

Duality, Deltas, and Derivatives Pricing

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Convex Duality and Mathematical Finance

- I believe that convex duality was introduced into mathematical finance in the paper Karatzas, 1., Lehoczsky, J.P., SHREVE, S. E. and Xu, G. L. (1991). Martingale and duality methods for utility maximization in an incomplete market. SIAM J. Control Optimization 29 702-730.
- Since then it has been used extensively in the control context, by many of the people in this room.
- In this talk, we explore an application of convex duality in the traditional option pricing context. Since option payoffs are convex in both their underlying and their strike price, it is somewhat surprising that this work has not been explored previously to my knowledge (references are welcome).

• Option prices are subject to no arbitrage constraints of two types:

- fixed strike and maturity
- 2 varying strike or maturity
- Feeding a positive implied variance rate to Black Scholes formulas eliminates the first type of arbitrage, but not the second.
- For example, if a positive term structure of implied variance rates is fed to the Black Scholes formula for a call, then each generated call price in the term structure is above intrinsic value and below the spot price of its underlying.
- However, if the positive term structure of implied variances declines too fast in term, then a calendar spread arbitrage is generated.

- Ruling out all calendar spread arbitrages implies that the term structure of implied variance rates cannot decline too fast in term at each strike.
- Similarly, ruling out all vertical spread arbitrages implies that the strike structure of implied variance rates cannot rise or fall too fast in strike at each maturity.
- Similarly, ruling out all butterfly spread arbitrages implies that implied variance rates cannot be too concave in strike at each maturity.
- Might there be an alternative to an implied variance rate surface for which no arbitrage just requires positivity?

Alternatives to Quoting Implied Variance Rate

• There at least three simple alternatives to quoting an implied variance rate:

- Dupire Risk 1994: quote one positive function of strike and maturity to generate an arbitrage-free implied vol surface
- Schweitzer Wissel F&S 2008: quote one positive function of just moneyness to generate an arbitrage-free implied vol smile
- This paper: quote one positive function of just moneyness to generate an arbitrage-free implied vol smile or quote one positive function of just moneyness and one positive function of just term to generate an arbitrage-free implied vol surface.
- In the second case, our positive function of just term controls the level of ATM implied vol at each term, while our second positive function of just moneyness controls the implied vol skew across all terms.

Standing Assumptions

- For convenience, we adopt a currency options perspective, but our results apply to other underlyings.
- We assume zero interest rates, but our results easily extend to deterministic interest rates in the usual way.
- We assume no default by FX option issuers. However, countries can default or hyper-inflate, sending the value of their currency in any other currency from some positive number to zero. Negative currency values are disallowed.
- The analysis presented here is purely static, i.e. calendar time is frozen at zero. The currencies need only trade at pricing time t = 0 and at option maturity date $T \ge 0$. However, we have developed dynamic versions for a continuous time interval [0, T].
- The only problem to be solved is to quote some alternative to option prices or implied variance rates. Eliminating all arbitrages should require non-negativity of our alternative, but no other constraint.

- The pricing currency is defined as the currency in which an FX option's premium is denominated.
- When an FX option is exercised, the long position pays a fixed amount of one currency and receives a fixed amount of another currency.
- The pricing currency can be either the paid currency (call), the received currency (put), or neither (exchange option).
- We develop a notation that covers all 3 cases.

- Start by picking a pricing currency.
- Let N_+ denote the spot price in the specified pricing currency of the contract received in the optional exchange.
- Let *N*₋ denote the spot price in the same pricing currency of the contract delivered.
- The intrinsic value of the option contract is $(N_+ N_-)^+$.
- We do an example on the next slide.

