LOCAL MINIMIZERS OF THE ANISOTROPIC ISOPERIMETRIC PROBLEM ON CLOSED MANIFOLDS

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Abstract. Local minimizers for the anisotropic isoperimetric problem in the small-volume regime on closed Riemannian manifolds are shown to be geodesically convex and small smooth perturbations of tangent Wulff shapes, quantitatively in terms of the volume.

1. Introduction

Perimeter-driven variational problems play a central role across analysis and geometry, serving as a tool to investigate the geometry of Riemannian manifolds and constituting the foundation of various models of real-world phenomena in materials science and physics. Describing the shape of local and global minimizers is a central aim in study of these problems.

For the isoperimetric problem on a general closed Riemannian manifold, the qualitative description of the shape of small-volume global minimizers goes back to work of Kleiner as described in [Tom93]; see also [MJ00, Ros05, Nar09, Fal10] and references therein among the many works dedicated to this topic. In the realm of physically motivated models, small-volume capillary droplets on a flat surface were shown to be asymptotically spherical in [Fin82, Tam84] (see also [Fin86]), and the shape of crystalline materials interacting with a convex potential was addressed in [McC98, FM11, DPG22]. In [FM11], Figalli and Maggi developed an approach to describing global minimizers in the small-volume regime *quantitatively* in terms of the volume, based on the quantitative Euclidean isoperimetric inequality [FMP08, FMP10, CL12]. The technique was further developed to prove a quantitative description of small-volume capillary droplets in a container in [MM16]. Quantitative isoperimetry has also been crucial in explicitly characterizing small-mass global minimizers for the liquid drop model in nuclear physics [KM13, KM14, BC14].

Recent years have seen a surge of interest in the shape of *local* minimizers, and more generally of critical points, of geometric variational problems. At the forefront of this line of research was work of Ciraolo and Maggi [CM17], where the authors prove and apply a quantitative version of Alexandrov's theorem to show that any volume-constrained local minimizer of a capillarity-type energy consisting of perimeter plus potential energy with sufficiently small volume is quantitatively close to a ball; see also [KM17, Bel, CFMN18]. For anisotropic surface energies interacting with an external potential in $\mathbb{R}^n$, it was shown in [DMMN18] that small-volume local minimizers that are assumed *a priori* to be smooth and to satisfy scale-invariant diameter bounds are quantitatively close to a Wulff shape. Very recently, a qualitative description of small-volume local minimizers of the capillary droplet problem in a container was derived [DW22], also under a priori smoothness and diameter assumptions. These results are all based on quantitative or qualitative stability of Heintze-Karcher-type inequalities.

In this paper, we use a different approach to the study of local minimizers. We focus on the context of volume-constrained local minimizers of anisotropic surface energies on a closed Riemannian manifold, though we expect that the ideas will apply in other settings. This approach combines compactness, via achieving scale-invariant diameter bounds, and the rigidity of critical points of the blow-up problem.

Fix a closed Riemannian manifold (M, g) of dimension $n \geq 2$, $\alpha \in (0, 1)$, and a $C^{2,\alpha}$ *elliptic integrand* F on M. More precisely, let $F : TM \to \mathbb{R}$ be a function whose restriction to the unit tangent bundle is $C^{2,\alpha}$ and whose restriction $F_{x_0}(\cdot) := F(x_0, \cdot) : T_{x_0}M \to \mathbb{R}$ is a convex, positively one-homogeneous function with $F^2_{x_0}$ uniformly convex, $C^{2,\alpha}$, and positive except at the origin for each $x_0 \in M$; see Section 2.3. The associated *anisotropic surface energy* of a set of finite perimeter $\Omega \subset M$ is

$$\mathcal{F}(\Omega) = \int_{\partial^*\Omega} F\big(x, \nu_\Omega(x)\big)\, d\mathcal{H}^{n-1}_g.$$

A set of finite perimeter Ω is said to be a *volume-constrained* ϵ_0*-local minimizer of* $\mathcal{F}$ if $\mathcal{F}(\Omega) \le \mathcal{F}(E)$ for any competitor $E \subset M$ with

$$|E|_g = |\Omega|_g = v \qquad \text{and} \qquad E\Delta\Omega \subset U_g(\partial\Omega, \epsilon_0 v^{1/n})$$

Here, for any $r > 0$ and measurable set E we let $U_g(E, r)$ denote the tubular neighborhood

$$U_g(E, r) = \{x \in M : d_g(x, y) < r\}\,. \tag{1.1}$$

Volume-constrained ϵ_0-local minimality is scaling invariant: if Ω is a volume constrained ϵ_0-local minimizer of $\mathcal{F}$ in (M, g) with volume v, then it is a volume constrained ϵ_0-local minimizer of $\mathcal{F}$ in $(M, v^{-2/n}g)$ with volume 1. Restricting F at a point $x_0 \in M$ induces a translation invariant elliptic integrand F_{x_0} and a corresponding anisotropic surface energy $\bar{\mathcal{F}}_{x_0}$ defined for sets of finite perimeter in $T_{x_0}M$. The surface energy $\bar{\mathcal{F}}_{x_0}$ is minimized for any volume constraint by a translation or dilation of the unit-volume *Wulff shape* $K_{x_0} \subset T_{x_0}M$ corresponding to $\bar{\mathcal{F}}_{x_0}$; again see Section 2.3.

We prove that a volume-constrained ϵ_0-local minimizer of $\mathcal{F}$ for sufficiently small volume is a small $C^{2,\alpha}$ perturbation of the image of a Wulff shape K_{x_0} under the exponential map at a point $x_0 \in M$. The estimates are quantitative with respect to the volume constraint.

