Phase Field Models and Diffusion Generated Motion

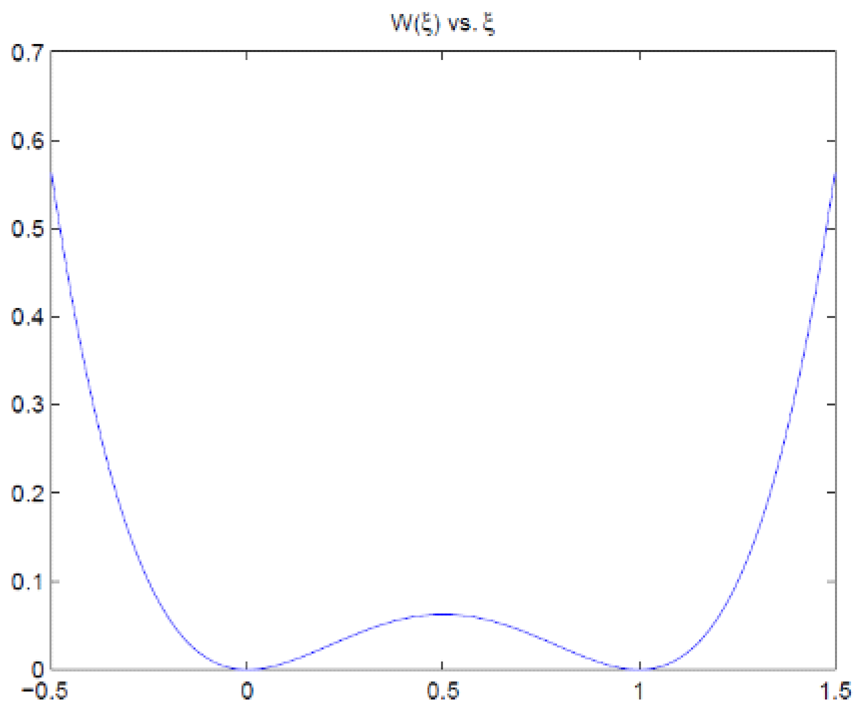
in Image Processing and Computer Vision Applications

Approximation of Perimeter

- Modica-Mortola Energies:

$$E_\varepsilon(u) = \int_\Omega \varepsilon|\nabla u|^2 + \frac{1}{\varepsilon}W(u)\,dx$$

where $W(\xi)$ is a double well potential with equidepth wells:



Approximation of Perimeter

$$E_\varepsilon(u) = \int_\Omega \varepsilon|\nabla u|^2 + \frac{1}{\varepsilon}W(u)\,dx$$

- The term

$$\frac{1}{\varepsilon}\int_\Omega W(u)\,dx$$

acts as a penalty term: Imposes the constraint

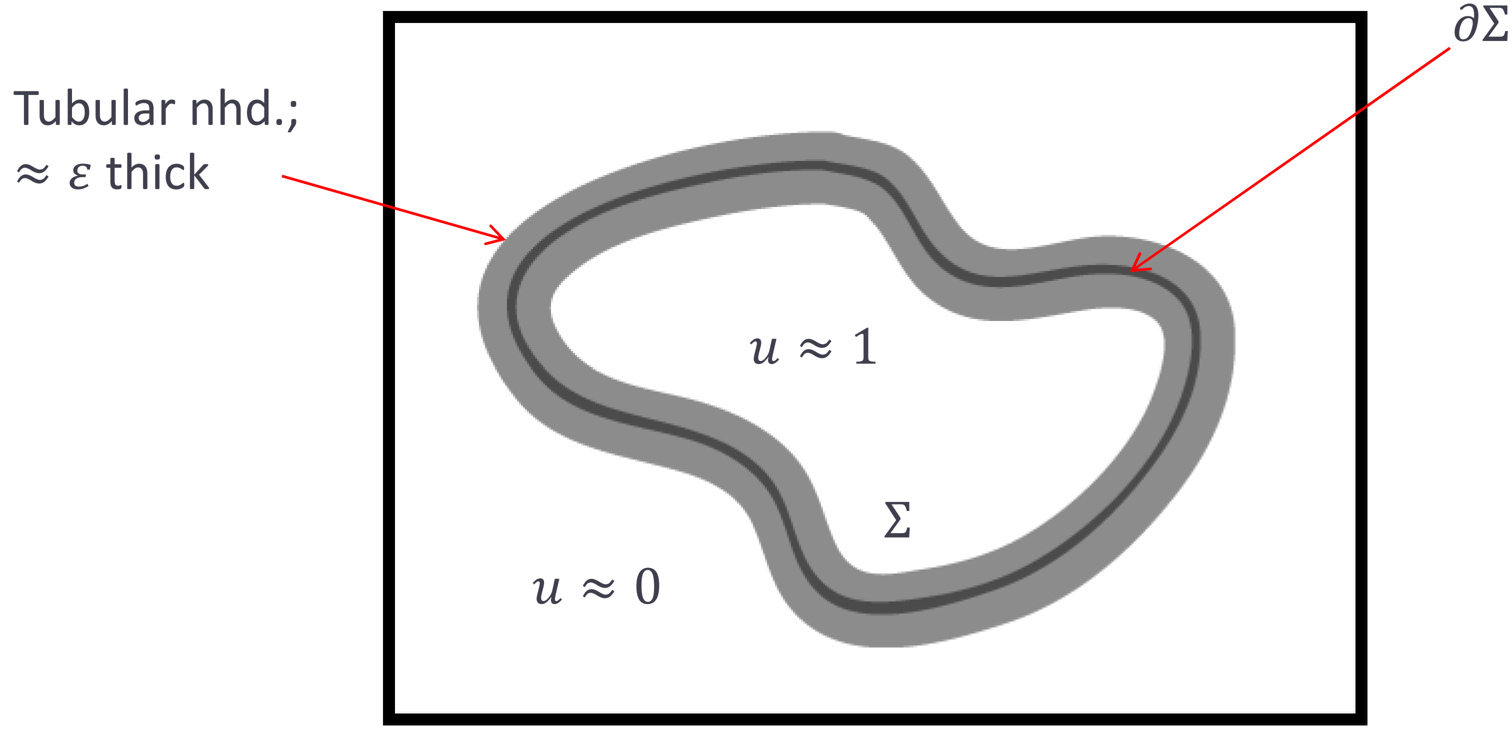
$$u(x) \in \{0,1\} \text{ for all } x \in \Omega.$$

- The term

$$\varepsilon\int_\Omega |\nabla u|^2\,dx$$

incurs a cost for transitions between $u = 0$ and $u = 1$.

Approximation of Perimeter



u needs to make a transition between 0 and 1 on a tubular nhd. of thickness $O(\varepsilon)$.

Approximation of Perimeter

- In the tubular nhd. T,

$$|\nabla u| \approx \frac{1}{\varepsilon}$$

- Size of tubular nhd. T:

$$\approx L\varepsilon \ \text{ where } \ L = \mathrm{Per}(\Sigma).$$

- Therefore,

$$\int_T \varepsilon |\nabla u|^2 \, dx \approx \varepsilon \left(\frac{1}{\varepsilon}\right)^2 L\varepsilon = L.$$

- In reality, both terms $\int \varepsilon |\nabla u|^2 \, dx$ and $\int \frac{1}{\varepsilon} W(u) \, dx$ contribute to $E_\varepsilon(u)$.
- We have:

$$E_\varepsilon(u) \to \gamma \cdot \mathrm{Per}(\Sigma)$$

Approximation of Perimeter

- The constant γ depends on details of $W(\xi)$.
- Consider the 1D version of the problem:

$$\min_{\substack{u \\ u(-\infty)=0 \\ u(\infty)=1}} \int_{-\infty}^{\infty} \varepsilon(u')^2 + \frac{1}{\varepsilon} W(u)\, dx$$

- Minimizer:

$$-2\varepsilon u'' + \frac{1}{\varepsilon} W'(u) = 0$$

- Integrate once:

$$u' = \frac{1}{\varepsilon}\sqrt{W(u)}$$

Approximation of Perimeter

- That gives:

$$W(u) = \varepsilon^2 (u')^2$$

so that

$$E_\varepsilon = 2 \int_{-\infty}^{\infty} \sqrt{W(u)} u' \, dx$$

$$= 2 \int_0^1 \sqrt{W(\xi)} \, d\xi$$

so that

$$\boxed{\gamma = 2 \int_0^1 \sqrt{W(\xi)} \, d\xi \, .}$$

Approximation of Perimeter

$$E_\varepsilon(u) = \int_\Omega \varepsilon |\nabla u|^2 + \frac{1}{\varepsilon} W(u)\, dx$$

- Gradient descent in L^2 gives: Allen-Cahn equation.

$$u_t = \Delta u - \frac{1}{\varepsilon^2} W'(u).$$

- We would expect it to be related to mean curvature motion.
- Indeed, consider the ansatz

$$u(x,t) = \phi\left(\frac{d_{\Sigma(t)}(x)}{\varepsilon}\right)$$

where ϕ solves $\phi'' - W'(\phi) = 0;\ \phi(-\infty) = 0;\ \phi(\infty) = 1;\ \phi(0) = \frac{1}{2}$.

