Sigma_k} * G_{\delta t}$$

2. Step 2: Dilation.

$$u_{k+1} \to u_{k+1} + \lambda\sqrt{\delta t}\big((f - c_2)^2 - (f - c_1)^2\big)$$

3. Step 3: Thresholding.

$$\Sigma_{k+1} = \left\{x : u_{k+1} \geq \frac{1}{2}\right\}$$

- Doesn't work well: Time splitting error too large for large λ.

Threshold Dynamics for Mumford-Shah

- Joint work with Richard Tsai.
- Time split the phase field approximation of P.C. Mumford-Shah:

$$u_t = \varepsilon \Delta u - \frac{1}{\varepsilon} W'(u) - \lambda (f - c_1)^2 u + \lambda (f - c_2)^2 (1 - u)$$

1. Step 1: Solve a linear PDE:

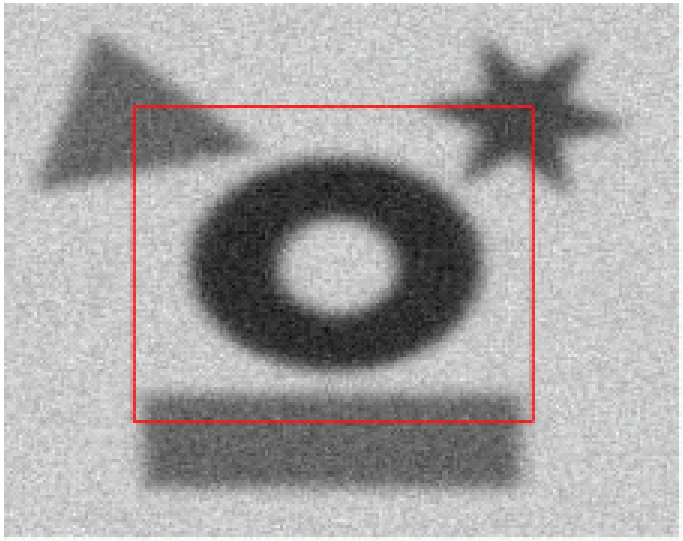
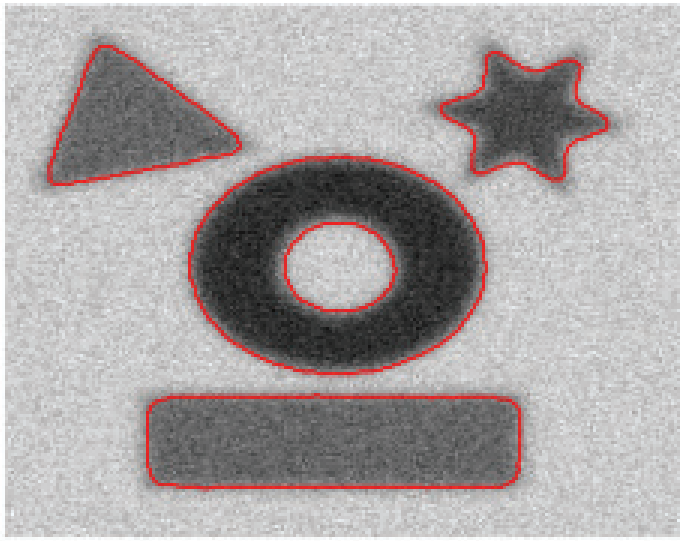
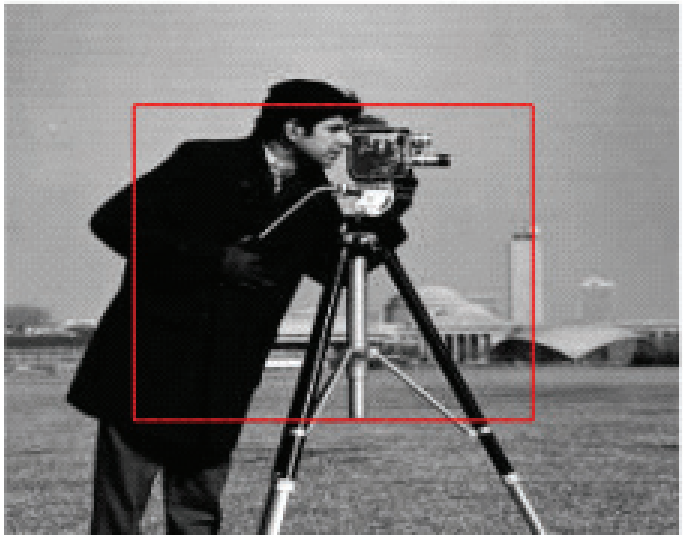
$$v_t = \Delta v - \frac{\lambda}{\sqrt{\pi \delta t}} \left(v (c_1 - f)^2 + (v - 1)(c_2 - f)^2 \right)$$

$$v(x, 0) = u_n(x)$$

2. Step2: Threshold:

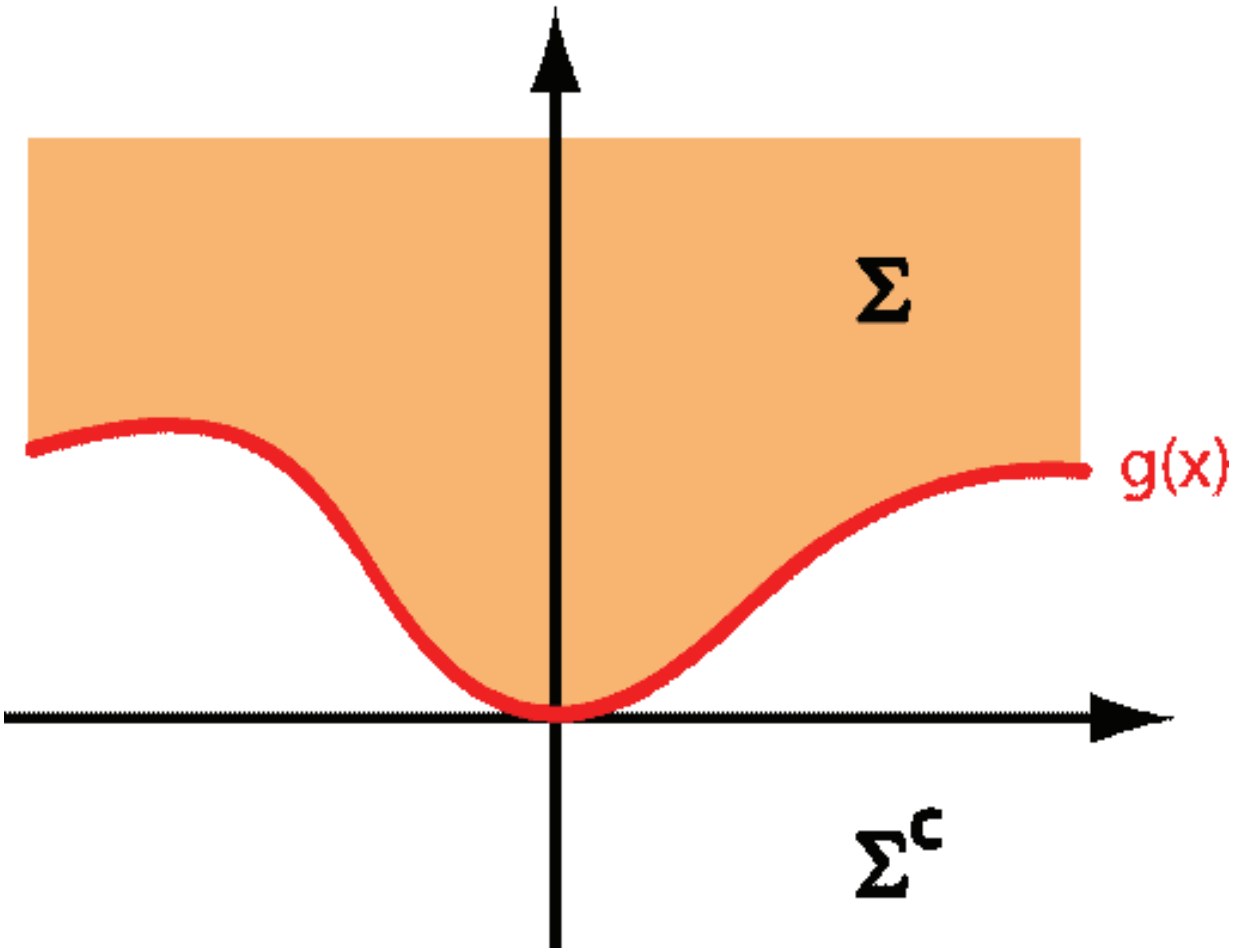
$$u_{n+1}(x) = \begin{cases} 0 & \text{if } \; v(x, \delta t) \leq \frac{1}{2}, \\ 1 & \text{if } \; v(x, \delta t) > \frac{1}{2}. \end{cases}$$

Threshold Dynamics for Mumford-Shah


These computations take just a few iterations.

Threshold Dynamics: Extensions



Threshold Dynamics: Extensions

- For the elastica energy

$$\int_{\partial\Sigma} \kappa^2 \, d\sigma$$

- L^2 gradient descent is known as Willmore flow.
- Normal speed of the curve:

$$v_n = \boxed{W = -\kappa_{ss} - \frac{1}{2}\kappa^3.}$$

- Extension of threshold dynamics to Willmore flow:
  - Grzibovskis, Heintz. *A convolution thresholding scheme for the Willmore functional*. Interfaces and Free Boundaries (2008).

Threshold Dynamics: Extensions

- Following Mascarenhas (1992), Ruuth (1996), let

$$F(x, y, t) = (\mathbf{1}_{\Sigma} * G_t)(x, y)$$

- Express the convolution in terms of $g(\xi)$.
- Taylor expand $g(\xi)$ at $\xi = 0$:

$$F(0, y, t) = \frac{1}{2} - \frac{1}{2\sqrt{t}} y t^{-\frac{1}{2}} + \frac{1}{2\sqrt{t}} g''(0) t^{\frac{1}{2}} + O\left(t^{\frac{3}{2}}\right)$$

provided that $y = O(t)$ as $t \to 0$.

Threshold Dynamics: Extensions

- If we continue Mascarenhas' expansion, we find:

$$F(0,y,t) = \frac{1}{2} - \frac{1}{2\sqrt{t}} y t^{-\frac{1}{2}} + \frac{1}{2\sqrt{t}} g''(0) t^{\frac{1}{2}}$$

$$+ \frac{1}{4\sqrt{\pi}} g^{(4)}(0) t^{\frac{3}{4}} - \frac{5}{8\sqrt{\pi}} (g''(0))^3 t^{\frac{3}{2}} + O(t^{\frac{5}{2}})$$

provided once again that $y = O(t)$.