Theorem 1.1. *Let (M, g) be a closed Riemannian manifold of dimension $n \ge 2$, fix $\alpha \in (0, 1)$, and let $\mathcal{F}$ be an anisotropic surface energy corresponding to a $C^{2,\alpha}$ elliptic integrand F. For any $\epsilon_0 > 0$ and $\kappa > 0$, there exist $v_0(g, F, \epsilon_0, \kappa) \in (0, |M|)$ and $C = C(g, F, \alpha, \epsilon_0, \kappa) > 0$ such that the following holds.*

Let Ω_v be a volume-constrained ϵ_0-local minimizer of $\mathcal{F}$ with volume $v \in (0, v_0]$ and $\mathcal{F}(\Omega_v) \le \kappa v^{(n-1)/n}$. Then Ω_v is a geodesically convex $C^{2,\alpha}$-domain. Moreover, there is a point $x_0 \in M$ such that

$$\frac{d_{H,g}\Big(\partial\Omega_v, \exp_{x_0}(\partial v^{1/n}K_{x_0})\Big)}{v^{1/n}} < Cv^{1/2n^2} \qquad \text{and} \qquad \frac{\Big|\Omega_v \Delta \exp_{x_0}(v^{1/n}K_{x_0})\Big|_g}{v} < Cv^{1/2n}\,. \tag{1.2}$$

Let us make a few remarks about Theorem 1.1.

Remark 1.2. The property of Ω_v being of class $C^{2,\alpha}$ follows from (1.2) and the regularity theory for minimizers of anisotropic surface energies (see Section 2.4). Actually, arguing as in the proof of [FM11, Theorem 2], one can strengthen the quantitative Hausdorff estimates (1.2) and prove that $\partial\Omega_v$ is locally a $C^{2,\alpha}$ graph, with quantitative estimates in v. By Schauder estimates, if F is $C^{k,\alpha}$, one can upgrade the quantitative regularity of Ω_v to $C^{k,\alpha}$.

Remark 1.3. The smallness of the volume v_0 in Theorem 1.1 depends on the constant κ of the anisotropic surface energy bound $\mathcal{F}(\Omega_v) \le \kappa v^{(n-1)/n}$. This dependence is crucial at several points in our proof. A posteriori, however, Theorem 1.1 implies the *improved* bound $\mathcal{F}(\Omega_v) \le Cv^{(n-1)/n}$, where $C = C(F, g) > 0$ depends only on the maximum (in x) of the anisotropic surface energies of the Wulff shapes K_x. Hence it is conceivable that one could upgrade Theorem 1.1, removing the assumption on the anisotropic perimeter bound and obtaining constants v_0 and C that are uniform in κ. We did not succeed in doing so and we leave it as interesting open question.

Remark 1.4. In the case of *global* minimizers of $\mathcal{F}$, one may hope to prove a more refined statement than the one in Theorem 1.1, namely providing information about the point $x_0 \in M$ at which a global minimizer is centered. We expect a global minimizer is centered at a point x_0 where $\bar{\mathcal{F}}_{x_0}(K_{x_0})$ is minimized. This question becomes intriguing if one considers an integrand F such that $\bar{\mathcal{F}}_{x_0}(K_{x_0})$ is constant. Here we expect global minimizers to be centered at a point maximizing a weighted average of the sectional curvatures of (M, g) that is compatible with the anisotropy of F, playing the role of the scalar curvature for the isoperimetric problem. We leave this point as an interesting open problem.

Remark 1.5. For global minimizers, a slightly weaker version of Theorem 1.1, which is stated in Theorem 5.1, can be generalized to every convex integrand that is C^1 in the x variable. Indeed, we just need to replace the use of the Alexandrov-type theorem of the first author, Kolasiński, and Santilli [DRKS20, Corollary 6.8] in the proof of the intermeditate Theorem 5.5 with the uniqueness (up to translations) of the Wulff shape among global minimizers [Tay78, FM91, BM94, MS86]. The rest of the proof of Theorem 5.1 remains unchanged.

Let us discuss the proof of Theorem 1.1. First, a qualitative form of Theorem 1.1 requires three key ingredients: (1) a diameter bound, (2) a compactness argument, and (3) a classification of local minimizers in the blow-up limit. Ingredient (3) was proven in [DRKS20], where the first author, Kolasiński, and Santilli showed that Wulff shapes are the only critical points of the blowup problem among sets of finite perimeter. The main contributions of this paper are ingredients (1) and (2).

For ingredient (1), in the context of the standard perimeter functional, a sequence of volume-constrained local minimizers with a scale-invariant perimeter bound and volume $v_k \to 0$ can be shown to have uniformly bounded (constant) mean curvature with respect to the rescaled metrics $h = v_k^{-2/n} g$ via the Heitze-Karcher inequality (see [MJ00, Theorem 2.2] for global minimizers). The *area monotonicity formula* then implies a uniform diameter bound. For any anisotropic surface energy that is not an affine transformation of the perimeter, however, no monotonicity formula is available [All74]!

Establishing uniform density estimates is another known technique for achieving diameter bounds, but it also falls short in this setting. An argument of Almgren [Alm76] (see also [GMT83, Mor03]) shows that Ω_v is a quasi-minimizer of $\mathcal{F}$ and satisfies density estimates, but with constants that depend on the set itself and in particular are not uniform in v in a scale invariant sense. Uniform quasi-minimality and thus uniform density estimates can be achieved by scaling for some volume-constrained problems in Euclidean space, see e.g. [FJ14, Neu16], but this technique is clearly specific to $\mathbb{R}^n$. (*After* proving the diameter bound, we apply this technique in charts to prove in Lemma 5.4 that local minimizers as in Theorem 1.1 satisfy uniform density estimates in a scale invariant sense.)

Another approach to obtaining a diameter bound was shown in [FM13] and [MM16], for global minimizers of weighted Euclidean isoperimetric problems and capillary drops in a container in $\mathbb{R}^n$ respectively. Here the idea is to partition a global minimizer E_v with a well-chosen collection of cubes $\{Q_i\}_{i=1}^N$. Applying the isoperimetric inequality to each element of the partition they bound from below the sum of the energies of the partition elements. On the other hand, by global minimality, this sum can be bounded from above by the same constant, obtaining an estimate of the following type:

$$0 \le \sum \left(\frac{|\Omega_v \cap Q_i|_g}{v} \right)^{(n-1)/n} - 1 \le \epsilon(v) \tag{1.3}$$

The concavity of the function $t \mapsto t^{(n-1)/n}$ immediately implies that for any $1 \le L \le N$,

$$\left(\sum_{i=1}^{L} \frac{|\Omega_v \cap Q_i|_g}{v} \right)^{(n-1)/n} + \left(\sum_{i=L+1}^{N} \frac{|\Omega_v \cap Q_i|_g}{v} \right)^{(n-1)/n} - 1 \le \epsilon(v),$$

from which one deduces that $|\Omega_v \cap Q_i| \ge 1 - \epsilon$ for some i and for v small enough. Then a classical use of coarea formula provides a standard differential inequality which allows to prove a diameter bound. We remark that in doing so, in order to make $\epsilon(v)$ small enough, the radius of the cubes should be optimized at a scale $v^{1/2n}$. However this provides a suboptimal diameter bound that in the scale invariant sense blows up as $v \to 0$.

