Achieving Efficiency in Dynamic Contribution Games

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Motivation

- Team problems are ubiquitous in modern economies.
  - But such environments are susceptible to the free-rider problem.
  - Olson (1965), Hardin (1968), Alchian and Demsetz (1972), ...
- Collaboration is often geared towards achieving a particular objective.
  - *i.e.*, towards completing a "project".
  - *Examples:* startups, joint R&D and new product development projects.
- *Our Question:* How to restore efficiency?
  - Relax assumption that agents are completely cash constrained.
  - In most relevant examples, parties have some "skin in the game".

Our Framework

- We study a dynamic contribution game to a joint project.
- Model in a nutshell:
  1. At every t, each of n agents chooses his costly effort $a_{i,t}$;
  2. state of project progresses at a rate that depends on $\sum_i a_{i,t}$; and
  3. it generates a payoff once its state reaches a threshold.
- *Objective:* Mechanism that implements efficiency as outcome of MPE.
  - Specifies flow payments to be paid by the agents and their terminal rewards.
  - Payments placed in a savings account that accumulates interest.

Outline of Results

1. Incentives are plagued by two kinds of inefficiency.
   - Each agent gets only a fraction of project's payoff $\Rightarrow$ effort is inefficiently low.
   - Strategic complementarity $+$ positive externalities $\Rightarrow$ agents front-load effort.

2. Efficient mechanism specifies:
   - Flow payments that increase in progress to *kill* front-loading.
   - Terminal rewards that make each agent full residual claimant.

3. Properties / Limitations:
   - Mechanism is budget balanced.
   - Gives the agents (in aggregate) the first-best payoff.
   - Implementable iff the agents have sufficient cash reserves.

Relevance & Implications

- Mechanism resembles the incentives structure in startups.
  - Might explain why freerider problem less severe than theory predicts.
- Entrepreneurs typically take salary below the market rate (if any).
  - Flow payments can be viewed as salary differential.
  - As startup value increases, the value of their outside option increases. So, the foregone salary increases with progress.
- Value harvested primarily when acquired by larger firm / goes public.
  - Profits are often re-invested.
  - Founders and initial employees receive shares, if vested.

Related Literature

- **Moral Hazard in Teams:** Free-rider problem and restore efficiency.
  - Holmström (1982): Budget breaker.
  - Legros & Matthews (1993): Mixed strategies and unlimited liability.
- **Dynamic Contribution Games:**
  - Admati & Perry (1991) ; Marx & Matthews (2000)
  - Yildirim (2007) ; Kessing (2007)
  - Georgiadis (2015) ; Georgiadis et. al. (2014)

Outline

1. Introduction
2. Model
3. Benchmark Analysis
   - Characterize MPE and first best outcome.
   - Shirking and front-loading effects.
4. Efficient Mechanism
   - Characterize the efficient mechanism with unlimited liability.
   - Provide conditions for implementability without unlimited liability.
   - Optimal mechanism when these conditions are not satisfied.
5. Incorporate uncertainty and time-dependence.
   - Characterize the efficient time-dependent mechanism.
6. Other applications where our mechanism can be applied.

Model Setup

- The team consists of n agents. Agent i
  - has cash reserves w_i, and outside option $\bar{u}_i$;
  - discounts time at rate $r > 0$;
  - at time t, *privately* exerts effort $a_{i,t}$ *at cost* $c_i(a_{i,t})$, *where* $c_i', c_i'', c_i''' \geq 0$, $c_i(0) = c_i'(0) = 0$ and $\lim_{a\to\infty} c_i'(a) = \infty$;
  - receives lump-sum $\alpha_i V$ upon completion of the project.

- Project starts at q_0, it evolves according to

$$dq_t = \left(\sum_{i=1}^{n} a_{i,t}\right) dt\,,$$

and it is completed at the first time τ such that $q_\tau = Q$.

- Assume Markov strategies; *i.e.*, efforts at t depend only on q_t.