- Error term in curvature motion to leading order is:

$$\left(\frac{1}{4\sqrt{\pi}} g^{(4)}(0) - \frac{5}{8\sqrt{\pi}} (g''(0))^3 \right) t.$$

Threshold Dynamics: Extensions

- At the origin,

$$g''(0) = g_{ss}(0) = \kappa(0),$$

and

$$g^{(4)}(0) = \kappa_{ss} + 3\kappa^3.$$

Therefore,

$$W = -g^{(4)}(0) + \frac{5}{2}\kappa^3(0).$$

That means:

$$F(0, y, t) = \frac{1}{2} - \frac{1}{2\sqrt{\pi}} y t^{-\frac{1}{2}} + \frac{1}{2\sqrt{\pi}} \kappa t^{\frac{1}{2}} + \frac{1}{2\sqrt{\pi}} W t^{\frac{3}{2}} + O\left(t^{\frac{5}{2}}\right).$$

Threshold Dynamics: Extensions

- Grzibovskis & Heintz on Willmore flow:
  - Take convolutions using two different Gaussians,
  - Take the correct linear combination between the two convolutions so that lower order curvature terms cancel out.
  - Threshold.
- Algorithm:

1. Form the convolution:

$$A(x,y) = (2\delta t)^{\frac{1}{4}}\mathbf{1}_{\Sigma^k} * \left(2G_{\frac{\sqrt{2\delta t}}{4}} - \frac{1}{2}G_{4\sqrt{2\delta t}}\right)$$

2. Set:

$$\Sigma^{k+1} = \left\{(x,y) : A(x,y) \geq \frac{3(2\delta t)^{\frac{1}{4}}}{4}\right\}$$

Threshold Dynamics: Extensions

- Joint work with Ruuth and Tsai:
  - Esedoglu, Ruuth, Tsai. *Threshold dynamics for disocclusion and shape reconstruction.* Proc. of ICIP (2005).
  - Esedoglu, Ruuth, Tsai. *Threshold dynamics for high order geometric motions.* Interfaces and Free Boundaries (2008).
- Extensions to:
  - Willmore + Lower order terms:

$$v_n = c_0(x, y, t) + c_{1(x,y,t)}\kappa + W$$

  - Surface diffusion:

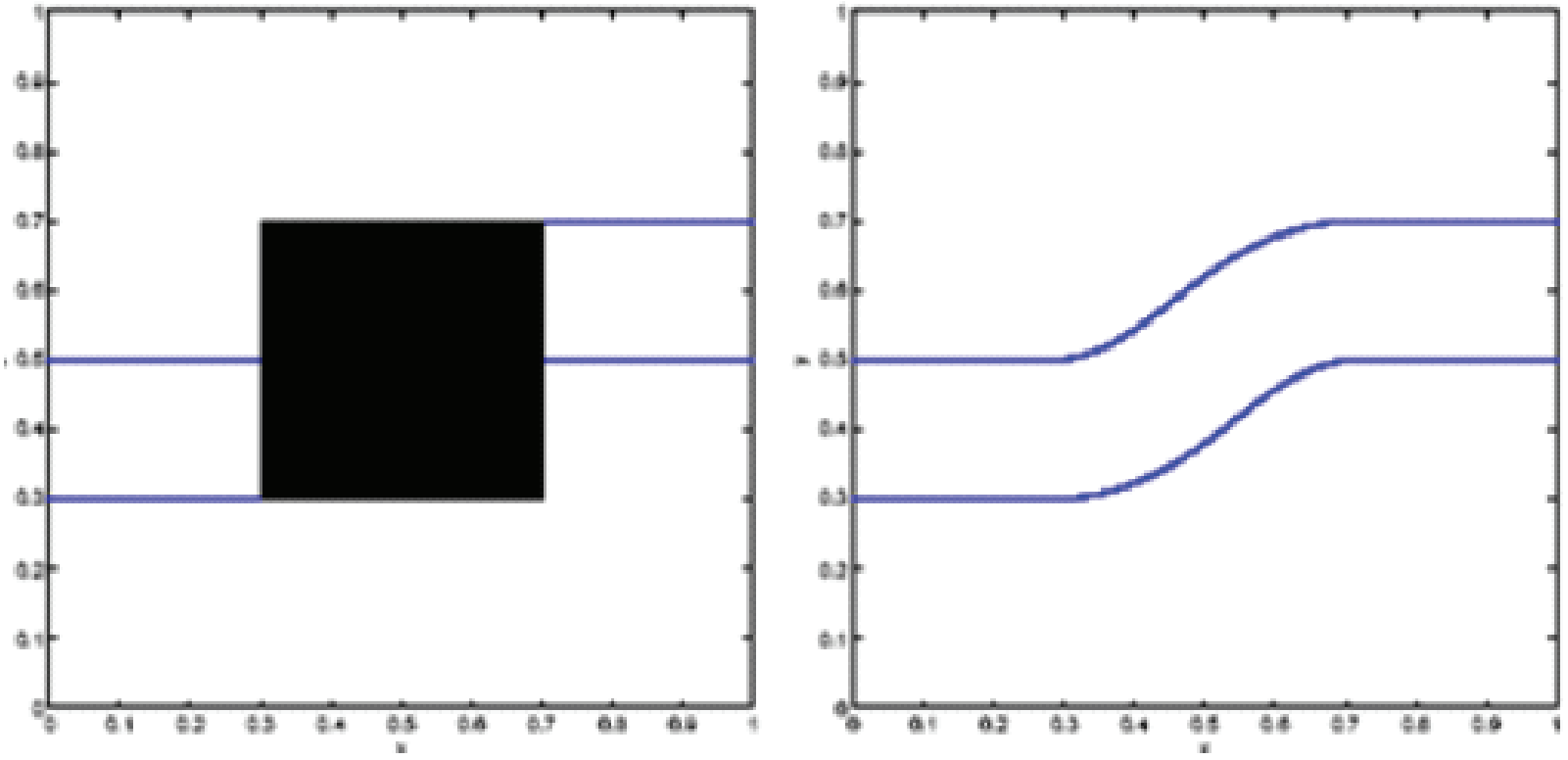
$$v_n = \kappa_{ss}.$$

- Applications to inpainting.

Threshold Dynamics: Extensions

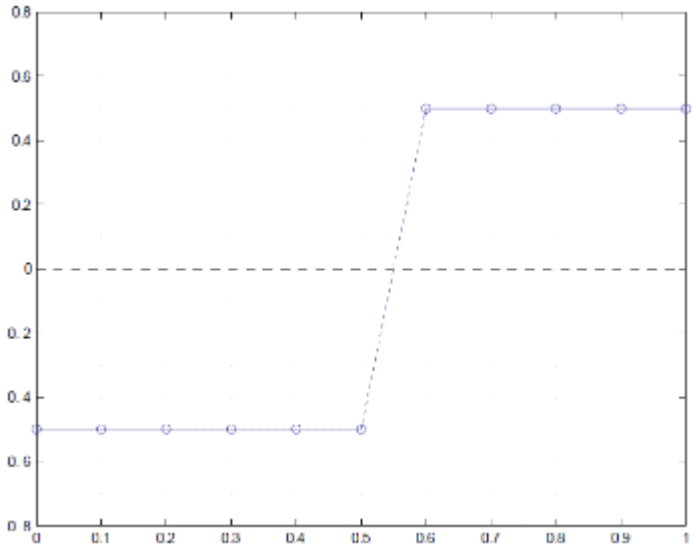
An inpainting example that involves the normal speed:

$$v_n = W + \alpha\kappa + \beta(x, y).$$



Distance Function Dynamics

- Joint work with Ruuth and Tsai (2009):
  - Esedoglu, Ruuth, Tsai. *Diffusion generated motion using the signed distance function.* Journal of Computational Physics (2009).
- Motivation:
  - Threshold dynamics is very inaccurate on uniform grids.
  - Cannot interpolate to locate interface at subgrid resolution.



Distance Function Dynamics

- New version of the algorithm:
  - Represent Σ not by $\mathbf{1}_\Sigma(x)$, but by its signed distance function.

$$d_\Sigma(x) = \begin{cases} \inf_{y \in \Sigma^c} |x - y| & \text{if } x \in \Sigma, \\ -\inf_{y \in \Sigma} |x - y| & \text{if } x \in \Sigma^c. \end{cases}$$

- Alternate two steps:

1. Convolution:

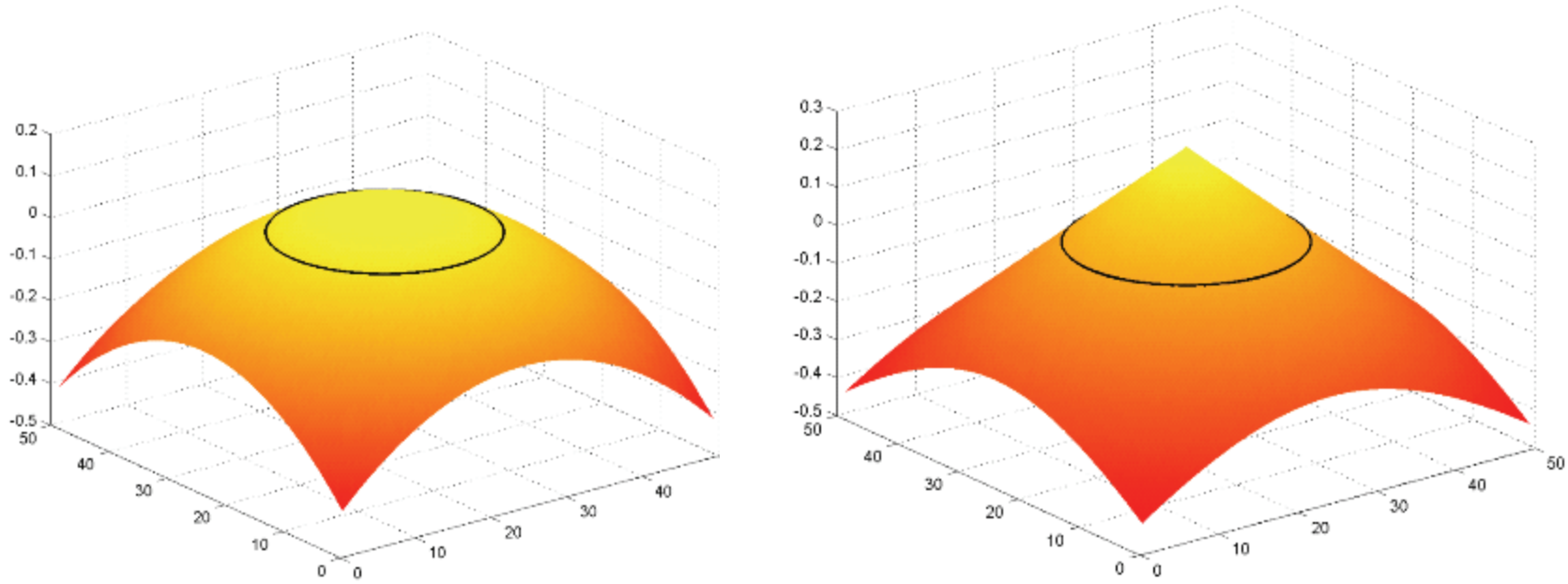
$$u(x) = G_{\delta t} * d_\Sigma$$

2. Redistancing:

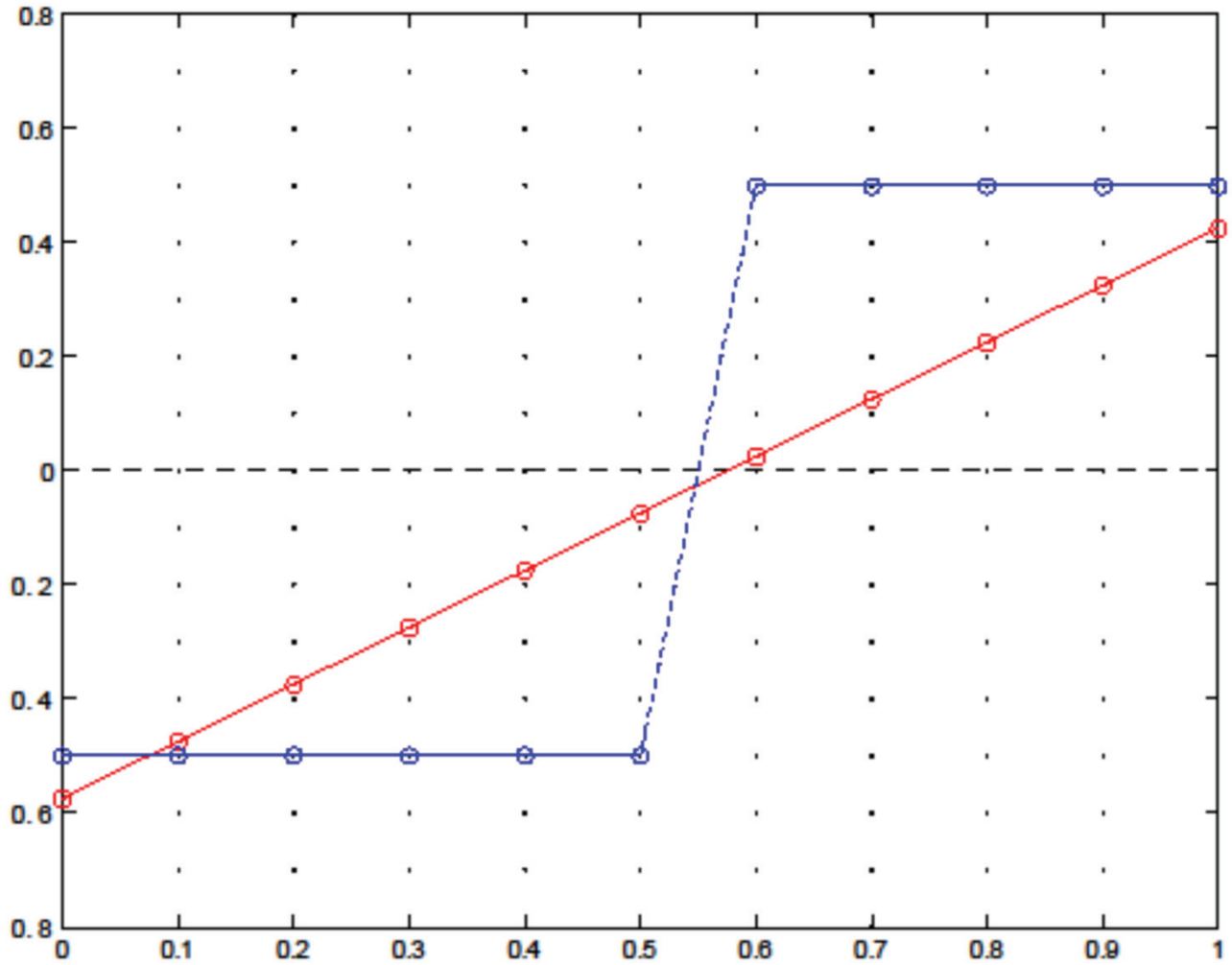
$$d_{\Sigma^{k+1}}(x) = \mathbf{Redist}(u) := \begin{cases} \inf_{\{y : u(y) > 0\}} |x - y| & \text{if } u(x) > 0, \\ -\inf_{\{y : u(y) < 0\}} |x - y| & \text{if } u(x) < 0. \end{cases}$$