This sandwiching argument relies on having precisely the constant 1 in (1.3). Underpinning this is the fact that the energy of a small-volume global minimizer is asymptotically equal to the isoperimetric constant of the blow up problem. Thus, this approach is confined to the setting of global minimizers.

For the aforementioned reasons, we need to take a new approach to obtain our scale invariant diameter bound for volume-constrained ϵ_0-local minimizers of $\mathcal{F}$ with small volume (Section 4). It is possible to adapt the ideas of [FM13, MM16] to obtain a similar partition of a local minimizer Ω_v on the compact manifold. However, as explained above, one cannot obtain the constant 1 in (1.3) (or equivalently, make the right-hand side of (1.3) small). To overcome this problem we use a general concavity lemma (Lemma 4.2) for sequences of real numbers, which has been previously utilized in concentration compactness arguments for isoperimetric type problems; see for instance [GNR22, CTG21, CGOS18, NPST22]. We apply this lemma to show that most of the volume of Ω_v is contained in the union of J_0 balls of radius $v^{1/n}$, where J_0 depends just on n, F, and the anisotropic isoperimetric ratio of the finite perimeter set. From this point, we can use Ω_v intersected with the J_0 balls as a competitor, to deduce a standard differential inequality which allows us to obtain the diameter bound. Beyond the fact that this argument works for *local* minimizers, this approach provides a diameter bound that does not blow up in a scale invariant sense.

The diameter bound is the starting point for ingredient (2), allowing us in Section 5 to pull back and rescale a sequence of local minimizers in (J_0) charts and obtain L^1 convergence to a set E. Using the rigidity theorem [DRKS20, Corollary 6.8] (Ingredient (3)), we deduce the limiting set is a *translation* of a tangential Wulff shape. This translation, whose modulus a priori could be much larger than the natural length scale $v^{1/n}$, leads to serious difficulties in transmitting this information about the blowup back to the local minimizers on the manifold. Hence we need to compare the shape of the limiting translated Wulff shape and a tangent Wulff shape at a different appropriately chosen point. Since the integrand F is not autonomous, that is, it is x-dependent, careful analysis is needed to carry this out in Section 3.

The final step (Section 6) is to provide a quantitative version of the closeness of the previous step, meaning that the closeness is not only scale invariant, but will actually decay quantitatively as a power of the volume. The key ingredient for this is the quantitative Wulff inequality of Figalli, Maggi, and Pratelli [FMP10].

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2. Preliminaries

In this section we introduce definitions and notation and prove some preliminary results that will be needed in the remainder of the paper.

2.1. **Basic Notation.** Consider a smooth Riemannian manifold (M, g) of dimension $n \geq 2$. Let $B_g(x,r) = \{y \in M; d_g(x,y) < r\}$ denote the geodesic ball of radius $r > 0$ centered at $x \in M$. Recall the notation $U_g(E,r) = \{y \in M : d_g(x,E) < r\}$ introduced in (1.1) for the tubular neighborhood of a set $E \subset M$. The Hausdorff distance between sets $\Sigma, \Sigma' \subset M$ is defined by

$$d_{H,g}(\Sigma, \Sigma') = \inf\{r > 0 : \Sigma \subset U_g(\Sigma', r) \text{ and } \Sigma' \subset U_g(\Sigma, r)\}.$$

For the k dimensional Hausdorff measure with respect to the metric g, we write $\mathcal{H}^k_g(\cdot)$. When $k = n$ we simply write $|\cdot|_g = \mathcal{H}^n_g(\cdot)$ since $\mathcal{H}^n_g$ agrees with the standard volume measure induced by g. We call a set measurable if it is $\mathcal{H}^n_g$-measurable. It is worth noting how these quantities behave under rescaling the metric g: if $h = r^{-2}g$ for $r > 0$, then for $x \in M, E \subset M$, and $\rho > 0$ we have

$$B_h(x,\rho) = B_g(x, r\rho), \quad U_h(E,\rho) = U_g(E, r\rho), \quad d_{H,h}(\Sigma,\Sigma') = \frac{d_{H,g}(\Sigma,\Sigma')}{r}, \quad \mathcal{H}^k_h(E) = \frac{\mathcal{H}^k_g(E)}{r^k}.$$

We denote the Euclidean metric by g_{euc}.

We let $\mathrm{inj}_g M > 0$ denote the injectivity radius of M, i.e. the supremum over $r > 0$ such that the exponential map $\exp_x : T_xM \to M$ is a diffeomorphism from $B_{g_x}(0,r)$ to $B_g(x,r)$ for all $x \in M$.

Let $E\Delta G = (E \setminus G) \cup (G \setminus E)$ be the symmetric difference between sets, which we note satisfies the triangle inequality-type property

$$E\Delta G \subset (E\Delta E') \cup (E'\Delta G). \tag{2.1}$$

A sequence of measurable sets $\{E_i\}$ converges in L^1 to E if $|E_i\Delta E|_g \to 0$.

2.2. **Sets of finite perimeter.** We work in the framework of sets of finite perimeter. A measurable set $E \subset M$ is a set of finite perimeter if

$$P_g(E) = \sup\left\{\int_E \mathrm{div}_g T(x)\, d\mathcal{H}^n_g(x) : T \in \mathcal{X}_c(M), |T(x)|_g \le 1 \text{ for all } x \in M\right\} < +\infty.$$

Here $\mathcal{X}_c(M)$ denotes the space of smooth compactly supported vector fields on M.