Building Blocks: Agents' Payoff Functions

- Agent i's discounted payoff at t:

$$J_i(q_t) = e^{-r(\tau - t)}\alpha_i V - \int_t^\tau e^{-r(s-t)} c_i(a_{i,s})\, ds$$

- HJ(B) equation:

$$rJ_i(q) = \max_{a_i}\left\{ -c_i(a_i) + \left(\sum_{j=1}^n a_j\right) J_i'(q)\right\}$$

subject to the boundary condition $J_i(Q) = \alpha_i V$.

Markov Perfect Equilibrium

- First order condition:

$$\underbrace{c_i'(a_i)}_{\text{marg. cost}} = \underbrace{J_i'(q)}_{\text{marg. benefit}}$$

- Thus, $a_i(q) = f_i(J_i'(q))$, where $f_i(\cdot) = c_i'^{-1}(\cdot)$.

- In a MPE, each agent's payoff function satisfies

$$rJ_i(q) = -c_i\left(f_i\left(J_i'(q)\right)\right) + \left[\sum_{j=1}^{n} f_j\left(J_j'(q)\right)\right] J_i'(q)$$

subject to $J_i(Q) = \alpha_i V$ for all i.

Markov Perfect Equilibrium: Characterization

Proposition 1:

1. There exists $\underline{q} < Q$ such that the game has a unique MPE on $\left(\underline{q}, Q\right]$, where $\underline{q} = \inf\left\{q : J_i\left(q\right) > 0 \text{ for all } i\right\}$.
2. We have $J_i, J_i', J_i'' > 0$, and $e^{-rt}c_i'(a_i(t))$ decreases in t.

- $J_i''\left(q\right) > 0 \Rightarrow a_i'\left(q\right) > 0$ on $\left(\underline{q}, Q\right]$; *i.e.*, effort increases with progress.
  - Each agent incurs the cost of effort when it is exerted, but receives the reward only upon completion. So, his incentives are stronger, the closer project is to completion.
- Efforts are strategic complements in this game.
  - By raising effort today, an agent induces others to raise future efforts.

Markov Perfect Equilibrium: Proof

- We write the ODE system in the form

$$J_i(q) = G_i\left(J_1'(q), \ldots, J_n'(q)\right)$$

- By the Gale-Nikaido Inverse Function Theorem, $G = (G_1, \ldots, G_n)$ from $\mathcal{R}_+^n$ to $\mathcal{R}_+^n$ is invertible and the inverse mapping F is C^1. (All the principal minors of the Jacobian matrix are positive.)
- Then, for $\varepsilon > 0$, we choose $q_0 = q_\varepsilon$ close enough to Q, and construct Picard iterations J_i^N such that $J_i^N \geq \varepsilon$.
- Convexity: differentiating the ODE for J_i, we get

$$rJ_i' = -J_i' f_i' J_i'' + J_i' \sum_{j=1}^{n} f_j'\left(J_j'\right) J_j'' + J_i'' \sum_{j=1}^{n} f_j\left(J_j'\right)$$

- By Kaykobad (1985), a sufficient condition is $J_i' > 0$ and

$$\sum_{j=1}^{n} f_j\left(J_j'\right) > \sum_{j \neq i} J_j' f_j'\left(J_j'\right)$$

First Best Outcome

- If efforts are chosen by a social planner, then her payoff satisfies

$$r\bar{S}(q) = \max_{a_1,..,a_n} \left\{ -\sum_{i=1}^{n} c_i(a_i) + \left(\sum_{i=1}^{n} a_i\right) \bar{S}'(q) \right\}$$

subject to $\bar{S}(Q) = V$.

- First order condition: $c_i'(a_i) = \bar{S}'(q) \Longrightarrow \bar{a}_i(q) = f_i\left(\bar{S}'(q)\right)$.

Proposition 2:

1. There exists $\underline{q}^s < Q$ such that the planner's problem has a unique solution on $\left(\underline{q}^s, Q\right]$.
2. We have $\bar{S}, \bar{S}', \bar{S}'' > 0$, and $e^{-rt} c_i'(a_i(t))$ is constant.