- We have:

$$\begin{aligned}\Delta u &= \frac{1}{\varepsilon^2}\phi''\left(\frac{d_{\Sigma(t)}(x)}{\varepsilon}\right)\overset{=1}{\boxed{\left|\nabla d_{\Sigma(t)}\,(x)\right|^2}} + \frac{1}{\varepsilon}\phi'\left(\frac{d_{\Sigma(t)}(x)}{\varepsilon}\right)\Delta d_{\Sigma(t)} \\ &= \frac{1}{\varepsilon^2}\phi''\left(\frac{d_{\Sigma(t)}(x)}{\varepsilon}\right) + \frac{1}{\varepsilon}\phi'\left(\frac{d_{\Sigma(t)}(x)}{\varepsilon}\right)\kappa(x,t).\end{aligned}$$

Approximation of Perimeter

and

$$u_t = \frac{1}{\varepsilon}\phi'\left(\frac{d_{\Sigma(t)}(x)}{\varepsilon}\right)\frac{\partial}{\partial t}d_{\Sigma(t)}(x)$$

These cancel since
ϕ solves 1d problem.

Substituting into the PDE:

$$\frac{1}{\varepsilon}\phi'\left(\frac{d_{\Sigma(t)}(x)}{\varepsilon}\right)\frac{\partial}{\partial t}d_{\Sigma(t)}(x) = \boxed{\frac{1}{\varepsilon^2}\phi''\left(\frac{d_{\Sigma(t)}(x)}{\varepsilon}\right)} + \frac{1}{\varepsilon}\phi'\left(\frac{d_{\Sigma(t)}(x)}{\varepsilon}\right)\kappa(x,t)$$
$$\boxed{-\frac{1}{\varepsilon^2}W'\left(\phi'\left(\frac{d_{\Sigma(t)}(x)}{\varepsilon}\right)\right)}.$$

Normal speed
of the curve.

$$\Rightarrow \boxed{\frac{\partial}{\partial t}d_{\Sigma(t)}(x)} = \kappa(x,t).$$

- Hence, if the curve moves by mean curvature motion, the ansatz solves the PDE very well right on the curve.

Approximation of Perimeter

Application: P.C. Mumford Shah

- Piecewise constant Mumford-Shah with two regions:

$$E(\Sigma, c_1, c_2) = \mathrm{Per}(\Sigma) + \lambda \left\{ \int_\Sigma (f - c_1)^2 \, dx + \int_{\Omega \setminus \Sigma} (f - c_2)^2 \, dx \right\}$$

- Phase-Field approximation:

$$\begin{aligned} E_\varepsilon(u, c_1, c_2) = \int_\Sigma \varepsilon |\nabla u|^2 + \frac{1}{\varepsilon} W(u) \\ + \lambda\{(f - c_1)^2 u^2 + (f - c_2)^2 (1 - u)^2\} \, dx \end{aligned}$$

- Gradient descent:

$$\begin{aligned} u_t = \varepsilon \Delta u - \frac{1}{\varepsilon} W'(u) - \lambda (f - c_1)^2 u \\ + \lambda (f - c_2)^2 (1 - u) \end{aligned}$$

Application: P.C. Mumford-Shah

$$u_t = \varepsilon \Delta u - \frac{1}{\varepsilon} W'(u) - \lambda (f - c_1)^2 u$$
$$+ \lambda (f - c_2)^2 (1 - u)$$

- Discretization:

$$\frac{u^{n+1} - u^n}{\delta t} = \varepsilon \Delta u^{n+1} - \frac{1}{\varepsilon} W'(u^n) - \lambda (f - c_1)^2 u^n$$
$$+ \lambda (f - c_2)^2 (1 - u^n)$$

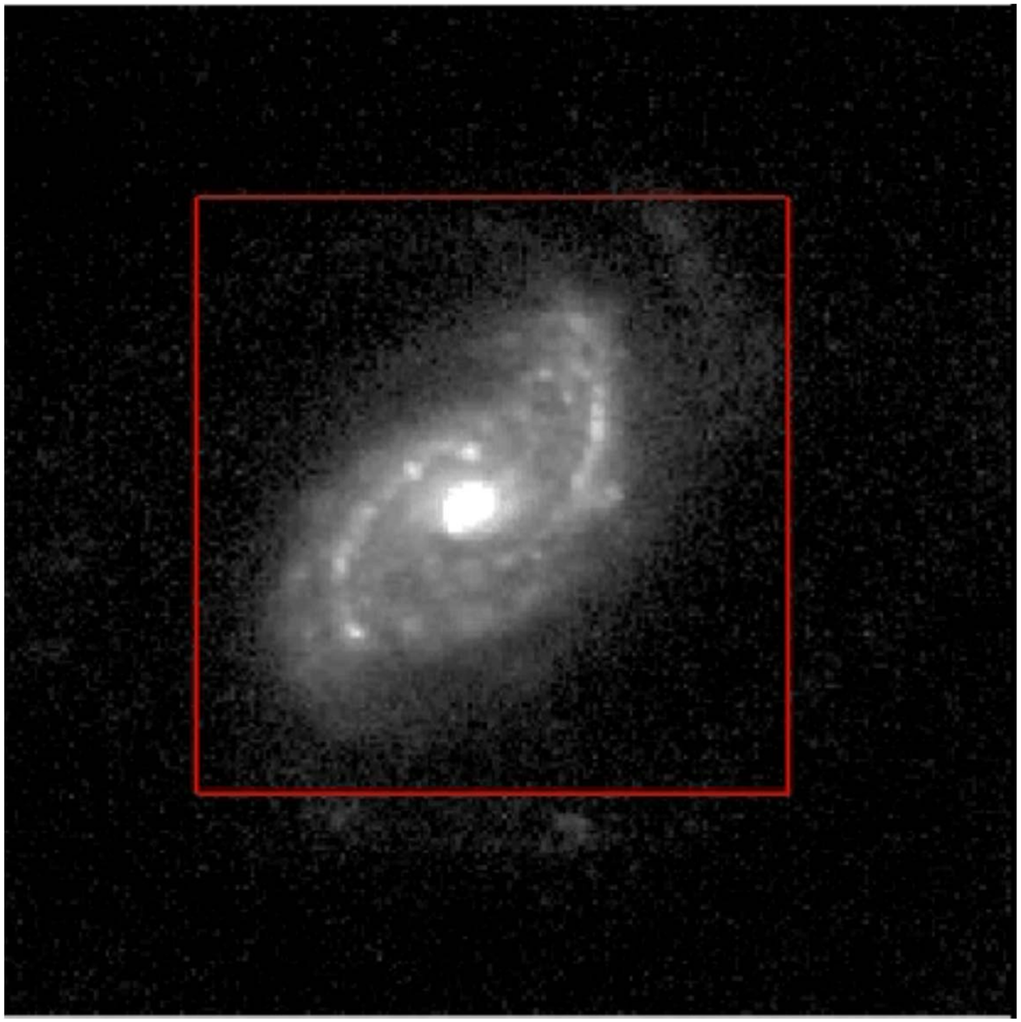
and use FFT to solve for u^{n+1} at every time step, or

$$\frac{u^{n+1} - u^n}{\delta t} = \varepsilon \Delta u^{n+1} - \frac{1}{\varepsilon} W'(u^n) - \lambda (f - c_1)^2 u^{n+1}$$
$$+ \lambda (f - c_2)^2 (1 - u^{n+1})$$

and use e.g. PCG to solve for u^{n+1}.