For any set of finite perimeter $E \subset M$, by the Riesz representation theorem for bounded linear functionals on $\mathcal{X}_c(M)$ ([MPPP07, Section 1.2], [Vol10, Theorem 2.36]), there is a finite Radon measure $|D1_E|$ on M and a $|D1_E|$-measurable vector field $\nu_E : M \to TM$ with $|\nu_E(x)|_g = 1$ for $|D1_E|$-a.e. $x \in M$, such that the distributional gradient $D1_E$ has the representation $D1_E = \nu_E|D1_E|$ (in other words, $\int_E \mathrm{div}_g T\, d\mathcal{H}^n = \int_M \langle T, \nu_E\rangle_g\, d|D1_E|$ for any $T \in \mathcal{X}_c(M)$). Note that ν_E depends on the metric g, though we suppress this dependence in the notation when there is no ambiguity.

The reduced boundary $\partial^* E$ of E is defined by

$$\partial^* E = \{x \in \mathrm{spt}|D1_E| \ : \ |\nu_E(x)|_g = 1\}.$$

It is easy to show that $\mathrm{spt}|D1_E|$ and thus $\partial^* E$ are contained in the topological boundary ∂E. Note that $D1_E = \nu_E|D1_E|\llcorner\partial^* E$. By the De Giorgi Structure Theorem (see [Mag12, Theorem 15.9] in the Euclidean case; the proof can be adapted to the setting of Riemannian manifolds), $\partial^* E$ is an $\mathcal{H}^{n-1}$-rectifiable set and $D1_E = \nu_E \mathcal{H}^{n-1}_g\llcorner\partial^* E$.

We denote with $E^{(1)}$ the set of points of density 1 for E, i.e.

$$E^{(1)} := \left\{x \ : \ \lim_{r\to 0} \frac{|B_g(x,r) \cap E|_g}{|B_g(x,r)|_g} = 1\right\}.$$

Sets of finite perimeter enjoy a useful compactness property with respect to the L^1 topology: if $\{E_i\}$ is a sequence of sets of finite perimeter in M with $E_i \subset A$ for a compact set A and $\sup_i P(E_i) < +\infty$, then up to a subsequence, $E_i \to E$ in L^1 for a set of finite perimeter $E \subset A$.

All of these properties of sets of finite perimeter are invariant by modification of the set on an $\mathcal{H}^n_g$-null set. By [Mag12, Prop. 12.19], we can modify E on a $\mathcal{H}^n_g$-null set to ensure that

$$\overline{\partial^* E} = \{x \in M : 0 < |E \cap B_g(x,r)|_g < |B_g(x,r)|_g \text{ for all } r > 0\} = \partial E\,. \tag{2.2}$$

(The first identity always holds, while the second holds after a measure zero modification.) In the sequel we will tacitly assume that every set of finite perimeter has been cleaned up in this way.

2.3. **Elliptic integrands.** Let $F : TM \to \mathbb{R}$ be an anisotropic integrand on (M, g) as defined in the introduction. Given a diffeomorphism $\psi : U \to V \subset M$, define the pulled-back integrand

$$\psi^* F : TU \to \mathbb{R} \qquad \text{by} \qquad \psi^* F(x,\nu) = F(\psi(x), d\psi_x \nu). \tag{2.3}$$

When no confusion can arise, we will write $\psi^* F = F^*$; this should not be confused with the dual integrand F_* used in Appendix 3. Let $r_0 = \mathrm{inj}_g(M)/2$. The $C^{2,\alpha}$ regularity of F in particular implies that for any $x_0 \in M$ if we take $\psi : B_{g_{euc}}(0r_0) \to B_g(x_0, r_0)$ to be the normal coordinate map, then

$$\sup\left\{\|\psi^* F(\ \cdot\ ,\nu)\|_{C^1(B_{g_{euc}}(0,r_0))} \ : \ \nu \in S^{n-1}\right\} \le C(n, g, F). \tag{2.4}$$

For the majority of the paper we will only use this C^1 regularity of F in x and will only use the higher regularity in x when applying the ϵ-regularity theorem and Schauder estimates in Section 6.

For $x_0 \in M$, the restricted integrand $F_{x_0} : T_{x_0}M \to \mathbb{R}$ defined by $F_{x_0}(\nu) = F(x_0, \nu)$ gives rise to a translation invariant surface energy for sets of finite perimeter $E \subset T_{x_0}M$:

$$\bar{\mathcal{F}}_{x_0}(E) = \int_{\partial^* E} F_{x_0}(\nu_E(x))\, d\mathcal{H}^{n-1}_{g_{x_0}}(x).$$

Here the Hausdorff measure and measure theoretic outer unit normal are taken with respect to the metric $g_{x_0}(\cdot,\cdot)$ on $T_{x_0}M$. Among sets of a fixed volume in $T_{x_0}M$, the energy $\bar{\mathcal{F}}_{x_0}$ is uniquely minimized by translations and dilations of the volume-1 *tangent Wulff shape* $K_{x_0} \subset T_{x_0}M$ defined by

$$K_{x_0} := \frac{\hat{K}_{x_0}}{|\hat{K}_{x_0}|_{g_{x_0}}} \qquad \text{where } \hat{K}_{x_0} := \big\{ y \in T_{x_0}M \ : \ g_{x_0}(y,\nu) < F_{x_0}(\nu) \text{ for all } \nu \in T_{x_0}M \setminus \{0\} \big\}\,; \tag{2.5}$$

see [Tay78, FM91, BM94, MS86]. Note that $0 \in K_{x_0}$. This minimality property is stated in scale-invariant form as the *Wulff inequality*:

$$\bar{\mathcal{F}}_{x_0}(E) \geq n|E|_{g_{x_0}}^{\frac{n-1}{n}} |\hat{K}_{x_0}|_{g_{x_0}}^{1/n} = \left(\frac{|E|_{g_{x_0}}}{|\hat{K}_{x_0}|_{g_{x_0}}} \right)^{\frac{n-1}{n}} \bar{\mathcal{F}}_{x_0}(\hat{K}_{x_0}) = |E|_{g_{x_0}}^{\frac{n-1}{n}} \bar{\mathcal{F}}_{x_0}(K_{x_0}) \tag{2.6}$$