- Assume that the project is socially desirable; *i.e.*, $\bar{S}(0) > 0$ or $\underline{q}^s < 0$.

Comparison: MPE vs First Best

Details

- *Two sources of inefficiency:* In the MPE,
  1. the agents exert less effort ; and
  2. they front-load their effort relative to first-best outcome.

1. In the first-best (MPE), incentives are driven by V $(\alpha_i V < V)$.
   - Because the agents ignore their externality on the other agents.
2. Each agent has incentives to front-load effort (*i.e.*, work harder early on) to induce others to raise future efforts, which renders him better off.
   - *Formally:* discounted marginal cost of effort decreases in t.

Achieving efficiency

- We consider the set of mechanisms which specify:
    - Upfront payment $P_{i,0} \geq 0$ for each agent i due before work commences.
    - Schedule of flow payments $h_i(q_t) \geq 0$ due while project is in progress.
    - Reward $\tilde{V}_i$ upon completion of the project.

$\star$ Payments are placed in a savings account accruing interest at rate r.

- *Objectives:* Efficiency, budget balance, individual rationality.

- Roadmap:
    1. Assume $w_i = \infty$ for all i and characterize efficient mechanism.
    2. Assume $w_i < \infty$ and provide conditions for implementability.
    3. Characterize optimal mechanism when conditions are not satisfied.
    4. Extend model to incorporate uncertainty & time-dependence.

Incentivizing Efficient Actions

- Given set of flow payments $\{h_i(q)\}_{i=1}^n$, agent i's payoff satisfies

$$r\hat{J}_i(q) = \max_{a_i}\left\{-c_i(a_i) + \left(\sum_{j=1}^n a_j\right)\hat{J}_i'(q) - h_i(q)\right\}$$

  - FOC: $c_i'(a_i) = \hat{J}_i'(q) \Longrightarrow a_i = f_i\left(\hat{J}_i'(q)\right)$

- For efforts to be efficient, we need: $\hat{J}_i'(q) = \bar{S}'(q)$ for all i and q.

- Then $\hat{J}_i(q) = \bar{S}(q) - p_i$, where p_i is a constant TBD.
  - Upon completion, agent i must receive $\hat{J}_i(Q) = V - p_i$.

- Using $\hat{J}_i(\cdot)$ and $\bar{S}(\cdot)$, we back out the flow payments function

$$h_i(q) = \sum_{j\neq i} c_j\left(f_j\left(\bar{S}'(q)\right)\right) + rp_i\,.$$

Budget Balance (on the Equilibrium Path)

- Each agent i's ex-ante discounted payoff is
$$\hat{J}_i(0) - P_{i,0} = \bar{S}(0) - (p_i + P_{i,0}) \ .$$
- Budget balance requires that
$$\begin{aligned} \sum_{i=1}^{n} \left[\bar{S}(0) - (p_i + P_{i,0})\right] &= \bar{S}(0) \\ \Longrightarrow \sum_{i=1}^{n} (P_{i,0} + p_i) &= (n-1)\,\bar{S}(0) \end{aligned}$$
- Total discounted cost of payments along eq'm path
$$e^{-r\bar{\tau}} \left[(n-1)\,V - \sum_{i=1}^{n} p_i\right]$$
  - Minimized when $\sum_{i=1}^{n} p_i = (n-1)\,\bar{S}(0)$ and $P_{i,0} = 0$ for all i.
  - Individual p_i's will be pinned down by relative bargaining powers.

An Efficiency Result

Lemma 1

Suppose that each agent i makes flow payments given by

$$h_i(q) = \sum_{j \neq i} c_j\left(f_j\left(\bar{S}'(q)\right)\right) + rp_i\,,$$

and receives $V - p_i$ upon compl'n, where $\sum_{i=1}^{n} p_i = (n-1)\,\bar{S}(0)$. Then:

- $\exists$ a MPE in which each agent exerts the efficient effort $\bar{a}_i(q)\ \forall q$.
- $h_i'(q) > 0$ for all i and q; *i.e.*, flow payments are increasing in q.