- NOTE: Stability restriction on δt: $\boxed{\delta t \leq O(\varepsilon).}$

Application: P.C. Mumford-Shah



Application: P.S. Mumford-Shah

- Piecewise smooth Mumford-Shah with two regions:

$$E(\Sigma, c_1, c_2) = \mathrm{Per}(\Sigma) + \lambda\left\{\int_{\Sigma} (f - c_1(x))^2\, dx + \int_{\Omega\setminus\Sigma} (f - c_2(x))^2\, dx\right\}$$
$$+\alpha_1 \int_{\Omega} |\nabla c_1|^2\, dx + \alpha_2 \int_{\Omega} |\nabla c_2|^2\, dx\,.$$

- u updates are the same.
- Update c_1 and c_2 as:

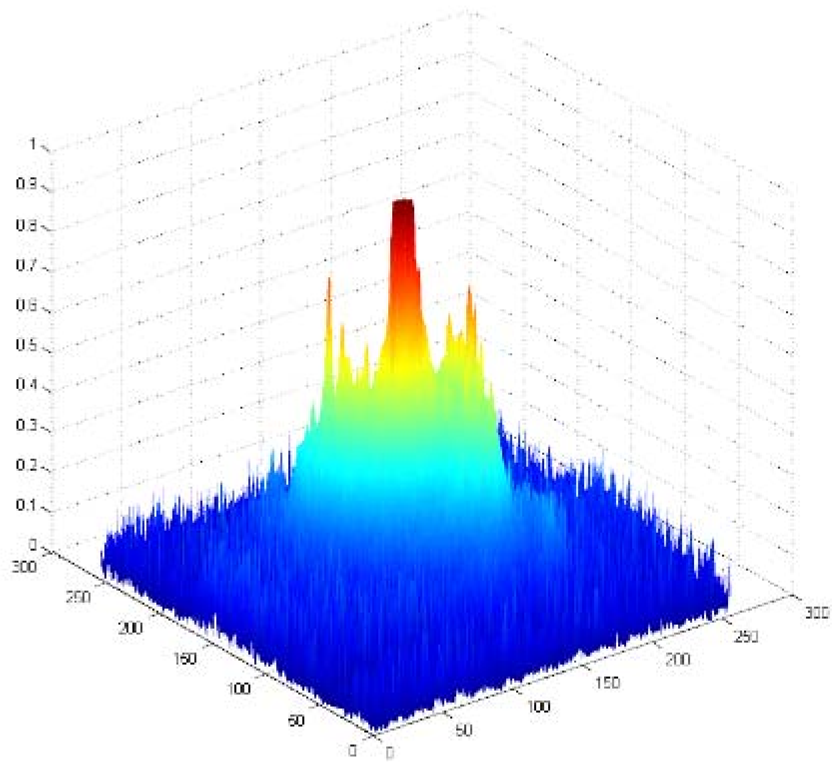
$$\alpha_1 \Delta c_1 + \lambda u^2 (f - c_1) = 0$$

and

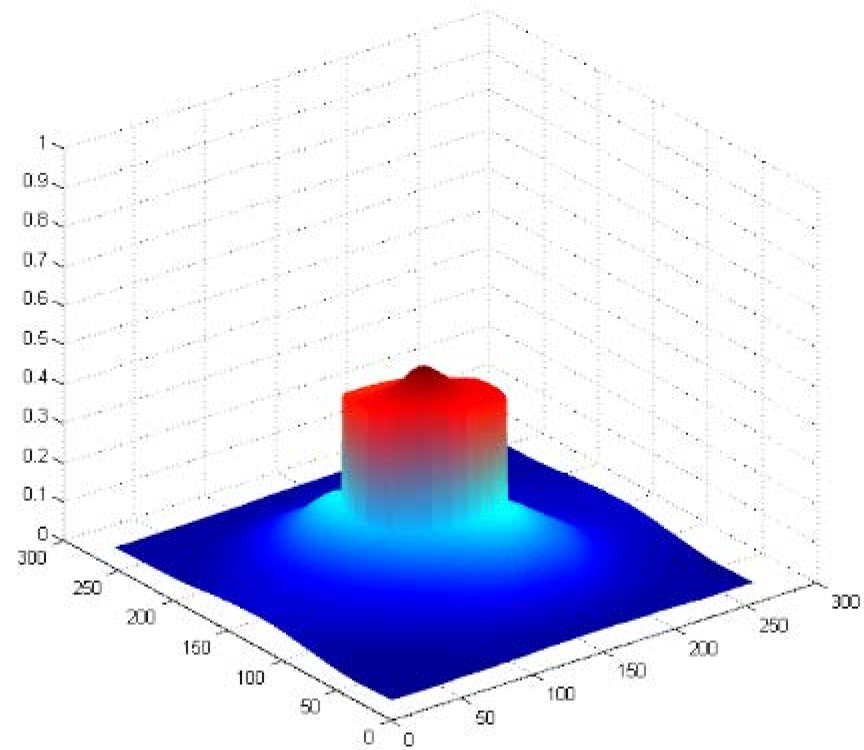
$$\alpha_2 \Delta c_2 + \lambda (1-u)^2 (f - c_2) = 0.$$

Note: From thesis of Catherine Kublik.

Application: P.S. Mumford-Shah



Galaxy image

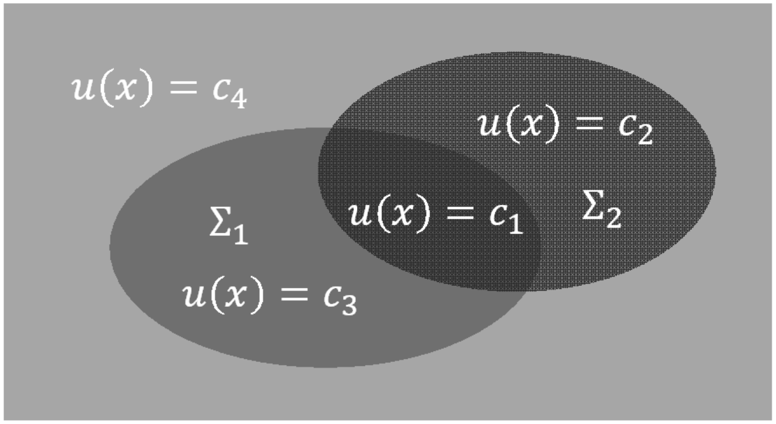


Its piecewise smooth approximation

Multi-Phase P.C. Mumford-Shah

- Easy generalization to > 2 regions:

$$E(\Sigma_1, \Sigma_2, \vec{c}) = \mathrm{Per}(\Sigma_1) + \mathrm{Per}(\Sigma_2) + \lambda \int_{\Sigma_1 \cap \Sigma_2} (c_1 - f)^2 \, dx$$
$$+\lambda \int_{\Sigma_1^c \cap \Sigma_2} (c_2 - f)^2 \, dx + \lambda \int_{\Sigma_1 \cap \Sigma_2^c} (c_3 - f)^2 \, dx$$
$$+\lambda \int_{\Sigma_1^c \cap \Sigma_2^c} (c_3 - f)^2 \, dx \, .$$

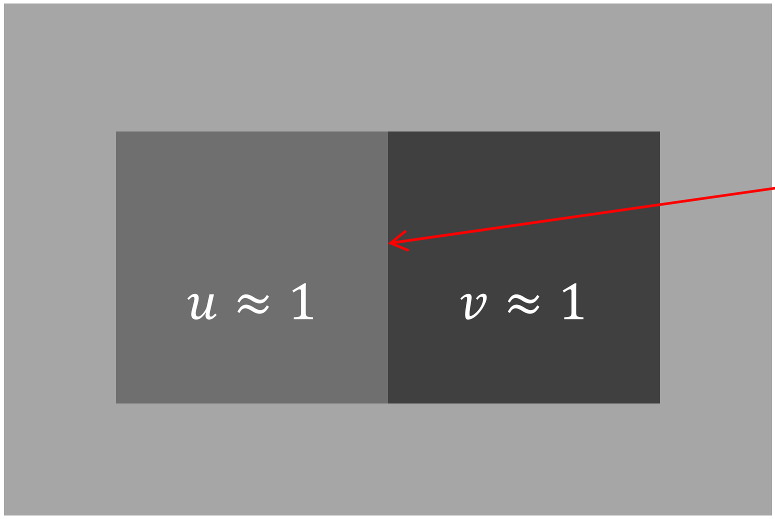


Multi-Phase P.C. Mumford-Shah

- Perimeter terms:

$$E_\varepsilon(u,v) = \int_\Omega \varepsilon|\nabla u|^2\,dx + \frac{1}{\varepsilon}W(u)\,dx + \int_\Omega \varepsilon|\nabla v|^2\,dx + \frac{1}{\varepsilon}W(v)\,dx$$

- An undesirable feature:



Unequal weights for some edges.