- For example, suppose the pricing currency is dollars.
- Suppose an option allows the long party to exchange 2 pounds for 3 Euros.
- Suppose 2 pounds costs 3 dollars (\$1.50 per pound). Then $N_{-} = 3$ dollars.
- Suppose 3 Euros costs 4 dollars (\$1.33 per Euro). Then $N_+ = 4$ dollars.
- The intrinsic value of the option contract is $(N_+ N_-)^+ = (\$4 \$3)^+ = \$1$.
- If the pricing currency is pounds instead, then the long party has 3 calls, each allowing the exchange of 2/3 of a pound for one Euro.
- If the pricing currency is Euros instead, then the long party has 2 puts, each allowing the exchange of one pound for 3/2 Euros.

- The intrinsic value of an option depends on the pricing currency.
- Given that some pricing currency has been selected, different option contracts have different intrinsic value representations:

1 Call:
$$(S - K)^+$$
 or Put: $(K - S)^+$
2 Exchange Option: $(n_+S_+ - n_-S_-)^+$
3 FX Option: $(N_+ - N_-)^+$

• We use only the last representation.

- Notice that the intrinsic value of the option contract, $(N_+ N_-)^+$, depends on the values N_+ - and N_- of the two currencies in some pricing currency, but is independent of the option maturity date T.
- In contrast, the option premium P depends not only on the values N_+ and N_- , but it also depends on the option maturity date $T \ge 0$.
- There are many no-arbitrage constraints on the option pricing function $P(N_+, N_-, T) : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \mapsto \mathbb{R}^+$.
- The main goal of this talk is to present an unconstrained non-negative alternative to $P(N_+, N_-, T)$, which respects all of these no arbitrage constraints.

Linear Homogeneity of our Option Pricing Function

- For each fixed T > 0, consider our option pricing function $P(N_+, N_-, T) : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \mapsto \mathbb{R}^+$ as a function of its first two arguments.
- In our static setting, simultaneously dilating N₊ and N₋ by the same positive scale factor λ > 0 causes the option price to dilate accordingly:

$$P(\lambda N_+, \lambda N_-, T) = \lambda P(N_+, N_-, T), \qquad \forall \lambda \ge 0.$$

• This is a consequence of linear pricing. We are not assuming that returns are independent of price or equivalently that the market dynamics are sticky delta. Recall we are not in a dynamic setting.

Implications of Linear Homogeneity

• The linear homogeneity of our option pricing function $P(N_+, N_-, T) : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \mapsto \mathbb{R}^+$ in N_+ and N_- implies the following Euler representation:

$$P(N_+, N_-, T) = N_+ P_1(N_+, N_-, T) + N_- P_2(N_+, N_-, T),$$

where the dimensionless quantities P_1 and P_2 are first partial derivatives of P w.r.t. N_+ and N_- respectively.

• Each partial derivative can be interpreted up to sign as a probability of the option finishing in-the-money:

$$P_1(N_+, N_-, T) = \mathbb{Q}_+ \{N_+ \tau > N_- \tau\} \equiv \Delta_+ \\ -P_2(N_+, N_-, T) = \mathbb{Q}_- \{N_+ \tau > N_- \tau\} \equiv \Delta_-.$$

 We refer to Δ₊ and Δ₋ as option deltas, but in our static setting, they arise due to changing contractual specifications, rather than changing market prices. In some model types, e.g. sticky delta, Δ₊ is the delta due to spot price change when the pricing currency is the received currency.

Convex Duality

- Recall that $P(N_+, N_-, T) = N_+ P_1(N_+, N_-, T) + N_- P_2(N_+, N_-, T)$.
- Since $N_- > 0$, we can solve the Euler representation for $-P_2(N_+, N_-, T)$:

$$-P_2(N_+, N_-, T) = \frac{N_+}{N_-}P_1(N_+, N_-, T) - \frac{P(N_+, N_-, T)}{N_-}$$

• Let $\hat{P}(R, T) \equiv \frac{P(N_+, N_-, T)}{N_-}, R \equiv \frac{N_+}{N_-}$. Since $P(N_+, N_-, T) = N_- \hat{P}(R, T)$, diff'g w.r.t. $N_+ \Rightarrow P_1(N_+, N_-, T) = \hat{P}_1(R, T)$. Substituting in:

$$-P_2(N_+, N_-, T) = R\hat{P}_1(R, T) - \hat{P}(R, T).$$

If the RHS is written as a function of Δ₊ ≡ P₁(N₊, N₋, T) = P̂₁(R, T), then it becomes the Legendre transform of the convex function P̂(R, T). We denote the LHS by -P₂(N₊, N₋, T) = D₋(Δ₊, T) : [0, 1] × ℝ⁺ ↦ [0, 1].

