Equilibria in Incomplete Stochastic Continuous-time Markets:

existence and uniqueness under "smallness"

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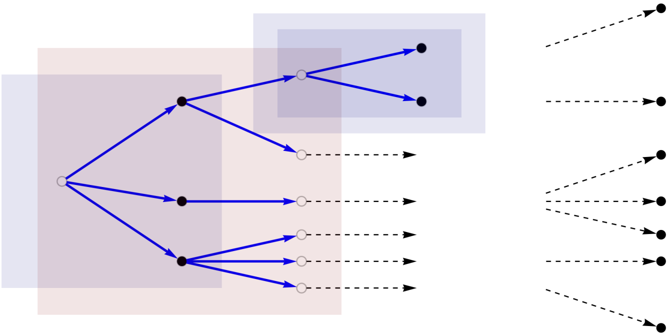
Steve's advice

"Always work on the easiest problem you cannot solve."

Stochastic finance economies



Agents, Filtration, Preferences, Endowments, Assets

Discrete-time theory

- Walras 1874,
- Arrow-Debreu '54, McKenzie '59,
- Radner '72 extends the classical Arrow-Debreu model.
- Hart '75 gives a non-existence example.
- Duffie-Shafer '85, '86 show that an equilibrium exists for *generic* endowments
- Cass, Drèze, Geanakoplos, Magill, Mas-Colel, Polemarchis, Stieglitz, and others

Continuous-time theory

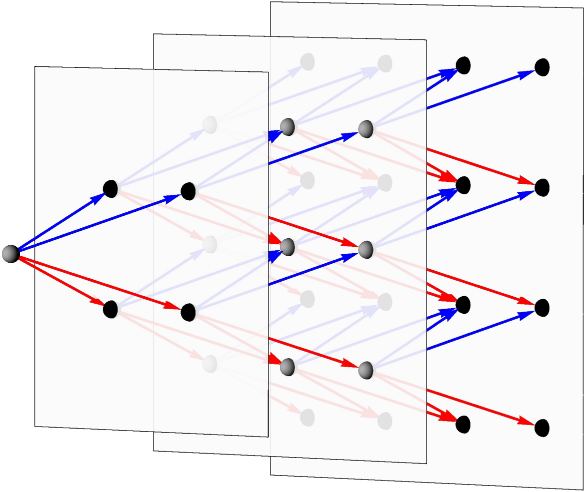
Complete Markets

- Merton '73
- Duffie-Zame '89, Araujo-Monteiro '89,
- Karatzas-Lakner-Lehoczky-Shreve **'91,** Karatzas-Lehoczky-Shreve **'90, '91**

Incomplete markets

- Basak, Cheridito, Christensen, Choi, Cuoco, He, Horst, Kupper, Larsen, Munk, Zhao, Ž

An incomplete, short-lived-asset model



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Demand $\hat{\pi}^{\lambda,i} := \operatorname{argmax}_{\pi\in\mathcal{A}^\lambda} \mathbb{E}\left[U^i\Big(\int_0^T \pi_u\,dS_u^\lambda + E^i\Big)\right].$

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Goal Is there an *equilibrium market price of risk* λ? That is,

does there exist a process λ such that

the *clearing condition* $\sum_{i=1}^I \hat{\pi}^{\lambda,i} = 0$ holds.

A BSDE characterization

Set $\alpha^i = \delta^i / (\sum_j \delta^j)$, $G^i = E^i/\delta^i$ and define the **aggregator**

$$A[\boldsymbol{x}] = \sum_i \alpha^i x^i, \text{ for } \boldsymbol{x} = (x^i)_i.$$

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Theorem. If $G^i \in \mathbb{L}^\infty$, for all i, then a process $\lambda \in$ bmo is an equilibrium *if and only if* it admits a representation

$$\lambda = A[\boldsymbol{\mu}],$$

for some solution $(\boldsymbol{\mu}, \boldsymbol{\nu}, \boldsymbol{Y}) \in \mathrm{bmo} \times \mathrm{bmo} \times \mathcal{S}^\infty$ of the following *nonlinear (quadratic)* and *fully-coupled* BSDE system:

$$\begin{cases} dY_t^i = \mu_t^i \, dB_t + \nu_t^i \, dW_t + \left(\frac{1}{2}(\nu_t^i)^2 - \frac{1}{2} A[\boldsymbol{\mu}]_t^2 + A[\boldsymbol{\mu}]\mu_t^i \right) dt, \\ Y_T^i = G^i, \qquad i = 1, \dots, I, \end{cases}$$

where $\boldsymbol{\mu} = (\mu^i)_i$, $\boldsymbol{\nu} = (\nu^i)_i$ and $\boldsymbol{Y} = (Y^i)_i$.