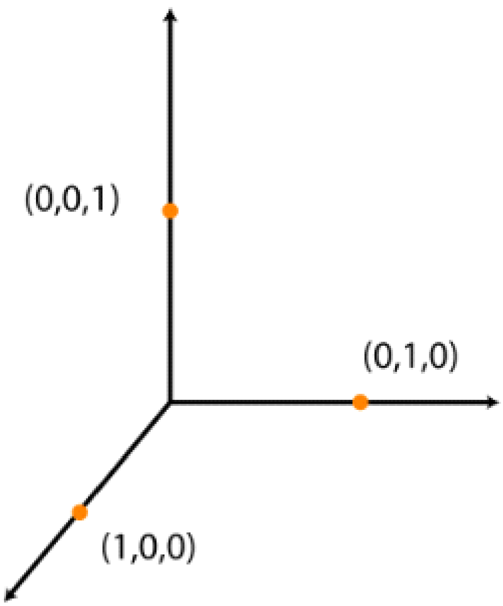
Multi-Phase Field

- Example: Three phases.
- Vectorial phase function: $u = (u_1, u_2, u_3)$.
- Energy:

$$\int_\Omega \varepsilon \sum_{j=1}^{3} \left|\nabla u_j\right|^2 + \frac{1}{\varepsilon} W(u)\, dx$$

- Potential:

$$W(\xi_1, \xi_2, \xi_3) = \prod_{j=1}^{3} \left|\xi - e^j\right|^2$$



Multi-Phase Field

- Alternatively, $u = (u_1, u_2)$.
- Energy:

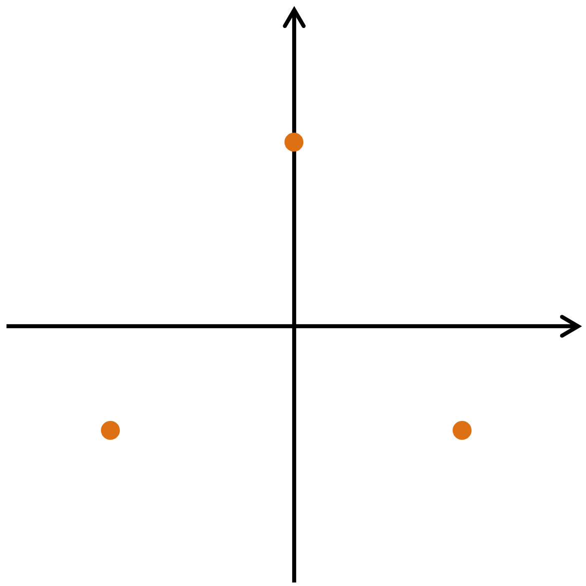
$$\int_\Omega \varepsilon \sum_{j=1}^{2} \left|\nabla u_j\right|^2 + \frac{1}{\varepsilon} W(u)\, dx$$

- Potential:

$$W(\xi) = \prod_{j=0}^{2} \left|\xi - \left(\cos\theta_j\,, \sin\theta_j\right)\right|^2$$

where

$$\theta_j = \frac{\pi}{2} + \frac{2\pi}{3} j$$



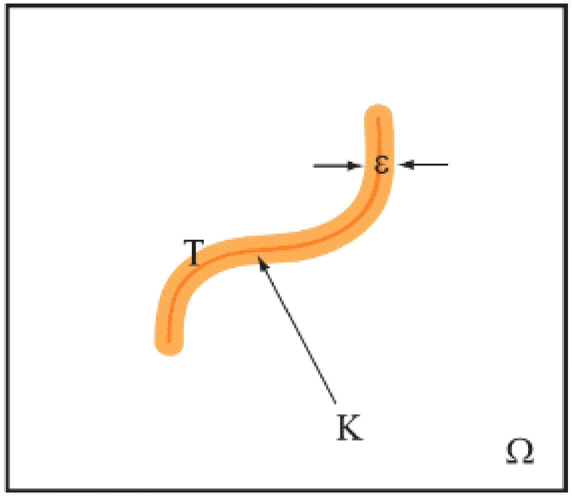
Phase Field for Full Mumford-Shah

$$\min_{\substack{u(x) \\ K\subset\Omega}} \int_{\Omega\setminus K} |\nabla u|^2 \, dx + \mu \, \mathrm{Length}(K) + \lambda \int_{\Omega} (f-u)^2 \, dx$$

- Ambrosio & Tortorelli (1992)

$$MS_\varepsilon(u,z) = \int_\Omega z^2|\nabla u|^2 + \alpha\left(\frac{(1-z)^2}{4\varepsilon} + \varepsilon|\nabla z|^2\right) + \lambda(f-u)^2 \, dx$$

- u : Piecewise smooth approximation.
- z : Keeps track of edges:
  - $z \approx 1$ away from edges (most of Ω).
  - $z \approx 0$ near edges (in an ε neighborhood of K).
- $\int_\Omega z^2|\nabla u|^2 \, dx \approx \int_{T_\varepsilon^c} |\nabla u|^2 \, dx.$
- $\int_\Omega \frac{(1-z)^2}{4\varepsilon} + \varepsilon|\nabla z|^2 \, dx \approx \mathrm{Length}(K).$



Phase Field for Full Mumford-Shah

- Gradient descent:

$$u_t = \nabla \cdot ((z^2{+}\delta)\nabla u) + \lambda(f-u)$$

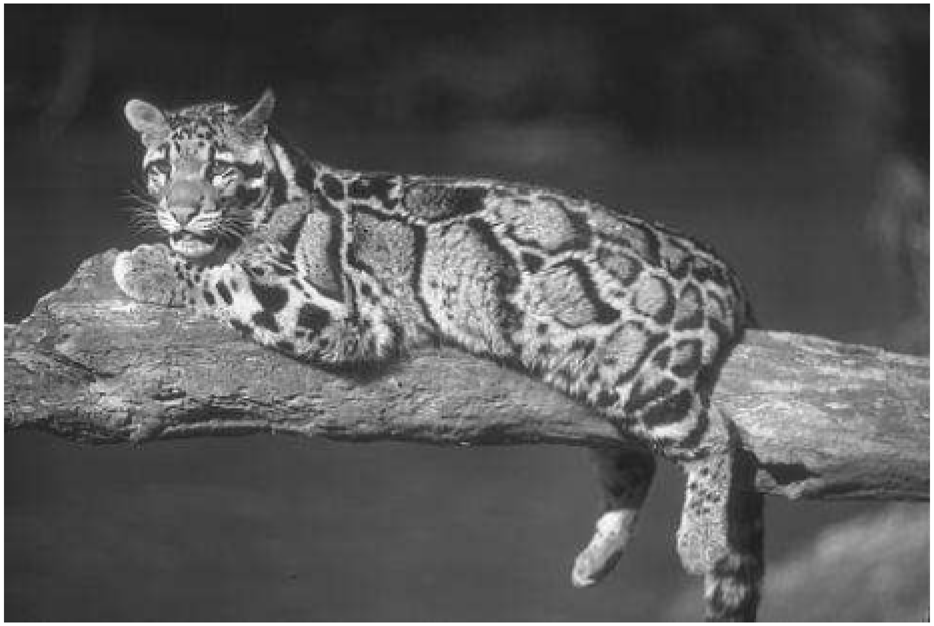
and

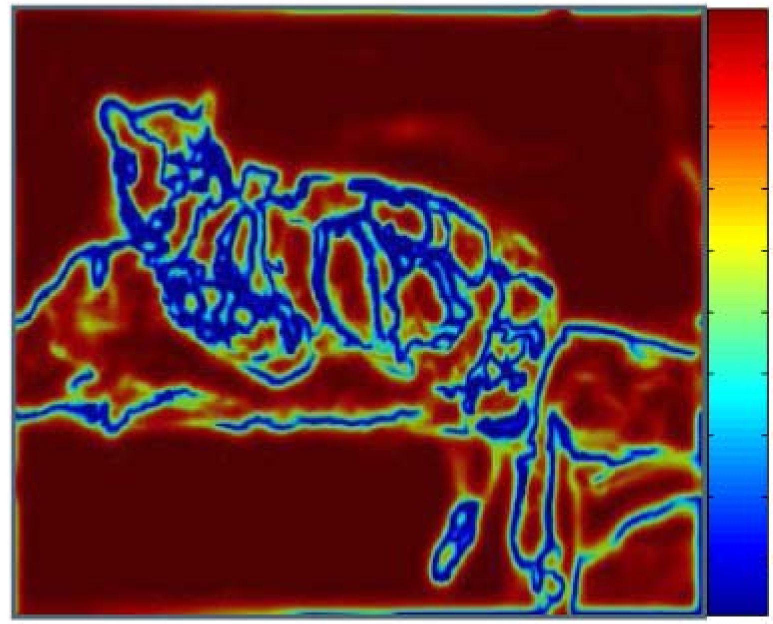
$$z_t = \varepsilon\Delta z + \frac{(1-z)}{4\varepsilon} - z|\nabla u|^2$$

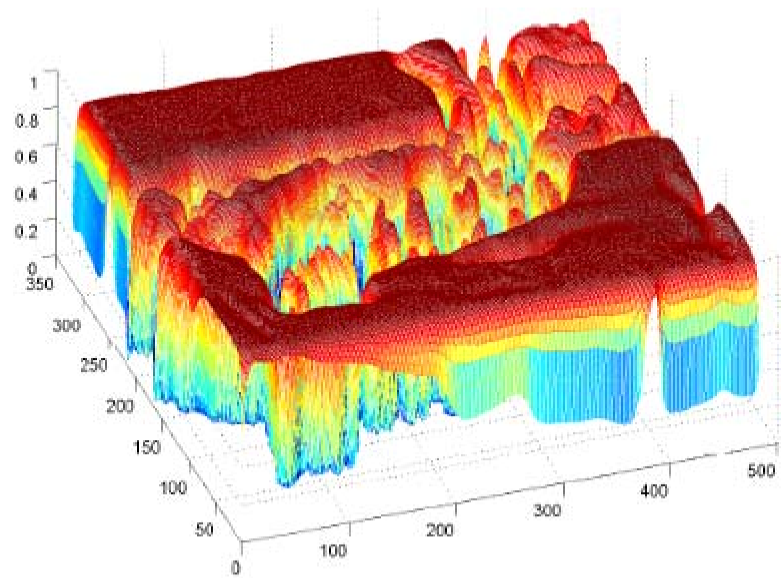
- NOTE:
  - u equation is linear in u.
  - z equation is linear in z.

    $\Rightarrow$ Lag z in the u update; lag u in the z update.
- Difficult to solve: Need to resolve ε-thick transitions in z.

Phase Field for Full Mumford-Shah







High Order Models

- Euler's elastica:

$$\int_{\partial\Sigma} \beta + \kappa^2 \, d\sigma$$

- Phase field approximation (De Giorgi):

$$\beta \int_{\Omega} \varepsilon |\nabla u|^2 + \frac{1}{\varepsilon} W(u) \, dx + \frac{1}{\varepsilon} \int_{\Omega} \left(\boxed{\varepsilon \Delta u - \frac{1}{\varepsilon} W'(u)} \right)^2 \, dx$$

RHS of Allen-Cahn
Expect to be related to κ.

High Order Models

- Curvature dependent term:

$$\frac{1}{\varepsilon}\int_{\Omega}\left(\varepsilon\Delta u-\frac{1}{\varepsilon}W'(u)\right)^2\,dx$$

- Recall the ansatz:

$$u=\phi\big(d_{\Sigma(t)}(x)\big)\quad\text{where}\quad\varepsilon\phi''-\frac{1}{\varepsilon}W'(\phi)=0.$$

- We had:

$$\varepsilon\Delta u-\frac{1}{\varepsilon}W'(u)=\varepsilon\phi'\big(d_{\Sigma(t)}(x)\big)\kappa$$

- Also note that:
  - $\phi'\big(d_{\Sigma(t)}(x)\big)$ is concentrated in a tubular nhd. T_ε of thickness ε around $\partial\Sigma$.
  - It has magnitude $\approx\frac{1}{\varepsilon}$ there.

High Order Models

- Therefore,

$$\frac{1}{\varepsilon}\int_{\Omega}\left(\varepsilon\Delta u-\frac{1}{\varepsilon}W'(u)\right)^2 dx=\frac{1}{\varepsilon}\int_{\Omega}\varepsilon^2\left(\phi'\left(d_{\Sigma(t)}(x)\right)\right)^2\kappa^2\,dx$$

$$\approx\frac{1}{\varepsilon}\int_{T_\varepsilon}\kappa^2\,dx\approx\int_{\partial\Sigma}\kappa^2\,d\sigma\,.$$

- L^2 gradient descent:

$$u_t=-\Delta\left(\Delta u-\frac{1}{\varepsilon^2}W'(u)\right)+\frac{1}{\varepsilon^2}W''(u)\left(\Delta u-\frac{1}{\varepsilon}W'(u)\right)$$

Highest order term is linear

High Order Models

- Discretization:

$$\frac{u^{k+1} - u^k}{\delta t} = -\Delta\left(\Delta u^{k+1} - \frac{1}{\varepsilon^2} W'(u^k)\right) + \frac{1}{\varepsilon^2} W''(u^k)\left(\Delta u^k - \frac{1}{\varepsilon^2} W'(u^k)\right)$$

- Or, let

$$v = \Delta u - \frac{1}{\varepsilon^2} W'(u)$$

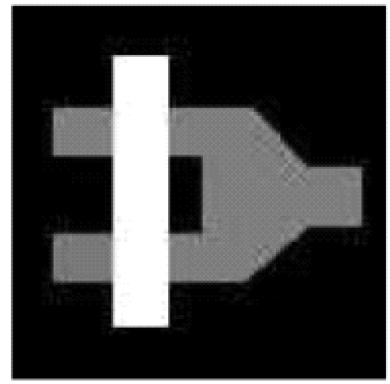
and evolve the system:

$$\boxed{\begin{aligned} u_t &= \Delta u - \frac{1}{\varepsilon^2} W'(u) - v \\ v_t &= \Delta v - \frac{1}{\varepsilon^2} W''(u) v \end{aligned}}$$

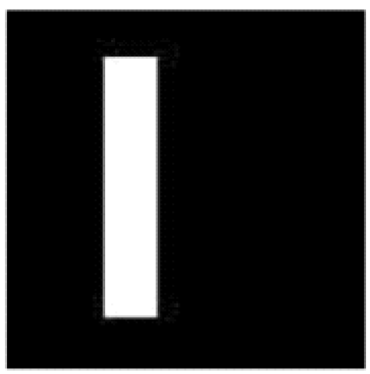
to stationary state.