Distortion Function

- Recall that $\Delta_{-} = \mathbb{Q}_{-}\{N_{+T} > N_{-T}\} = -P_2(N_+, N_-, T) = \mathcal{D}_{-}(\Delta_+, T) =:$ $R\hat{P}_1(R, T) - \hat{P}(R, T) = R \ge 0 \left[R\Delta_+ - \hat{P}(R, T)\right]$ where $\Delta_+ = \mathbb{Q}_+\{N_{+T} > N_{-T}\}$
- For each T > 0, \mathcal{D}_{-} is an increasing function of Δ_{+} mapping [0, 1] to [0, 1], 0 to 0, and 1 to 1, and so $\mathcal{D}_{-}(\cdot, T)$ is a *distortion function*.
- It follows that if one can somehow specify a convex distortion function $\Delta_{-} = \mathcal{D}_{-}(\Delta_{+}, T)$ directly at each T > 0, one can also generate the convex function linking the normalized option price $\hat{P}(R, T) = P(N_{+}, N_{-}, T)/N_{-}$ to the ratio $R \equiv N_{+}/N_{-}$ of its underlyings at each T > 0:

$$\hat{P}(R,T) = \Delta_+ \in [0,1] [R\Delta_+ - \mathcal{D}_-(\Delta_+,T)] = R\Delta_+(R,T) - \mathcal{D}_-(\Delta_+(R,T),T),$$

where $\Delta_+(R, T)$ is the inverse of the increasing function $\frac{\partial \mathcal{D}_-}{\partial \Delta_+}(\Delta_+, T) = R$.

The convexity of Δ₋ in Δ₊ implies the convexity of P̂ in R and hence the convexity of P in N₊ and N₋. As a result, the option pricing function produced would be free of butterfly spread arbitrage.

- The last slide showed that butterfly spread arbitrage can be avoided by specifying a convex distortion function: convexity of the distortion function $\Delta_{-} = \mathcal{D}_{-}(\Delta_{+}, T)$ in Δ_{+} leads to convexity of the normalized option price $\hat{P}(R, T) = P(N_{+}, N_{-}, T)/N_{-}$ in the ratio $R \equiv N_{+}/N_{-}$ of its underlyings at each T > 0.
- To also avoid calendar spread arbitrage, the normalized option price \hat{P} must be increasing in T at each price ratio $R \ge 0$. We show that this is equivalent to \mathcal{D}_- decreasing in T at each $\Delta_+ \in [0, 1]$. This is just a consequence of the order reversing property of convex conjugates.
- We say that a distortion function D₋(Δ₊, T) is arbitrage-free if it is both convex in Δ₊ ∈ [0, 1] and decreasing in T ≥ 0 with D(Δ₊, 0) = Δ₊.
- How can we specify an arbitrage-free distortion function?

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- The Black Scholes model is arbitrage-free. Hence, we can generate an example of an arbitrage-free distortion function by looking at say the put's distortion function in this model.
- In the Black Scholes model, $\ln R_T$ is normally distributed under both measures. For the put, $\Delta_- = N(-d_1)$, $\Delta_+ = N(-d_2)$, and the distortion function is $\Delta_- = N(N^{-1}(\Delta_+) \sqrt{\sigma^2 T})$, for $\sigma > 0$.
- This type of distortion function is called a Wang transform.
- One can easily check that Δ_{-} is convex in $\Delta_{+} \in [0, 1]$ and is decreasing in $T \geq 0$, so this distortion function is arbitrage-free as expected.