Distance Function Dynamics

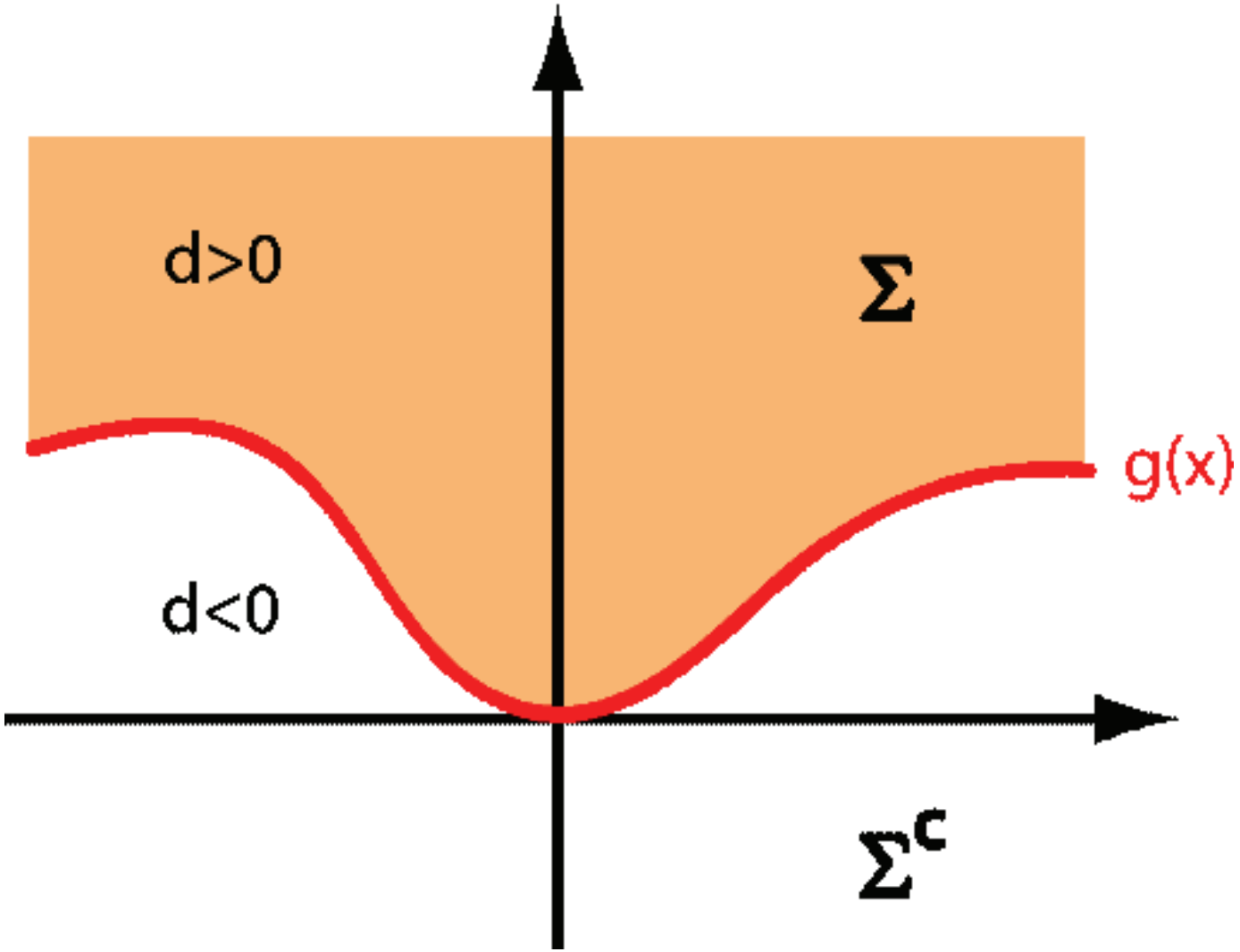
- Standard tool in level-set methods.
  1. Keeps 0-level set $\partial\Sigma = \{x : \phi(x) = 0\}$ of $\phi(x)$ fixed,
  2. Evolves $\phi(x,t)$ towards $d_\Sigma(x)$.
- $O(N \log N)$ algorithms exist: Fast marching, fast sweeping, etc.



Distance Function Dynamics



Distance Function Dynamics



Distance Function Dynamics

We must expand $d(x,y)$ at $(x,y)=(0,0)$. We know the following:

- Eikonal equation:

$$\boxed{d_x^2(x,y)+d_y^2(x,y)=1}$$

  in a neighborhood of $\partial\Sigma$.

- Boundary condition:

$$\boxed{d(x,g(x))=0}$$

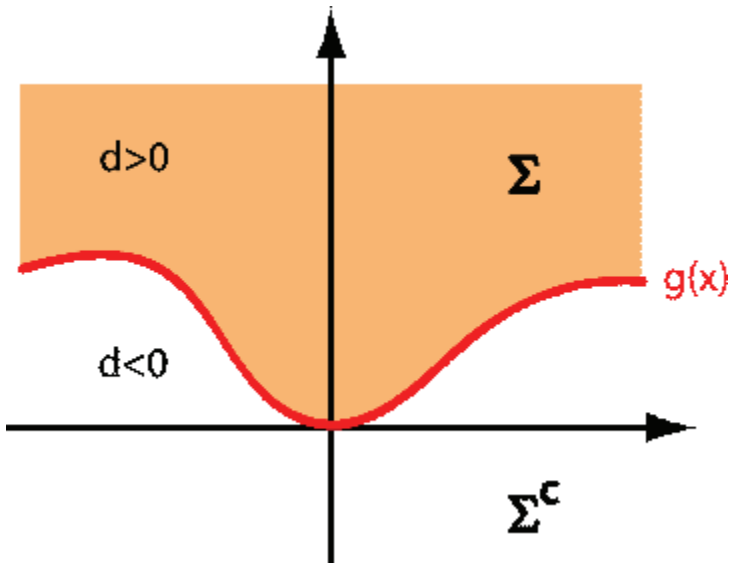
- Curvature and $d(x,y)$:

$$\boxed{d_{xx}(x,g(x))+d_{yy}(x,g(x))=\kappa(x)}$$

Distance Function Dynamics

Moreover, for sufficiently small y:

$$d(0,y) = y, \text{ so that}$$
$$d_y(0,y) = 1,$$
$$d_{yy}(0,y) = 0,$$
$$d_{yyy}(0,y) = 0,$$
$$\text{etc...}$$



Distance Function Dynamics

Let

$$A(x,y) = d_x^2(x,y) + d_y^2(x,y).$$

Then,

$$A_x = 2d_x d_{xx} + 2d_y d_{xy} = 0,$$
$$A_y = 2d_x d_{xy} + 2d_y d_{yy} = 0.$$

And,

$$A_{xx} = 2d_{xx}^2 + 2d_x d_{xx} + 2d_{xy}^2 + 2d_y d_{xxy} = 0.$$

Also, use:

$$d_{xx}(x, g(x)) + d_{yy}(x, g(x)) = \kappa(x).$$

Distance Function Dynamics

Differentiating these expressions many times, taking linear combinations, we arrive at

$$d_x(0, y) = 0,$$
$$d_{xy}(0, y) = 0,$$
$$d_{xyy}(0, y) = 0$$

for all sufficiently small y.

Distance Function Dynamics

Keep differentiating. In the same fashion, we get

$$
\begin{aligned}
d_{xxy}(0,0) &= -\kappa^2(0), \\
d_{xxx}(0,0) &= \kappa_x(0), \\
d_{xxxy}(0,0) &= -3\kappa(0)\kappa_x(0), \\
d_{xxyy}(0,0) &= 2\kappa^3(0), \\
d_{xxxx}(0,0) &= \kappa_{xx}(0) - 3\kappa^3(0), \\
&\text{etc...}
\end{aligned}
$$

Distance Function Dynamics

These lead to the following expansion of $d(x,y)$ at $(0,0)$:

Expansion of $d(x,y)$

$$\begin{aligned} d(x,y) = y & \\ & + \frac{1}{2}\kappa x^2 \\ & + \frac{1}{6}(\kappa_x x^3 - 3\kappa^2 x^2 y) \\ & + \frac{1}{24}\left((\kappa_{xx} - 3\kappa^3)x^4 - 12\kappa\kappa_x x^3 y + 12\kappa^3 x^2 y^2\right) \\ & + o\left((x^2+y^2)^2\right). \end{aligned}$$

Distance Function Dynamics

Expansion of $d(x,y)$ leads to an expansion for its convolution with the kernel $G_t(x)$:

Expansion of the Convolution

$$\begin{aligned}(G_t * d)(0,y) &= y \\ &\quad + \kappa t \\ &\quad - \kappa^2 ty + \frac{1}{2}\left(\kappa_{xx} - \kappa^3\right) t^2 \\ &\quad + o(t^2).\end{aligned}$$

Distance Function Dynamics

$$0 = (G_t * d)(0, y) \approx y + \kappa t \;\Rightarrow\; y \approx -\kappa t.$$