We set

$$\mathcal{C} = \sup\{P_{g_x}(K_x) : x \in M\}. \tag{2.7}$$

Define the quantities $\mathfrak{M} = \sup\{F(x,\nu) : x \in M, |\nu|_g = 1\}$ and $\mathfrak{m} = \inf\{F(x,\nu) : x \in M, |\nu|_g = 1\}$, which are positive and finite by the assumptions on F and the compactness of (M,g). Moreover, $\mathfrak{m}P_g(E) \leq \mathcal{F}(E) \leq \mathfrak{M}P_g(E)$ for any set of finite perimeter $E \subset M$ and $B_{g_{x_0}}(0,\mathfrak{m}) \subset K_{x_0} \subset B_{g_{x_0}}(0,\mathfrak{M})$. Estimating the isoperimetric profile above by taking geodesic ball competitors, we thus find that there exists $\bar{v} = \bar{v}(n,g) \in (0, |M|_g)$ such that

$$\mathcal{F}(E) \geq \frac{\mathfrak{m}}{2}\, n\omega_n^{1/n} |E|_g^{(n-1)/n} \qquad \text{for } E \subset M \ \text{ with } \ |E|_g \leq \bar{v}. \tag{2.8}$$

It is also useful to notice how the surface energy behaves under rescaling the metric. Setting $h = r^{-2}g$ for $r > 0$, the same integrand defines an anisotropic surface energy

$$\mathcal{F}_h(E) = \int_{\partial^* E} F(x, \nu_E^h(x))\, d\mathcal{H}^{n-1}_h(x) = \frac{\mathcal{F}(E)}{r^{n-1}}.$$

2.4. **Classical regularity results for local minimizers.** Let Ω_v be a volume-constrained ϵ_0-local minimizer of $\mathcal{F}$ with volume $v \in (0, |M|)$. A classical argument dating back to Almgren [Alm76] (see also [GMT83, Mor03]) shows that Ω_v is a quasi-minimizer of $\mathcal{F}$ and satisfies density estimates. Unlike the *uniform* quasi-minimality and density estimates we will ultimately derive in Lemma 5.4, these estimates depend on the set Ω_v itself, and thus are not be directly useful to proving Theorem 1.1. However, they show that $\mathcal{H}^n$-a.e. $x \in \partial E$ is a point of density strictly between 0 and 1; together with Federer's theorem [Mag12, Theorem 16.2] this implies that $\mathcal{H}^n(\partial\Omega_v \setminus \partial^*\Omega_v) = 0$ and thus, up to modifying Ω_v by an $\mathcal{H}^n$-null set, we can replace a local minimizer with an open set representative. In the remainder of the paper we will always take this representative.

In fact, the classical regularity theory, [Alm68, Bom82, SSA77, SS82, DS02] shows that $\partial\Omega_v$ is a $C^{1,\alpha}$ hypersurface outside of a singular set of $\mathcal{H}^{n-2}$ measure zero. These estimates are again not a priori uniform in v.

2.5. **Matching the volume constraint.** At several points throughout this paper, we will have a set whose volume is close, but not exactly equal, to a certain prescribed volume. To use it as a competitor for volume-constrained local minimality, we need to replace it with a set that exactly satisfies the volume constraint. The following technical lemma allows us to do this in such a way that the difference between the surface energies of the original set and the modified set is quantitatively controlled in terms of the volume error. For the proof we borrow some ideas from [MM16, Lemma 3.1].

Lemma 2.1. *For any $n \geq 2$, there exist $\eta_0 > 0$ and $c_n > 0$ depending only on n such that the following holds. Fix $D > 1$, $\eta \in (0, \eta_0)$ and let (N, h) be a closed Riemannian n-manifold with $\mathrm{inj}_h N > D$ and*

$$\|\psi^* h - g_{euc}\|_{C^1(B_{g_{euc}}(0,D))} \leq \frac{\eta}{D} \tag{2.9}$$

in normal coordinates ψ centered at any $x \in M$. Let $E \subset M$ be a measurable set with $||E|_h - 1| < \eta$ for which there exist finitely many disjoint open sets $\{V_i\}_{i=1}^K$ with $\mathrm{diam}_h(V_i) \leq D$ such that $U_h(E, 1) \subset \cup_{i=1}^K V_i$. Then there is a measurable set $\tilde{E} \subset \cup_{i=1}^K V_i$ such that

$$\tilde{E} \Delta E \Subset U_h(\partial E, c_n D\eta) \qquad \text{and} \qquad |\tilde{E}|_h = 1. \tag{2.10}$$

Moreover, if $\mathcal{F}$ is an anisotropic perimeter with integrand F satisfying

$$\|\psi^* F(\cdot, \nu) - \psi^* F(0, \nu)\|_{C^1(B_{g_{euc}}(0,D))} \leq 1 \tag{2.11}$$

for every x and ν, then

$$\mathcal{F}(\tilde{E}) \leq \mathcal{F}(E)\left(1 + c_n D\, ||E|_h - 1|\right). \tag{2.12}$$

Proof. Fix $\eta_0 > 0$ to be specified later in the proof and let $\eta \in (0, \eta_0)$ and $D > 1$. For each $k \in 1, \dots, K$, since $\mathrm{diam}_h V_k \leq D$, we can find $x_k \in V_k$ with $V_k \subset B_{g_{euc}}(x_k, D)$. Let $\psi_k : B_{g_{euc}}(0, D) \to B_h(x_k, D)$ be the normal coordinate map at x_k. For $\lambda \in [\frac{1}{2}, \frac{3}{2}]$, define the set $E_\lambda \subset M$ by

$$E_\lambda := \textstyle\bigcup_{k=1}^K \psi_k\left(\lambda\, \psi_k^{-1}(V_k \cap E)\right).$$

Note that $E_1 = E$ and that E_λ is well-defined for the full interval since we have assumed $U_h(E \cap V_k, 1) \subset V_k$. Moreover, letting $\lambda_+ = (1-\eta)^{-(n+1)/n}$ and $\lambda_- = (1-\eta)^{-(n+1)/n}$ and recalling $(1-\eta) \leq |E|_h \leq (1+\eta)$ by assumption, we compute in coordinates to find

$$|E_{\lambda_+}|_h \geq \lambda_+^n (1-\eta)^n |E|_h \geq \lambda_+^n (1-\eta)^{n+1} \geq 1, \text{ and}$$
$$|E_{\lambda_-}|_h \leq \lambda_-^n (1+\eta)^n |E|_h \leq \lambda_-^n (1+\eta)^{n+1} \leq 1.$$

The function $\lambda \mapsto |E_\lambda|_h$ is continuous, and thus we can find we can find $\lambda_0 \in [\lambda_-, \lambda_+] \subset [1 - c_n\eta, 1 + c_n\eta]$ such that $|E_{\lambda_0}|_h = 1$. We set $\tilde{E} = E_{\lambda_0}$.