Nonlinear Systems of BSDEs

- [Darling 95], [Blache 05, 06]: Harmonic maps.
- [Tang 03]: Riccati systems,
- [Tevzadze 08]: existence when terminal condition is small.
- [Delarue 02], [Cheridito-Nam 14]: generator $f + z\,g$, where both f and g are Lipschitz.
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Applications:

- [Bensoussan-Frehse 90], [El Karoui-Hamadène 03]: stochastic differential games.
- [Frei-dos Reis 11], [Frei 14]: relative performance.
- Counter example: bounded terminal condition, no solution.
- [Cheridito-Horst-Kupper-Pirvu 12]: equilibrium pricing.
- [Kramkov-Pulido 14]: large investor problem.

Existence and uniqueness "with cheating"

Theorem 0a. An equilibrium exists and is unique if $(G^i)_i$ is an (unconstrained) Pareto-optimal allocation. Then $\lambda \equiv 0$.

Note: in the exponential case, $\boldsymbol{G}$ is Pareto-optimal if and only if

$$G^i - G^j = c_{ij} \in \mathbb{R}, \text{ for all } i, j.$$

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Definition. $(G^i)_i$ is **in the Pareto domain of attraction** of **pre-Pareto** if there exists an equilibrium $\lambda \in \text{bmo}$ such that the allocation

$$\tilde{G}^i = G^i + \tfrac{1}{\delta^i} \hat{\pi}^{\lambda, G^i} \cdot S_T^{\lambda}, \; i = 1, \dots, I, \text{ is Pareto optimal.}$$

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Theorem 0b. An equilibrium exists if $(G^i)_i$ is pre-Pareto.

However, ...

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3. For λ, ν defined by

$$\exp(-\textstyle\sum_i \alpha^i G^i) \propto \mathcal{E}(-\lambda \cdot B - \nu \cdot W)_T,$$

   there exist $(y^i)_i \in \mathbb{R}^I$ and $(\varphi^i)_i \in \mathrm{bmo}^I$ such that

$$G^i - G^j = y^i - y^j + (\varphi^i - \varphi^j) \cdot B^\lambda_T, \qquad \text{for all } i, j.$$

In each of those cases, λ as above is the unique equilibrium.

Spaces

- $\mathrm{bmo}^2(\tilde{\mathbb{P}})$ - $||(m,n)||^2_{\mathrm{bmo}_2(\tilde{\mathbb{P}})} = ||m||^2_{\mathrm{bmo}(\tilde{\mathbb{P}})} + ||n||^2_{\mathrm{bmo}(\tilde{\mathbb{P}})}$.

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- EBMO - "Exponential" or "Entropic" BMO: the set of all $G \in \mathbb{L}^0$ such that

$$e^{-G} \propto \mathcal{E}(-m^G \cdot B - n^G \cdot W)_T,$$

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Note that:

- $G \in \mathrm{EBMO}$ iff

$$dX_t = m_t\, dB_t + n_t\, dW_t + \tfrac{1}{2}(m_t^2 + n_t^2)\, dt,\ \ X_T = G,$$

admits a (necessarily unique) solution $(m,n) \in \mathrm{bmo} \times \mathrm{bmo}$.

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- $\mathbb{L}^\infty$ embeds continuously into EBMO (under any $\text{bmo}^2(\tilde{\mathbb{P}})$).

The general "smallness" result

For an allocation $(G^i)_i$, with $G^i \in \text{EBMO}$, we define the **distance to Pareto optimality** $H((G^i)_i)$ by

$$H((G^i)_i) = \inf_{G \in \text{EBMO}} \max_i \left\| (m^{G^i} - m^G, n^{G_i} - n^G) \right\|_{\text{bmo}^2(\mathbb{P}^G)},$$

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Theorem. Assume that $(G^i)^+ \in \mathbb{L}^\infty$, $(G^i) \in \mathrm{EBMO}$ for all i. If

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THE GENERAL "SMALLNESS" RESULT

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Note: A similar result with "distance-to-Pareto" replaced by "distance--to-pre-Pareto" holds (mutadis mutandis). A different proof technique.

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provided all E^i have bounded Malliavin derivatives

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THANK YOU, and HAPPY BIRTHDAY, STEVE.