Algorithm

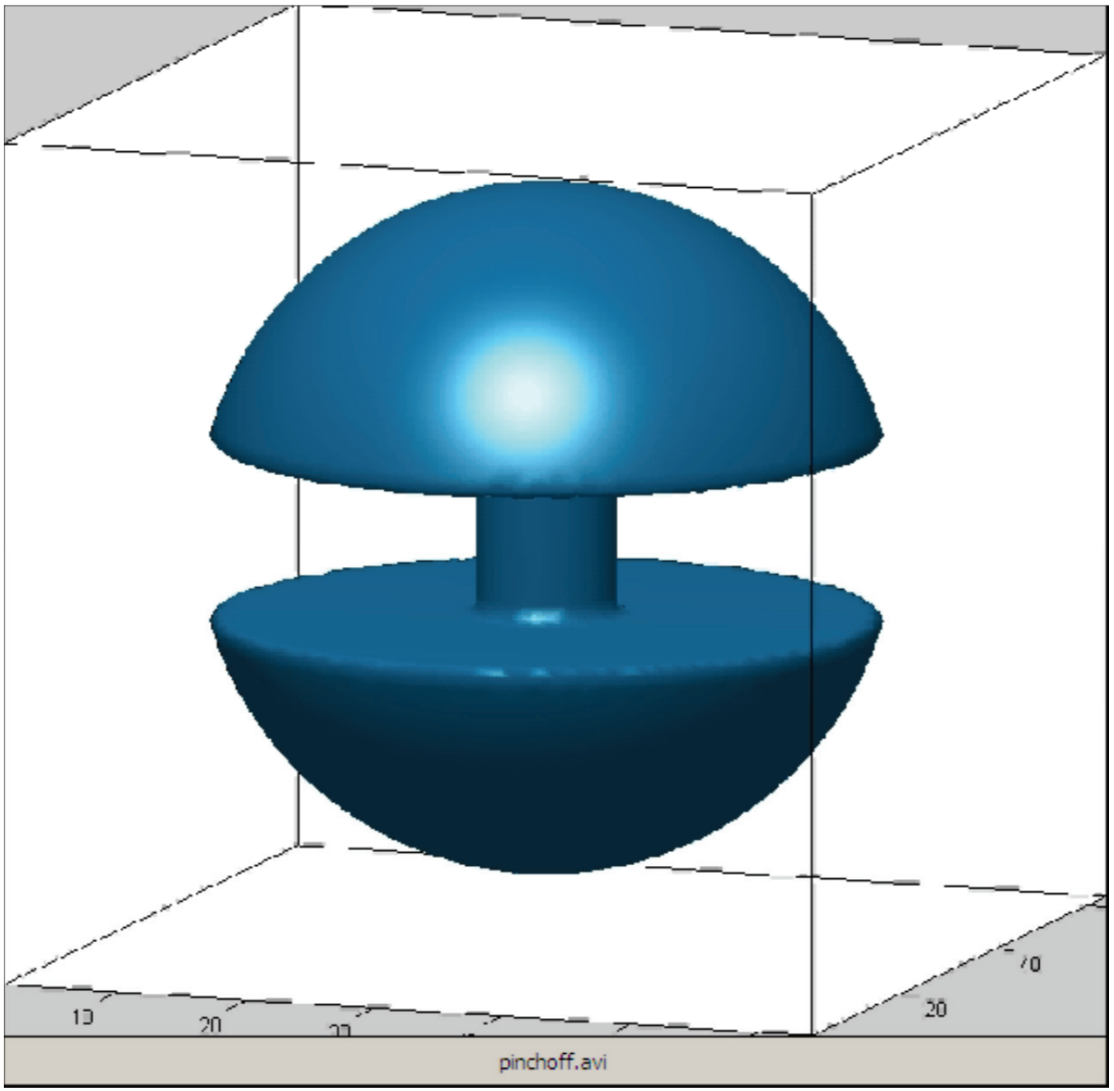
1. Convolution step: For $k = 0, 1, 2, \ldots$ let

$$L^k(x) = \left(G_{\delta t} * d^k\right)(x)$$

2. Redistancing step:

$$d^{k+1}(x) = \textbf{Redist}(L^k)$$

Distance Function Dynamics



Distance Function Dynamics

Notes:

- Expansions suggest first order accuracy in time.
- The scheme is unconditionally monotone:
  - Convolution with positive kernel preserves order,
  - Signed distance functions preserve order.
- Convergence to viscosity solution (in non-fattening situations) proven by Chambolle and Novaga.

Distance Function Dynamics

High order in time, multi-step algorithm for curvature motion:

Algoritm

1. Convolution step: For $k = 1, 2, \ldots$ form the functions

$$A_1(x) = G_{2\delta t} * d^{k-1},$$
$$A_2(x) = G_{\delta t} * d^{k}.$$

2. Redistancing step:

$$d^{k+1}(x) = \textbf{Redist}\left(\frac{1}{3}(4A_2 - A_1)\right).$$

Note:

- Second order accurate in time.
- One redistancing per time step.

Distance Function Dynamics

Convergence study with multistep algorithm:

Resolution	# of Time Steps	Relative Error	Order
32×32	10	0.092%	–
64×64	20	0.027%	1.77
128×128	40	0.0078%	1.79
256×256	80	0.0020%	1.96
512×512	160	0.00052%	1.94

Error in the radius of a shrinking circle.

Distance Function Dynamics

An algorithm for $v_n = f(\kappa)$

Algorithm

1. Convolution step: For $k = 0, 1, 2, \ldots$ let

$$L^k(x) = d^k(x) + f\left(\frac{(G_{N\delta t} * d^k)(x) - d^k(x)}{N\delta t}\right)\delta t$$

2. Redistancing step:

$$d^{k+1}(x) = \mathbf{Redist}(L^k)$$

Notes:

- This is an unconditionally monotone algorithm.

Distance Function Dynamics

- Recall from our expansions:

$$(G_t * d)(0, y) = y + \kappa t + O(t^2)$$

provided that $y = O(t)$, and

$$d(0, y) = y$$

for all small enough y.

Distance Function Dynamics

- Isolate K:

$$G_{Nt} * d - y = G_{Nt} * d - d \approx N\kappa t$$

so that

$$\kappa \approx \frac{G_{Nt} * d - d}{Nt}.$$

- Now we have

$$f(\kappa) \approx f\left(\frac{G_{Nt} * d - d}{Nt}\right)$$

Distance Function Dynamics

- Then:

$$y + tf(\kappa) \approx d + f\left(\frac{G_{Nt} * d - d}{Nt}\right) t.$$

- Let

$$A(d) = d + f\left(\frac{G_{Nt} * d - d}{Nt}\right) t.$$

- Claim: If $d_1 \geq d_2$, and $N \geq \mathrm{Lip}_f$, then

$$A(d_1) \geq A(d_2).$$

Distance Function Dynamics

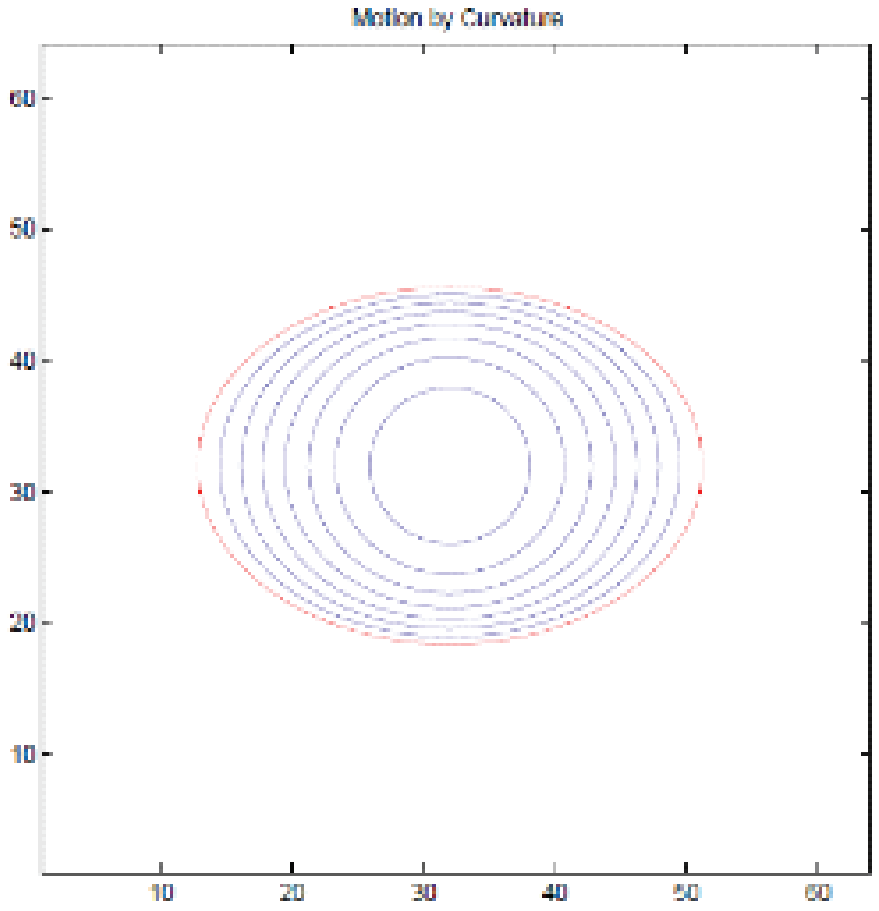
Example: Affine Invariant Curvature Motion

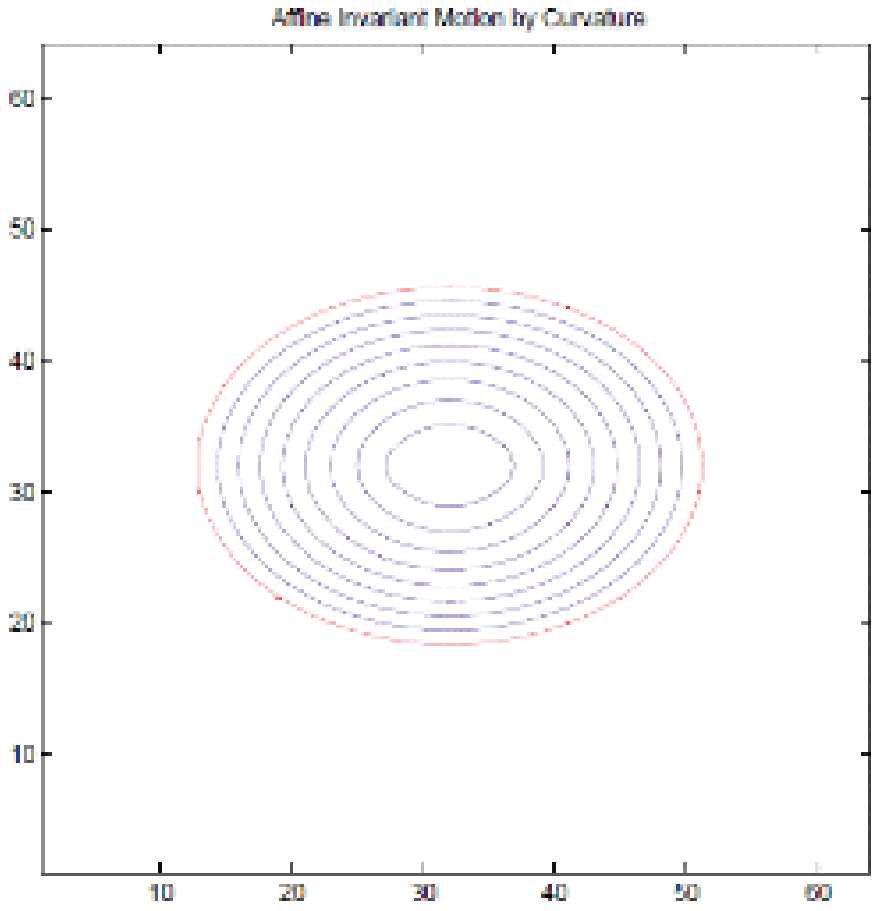
- Important in computer vision. Sapiro & Tannenbaum, others:

$$v_n = \kappa^{\frac{1}{3}}.$$

- Evolution commutes with affine transformations.
- In particular, an ellipse stays an ellipse with fixed eccentricity.
- Unfortunately, singular at $\kappa = 0$, but let's ignore that.

