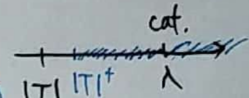
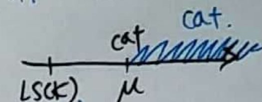
Categoricity results of abstract elementary classes (AECs)

(Morley, Shelah) Let T be a first-order theory. If $\mathrm{Mod}(T)$ is categorical in some $\lambda > |T|$, then $\mathrm{Mod}(T)$ is categorical in all $\lambda > |T|$.



Generalizations: $L_{\omega_1,\omega}$ (Shelah) assume WGCH, "few" models in $\aleph_n$ ($1 \le n < \omega$) $\Rightarrow$ categoricity transfers upwards



AECs (Boney / Shelah-Vasey) Let λ be a strongly compact cardinal, $LS(K) < \lambda$.

Categoricity in some $\mu > \lambda \Rightarrow$ categoricity in all $\mu' \geq \mu$.

(GV) Assume K has amalgamation, arbitrarily large models and tameness.
Then categoricity in a successor cardinal transfers upwards (above $LS(K)$).

(Vasey) Assume K has amalgamation, arbitrarily large models, tameness and K has primes.

Then categoricity transfers upwards.

Q: can we remove "successor" in (GV), or "primes" in Vasey's result?

Thm: Assume K has amalgamation over sets, arbitrarily large models, $LS(K)$-shortness. Then categoricity transfers upwards.

Fact: Complete first-order theories satisfy the above assumptions.

Definition An abstract elementary class $\underline{K} = \langle K, \leq_K \rangle$ $\langle \mathrm{Mod}(T),$ elem. substructures$\rangle$
K is a class of structures of the same underlying language $L = L(K)$ (finitary)
$\leq_K$ is a partial order on K.

Requirements: Let $M_1, M_2 \in K$. If $M_1 \leq_K M_2$, then $M_1 \subseteq_L M_2$.
If $N \cong M_1$, then $N \in K$.

If $M_1, M_2, N_1, N_2 \in K$ and f & g are isomorphisms,

$$\begin{tikzcd} M_2 \arrow[r, "g"] & N_2 \\ M_1 \arrow[u, "\leq_K"] \arrow[r, "f"] & N_1 \arrow[u, dotted, dash] \end{tikzcd}$$

then $N_1 \leq_K N_2$.

(Coherence) Let $M_1, M_2, M_3 \in K$
$M_1 \subseteq_L M_2$, $M_1 \leq_K M_3$, $M_2 \leq_K M_3$ then $M_1 \leq_K M_2$.

(Lowenheim-Skolem) $\exists \lambda \geq |L(K)| + \aleph_0$ $\forall M \in K$ $\forall A \subseteq |M|$
$\exists N \in K$, $|N| \supseteq A$, $N \leq_K M$, $\|N\| \leq |A| + \lambda$
We call min such λ the LS number ($LS(K)$).

(Chain) Let $\langle M_i : i<\alpha \rangle$ be a $\leq_K$-increasing chain.

Then $\bigcup_{i<\alpha} M_i \in K$ and each $M_i \leq_K \bigcup_{i<\alpha} M_i$.

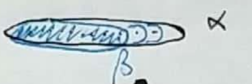
If in addition ~~each~~ $\exists N \in K$ s.t. each $M_i \leq_K N$,

then $\bigcup_{i<\alpha} M_i \leq N$.

Notation : K-embedding $f : M \to N$ if $f[M] \leq_K N$ and f is a monomorphism.

Let $\lambda \geq \aleph_0$.