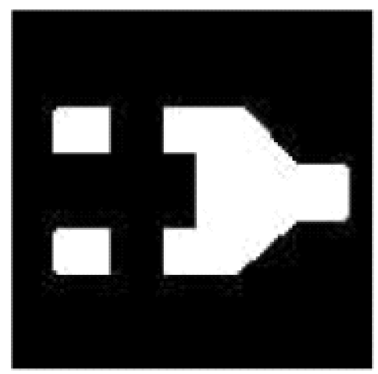
Application: Nitzberg-Mumford-Shiota

Original image:

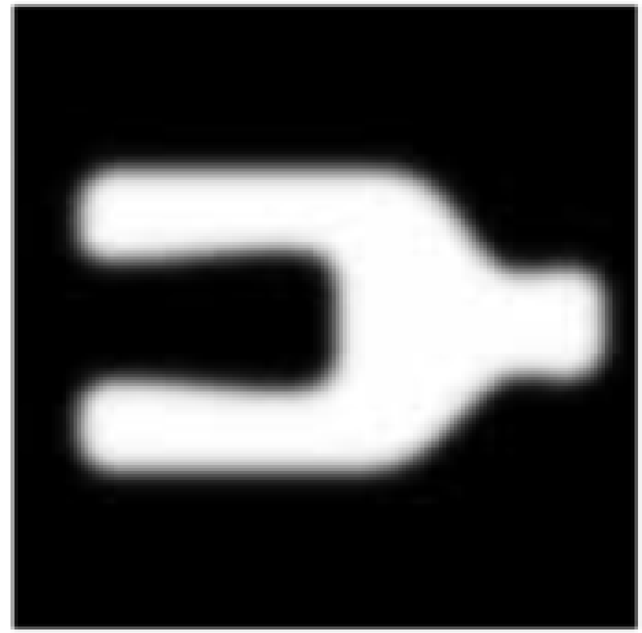


Regions taken as initial guess:



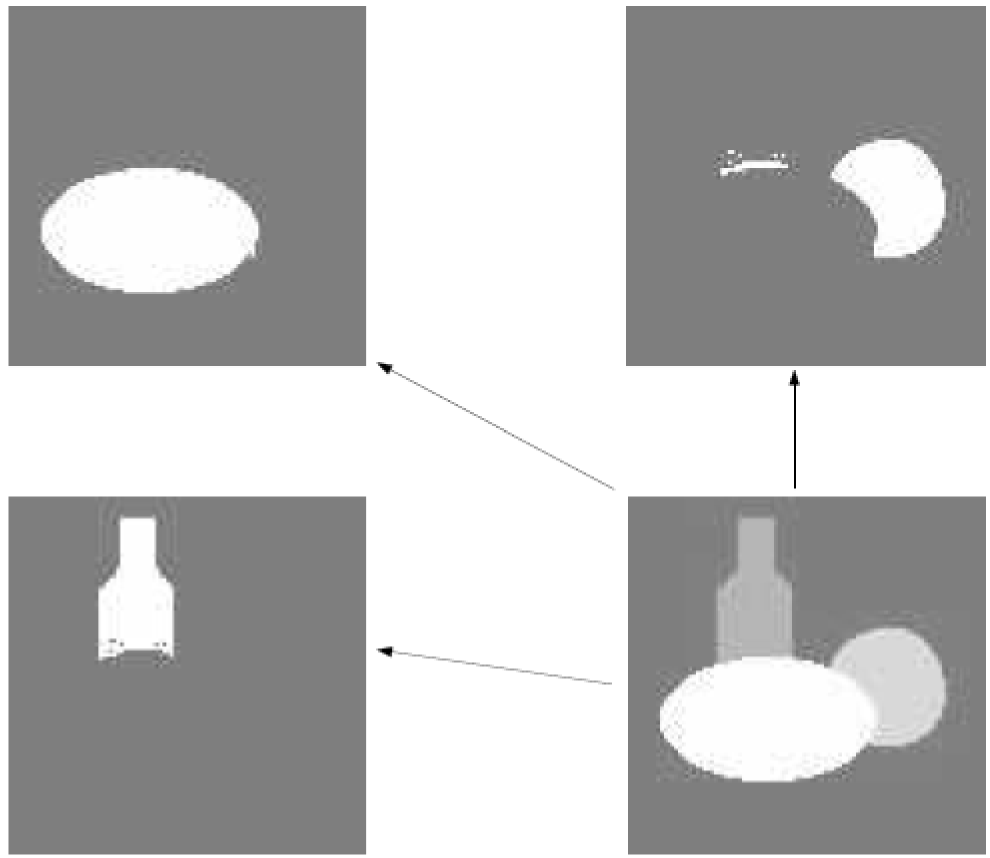


Application: Nitzberg-Mumford-Shiota



Stationary states found: Conclusion: A bar was in front of a fork.

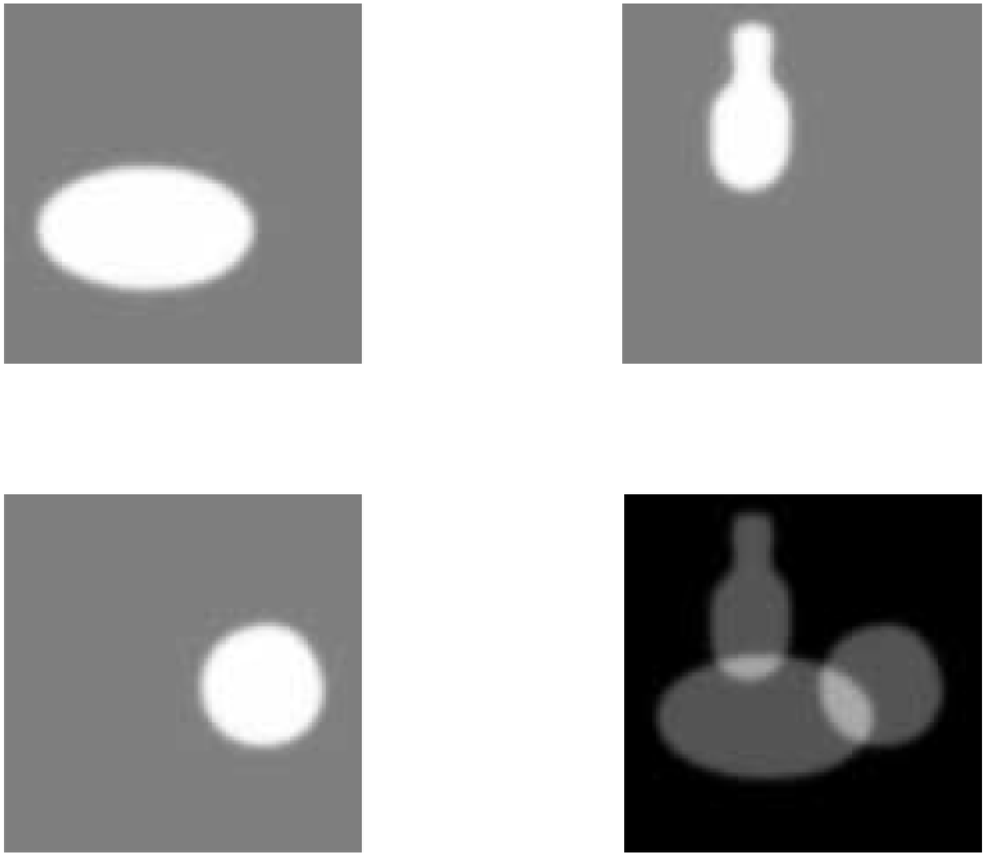
Application: Nitzberg-Mumford-Shiota



Standard segmentation of the given image on the lower right corner.

Joint work with R. March

Application: Nitzberg-Mumford-Shiota



Solution of the NMS model. Conclusion: An ellipse occludes a disk and a bottle shaped object.