In order to prove the remaining properties of $\tilde{E}$, we prove the following key fact:

$$|\lambda_0 - 1| \leq c_n ||E|_h - 1| \tag{2.13}$$

provided $\eta_0 > 0$ is small enough in terms of n. To this end, we let $\mathcal{J}(x) = \sqrt{\det h_{ij}(x)}$ be the volume form of h in the coordinates defined by ψ and let $G_k = \psi_k^{-1}(V_k \cap E) \subset \mathbb{R}^n$. We therefore have

$$|E_\lambda|_h - |E|_h = \sum_{k=1}^K \left\{ \int_{\lambda G_k} \mathcal{J}(x)\, dx - \int_{G_k} \mathcal{J}(x)\, dx \right\} = \sum_{k=1}^K \int_{G_k} \left\{\lambda^n \mathcal{J}(\lambda x) - \mathcal{J}(x)\right\} dx. \tag{2.14}$$

When $\lambda \in [1, 1 + c_n\eta]$, we add and subtract $\lambda^n \mathcal{J}(x)$ and use the fundamental theorem of calculus to find

$$\begin{aligned}\lambda^n \mathcal{J}(\lambda x) - \mathcal{J}(x) &\geq (\lambda^n - 1)\, \mathcal{J}(x) - \lambda^n\left(\mathcal{J}(\lambda x) - \mathcal{J}(x)\right)\\ &\geq (\lambda - 1)\, \mathcal{J}(x) - (\lambda - 1) \sup\left\{|\nabla \mathcal{J}(tx) \cdot x| : t \in [1 - c_n\eta, 1 + c_n\eta]\right\}\end{aligned}$$

for any fixed x. Since (2.9) guarantees that $\mathcal{J}(x) \geq 1 - \eta$ and that the term in brackets is bounded above by $1/2$ for all $x \in B(0, D)$ provided $\eta_0 > 0$ is small enough depending on n, we see that $\lambda^n \mathcal{J}(\lambda x) - \mathcal{J}(x) \geq c_n(\lambda - 1)\mathcal{J}(x)$. Applying this inequality to the right-hand side of (2.14) we find that (2.13) holds in this case. The analogous argument in the case when $\lambda \in [1 - c_n\eta, 1]$ shows that $\mathcal{J}(x) - \lambda^n \mathcal{J}(\lambda x) \geq (1 - \lambda)c_n\mathcal{J}(x)$, which applied to (2.14) implies (2.13) in this case as well.

Next, with (2.13) in hand, we show the containment

$$\tilde{E} \Delta E \subset U_h(\partial E, c_n D\eta). \tag{2.15}$$

Indeed, if $x \in E \setminus E_{\lambda_0}$, then for some $k \in \{1, \dots, K\}$, we have $y := \psi_k^{-1}(x) \in G_k$ but $y \notin \lambda_0 G_k$. Here we again let $G_k = \psi_k^{-1}(V_k \cap E) \subset \mathbb{R}^n$. So, $ty \in \partial G_k$ for some $t \in [1, \lambda_0]$ or $t \in [\lambda_0, 1]$, and thus by (2.13),

$$d_{g_{euc}}(y, \partial G_k) \leq |t - 1||y| \leq |\lambda_0 - 1||y| \leq c_n D\eta.$$

Hence by (2.9), we have $d_h(x,\partial E) \le c_n \eta D$ (up to doubling c_n). The analogous argument holds for any $x \in E_{\lambda_0} \setminus E$, and thus (2.15) holds.

Finally, we show the estimate for the anisotropic perimeter. Similarly to (2.14), we have

$$\begin{aligned}\mathcal{F}(E_\lambda) - \mathcal{F}(E) &= \sum_{k=1}^K \left\{ \int_{\partial^* \lambda G_k} \hat{F}(x,\nu_{G_k}(x))\, d\mathcal{H}^{n-1}_{g_{euc}} - \int_{\partial^* G_k} \hat{F}(x,\nu_{G_k}(x))\, d\mathcal{H}^{n-1}_{g_{euc}} \right\} \\ &= \sum_{k=1}^K \left\{ \int_{\partial^* G_k} \lambda^{n-1} \hat{F}(\lambda x, \nu_{G_k}(x)) - \hat{F}(x,\nu_{G_k}(x))\, d\mathcal{H}^{n-1}_{g_{euc}} \right\}\end{aligned} \tag{2.16}$$

where here we define the function $\hat{F}(x,\nu) = F^*(x,\nu)\sqrt{\det \tilde{h}_{ij}(x)}$ where $\tilde{h}_{ij}$ are the coefficients of the metric on $\partial^* E$ induced by h. Adding and subtracting terms and using $\lambda \in [1-c_n, 1+c_n\eta]$, (2.9), and (2.11), a Taylor expansion shows that for any fixed x and $\nu \in S^{n-1}$ that

$$\begin{aligned}\left|\lambda^n \hat{F}(\lambda x,\nu) - \hat{F}(x,\nu)\right| &\le \left|\lambda^{n-1} - 1\right| \hat{F}(x,\nu) + \lambda^{n-1}\sqrt{\det \tilde{h}_{ij}(\lambda x)}\, |F^*(\lambda x,\nu) - F^*(x,\nu)| \\ &\quad + \lambda^{n-1} F^*(x,\nu) \left| \sqrt{\det \tilde{h}_{ij}(\lambda x)} - \sqrt{\det \tilde{h}_{ij}(x)} \right| \le c_n D(\lambda - 1)\hat{F}(x,\nu),\end{aligned}$$

provided $\eta > 0$ is chosen sufficiently small depending on n. Combining this with (2.16) and recalling (2.13), we conclude that the estimate (2.12) holds. This completes the proof. □

3. Comparing projections of tangent Wulff shapes

The following proposition compares projections of translated tangent Wulff shapes via the exponential map at different points. Its proof is postposed until the appendix.