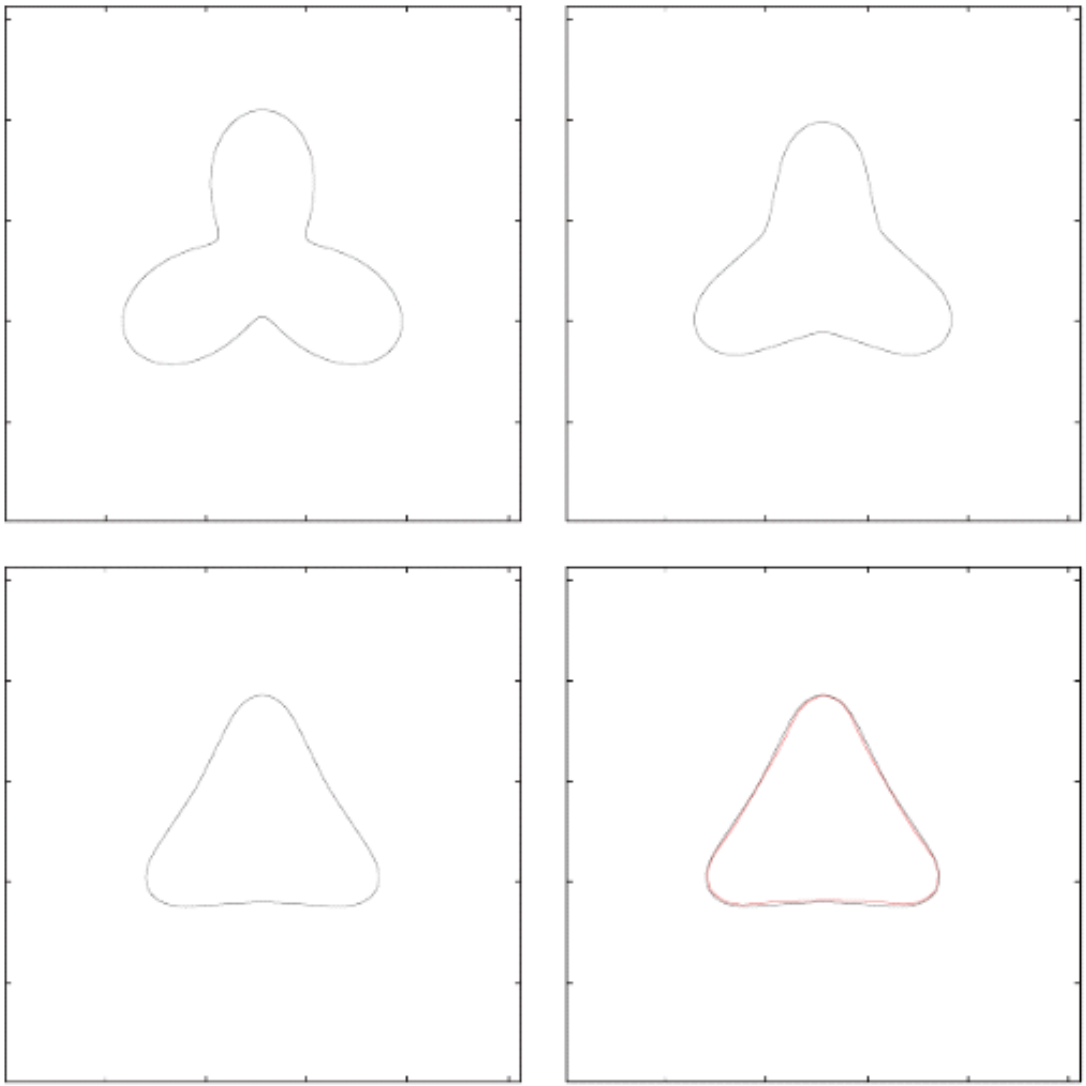
Distance Function Dynamics



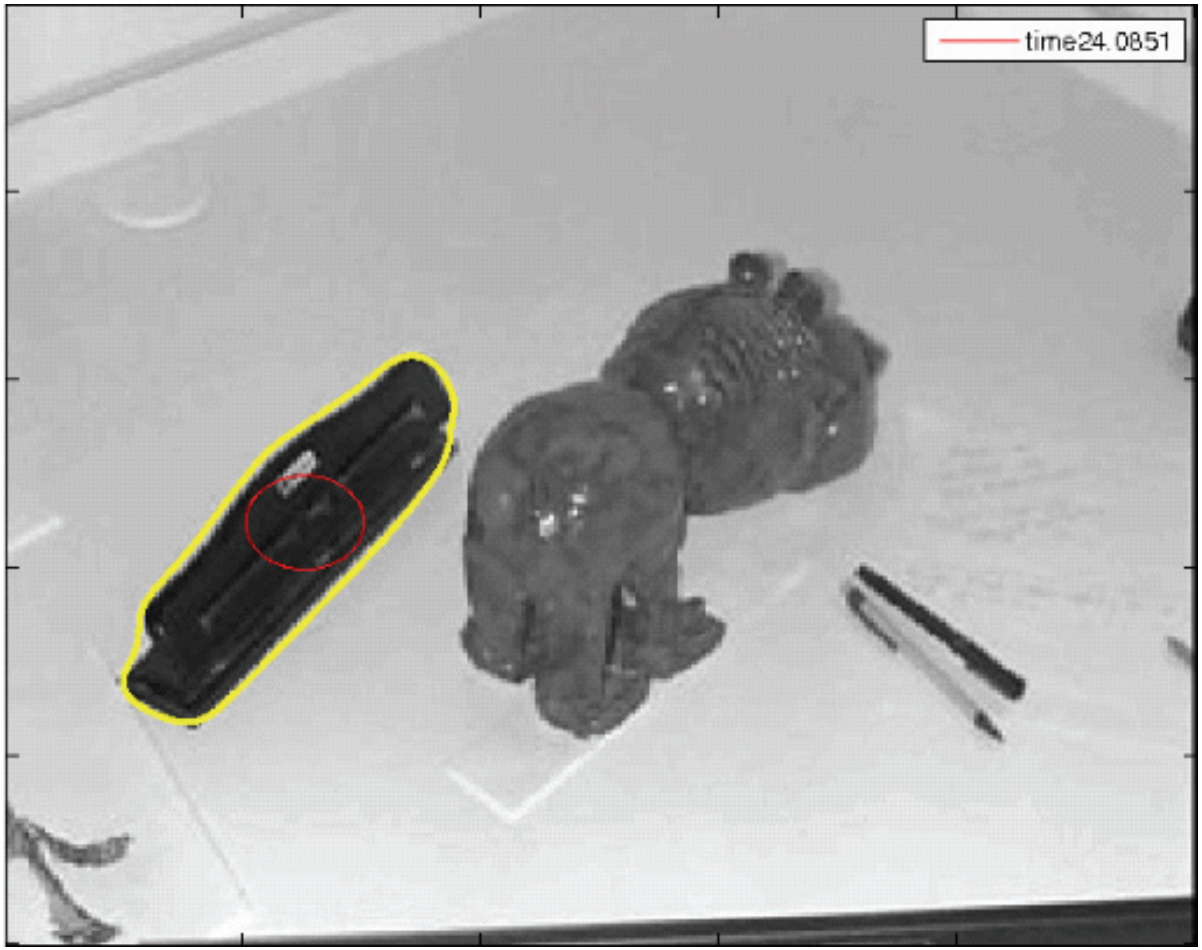


Distance Function Dynamics

Affine invariant curvature motion:

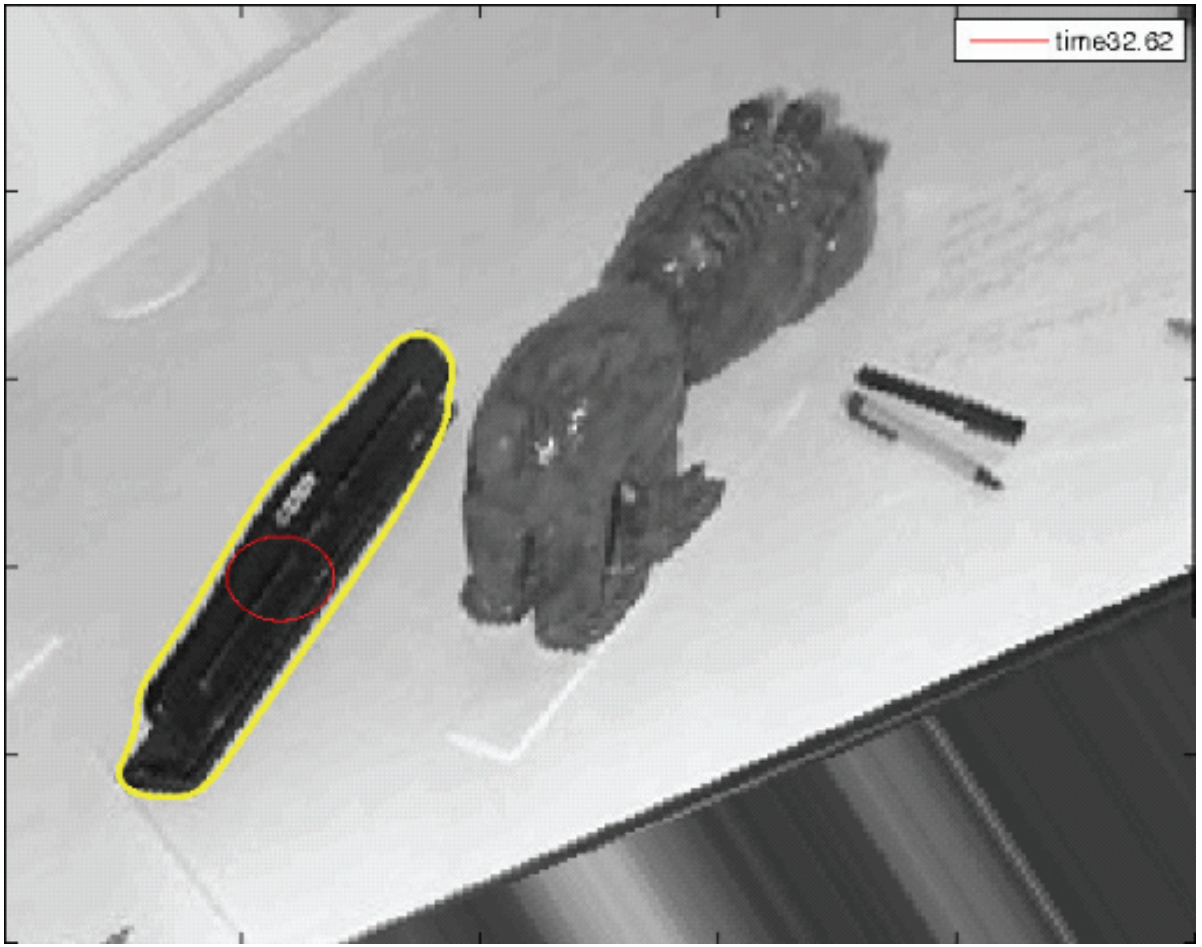


Distance Function Dynamics



Work of Catherine Kublik

Distance Function Dynamics



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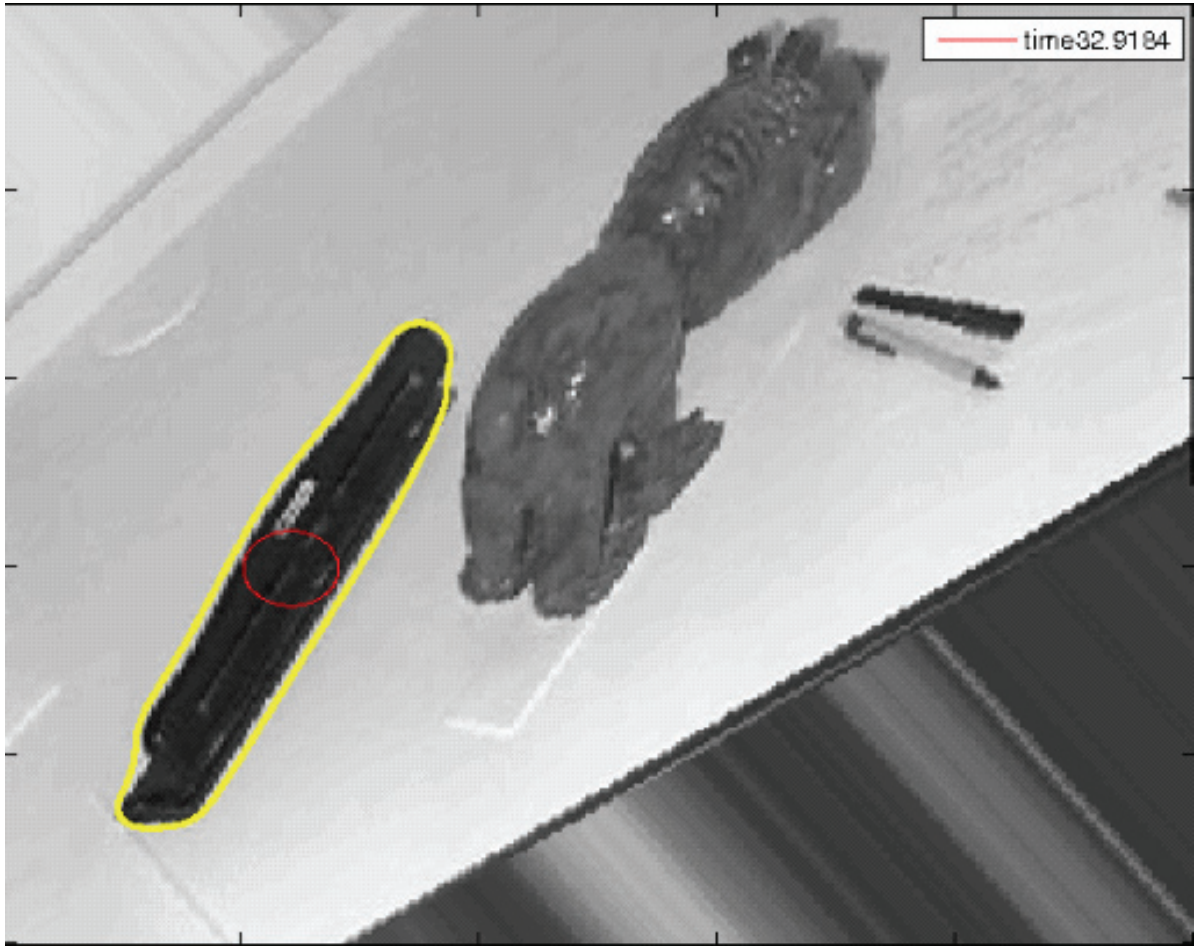