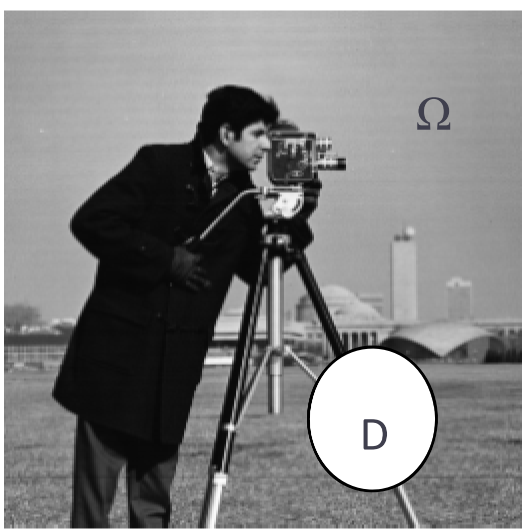
Application: Inpainting

- Mumford-Shah-Euler Model:

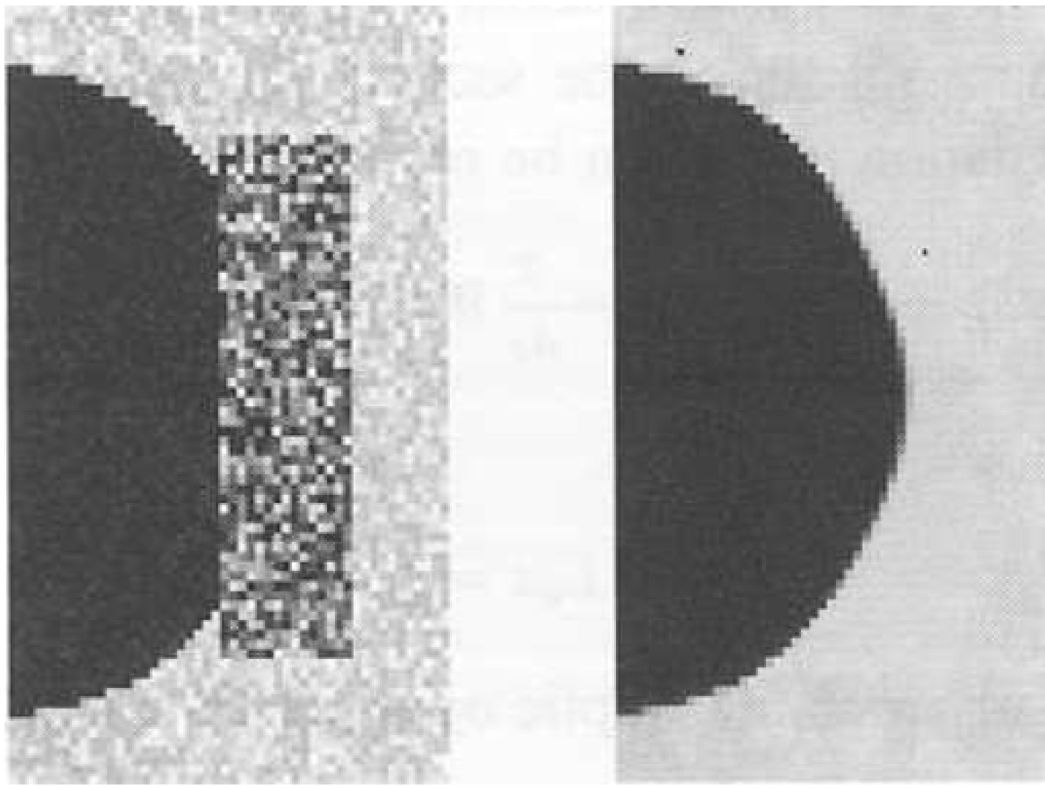
$$\int_{\Omega\backslash K} |\nabla u|^2 \, dx + \gamma \int_K \kappa^2 + \beta \, d\sigma + \lambda \int_{\Omega\backslash D} (f-u)^2 \, dx$$

- Phase-field approximation:

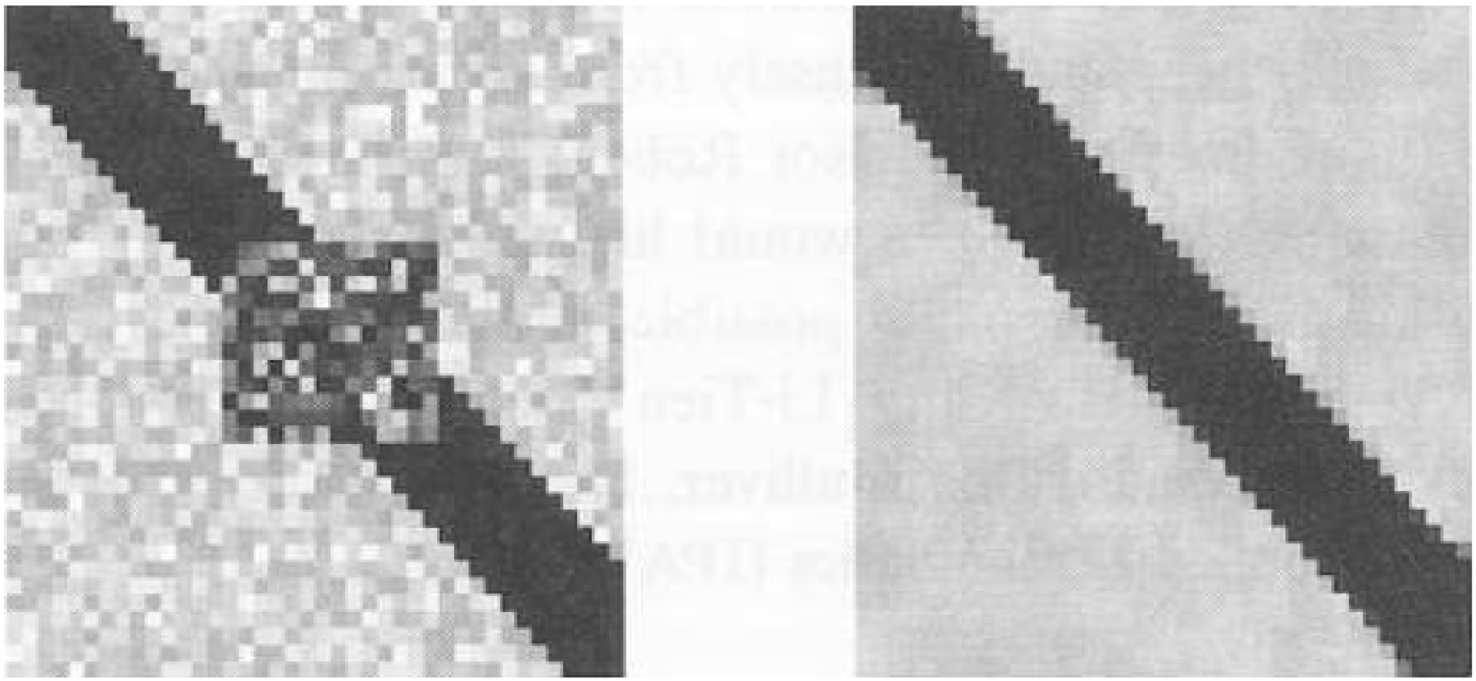
$$\int_\Omega z^2 |\nabla u|^2 \, dx + \beta\gamma \int_\Omega \varepsilon |\nabla z|^2 + \frac{(1-z)^2}{4\varepsilon} \, dx$$
$$+ \frac{\gamma}{\varepsilon} \int_\Omega \left(\varepsilon \Delta z + \frac{(1-z)}{4\varepsilon} \right)^2 \, dx$$
$$+ \lambda \int_{\Omega\backslash D} (f-u)^2 \, dx$$



Application: Inpainting



Application: Inpainting



Inpainting of Binary Images

- Joint work with A. Bertozzi and A. Gilette:
- Motivated by the energy:

$$\int_{\Omega} \varepsilon|\nabla u|^2 + \frac{1}{\varepsilon} W(u)\, dx + \lambda \int_{\Omega \setminus D} (f-u)^2\, dx$$