Proposition 3.1. *There exist $C = C(g,F)$ and $\rho_0 = \rho_0(g,F) > 0$ such that the following holds. Fix $\rho \in [0,\rho_0)$, choose $x_0, x_1 \in M$ with $d_g(x_0,x_1) < \rho$, and let $K_{x_0} \subset T_{x_0}M$ and $K_{x_1} \subset T_{x_1}M$ be defined as in (2.5). Then for any $0 < r < \rho_0$, letting $z_1 = \exp^{-1}_{x_0}(x_1)$, we have*

$$\begin{aligned}d_{H,g}\big(\exp_{x_1}(\partial r K_{x_1}),\ \exp_{x_0}(\partial r K_{x_0} + z_1)\big) &< C\rho\, r, \\ d_{H,g}\big(\exp_{x_1}(r K_{x_1}),\ \exp_{x_0}(r K_{x_0} + z_1)\big) &< C\rho\, r.\end{aligned}$$

This section is dedicated to proving Proposition 3.1. As in the remainder of the paper we assume that F is a $C^{2,\alpha}$ elliptic integrand. However, we note that the proof of Proposition 3.1 only requires that F has C^1 dependence on x and for each $x_0 \in M$, $F(x_0,\cdot)$ is a convex one-homogenous function that is positive expect a the origin, with no smoothness or ellipticity needed.

Together with the metric g, the integrand F induces a dual integrand $F_* : TM \to \mathbb{R}$ via

$$F_*(x,z) = \sup\big\{ g_x(z,\nu) \,:\, \nu \in T_xM,\ F(x,\nu) \le 1 \big\}.$$

Given a pair of points $x_0, x_1 \in M$, let

$$d_{F_*}(x_0,x_1) = \inf\left\{ \int_0^1 F_*(\gamma(t),\dot\gamma(t)) \;:\; \gamma : [0,1] \to M,\ \gamma(0) = x_0,\ \gamma(1) = x_1 \right\}.$$

If F is not symmetric, we may have $d_{F_*}(x_0,x_1) \neq d_{F_*}(x_1,x_0)$. Nonetheless we may consider the F_*-balls

$$B_{F_*}(x_0,\rho) = \{x_1 \in M : d_{F_*}(x_0,x_1) < \rho\}\,.$$

Recall the quantities $\mathfrak{m}$ and $\mathfrak{M}$ defined in Section 2.3 and note that $\mathfrak{m}^{-1} = \sup\{F_*(x,z) : x \in M, g_x(z,z) = 1\}$ and $\mathfrak{M}^{-1} = \inf\{F_*(x,z) : x \in M, g_x(z,z) = 1\}$ and therefore

$$\begin{aligned}\mathfrak{M}^{-1} d_g(x_0,x_1) \le d_{F_*}(x_0,x_1) &\le \mathfrak{m}^{-1} d_g(x_0,x_1), \\ B_g(x_0,\mathfrak{m}\rho) \subset B_{F_*}(x_0,\rho) &\subset B_g(x_0,\mathfrak{M}\rho)\,.\end{aligned} \tag{3.1}$$

Proposition 3.1 will follow from the next lemma and the triangle inequality.

Lemma 3.2. *There exist $C = C(g,F) > 0$ and $\rho_0 = \rho_0(g,F) > 0$ such that the following holds. Fix $x_0, x_1 \in M$ with $d_g(x_0,x_1) \le \rho \le \rho_0$. Then for all $0 < r \le \rho_0$, letting $z_1 = \exp_{x_0}^{-1}(x_1)$, we have*

$$d_{H,g}\big(\exp_{x_0}\left(\partial rK_{x_0} + z_1\right),\ \partial B_{F_*}(x_1,r)\big) \le C\rho\, r,$$
$$d_{H,g}\big(\exp_{x_0}\left(rK_{x_0} + z_1\right),\ B_{F_*}(x_1,r)\big) \le C\rho\, r.$$

Before proving Lemma 3.2, let us see how it implies Proposition 3.1.

Proof of Proposition 3.1. Let $\rho_0 > 0$ be chosen according to Lemma 3.2. Fix $r, \rho \le \rho_0$ and choose $x_0, x_1 \in M$ with $d_g(x_0,x_1) \le \rho$. First, we apply Lemma 3.2 to find

$$d_{H,g}(\exp_{x_0}\left(\partial \rho K_{x_0} + z\right),\ \partial B_{F_*}(x_1,\rho)) \le C\rho\, r,$$
$$d_{H,g}\big(\exp_{x_0}(\rho K_{x_0} + z),\ B_{F_*}(x_1,\rho)\big) \le C\rho\, r.$$

Next, apply Lemma 3.2 with the roles of both x_0 and x_1 played by x_1 (and thus $z_1 = 0 \in T_{x_1}M$) to find

$$d_{H,g}\big(\exp_{x_1}\left(\partial rK_{x_1}\right),\ \partial B_{F_*}(x_1,r)\big) \le C\rho\, r,$$
$$d_{H,g}\big(\exp_{x_0}\left(rK_{x_1}\right),\ B_{F_*}(x_1,r)\big) \le C\rho\, r\,.$$

We apply the triangle inequality to conclude the proof. □

We need three preparatory lemmas to prove Lemma 3.2. The first transfers the assumed regularity on F in the variable x to regularity of F_* in x. Let $r_0 = \mathrm{inj}_g(M)/2$. Fix $x_0 \in M$ and let $\psi = \exp_{x_0}$ and consider the pulled-back integrand $\psi^*F : B_{g_{x_0}}(0,r_0) \times T_{x_0}M \to \mathbb{R}$ defined as in (2.3). The regularity (2.4) of F in x implies, in particular, that for all $\nu \in T_{x_0}M$ with $g_{x_0}(\nu,\nu) = 1$ and $\rho < r_0$ we have

$$\|\psi^*F(\,\cdot\,,\nu) - \psi^*F(0,\nu)\|_{C^0(B(0,\rho))} \le \mathsf{C}\,\rho \tag{3.2}$$

for a constant $\mathsf{C} = \mathsf{C}(F,g)$. Define the pulled-back dual integrand $\psi^*F_* : B_{g_{x_0}}(0,r_0) \times T_{x_0}M \to \mathbb{R}$ by

$$\psi^*F_*(y,z) = F_*(\psi(y), d\psi_y(z)).$$

Lemma 3.3. *There exists $\rho_2 = \rho_2(g,F)$ and $C = C(g,F)$ such that the following holds. Let $F : TM \to \mathbb{R}$ be an elliptic integrand satisfying (3.2). Then for each $x_0 \in M$ and $z \in T_{x_0}M$ with $g_{x_0}(z,z) = 1$, and $\rho < \rho_2$, we have*

$$\|\psi^*F_*(\,\cdot\,,z) - \psi^*(F_*(0,z))\|_{C_0(B(0,\rho))} \le C\,\rho. \tag{3.3}$$

Here we let $\psi = \exp_{x_0} : T_{x_0}M \to M$.