Work of Catherine Kublik

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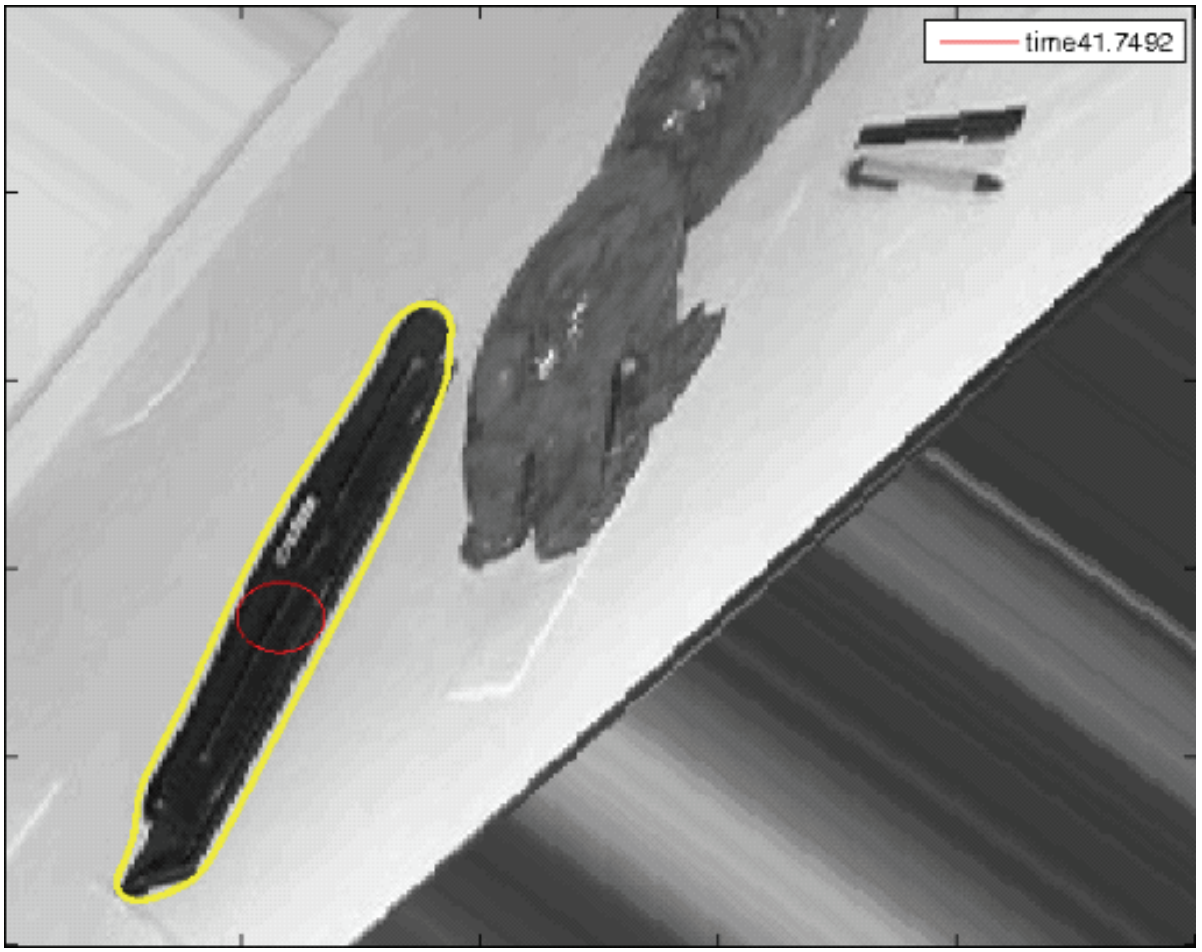

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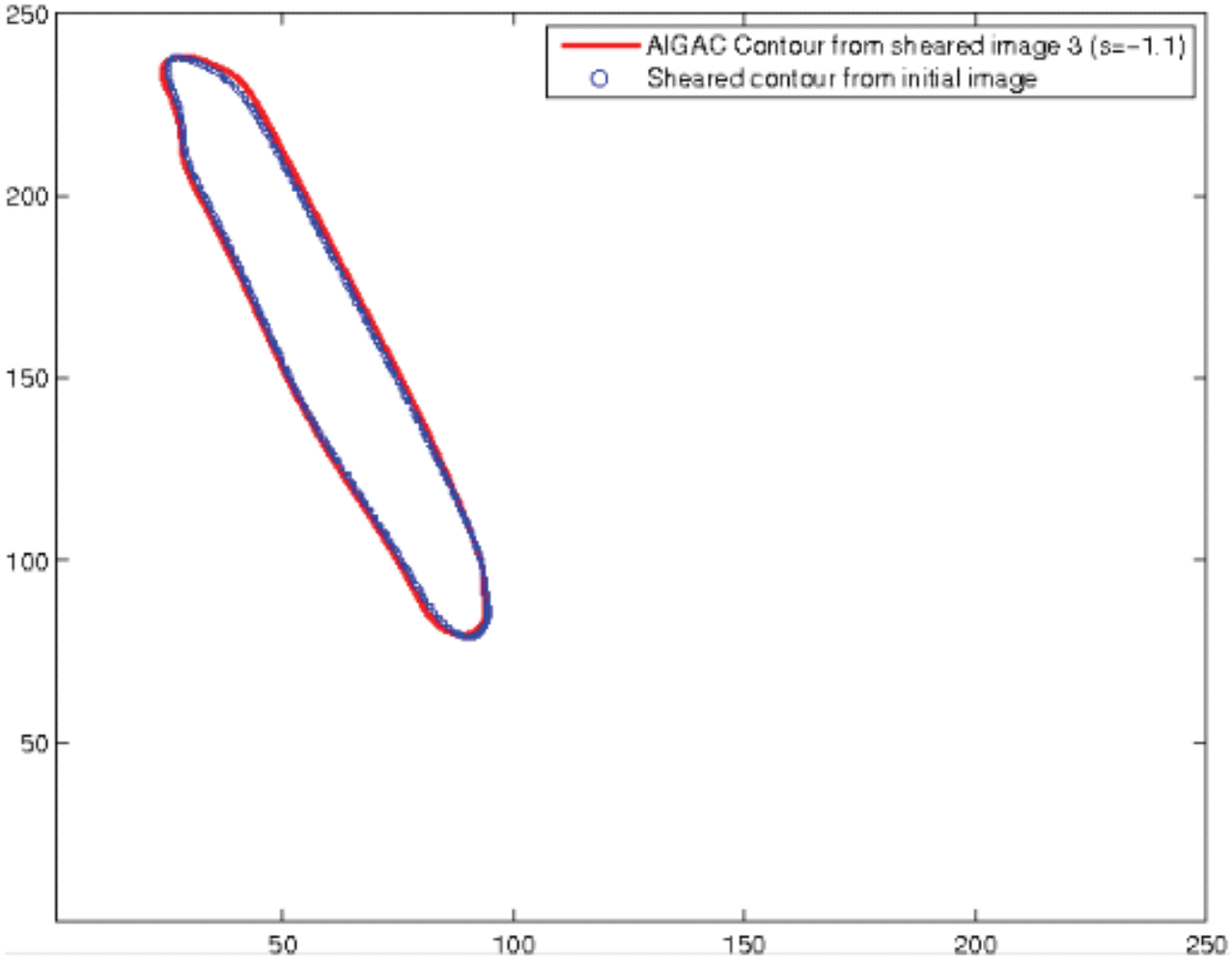


Work of Catherine Kublik

Distance Function Dynamics

High order motions via distance functions:

To generate 4th order flows, we cancel out the $O(t)$ terms in the expansion:

$$\Rightarrow \text{Take linear combinations of } G_{\sigma_1} * d \text{ and } G_{\sigma_2} * d$$

Example: Willmore flow. Recall:

$$\begin{aligned} W &= g_{xxxx} - \frac{5}{2}K^3 \\ &= K_{xx} - \frac{5}{2}K^3 \end{aligned}$$

Distance Function Dynamics

- Based on our expansions, we get:

$$\left(2G_{\sqrt{t}} - G_{2\sqrt{t}}\right) * d = y - t\left(K_{xx} - K^3\right) + o(t)$$

  at $x = 0$.

- We need to compensate by a factor of K^3:

$$\frac{1}{\sqrt{t}}\left(G_{\sqrt{t}} * d - d\right)^3 = K^3 t + o(t).$$

Distance Function Dynamics

We get an algorithm for Willmore flow:

Algorithm

1. Construct the signed distance function $d_0(x)$ to $\partial\Sigma_0$.
2. For $n = 0, 1, 2, \ldots$ let

$$L_n(x) = \left(G_{2\sqrt{t}} - 2G_{\sqrt{t}}\right) * d_n + \frac{C}{\sqrt{t}}\left(G_{\sqrt{t}} * d_n - d_n\right)^3$$

3. Construct the signed distance function $d_{n+1}(x)$ to the 0-level set of the function $L_n(x)$; go back to Step 2.

- Works in principle.
- Unfortunately, does not work very well in practice.

Dijkstra's Algorithm

Dijkstra's algorithm is formulated on graphs. We introduce some notation:

- Let $\{v_j\}$ denote the vertices of the graph.
- For each j, let S_j denote the star of the vertex v_j. These are all vertices connected to v_j by edges of the graph.
- For each j and $i \in S_j$, let $e_{j,i}$ denote the edge of the graph connecting v_j to v_i.
- For each j and $i \in S_j$, we will denote with $w_{j,i} \in \mathbb{R}$ the weight of the edge $e_{j,i}$.

Dijkstra's Algorithm

- We say that Γ is a path on the graph if it is an ordered list of vertices $\Gamma = (v_{i_1}, v_{i_2}, \ldots, v_{i_n})$ such that

$$i_{j+1} \in S_{i_j} \text{ for each } j = 1, 3, \ldots, n-1.$$

- We define the length of the path Γ to be

$$L(\Gamma) = \sum_{j=1}^{n-1} w_{j\,j+1}.$$

- Let's write $\mathscr{P}(v_i, v_j)$ to denote all the paths on the graph that start at vertex v_i and end at vertex v_j.

Dijkstra's Algorithm

- Given two vertices v_i and v_j on the graph, we denote by $d(v_i, v_j)$ the distance between v_i and v_j, defined as follows:

$$d(v_i, v_j) = \min_{\Gamma \in \mathscr{P}(v_i, v_j)} L(\Gamma).$$

- We will also denote the distance function from the node v_i as

$$d_{v_i}(v_j) = d(v_i, v_j)$$

so that, in particular, $d_{v_i}(v_i) = 0$.

Dijkstra's Algorithm

- Known nodes are at which the distance function has been computed for sure.
- At the beginning of the algorithm, only the initial node v_i belongs to known nodes:

$$\mathsf{K} = \{i\} \text{ initially.}$$

- Trial nodes are the immediate neighbors of the known nodes:

$$\mathsf{T} = \bigcup_{j \in \mathsf{K}} S_j$$

so that

$$\mathsf{T} = S_i \text{ initially.}$$

- Far nodes are all the remaining nodes. They are assigned an temporary value of ∞ as their distances.