Black Scholes Distortion F'n $\Delta_{-} = N(N^{-1}(\Delta_{+}) - \sqrt{\sigma^{2}T})$



- At $\sigma^2 T = 0$, $\Delta_- = \Delta_+$.
- At any σ²T > 0, Δ₋ is increasing in Δ₊, maps [0, 1] to [0, 1], 0 to 0, and 1 to 1, and hence is a distortion function.
- This distortion function is convex in $\Delta_+ \in [0, 1]$ for each T > 0, and decreasing in T > 0 for each $\Delta_+ \in [0, 1]$, so it is arbitrage-free.
- Can we generalize this arbitrage-free distortion function to some non-Black Scholes models?

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Understanding OTC FX Option Quoting

- Recall that for a put in the Black Scholes model, the distortion function is the Wang transform $\Delta_{-} = N(N^{-1}(\Delta_{+}) \sqrt{\sigma^{2}T})$, where σ^{2} is a positive constant independent of Δ_{+} and T.
- To leave the Black Scholes world, one can make σ² be a positive function of Δ₊ and/or *T*, as is done in OTC FX options markets.
- However, an overly concave dependence of σ² > 0 on Δ₊ ∈ [0,1] can produce cross-strike arbitrage, while an overly *increasing* dependence of σ² > 0 on T > 0 can produce cross-maturity arbitrage.
- Even if the positive implied variance surface σ²(Δ₊, T) is arbitrage-free, one may not be able to produce option prices as a function of N₊ and N₋.
- Is there another way to generalize the top equation, stay arbitrage-free, and be able to relate option prices to N_+ , N_- , and T?

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Constant Self-quanto Shift

- Recall again that for a put in the Black Scholes model, the distortion function is the Wang transform $\Delta_{-} = N(N^{-1}(\Delta_{+}) \sqrt{\sigma^{2}T})$, where $\sigma^{2} > 0$ is independent of Δ_{+} and T.
- In the Black Scholes model, $-d_2$ is a natural moneyness measure under \mathbb{Q}_+ measure since $-d_2 = 0$ corresponds to $E^{\mathbb{Q}_+} \ln S_T = \ln K$. Switching the measure from \mathbb{Q}_+ to \mathbb{Q}_- shifts the natural moneyness measure down to $-d_1 = -d_2 - \sqrt{\sigma^2 T}$ since $-d_1 = 0$ corresponds to $E^{\mathbb{Q}_-} \ln S_T = \ln K$.
- Outside Black Scholes, we will construct moneyness variables z_+ and z_- which differ by a constant at each fixed maturity and which correspond to $-d_2$ and $-d_1$ respectively.
- If Ω is a CDF of some random variable $Z_T \in \mathbb{R}$, then $\Delta_- = \Omega(\Omega^{-1}(\Delta_+) - \tau(T))$, with $\tau(T) \ge 0$, is a distortion function, which generalizes the Wang Transform.
- We know that the derivative of Δ₋ w.r.t. Δ₊ is R ≥ 0. To ensure that Δ₋ is convex w.r.t. Δ₊, we need to restrict Ω so that R is increasing in Δ₊.

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Sufficient Condition for Convexity