- Mechanism must neutralize the two inefficiencies:
  1. Each agent is (essentially) made a full residual claimant.
  2. $h_i(\cdot)$ increases at a rate s.t. agent's benefit from front-loading is offset by the cost associated with larger flow payments in the future.

A Potential Problem

- Mechanism in Lemma 1 is budget balanced on the eq'm path.
- What about off the eq'm path?
  - Suppose an agent deviates at some t.
  - Then amount in savings account at τ may be $\gtrless (n-1)\left[V - \bar{S}(0)\right]$.
  - If a 3^{rd} party can't pay each agent $V - p_i$ upon completion (irrespective of the balance in savings account), then incentives will be affected.
- Would like the mechanism to be budget balanced on & off eq'm path.

Strict Budget Balance

- Let

$$H_t = \sum_{i=1}^{n} \int_0^t e^{r(t-s)} h_i(q_s)\, ds$$

denote the balance in savings account at time t.

Lemma 2:

Suppose that each agent i receives $\beta_i (V + H_\tau)$ upon completion. Then:

1. $\nexists$ no flow payment functions $h_i(\cdot)$ that lead to the efficient outcome.
2. Optimal mechanism is equivalent to mechanism with $\sum_{i=1}^{n} h_i(q) = 0$.

- Takeaways:
  - If we want efficiency, we must give up strict budget balance.

Second Best

- It turns out a small modification suffices to ensure that
  1. the mechanism is budget balanced on the eq'm path ; and
  2. it never results in a budget deficit (but possibly a budget surplus).

Proposition 4:

Consider the previous mechanism, except upon completion, each agent receives $\min\left\{V - p_i,\, \beta_i\left(V + H_\tau\right)\right\}$, where $\beta_i = \frac{V - p_i}{V + (n-1)\left[V - \bar{S}(0)\right]}$.

- There exists a MPE in which each agent exerts the efficient effort level.
- The agents' total discounted payoff is equal to FB discounted payoff.

- With strict budget balance, agents have incentives to shirk, let the balance grow, and collect a bigger reward upon completion.
- Capping each agent i's reward at $V - p_i$ eliminates these incentives.

Limited (but Sufficient) Cash Reserves

- Suppose that each agent has cash reserves $w_i < \infty$ at $t = 0$.
- Two problems:
  1. Mechanism must specify what happens if an agent runs out of cash.
  2. Efficient mechanism may not be implementable.

Limited Cash Reserves: Implementability Condition

Proposition 5:

- The mechanism in Prop. 4 is implementable iff $\exists \{p_i\}_{i=1}^n$ s.t

$$\sum_{i=1}^n p_i = (n-1)\,\bar{S}(0) \qquad \text{(BB)}$$

$$\int_0^{\bar{\tau}} e^{-rt} \sum_{j \neq i} c_j\left(f_j\left(\bar{S}'(q_t)\right)\right) dt + p_i\left(1 - e^{-r\bar{\tau}}\right) \leq w_i \qquad \text{(Cash)}$$

$$\bar{u}_i \leq \bar{S}(0) - p_i \qquad \text{(IR)}$$

- If agents symmetric, implementable iff $w_i > e^{-r\bar{\tau}}\left(\frac{n-1}{n}\right)\left[V - \bar{S}(0)\right]$.

- Interpretation:
  1. Each agent must have sufficient cash to make payments.
  2. Each agent's IR constraint must be satisfied.

Limited Cash Reserves (Cont'd)

- Assume that an agent who runs out of cash receives 0 upon completion, while others' rewards are unchanged.
- Define $I_i(q) = \mathbf{1}_{\{\text{agent i has had cash at every } \tilde{q} < q\}}$.

Proposition 6:

- Suppose that each agent i makes flow payments $h_i(q)\, I_i(q)$, and receives $\min\{V - p_i,\, \beta_i(V + H_\tau)\}\, I_i(Q)$ upon completion.
- Assume that the conditions of Prop. 5 are satisfied.
- Then the resulting mechanism implements the efficient outcome.

- *Intuition:*
  - Mechanism must punish agent who runs out of cash.
  - But his "share" cannot be distributed to the other agents.