Dijkstra's Algorithm

- Dijkstra's algorithm progresses by switching one of the trial nodes from $\mathbf{T}$ to the known nodes class $\mathbf{K}$:
  - Step 1: Among the trial nodes in $\mathbf{T}$, find the one with the least temporary distance; call it v_{i_*}.
  - Step 2: Remove that node v_{i_*} from trial nodes group $\mathbf{T}$ and add it to known nodes group $\mathbf{K}$. Its temporary distance value becomes its final distance value:

$$\mathbf{K} \rightarrow \mathbf{K} \cup \{i_*\}, \mathbf{T} \rightarrow \mathbf{T} \setminus \{i_*\}, \text{ and } d_{v_i}(v_{i_*}) = \tilde{d}_{v_i}(v_{i_*}).$$

  - Step 3: Update the trial group $\mathbf{T}$: Since there is now a new known node (namely, v_{i_*}), there may be new trial nodes; add them to $\mathbf{T}$:

$$\mathbf{T} \rightarrow \mathbf{T} \cup (S_{i_*} \setminus \mathbf{K}) \text{ and } \mathbf{F} \rightarrow \mathbf{F} \setminus S_{i_*}.$$

Dijkstra's Algorithm

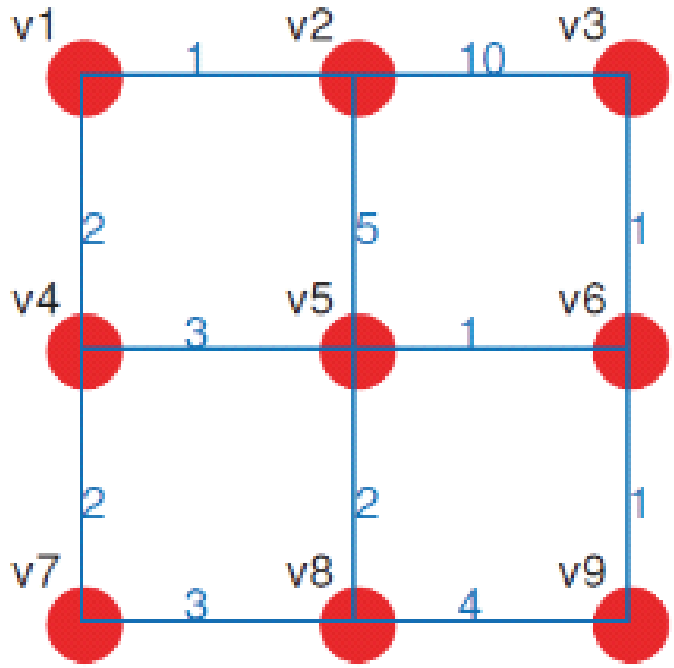
- Step 4: Update the temporary values of the trial neighbors of v_{i_*}:

  For each $j \in S_{i_*} \cap \mathbf{T}$, recompute $\tilde{d}_{v_i}(v_j) = \min_{k \in S_{v_j} \cap \mathbf{K}} \left(d_{v_i}(v_k) + w_{k,j} \right)$.

- The algorithm now repeats these steps, moving a single trial node from $\mathbf{T}$ to the set of known nodes $\mathbf{K}$ at every step; it therefore terminates in N steps, where N is the total number of nodes.

Dijkstra's Algorithm

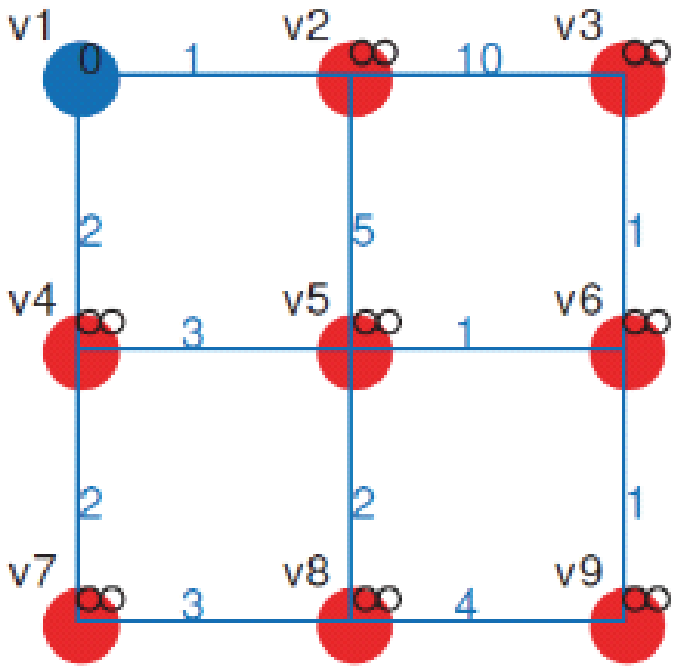
Example: We'll compute the distance function to node v_1 on the following graph:



RED = Far; GREEN = Trial; BLUE = Known.

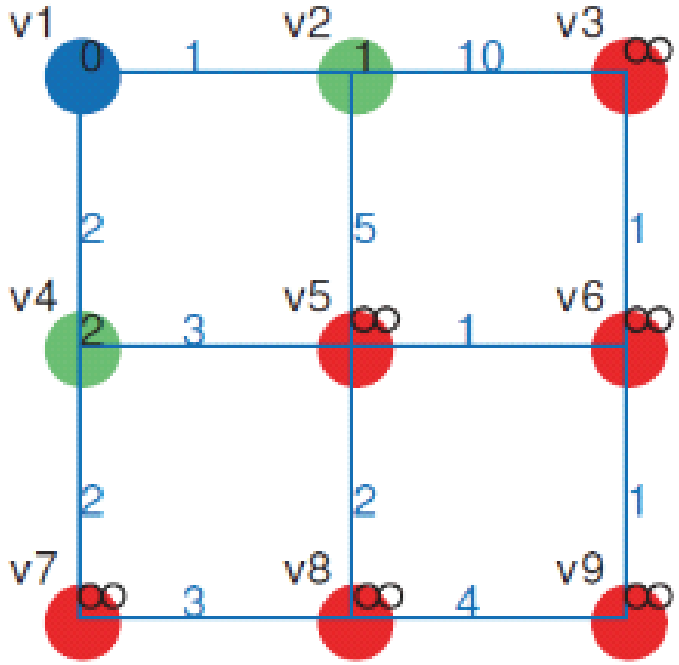
Dijkstra's Algorithm

In the beginning, only v_1 is known, whose distance is 0. All others are in the FAR group, with temp. values of ∞.



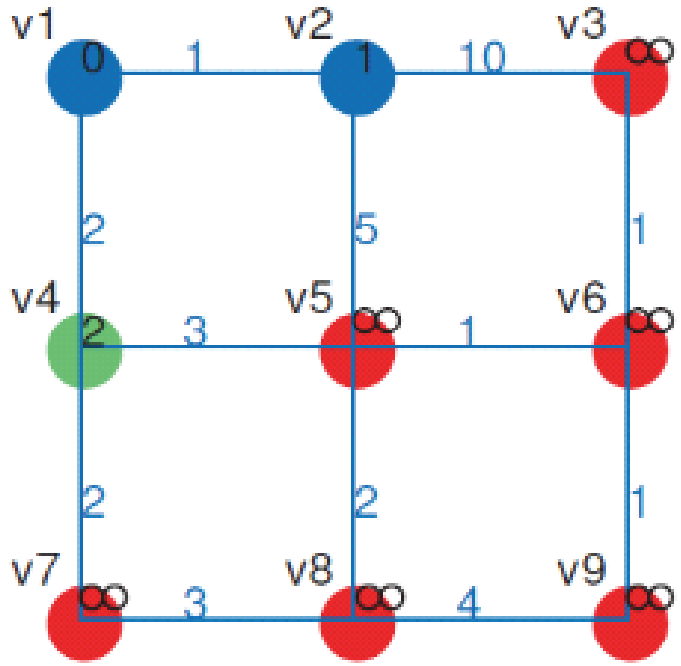
Dijkstra's Algorithm

The neighbors of v_1 are moved to the TRIAL group, and assigned temporary distance values.



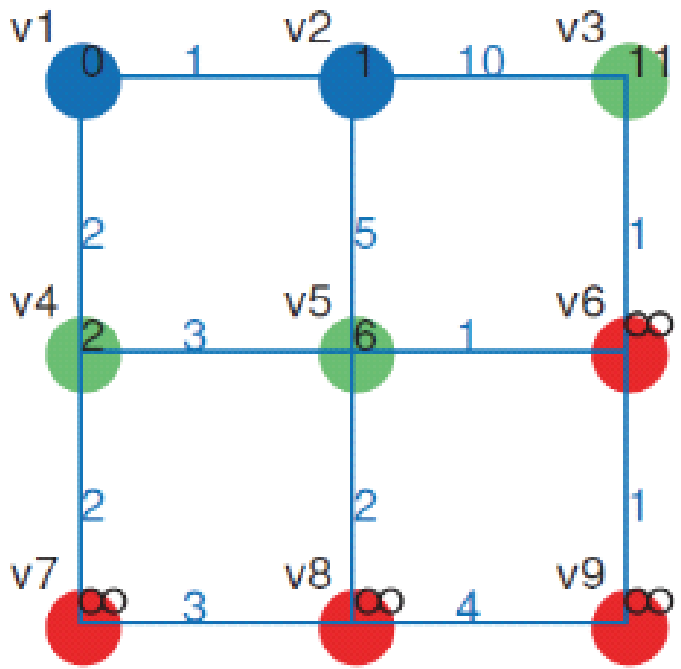
Dijkstra's Algorithm

v_2 is the smallest among the TRIAL nodes; it is therefore chosen and moved to KNOWN group, and its value is fixed.



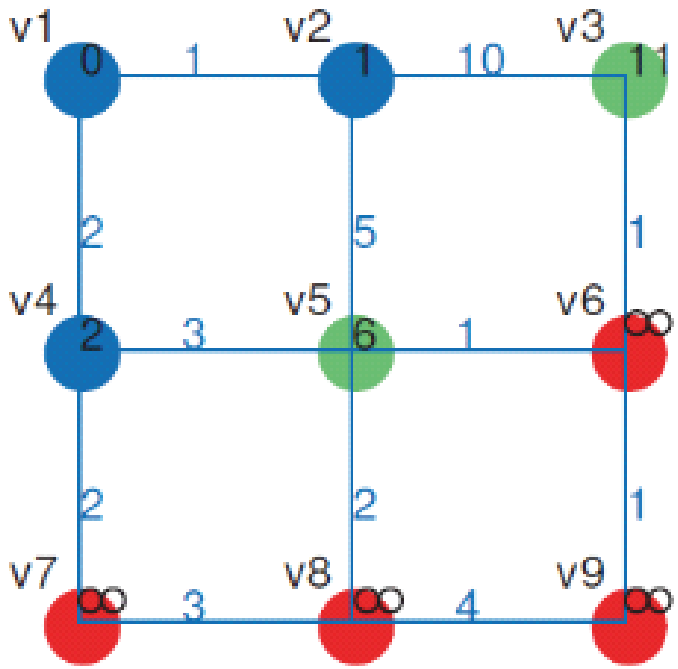
Dijkstra's Algorithm

The FAR neighbors of v_2, namely v_5 and v_3, are moved to the TRIAL group and assigned temporary distance values.



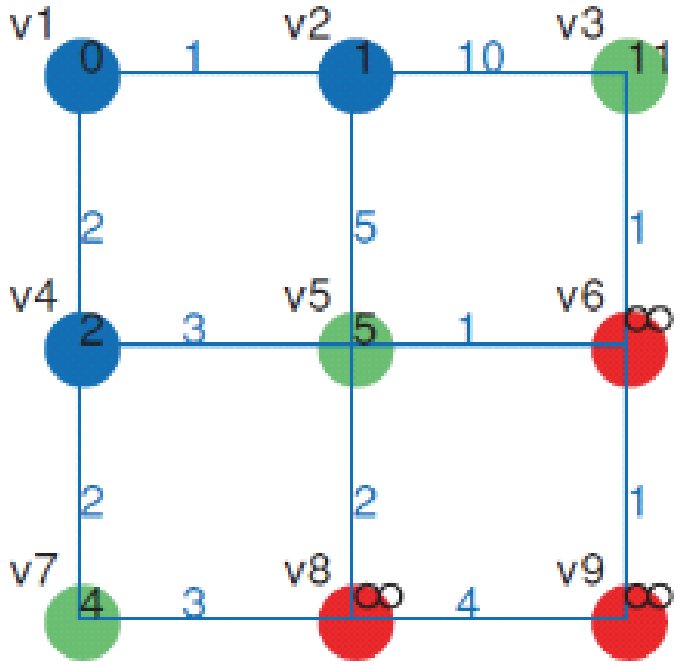
Dijkstra's Algorithm

$v4$ was the TRIAL node with smallest distance, therefore it got chosen and moved to the KNOWN group.