- Recall that Δ₋ = Ω(Ω⁻¹(Δ₊) − τ(T)), τ(T) ≥ 0, is a distortion function whenever Ω(z₋) is a CDF of some random variable Z_T ∈ ℝ.
- Also recall that this distortion function will be convex in Δ_+ if its derivative $\frac{\partial \Delta_-}{\partial \Delta_+} = R$ is increasing in Δ_+ .
- R will be increasing in Δ₊ if R and Δ₊ are both increasing functions of some third variable z_− ∈ ℝ at each T ≥ 0.
- Guessing that $z_{-} = \Omega^{-1}(\Delta_{-})$, the top equation implies that $\Delta_{+} = \Omega(z_{-} + \tau(T))$, so Δ_{+} is indeed increasing in $z_{-} \in \mathbb{R}$ at each $T \ge 0$.
- Differentiating Δ_- w.r.t. Δ_+ implies that $R = \frac{\Omega'(z_-)}{\Omega'(z_+)}$ where $z_+ \equiv z_- + \tau(T)$.
- We show that a sufficient condition for R to be increasing in z_{-} at each $T \ge 0$ is that the PDF $\Omega'(z_{-})$ is log concave in z_{-} .
- In this case, R is increasing in Δ_+ and hence Δ_- is convex w.r.t. Δ_+ .

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Sufficient Conditions for No Arbitrage

- The last slide showed that Δ₋ = Ω(Ω⁻¹(Δ₊) − τ(T)), τ(T) ≥ 0, is a convex distortion function if Ω'(z₋), z₋ ∈ ℝ, is a log concave PDF.
- The convexity of Δ₋ in Δ₊ leads to the convexity of the normalized option price P
 (R, T) = P(N₊, N₋, T)/N₋ in R. This leads to convexity of the option premium P in the currency values N₊ and N₋ at each T > 0, so there is no butterfly spread arbitrage.
- If in addition, τ(T) is increasing in T with τ(0) = 0, then Δ_− is decreasing in T ≥ 0 at each Δ₊ ∈ [0, 1]. This leads to P and P being increasing in T ≥ 0, so there is no calendar spread arbitrage either.
- All of the other no arbitrage constraints on the option premium are captured by the fact that our function linking Δ_- to Δ_+ at each $T \ge 0$ is a distortion function, i.e. an increasing function mapping [0,1] to [0,1], 0 to 0, and 1 to 1.

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Two Positive Functions Suffice

- The last slide showed that so long as the PDF $\Omega'(z_{-})$ is log concave in $z_{-} \in \mathbb{R}$ and $\tau(T)$ is increasing in $T \ge 0$ with $\tau(0) = 0$, then the generalized Wang Transform $\Delta_{-} = \Omega(\Omega^{-1}(\Delta_{+}) \tau(T)), \tau(T) \ge 0$, is an arbitrage-free distortion function.
- The log concavity of Ω'(z_) in z_ is equivalent to the convexity of h(z_) = − ln Ω'(z_) in z_. To generate a convex function h(z_) : ℝ → ℝ, pick a positive function p(z_) : ℝ → ℝ⁺, and integrate it twice in z_ ∈ ℝ.
- Similarly, to generate an increasing function τ(T) : ℝ⁺ → ℝ⁺s.t.τ(0) = 0, pick a positive function q(T) : ℝ⁺ → ℝ⁺, and integrate it once in T ≥ 0.
- If the positive function $p(z_{-})$ is chosen to be a positive constant $p_0 > 0$, then double integration leads to a quadratic function $h(z_{-})$, so the log concave PDF $\Omega'(z_{-}) = e^{-h(z_{-})}$ is Gaussian. Furthermore, $\ln R_T$ is linear in Z_T , so it is also Gaussian. Hence, flat $p(z_{-})$ implies Black Scholes as does flat implied variance. However, positive p(z) implies no cross-strike arbitrage, while positive implied variance can produce vertical and/or butterfly spread arbitrage.

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• We call z_{-} moneyness, while $x = \ln R = \ln(N_{+}/N_{-})$ is a log price relative. They are increasing in each other and mathematically related by:

$$x = \ln R = \ln \left(\frac{\Omega'(z_-)}{\Omega'(z_-+\tau)}\right) = h(z_-+\tau) - h(z_-),$$

where recall $h(z_{-}) \equiv -\ln \Omega'(z_{-})$ must be convex in $z_{-} \in \mathbb{R}$.