Cash Constraints

Details

- Efficient mechanism is implementable iff $w_i \geq \underline{w}_i$ for each i.
- What if the agents don't have sufficient cash reserves?

- Assuming symmetry, we characterize the optimal mechanism for arbitrary w.
    - "Optimal" $=$ mechanism that maximizes ex-ante total surplus.

- Optimal Mechanism:
    - Agents make payments while they still have cash.
    - Upon completion, they share balance in account $+$ project payoff.
    - Similar properties as the efficient mechanism.

Stochastic case

- We extend our model by
  1. incorporating uncertainty in the evolution of the project; *i.e.*,

$$dq_t = \left(\sum_{i=1}^{n} a_{i,t}\right) dt + \sigma dW_t \text{ , and}$$

  2. considering mechanisms that depend on both q and t.
     - *i.e.*, flow payments $h_i(t, q)$ and terminal rewards $g_i(\tau)$.

- Assume $w_i = \infty$ for all i.

First Best Outcome

- The planner's problem satisfies

$$r\bar{S}(q) = \max_{a_1,..,a_n} \left\{ -\sum_{i=1}^{n} c_i(a_i) + \left(\sum_{i=1}^{n} a_i\right) \bar{S}'(q) + \frac{\sigma^2}{2} \bar{S}''(q) \right\}$$

subject to $\lim_{q \to -\infty} \bar{S}(q) = 0$ and $\bar{S}(Q) = V$.

- FOC: $c_i'(a_i) = \bar{S}'(q)$

- We know from Georgiadis (2015) that
  1. the problem admits a unique solution ; and
  2. effort increases with progress; *i.e.*, $a_i'(q) > 0$.

Analysis

- Agent i's discounted payoff satisfies

$$rJ_i - J_{t,i} = \max_{a_i} \left\{ -c_i\left(a_i\right) + \left(\sum_{j=1}^{n} a_j\right) J_{q,i} + \frac{\sigma^2}{2} J_{qq,i} - h_i \right\}$$

  subject to $\lim_{q \to -\infty} J_i\left(q\right) = 0$ and $J_i\left(t, Q\right) = g_i\left(t\right)$.

- FOC $c_i'\left(a_i\right) = J_{q,i}\left(t, q\right)$

- Efficiency requires $J_{q,i}\left(t, q\right) = \bar{S}'\left(q\right)$ for all i, t, q.

- Therefore,

$$J_i\left(t, q\right) = \bar{S}\left(q\right) - p_i\left(t\right) \text{ , where } p_i\left(t\right) = V - g_i\left(t\right)$$

Analysis (Cont'd)

- We now back out the flow payment functions

$$h_i(t,q) = \sum_{j \neq i} c_j\left(f_j\left(\bar{S}'(q)\right)\right) + r\left(V - g_i(t)\right) + g_i'(t)$$

- Budget balance requires that

$$\sum_{i=1}^{n} J_i(0,0) = \bar{S}(0) \Longrightarrow \sum_{i=1}^{n} g_i(0) = nV - (n-1)\bar{S}(0)$$

- We pick $g_i(\cdot)$ such that $\mathbb{E}\left[h_i(t,q)\right] = 0$, and so we solve

$$0 = \sum_{j \neq i} \mathbb{E}\left[c_j\left(f_j\left(\bar{S}'(\bar{q}_t)\right)\right)\right] + r\left(V - g_i(t)\right) + g_i'(t)$$

subject to $g_i(0)$, where $\sum_{i=1}^{n} g_i(0) = nV - (n-1)\bar{S}(0)$.

Efficient Mechanism

Proposition 7

Suppose each agent pays flow $h_i(t, q)$ and receives $g_i(\tau)$ upon compl'n.

1. There exists a MPE in which each agent exerts efficient effort level;
2. the agents' ex-ante discounted payoff = FB payoff; and
3. the mechanism is ex-ante budged-balanced.

- With uncertainty, to achieve efficiency,
  1. the agents must have unlimited cash; and
  2. there must be a 3^{rd} party to balance the budget.