Example : $\underline{K} = \langle$ wellorderings of order type $\leq \lambda^+$, by initial segments $\rangle$



$\beta \leq_K \alpha$

$LS(K) = \lambda$

Definition : K has amalgamation (over models) if for any $M_1 \leq_K M_2$, $M_1 \leq_K M_3$ then there is $N \in K$, $f : M_2 \to N$, $g : M_3 \to N$ s.t. the diagram is commutative.

$$\begin{tikzcd} M_2 \arrow[r, "f"] & N \\ M_1 \arrow[u] \arrow[r] & M_3 \arrow[u, "g"'] \end{tikzcd}$$

K has joint-embedding if for any $M_1, M_2 \in K$, there is $N \in K$, $f : M_1 \to N$ and $g : M_2 \to N$.

$$\begin{tikzcd} & N & \\ M_1 \arrow[ur, "f"] & & M_2 \arrow[ul, "g"'] \end{tikzcd}$$

K has arbitrarily large models if for any $\lambda > LS(K)$ there is $M \in K$, $\|M\| > \lambda$.

K has no maximal models if for any $M \in K$ there is $N \in K$ $N \geq_K M$ but $N \neq M$.

K has a monster model if it has amalgamation, joint-embedding and no maximal models.

We can think of "everything" to be inside the monster model.

K has amalgamation over sets if for any $A \subseteq |M_2| \cap |M_3|$, then there is $N \in K$, $f : M_2 \to N$ $g : M_3 \to N$ s.t. ~~$f[A] = g[A]$~~ f, g agree on A.

$$\begin{tikzcd} M_2 \arrow[r, "f"] & N \\ A \arrow[u, phantom, "\cup\!\!\mid"] \arrow[r, phantom, "\subseteq"] & M_3 \arrow[u, "g"'] \end{tikzcd}$$

In complete first-order theories, if we work in a monster model, then it satisfies amalgamation over sets

Proof idea: "Good-frames": nice relationship between types and models (one type and 3 models)

$a \downarrow^{M_2}_{M_0} M_1$

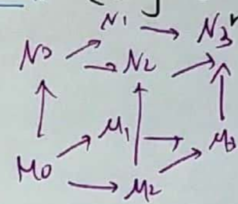
(Vasey #14) If ~~$\mathbf{K}$~~ K has a $[\mu_1, \mu_2^+]$-good frame and K is cat. in μ_1 and μ_2^+, then K is cat. $[\mu_1, \mu_2^+]$ (Amalgamation, tameness)

How to construct? Nonsplitting $\longrightarrow$ nonforking relation over $LS(K)^+$-saturated models

(Vasey #31) One can get a good frame over models of size $LS(K)$ if we assume cat in $LS(K)$ and WGCH. (using Jarden-Shelah's result).

"Better-frames": ~~types~~ sequences enumerate models

allow n-dimensional systems of models



How to construct? Use good frame above + shortness.

(or use WGCH as in Shelah-Vasey's result).

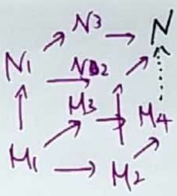
Fact: Better-frames with extra properties $\Rightarrow$ "excellent" $\Rightarrow$ prime extension $\Rightarrow$ primes

(amalgamation over sets, shortness)

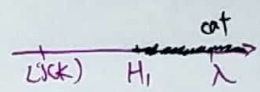
Actually, we can weaken amalgamation over sets to amalgamation over systems.

$$\begin{tikzcd} M_3 \arrow[r] & N \\ M_1 \arrow[u] \arrow[r] & M_2 \arrow[u] \end{tikzcd}$$

usual amalgamation



amalgamation over 3-systems.



Fact: amalgamation + tameness + arbitrarily large models

cat. in a successor cardinal $\lambda > LS(K)$ $\Rightarrow$ $\exists \chi < \beth_1 = \beth_{(2^{LS(K)})^+}$ s.t. cat in all $\lambda' \geq \min(\chi, \lambda)$

Proof idea: If not, there is M_χ of size χ omitting some type p_χ for each $\chi < \beth_1$.

Fix $p_\chi = p$ and consider $K_{\neg p} = \{M \in K : M \not\models p\}$

Can show that $K_{\neg p}$ is an AEC.