- Our new language for quoting options can accomodate a view that either or both currencies could hyper-inflate (at different times).
- If N_+ goes to zero, then the ratio $R \equiv N_+/N_-$ also goes to zero. As a result, $x = \ln R$ goes to negative infinity.
- If N_- goes to zero, then the ratio $R \equiv N_+/N_-$ goes to positive infinity. As a result, $x = \ln R$ also goes to positive infinity.



 $z_{\pm} \in (z_{\min}, z_{\max})$ under \mathbb{Q}_{\pm} measure. $x < 0 \Longrightarrow N_{-} < N_{+}$.



 $z_{\pm} \in (z_{\min}, z_{\max})$ under \mathbb{Q}_{\pm} measure. $x \to -\infty$: N_{-} is about to crash.



The hyperinflation event $N_{-T} = 0$ can't be seen under \mathbb{Q}_{-} -measure, but:

$$\mathbb{Q}_{+} \{N_{-T} = 0\} = \int_{z_{\min}}^{z_{\min}+\tau} e^{-h(z)} dz > 0.$$







The hyperinflation event $N_{+T} = 0$ can't be seen under \mathbb{Q}_+ -measure, but:

$$\mathbb{Q}_{-} \{N_{+T} = 0\} = \int_{z_{\max - \tau}}^{z_{\max}} e^{-h(z)} dz > 0.$$

Multi-Cross Extension





- Underlyings N_i (i = 1,..., n) are associated with n - 1-dimensional vectors z_i linked via constant shifts.
- PDF_i(z_i) = e^{-h(z_i)}, where h is a multi-dimensional convex function.
- $x_{ij} = h(\mathbf{z}_j) h(\mathbf{z}_i)$ is the log return of N_i relative to N_j .

Multi-Cross Extension





- Underlyings N_i (i = 1,..., n) are associated with n - 1-dimensional vectors z_i linked via constant shifts.
- PDF_i(**z**_i) = e^{-h(**z**_i)}, where h is a multi-dimensional convex function.
- The event that N₁ hyperinflates can't be seen by the Q₁ measure.

Multi-Cross Extension





- Underlyings N_i (i = 1,..., n) are associated with n - 1-dimensional vectors z_i linked via constant shifts.
- PDF_i(**z**_i) = e^{-h(**z**_i)}, where h is a multi-dimensional convex function.
- The event $N_{1T} = N_{3T} = 0$ can only be priced under the \mathbb{Q}_2 measure.

Roles of our Two Positive Functions

- Recall that our specification $\Delta_{-} = \Omega(\Omega^{-1}(\Delta_{+}) \tau(T))$ is a convex distortion function decreasing in T provided $\Omega'(z)$ is log concave and $\tau'(T) \ge 0$ with $\tau(0) = 0$.
- Also recall that our model is perfectly specified by a positive function $q(T), T \ge 0$ whose integral is $\tau(T)$ and a positive function $p(z_{-}), z_{-} \in \mathbb{R}$, which determines the log concave PDF $\Omega'(z_{-})$.
- Our positive function q(T) controls the level of ATM implied vol at each maturity T, while p(z_) p(0) controls the skew (i.e. departure from flatness) of implied volatility across moneyness.
- This separation of roles implies that the skew determined at one maturity is the same as the skew generated at another. This can be helpful when data is scarce, e.g. for predicting a term structure of default probabilities from short-term deep OTM put prices and from long-term credit default swap rates.

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- Convex duality was used to convert the option quotation problem into the problem of specifying a convex distortion function which is decreasing across maturities, i.e. an arbitrage-free distortion function.
- We examined the Black Scholes distortion function (AKA Wang Transform) and provided an arbitrage-free generalization called the constant quanto shift parametrization.
- Just as there is a cottage industry in generating formulas for implied vol surfaces, future research can investigate different or more general specifications of arbitrage-free distortion functions.
- We have examined whether our constant quanto shift parametrization can hold across different calendar times. We found that it can. This suggests the possibility of evolving either or both of our two positive functions as stochastic processes, HJM-style.
- As always, further work is needed...

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