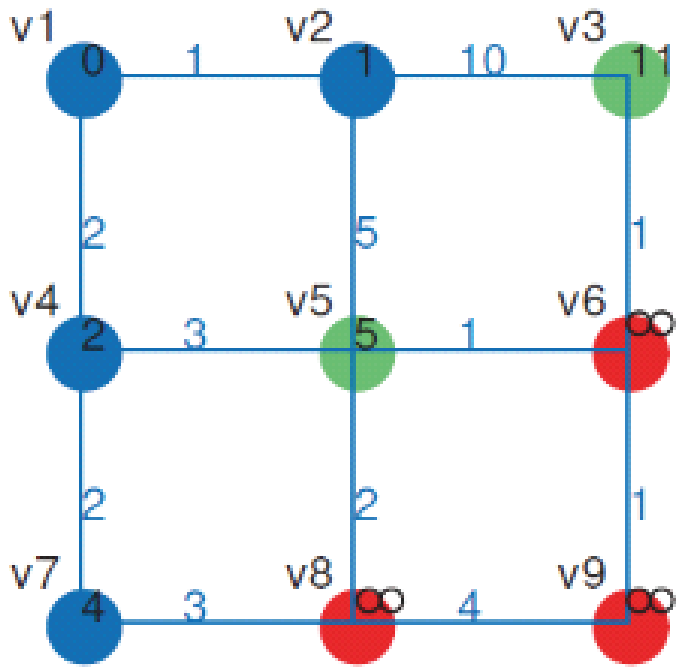
Dijkstra's Algorithm

The newly KNOWN node v_4 had one FAR neighbor, namely v_7. So v_7 got moved into the TRIAL group and assigned a temporary distance. Also, the temporary value of v_4's TRIAL neighbor v_5 got updated.



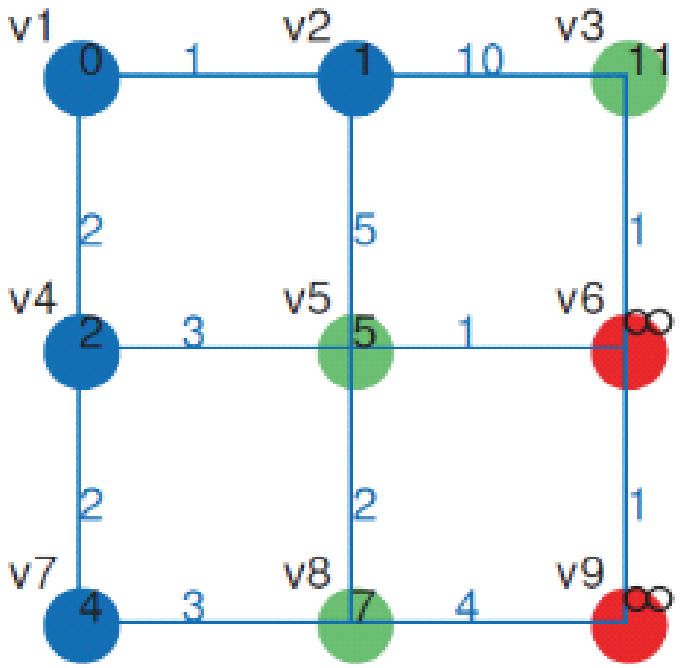
Dijkstra's Algorithm

Among the TRIAL nodes, v_7 had the smallest temporary value, so it got moved to KNOWN group and its distance is fixed.



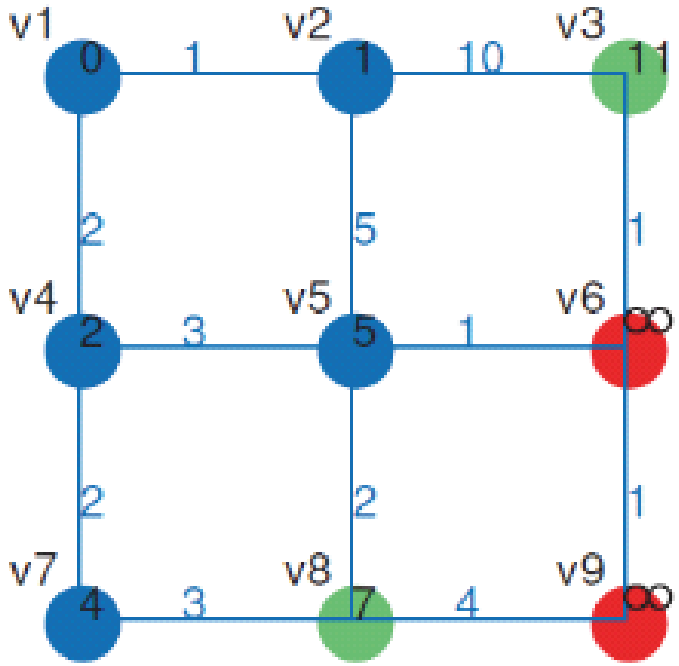
Dijkstra's Algorithm

The newly KNOWN node v_7 had a FAR neighbor v_8; so this neighbor got moved to TRIAL group and assigned a temporary value.



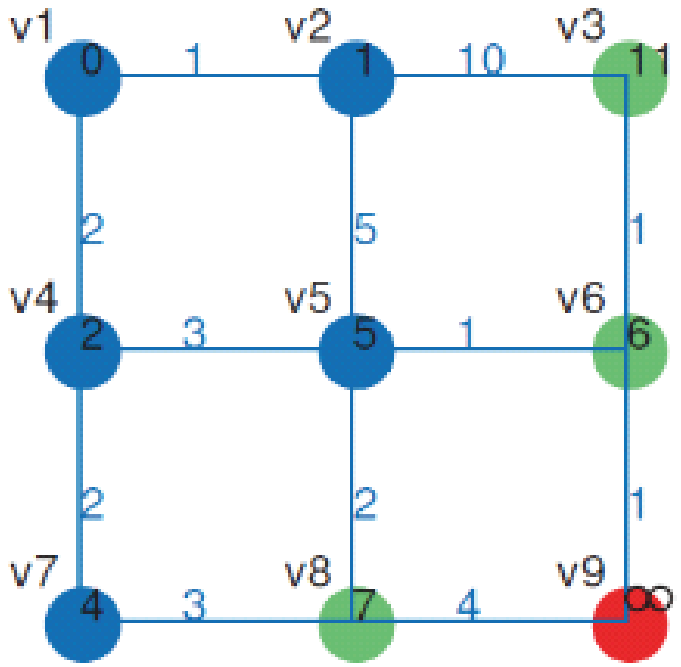
Dijkstra's Algorithm

Among the TRIAL nodes, v_5 had the smallest temporary value; so it got moved to KNOWN group, and its distance value is fixed.



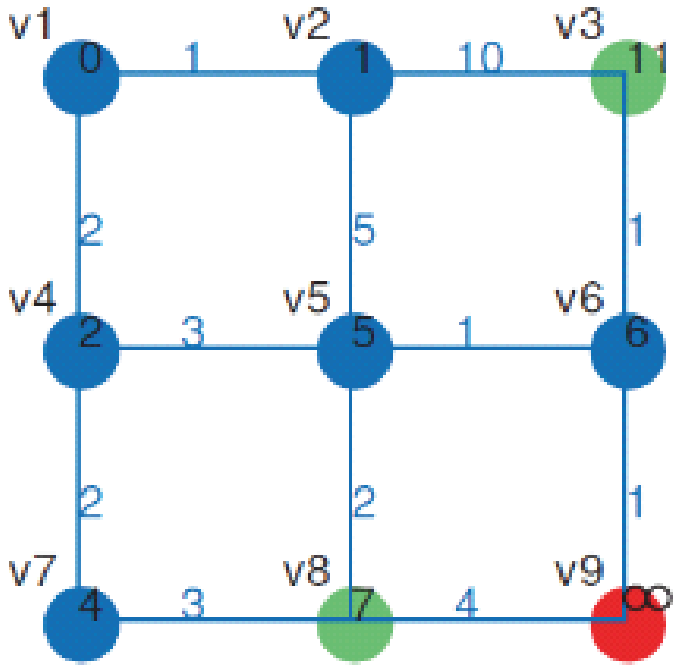
Dijkstra's Algorithm

The newly KNOWN node v_5 had a FAR neighbor v_6, which got moved to TRIAL group and assigned a temporary value; the TRIAL neighbor of v_5, namely v_8, did not need to be updated.



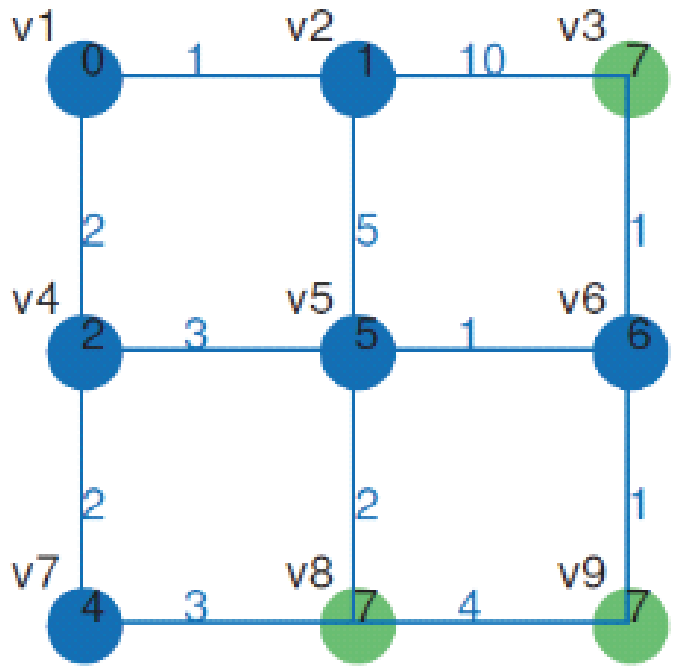
Dijkstra's Algorithm

Among the TRIAL nodes, v_6 has the smallest temp. value; so it got moved to KNOWN group and its distance is fixed.



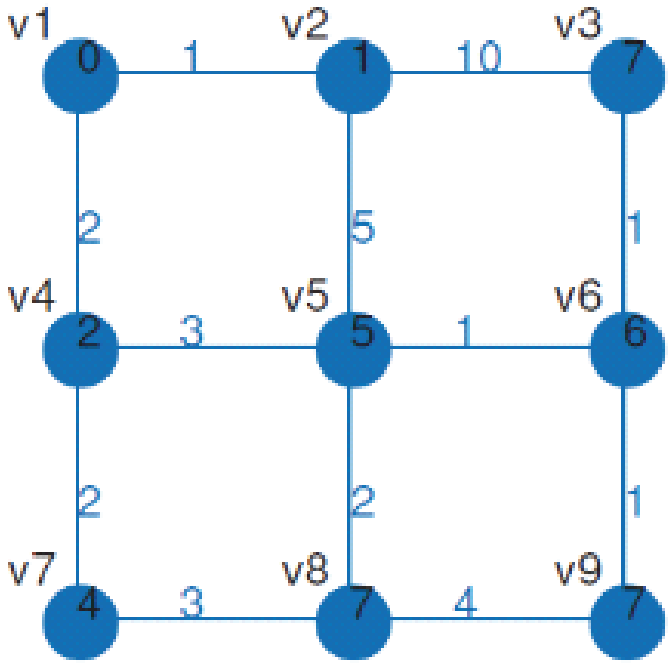
Dijkstra's Algorithm

The newly KNOWN node v_6 had a FAR neighbor v_9, which got moved to the TRIAL group and assigned a temp. value. The TRIAL neighbor v_3 of v_6 had its temp. value updated.



Dijkstra's Algorithm

Since all the remaining nodes are TRIAL nodes with minimal temp. value of 7 each, they can all be moved to KNOWN set. The algorithm is finished.



Fast Marching

- Tsitsiklis & Sethian.
- Use a discretization of

$$|\nabla u| = g$$

that gives the viscosity solution, e.g.

$$\max(\max(D_x^- u, 0), -\min(D_x^+ u, 0))^2 + \max\left(\max\left(D_y^- u, 0\right), -\min\left(D_y^+ u, 0\right)\right)^2 = g(x, y)$$

to update distance values of a grid point based on its neighbors during the course of Dijkstra's algorithm.

ALTERNATIVE: Fast marching of Osher, Tsai, Zhao, et. al.