- Gradient descent in L^2 for

$$\lambda \int_{\Omega \setminus D} (f-u)^2\, dx$$

- Gradient descent in H^{-1} for

$$\int_{\Omega} \varepsilon|\nabla u|^2 + \frac{1}{\varepsilon} W(u)\, dx$$

Inpainting of Binary Images

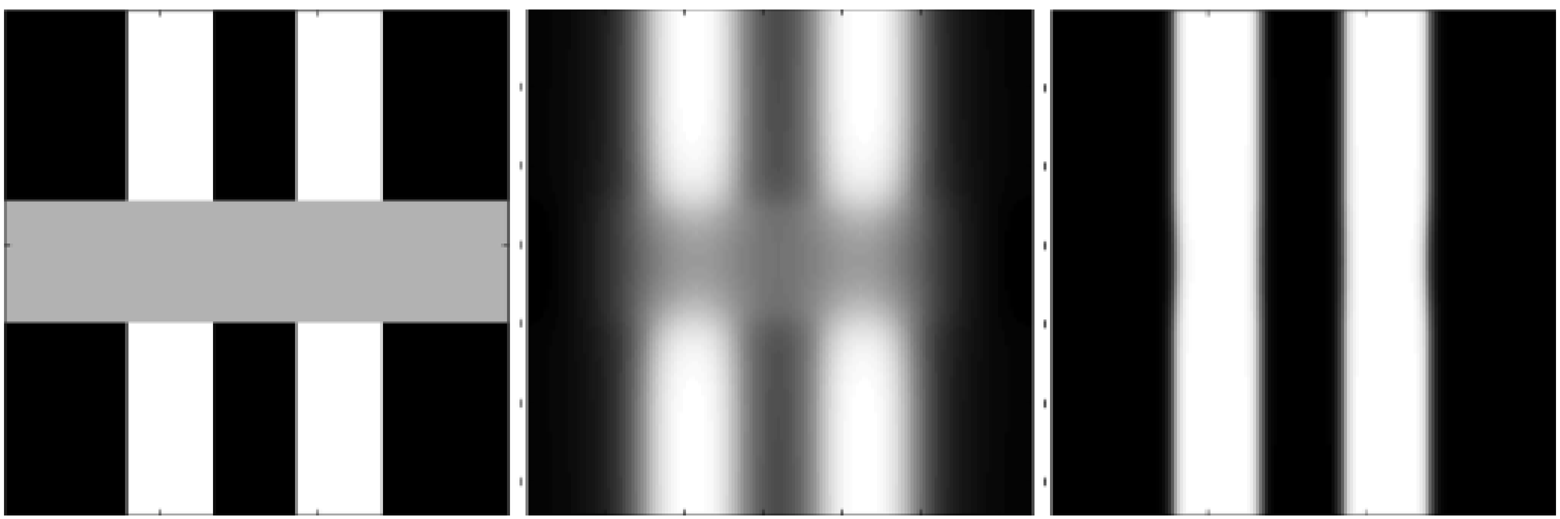
- Cahn-Hilliard based inpainting:

$$u_t = -\Delta\left(\varepsilon\Delta u - \frac{1}{\varepsilon}W'(u)\right) + \lambda \mathbf{1}_{\Omega\backslash D}\,(f-u)$$

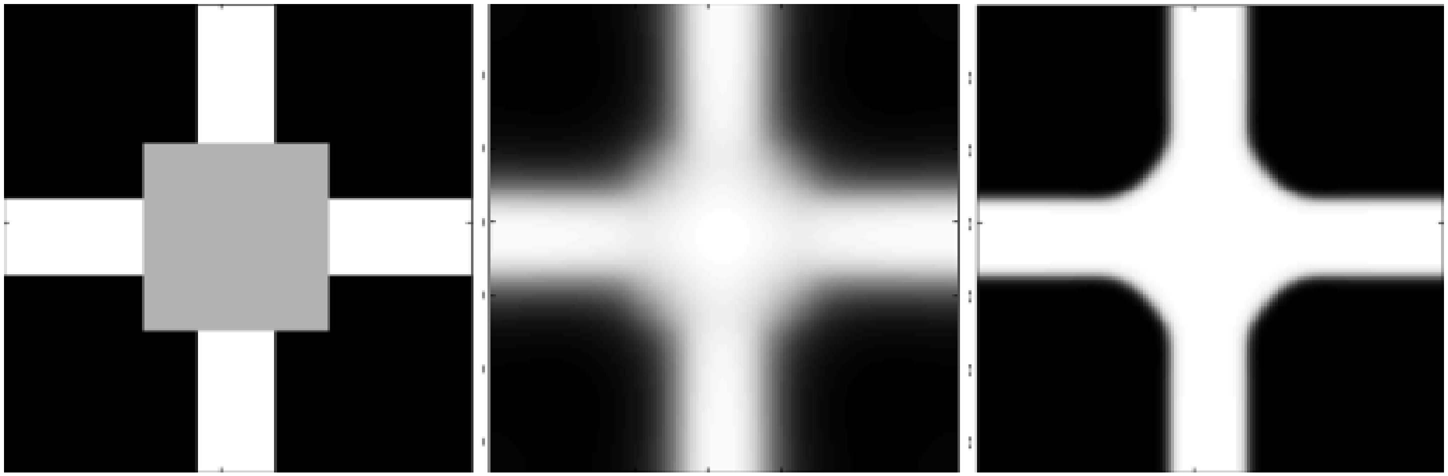
- Combined with continuation on ε:

$$\varepsilon_1 > \varepsilon_2 > \varepsilon_3 > \cdots$$

- Example:



Inpainting of Binary Images



Extension to all images: Do it layer by layer, add up:

by Chiu-Yen Kao

Convexity Splitting

- Suppose we want to implement gradient descent for

$$E(u) = E_1(u) + E_2(u)$$

where:

  - E_1 is convex, and
  - E_2 is concave.

- An unconditionally stable scheme (David Eyre):

$$\frac{u^{k+1} - u^k}{\delta t} = -\nabla E_1\left(u^{k+1}\right) - \nabla E_2\left(u^k\right)$$

i.e. treat

  - Convex term implicitly, and
  - Concave term explicitly.

Convexity Splitting

- <u>Unconditional stability:</u>

Taylor expand E_1:

$$E_1(u^k) = E_1(u^{k+1}) - \langle \nabla E_1(u^{k+1}), (u^{k+1} - u^k) \rangle + \frac{1}{2} \langle D^2 E_1 \xi, \xi \rangle$$

$$\geq E_1(u^{k+1}) - \langle \nabla E_1(u^{k+1}), (u^{k+1} - u^k) \rangle$$

Taylor expand E_2:

$$E_2(u^{k+1}) = E_2(u^k) + \langle \nabla E_2(u^k), (u^{k+1} - u^k) \rangle + \frac{1}{2} \langle D^2 E_2 \xi, \xi \rangle$$

$$\leq E_2(u^k) + \langle \nabla E_2(u^k), (u^{k+1} - u^k) \rangle$$

Convexity Splitting

Now look at energy at time step $k+1$:

$$E\left(u^{k+1}\right) = E_1\left(u^{k+1}\right) + E_2\left(u^{k+1}\right)$$

$$\leq E_1\left(u^k\right) + \langle \nabla E_1\left(u^{k+1}\right), \left(u^{k+1} - u^k\right)\rangle$$

$$+E_2\left(u^k\right) + \langle \nabla E_2\left(u^k\right), \left(u^{k+1} - u^k\right)\rangle$$

$$= E\left(u^k\right) - \delta t \left\|\nabla E_1\left(u^{k+1}\right)\right\|^2 - \delta t \left\|\nabla E_2\left(u^k\right)\right\|^2$$

$$\boxed{-2\delta t \langle \nabla E_1\left(u^{k+1}\right), \nabla E_2\left(u^k\right)\rangle}$$

$$\leq E\left(u^k\right).$$

Cauchy-Schwartz

Convexity Splitting

- Can be applied very generally.
- Given: Energy E.
- Add and subtract a convex term F:

$$E = (E - cF) + cF$$

- Choose $c > 0$ large enough so that

$$\boxed{E_2 := E - cF \text{ is concave.}}$$

- Let

$$\boxed{E_1 := cF}$$

which is of course convex.

- Crucial point: Choose F to be easy to invert!

Convexity Splitting

- Example: Cahn-Hilliard equation

$$u_t = -\varepsilon\Delta\left(\varepsilon\Delta u - \frac{1}{\varepsilon}W'(u)\right)$$

- It's gradient descent in H^{-1} for Modica-Mortola energy:

$$E = \int \varepsilon|\nabla u|^2 + \frac{1}{\varepsilon}W(u)\,dx$$

- Let

$$F = \int |\nabla u|^2 + u^2\,dx$$

- Add and subtract:

$$E_2 := E - cF \text{ and } E_1 := cF.$$

Convexity Splitting

- Resulting scheme:

$$\frac{u^{k+1} - u^k}{\delta t} = \boxed{\begin{array}{l} -\varepsilon\Delta\left(\Delta u^k - \frac{1}{\varepsilon}W'\left(u^k\right)\right) \\ +c\Delta^2 u^k - c\Delta u^k \end{array}} \; \boxed{-c\Delta^2 u^{k+1} + c\Delta u^{k+1}}$$

- Use FFT to solve

$$(I + c\delta t\Delta^2 - c\delta t\ \Delta)u^{k+1} = R.H.S.\left(u^k\right)$$