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Risk Measures, Orlicz Spaces and Mackey Topology

Conference in Honour of Steve Shreve

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Outline

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1 Why

Happy Birthday (50 years + some months)

2 Joint Work with

Teemu Pennanen (London)

3 Some Generalities on Risk Measures

$(\Omega, \mathcal{F}, \mathbb{P})$ atomless probability space

$L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ or L^∞, space of bounded random variables, modulo equality a.s.

$L^1(\Omega, \mathcal{F}, \mathbb{P})$ or L^1 space of integrable random variables modulo equality a.s.

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The duality (L^1, L^∞) plays a central role. The Mackey topology on L^∞ is the finest locally convex topology such that L^1 remains the dual space. It is also the topology of uniform convergence on weakly compact sets of L^1. The Dunford-Pettis Theorem says that relatively weakly compact sets in L^1 are precisely the sets that are uniformly integrable.

Because of the “de la Vallée-Poussin Theorem" it is also the initial topology of the imbeddings

$$L^\infty \to L^\Phi,$$

where L^Φ is the Orlicz space

$$\{\xi \mid \text{ there is } \lambda > 0 \text{ with } \mathbb{E}[\Phi(\lambda|\xi|)] < \infty\},$$

equipped with the Luxemburg norm:

$$\|\xi\|_\Phi = \inf\left\{\lambda > 0 \mid \mathbb{E}\left[\Phi\left(\frac{|\xi|}{\lambda}\right)\right] \leq 1\right\}.$$

$\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ is a convex function with $\Phi(0) = 0$ and $\lim_{x\to+\infty} \frac{\Phi(x)}{x} = \infty$. Most of the time we use strictly increasing functions Φ. L^Φ is not always the closure of L^∞. The closure of L^∞ is called the Orlicz heart and is equal to:

$$L^{(\Phi)} = \{\xi \mid \text{ for all } \lambda > 0 \text{ we have } \mathbb{E}[\Phi(\lambda|\xi|)] < \infty\},$$

Clearly $L^{(\Phi)} \subset L^\Phi$.

A monetary utility function is defined as a function $u\colon L^\infty \to \mathbb{R}$ such that

- $u(0) = 0$ and $\xi \geq 0$ implies $u(\xi) \geq 0$
- u is concave
- for all $a \in \mathbb{R}$, $u(\xi + a) = u(\xi) + a$
- if $\sup_n \|\xi_n\|_\infty < \infty$ and $\xi_n \to \xi$ in probability, then $u(\xi) \geq \limsup u(\xi_n)$ (Fatou property).

Using the Krein-Smulian-Banach-Dieudonné Theorem we get that the set

$$\mathcal{A} = \{\xi \mid u(\xi) \geq 0\}$$

is convex and Mackey closed in L^∞. It is of course weak* closed as well.

Using convex duality theory we can describe the function u using its Fenchel-Legendre conjugate, denoted c (Föllmer-Schied).

This is defined on the set, $\mathcal{P}$, of all probabilities absolutely continuous with respect to $\mathbb{P}$. We have

- $c \colon \mathcal{P} \to \overline{\mathbb{R}_+}$
- c is convex and lower semi continuous
- $\inf_{\mathbb{Q}\in\mathcal{P}} c(\mathbb{Q}) = 0$
- $c(\mathbb{Q}) = \sup_{\xi\in\mathcal{A}} \mathbb{E}_{\mathbb{Q}}[-\xi]$
- $u(\xi) = \inf_{\mathbb{Q}\in\mathcal{P}} \left(\mathbb{E}_{\mathbb{Q}}[\xi] + c(\mathbb{Q})\right)$

The sets $\{\mathbb{Q} \mid c(\mathbb{Q}) \le m\}$ are convex and closed in L^1.

The stronger property: if $\sup_n \|\xi_n\|_\infty < \infty$ and $\xi_n \to \xi$ in probability, then $u(\xi) = \lim u(\xi_n)$ is called the Lebesgue property and is equivalent to: for all $0 \le m < \infty$:

$$\{\mathbb{Q} \mid c(\mathbb{Q}) \le m\},$$

is weakly compact. This is the link to Orlicz spaces.

4 Work of Cheridito-Li

If u satisfies the Lebesgue property then there is an Orlicz space, L^Φ, such that u can be extended to $L^{(\Phi)}$ and $u\colon L^{(\Phi)} \to \mathbb{R}$ is continuous. If we replace Φ by $\Psi(x) = \Phi(x^2)$, then we have $L^\Psi \to L^{(\Phi)}$ and hence we can extend u to an Orlicz space.

There is a natural Riesz space, $L^{(u)}$, on which u can be defined and on which u is locally Lipschitz. This space contains an Orlicz heart $L^{(\Phi)}$.

The Mackey topology on closed balls of L^∞ is the convergence in probability (this result goes back to Grothendieck). The above result can be phrased as follows: If u is continuous on closed balls for the Mackey topology, then it is continuous on L^∞ for the Mackey topology.

How general is this result?

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5 General Questions for convex functions on dual spaces

E is a Banach space and E^* is its topological dual. The main interest is the Mackey topology $\tau(E^*, E)$.

- When is the Mackey topology metrisable on bounded sets of E^*
- Is Mackey sequential continuity the same as Mackey continuity. This question must be restricted to bounded sets.

- Let f be a convex function $f\colon E^* \to \mathbb{R}$, when does Mackey continuity on bounded sets implies Mackey continuity on E^*?

The Banach-Dieudonné-Krein-Smulian theorem immediately implies that f is lower semi continuous for the weak* topology on E^*. So we must only look for the upper semi continuity of f. It also implies that there is a function $g\colon E \to \overline{\mathbb{R}}$, convex and proper such that $f = g^*$.

The first question was solved by Schlüchtermann and Wheeler (1988). The Mackey topology on bounded sets of E^* is metrisable if and only if the space E is strongly weakly compactly generated (SWCG). This means that there is a weakly compact set K such that for each weakly compact set $L \subset E$ and every $\epsilon > 0$, there is a number m such that $L \subset mK + B_\epsilon$, where B_ϵ is the ball of radius ϵ in E. For $L^\infty = (L^1)^*$ the Dunford-Pettis theorem tells us that L^1 is SWCG. Spaces such as c_0, $C[0,1]$ are NOT SWCG.

Other spaces that are SWCG are: H^1 space of holomorphic functions (Hardy space), H^1 space of martingales, the Orlicz spaces L^Φ where Φ satisfies:

$$\lim_{x\to\infty} \frac{\Phi(x)}{x\Phi'(x)} = 1.$$

In this case we have $K \subset L^\Phi$ is relatively weakly compact if and only if $\{\Phi(|\xi|) \mid \xi \in K\}$ is uniformly integrable. This applies to L^Φ with $\Phi(x) = (x+1)\log(x+1) - x$ (the $L\log L$ space).

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The last question is more delicate to handle. We could show that a convex function $f\colon E^* \to \mathbb{R}$, that is Mackey continuous on bounded sets is also Mackey continuous on E^*.

The question on sequential continuity also has an answer (and counterexamples). If E is SWCG, then Mackey sequential continuity implies Mackey continuity.