Corollary

- Suppose that $\sigma = 0$; *i.e.*, the project is deterministic.
- Then the efficient mechanism specifies 0 flow payments on the FB path, and each agent receives $\alpha_i V$ upon completion.

Other Applications

1. Dynamic extraction of a common resource:
   - A group of n agents extract a common resource over time.
   - $dX_t = -\sum_{i=1}^{n} c_{i,t} dt$, where $X_0 > 0$, and game ends when $X_\tau = 0$.
   - each agent obtains flow utility $u_i(c_{i,t})$ from extracting at rate $c_{i,t}$.
   - Efficient mechanism specifies that each agent pays an "entry" fee and receives a subsidy $s_i(X)$, where $s_i'(X) < 0$.

2. Dynamic experimentation:
   - Consider a good news, Poisson experimentation model.
   - Each of n agents experiment in pursuit of a breakthrough.
   - Experimentation is inefficiently low.
   - A central authority can restore efficiency by offering subsidies $s_i(p)$ and a prize to the agent who achieves the breakthrough.

Conclusions

- Propose a mechanism for dynamic contribution games that induces agents to always choose the efficient actions in a MPE.
  - Flow payments that increase in progress to *kill* front-loading.
  - Terminal rewards that make each agent full residual claimant.
  - Mechanism reminiscent of incentive structures in startups.
- Future work:
  - Optimal mechanism with uncertainty and cash constraints.
  - Applications (dynamic games with many agents and externalities).
  - Test mechanism empirically.

Limited cash reserves: stochastic case

- $X(0)$ – an agent's initial wealth

$$dX(t) = [rX(t) - h(t)]dt$$

Suppose the agent has receives $g(\tau)$ at completion time τ. The agent's value is

$$J_t = \sup_a E_t \left[e^{-r(\tau - t)}(X(\tau) + g(\tau)) - \int_t^\tau e^{-r(s-t)} c(a(s))ds \right]$$

Optimal h and g are the solutions to the following "planner's" problem:

$$L_t = \sup_{a,h,g} E_t \left[e^{-r(\tau - t)} \left(ng(\tau) + nX(\tau) \right) - n \int_t^\tau e^{-r(s-t)} c(a(s))ds \right]$$

Limited cash reserves: stochastic case

- The optimization is performed under the following constraints:

- **(i) Cash constraint:** $h(t) = 0, g(t) \geq 0$ for $t \geq \tau_0$
- **(ii) Budget balance:** $E\left[e^{-r\tau}\left(ng(\tau) + nX(\tau) - V\right)\right] = nX(0)$
- **(iii) Incentive compatibility:** The action process a is optimal for each agent i, if everyone else plays the same action.
- **(iv) Participation constraint:** $J_0 \geq 0$

Here, τ_0 is the first time X hits zero.

Comparison: MPE vs First Best (1/3)

- *Two sources of inefficiency:* In the MPE,
  1. the agents exert less effort ; and
  2. they front-load their effort relative to first best outcome.

Proposition 3:

- Eq'm effort is inefficiently low; *i.e.*, $a_i(q) < \bar{a}_i(q)$ for all i and $q > \underline{q}$.

- In equilibrium, incentives are driven by $\alpha_i V < V$.
  - Because agents ignore their externality on the other agents.

Comparison: MPE vs First Best (2/3)

- To illustrate the front-loading effect, we use the maximum principle of optimal control.
- Social Planner's Hamiltonian:

$$\bar{H}_t = -\sum_{i=1}^{n} e^{-rt} c_i\left(a_{i,t}\right) + \lambda_t^{fb}\left(\sum_{i=1}^{n} a_{i,t}\right)$$

- Optimality and adjoint equations are

$$\frac{d\bar{H}_t}{da_{i,t}} = 0 \quad \text{and} \quad \dot{\lambda}_t^{fb} = -\frac{dH_t}{dq},$$

or equivalently,

$$e^{-rt} c_i'\left(\bar{a}_{i,t}\right) = \lambda_t^{fb} \quad \text{and} \quad \dot{\lambda}_t^{fb} = 0$$

- Therefore, $e^{-rt} c_i'\left(\bar{a}_{i,t}\right) =$ constant for all t in the first best outcome.
- *Intuition:* Efficient to smooth efforts over time (because $c_i'' > 0$).