WTS it has arbitrarily large models (will imply cat. fails)

$\hookrightarrow$ use Facts ①: $K_{\neg p}$ is $PC_{\mu, 2^\mu}$ ($\mu = LS(K)$)

② Let K^* be an EC-class and $EC_{\lambda,\kappa}$ $\forall \alpha < \delta(\lambda, \kappa)$ $\exists M_\alpha \in K^*$ $\|M_\alpha\| > \beth_\alpha$, then K^* has arbitrarily large models.

threshold cardinal for an infinite dec. seq. to exist in a $EC_{\lambda,\kappa}$-class.

Work in a weak model of set theory + Erdős-Rado theorem

③ ~~$*$~~ $\delta(\lambda, \kappa) \leq (2^\lambda)^+$.

Let $\lambda \geq \aleph_0$.

Examples: We construct AECs K_α (for each $\alpha < (2^\lambda)^+$) (ordered by L-substructures)

s.t. K_α is cat. in $\cancel{(\beth_\alpha)^+}$ $(\beth_\alpha(\lambda))^+$ but not anywhere below, $LS(K_\alpha) = \lambda$.

cat.

H_1

But it will fail amalgamation!

? can we lower the bound H_1?

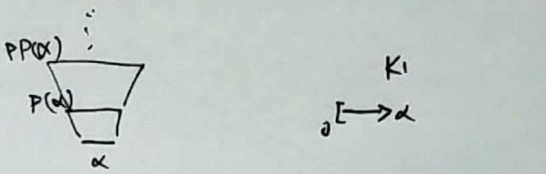
Let K_α be an AEC with two sorts, K^0_α and K^1_α.

$K^{\alpha 1}$ will be the ordinal α. (possible)

K^0 builds the cumulative hierarchy over α.

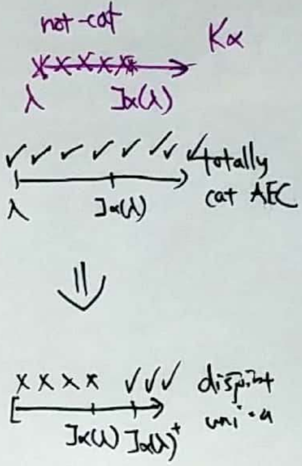
(with rank function to K^1)

Biggest model $V_\alpha(\alpha)$



$\Rightarrow \quad \|K_\alpha\| \leq \beth_\alpha(|\alpha|) = \beth_\alpha(\lambda)$

Take the "disjoint union" of K_α with a totally categorical AEC.



$V_\alpha(\alpha)$ $M_2 := \begin{cases} K^0 & \{0,1\} \; \{0,1,2\} \; \{0,3\} \\ K^1 & 0,1,2,3,4,5,6,\ldots \end{cases}$

$V_\alpha(\alpha) = M_3 := \begin{cases} K^0 & \{0,1,3\} \\ K^1 & 0,1,2,3 \end{cases}$

$M_1 := \begin{cases} K^0 & \{0,1\} \\ K^1 & 0,1,2 \end{cases}$

N

f, id. g, id.

$\{0,1\}$ $\quad$ $0,1,2,3$ $\qquad$ $\{0,1,3\}$ $\quad$ $0,1,2,3$ $\qquad$ $\{0 \ldots \alpha\}$

$\{0,1\}$ $\quad$ $0,1,2$

$f: V_\alpha(\alpha) \to N = V_\alpha(\alpha)$

It must map K^1 identically.

$P(\alpha)$ $\{0,1\}$ $\qquad$ $P(\alpha)$ $\{0,1\}$

$\alpha \xrightarrow{\text{id}} \alpha$

$0 \in \{0,1\}$ $\quad 1 \in \{0,1\}$ $\quad 2 \notin \{0,1\}$

$\{0,1\} = f(\{0,1\}) = g(\{0,1\}) = \{0,1,3\}$

contradiction.

Q: Can we find examples where each K^α has amalgamation?

Vasey proved that (w/ some assumptions) cat. spectrum can be of the form

$[\, LS(K)^{+m}, LS(K)^{+n} \,]$ where $m \leq n < \omega$

Examples: code the successors of a cardinal. (but without amalgamation).