Comparison: MPE vs First Best (3/3)

Return

- Agent *i*'s Hamiltonian (MPE):

$$H_{i,t} = -e^{-rt} c_i\left(a_{i,t}\right) + \lambda_{i,t}\left(\sum_{j=1}^{n} a_{j,t}\right)$$

- Optimality equation: $e^{-rt} c_i'\left(a_{i,t}\right) = \lambda_{i,t}$
- Adjoint equation: $\dot{\lambda}_{i,t} = -\sum_{j \neq i} \frac{\lambda_{i,t} \dot{a}_{j,t}}{\sum_{l=1}^{n} a_{l,t}} < 0$
  - Inequality follows because $\lambda_{i,t} > 0$ and $a_i'(q) > 0 \Rightarrow \dot{a}_{j,t} > 0$.
- So $e^{-rt} c_i'\left(a_{i,t}\right)$ decreases in t; *i.e.*, in MPE, agents front-load effort.
- *Intuition:* Each agent has incentives to front-load effort to induce others to raise future efforts (which renders him better off).

Cash Constraints

- Efficient mechanism is implementable iff $w_i \geq \underline{w}_i$ for each i.
- What if the agents don't have sufficient cash reserves?

- We characterize the optimal mechanism for arbitrary w.
  - "Optimal": Upfront & flow payments that induce strategies which maximize the agents' total surplus.
  - Challenging problem! Need to solve a PDE + a fixed point problem.

- *Simplification:* Symmetric agents & focus on symmetric mechanisms.
  - *i.e.*, $\alpha_i = \frac{1}{n}$, $w_i = w_j$, $c_i(\cdot) \equiv c_j(\cdot)$, $P_{i,0} = P_{j,0}$, $h_i(\cdot) \equiv h_j(\cdot)\ \forall i, j$.

Solution Approach

- Given arbitrary upfront & flow payments, there is some state $q_\emptyset$ (and corresponding time $t_\emptyset$) when the agents run out of cash.

- For $q \geq q_\emptyset$, the game is similar to the cash constrained case.
  - *Only difference:* Upon completion, each agent receives reward K (TBD).
- For $q \leq q_\emptyset$, need to choose payments to solve

$$L = \max_{h(\cdot)} \left\{ e^{-r(\tau - t)} V - \int_t^\tau e^{-r(s-t)} n\, c\,(a_s)\, ds \right\}$$

  where effort a_t depends on $h(\cdot)$.
- In eq'm, mechanism has $P_{t_\emptyset} = w$ and $nK = nP_\tau e^{r\tau} + V$, where

$$P_t = P_0 + \int_0^t e^{-rs} h\,(q_s)\, ds$$

  is per-agent balance in savings account at t discounted to $t = 0$.

4-Step Characterization Approach

1. Guess value for triplet $\{t_\emptyset, q_\emptyset, K\}$, and characterize the MPE on $[q_\emptyset, Q]$; *i.e.*, after the agents have run out of cash.
   - *Note:* $K \geq \frac{V}{n}$ is each agent's reward upon completion.
2. Characterize the MPE for arbitrary smooth $h(\cdot)$ on $[0, q_\emptyset]$; *i.e.*, while the agents still have cash in hand.
3. Solve the planner's problem who chooses upfront & flow payments to maximize the agents' total discounted payoff $L(q)$.
4. Solve a fixed point problem to pin down $\left\{t_\emptyset^*, q_\emptyset^*, K^*\right\}$.

Results

Return

- Proposition 8, assuming that a solution to the fixed point problem exists, characterizes optimal mechanism for given initial wealth w.

Remark

The total discounted payoff under the optimal mechanism increases in w.

- Feasibility set is larger: any mechanism that is implementable with w, is also implementable with $w' > w$.