Proof. Choose $\rho_2 \le \mathrm{inj}_g M/2$ small enough depending on g such that

$$(1-\rho)g_{x_0} \le \psi^*g \le (1+\rho)g_{x_0} \qquad \text{in } B_{g_{x_0}}(0,\rho) \text{ for all } \rho < \rho_2. \tag{3.4}$$

For any $z \in T_{x_0}M$ with $g_{x_0}(z,z) = 1$, choose $\nu_z \in T_{x_0}M$ such that $F(x_0,\nu_z) = 1$ and

$$g_{x_0}(z,\nu_z) = F_*(x_0,z) = \psi^*F_*(0,z). \tag{3.5}$$

So, choosing any $y \in B_{g_{x_0}}(0,\rho) \subset T_{x_0}M$, the assumption (3.2) implies that $\psi^*F(y,\nu_z) \le 1 + \mathsf{C}\rho$ with C as in (3.2). So, using $\overline{\nu}_z := \nu_z/\psi^*F(y,\nu_z)$ as a competitor in the definition of $\psi^*F_*(y,z)$, we have

$$\psi^*F_*(y,z) = \sup\left\{(\psi^*g)_y(z,\nu) : \psi^*F(y,\nu_z) \le 1\right\} \ge (\psi^*g)_y\left(z,\bar{\nu}_z\right) = \frac{(\psi^*g)_y(z,\nu_z)}{\psi^*F(y,\nu_z)} \ge \frac{(\psi^*g)_y(z,\nu_z)}{1+\mathsf{C}\rho}.$$

By (3.4) and (3.5), we have $(\psi^*g)_y(z,\nu_z) \ge (1-\rho)g_{x_0}(z,\nu_z) = (1-\rho)\psi^*F_*(y,z)$, and thus

$$\frac{\psi^*F_*(y,z)}{\psi^*F_*(0,z)} \ge \frac{1-\rho}{1+\mathsf{C}\rho} \ge 1 - 2(1+\mathsf{C})\rho,$$

where the final inequality holds for ρ small enough depending on C and thus on $F.g$. The same argument holds with the roles of 0 and y swapped. Together these inequalities along with (3.1) show that

$$|\psi^*F_*(0,z) - \psi^*F_*(y,z)| \le 4(1+\mathsf{C})\,\psi^*F_*(0,z)\rho \le \frac{4(1+\mathsf{C})}{\mathfrak{m}}\,\rho\,.$$

This proves the lemma. □

The next simple lemma will allow us to pull back (almost) F_*-geodesics via the exponential map.

Lemma 3.4. *Fix $x_1, x_2 \in M$ and let $\hat\gamma : [0,1] \to M$ be a curve with $\hat\gamma(0) = x_1$ and $\hat\gamma(1) = x_2$ such that $\int_0^1 F_*(\hat\gamma(t), \dot{\hat\gamma}(t))\, dt \le 2 d_{F_*}(x_1, x_2)$. Then $d_g(x_1, \hat\gamma(t)) \le \frac{2\mathfrak{M}}{\mathfrak{m}} d_g(x_1, x_2)$ for all $t \in [0,1]$.*

Proof. Let $\hat\gamma : [0,1] \to M$ be a curve as in the statement of the lemma.. Recalling that $d_{F_*}(x_1, x_2) \le \mathfrak{m}^{-1} d_g(x_1, x_2)$ by (3.1), we see that for any $t \in [0,1]$,

$$d_g(x_1, \hat\gamma(t)) \le \int_0^t |\dot{\hat\gamma}(t)|_g\, dt \le \mathfrak{M} \int_0^t F_*(\hat\gamma(t), \dot{\hat\gamma}(t))\, dt \le \mathfrak{M} \int_0^1 F_*(\hat\gamma(t), \dot{\hat\gamma}(t))\, dt \le \frac{2\mathfrak{M}}{\mathfrak{m}} d_g(x_1, x_2).$$

□

Lemmas 3.3 and 3.4 will be used to prove the following lemma.

Lemma 3.5. *There exist $\rho_1 = \rho_1(g, F) > 0$ and $C = C(g, F) > 0$ such that for all $x_0 \in M$, $\rho < \rho_1$, and $z_1, z_2 \in B_{g_{x_0}}(0, \rho) \subset T_{x_0}M$, we have*

$$(1 - C\rho) F_*(x_0, z_2 - z_1) \le d_{F_*}\big(\exp_{x_0}(z_1), \exp_{x_0}(z_2)\big) \le (1 + C\rho) F_*(x_0, z_2 - z_1). \tag{3.6}$$

In particular, up to further decreasing ρ_1 depending on the same parameters,

$$\begin{aligned} &\Big| d_{F_*}\big(\exp_{x_0}(z_1), \exp_{x_0}(z_2)\big) - F_*\big(x_0, z_2 - z_1\big)\Big| \\ &\qquad \le C\rho \min\Big\{ F_*\Big(x_0, z_2 - z_1\Big), d_{F_*}\Big(\exp_{x_0}(z_1), \exp_{x_0}(z_2)\Big)\Big\} \end{aligned} \tag{3.7}$$

Proof. Let $\rho_2 = \rho_2(g, F)$ be chosen according to Lemma 3.3. Let ρ_1 be a fixed constant to be specified later in the proof, small enough such that $\rho_1 \le (1 + \frac{\mathfrak{m}}{4\mathfrak{M}})\rho_2$. Let $\rho < \rho_1$ and fix $z_1, z_2 \in B_{g_{x_0}}(0, \rho) \subset T_x M$. We prove the second inequality in (3.6) first. With the usual shorthand $\psi = \exp_{x_0}$, we have

$$d_{F_*}(\psi(z_1), \psi(z_2)) \le \inf \left\{ \int_0^1 \psi^* F_*\left(\gamma(t), \dot\gamma(t)\right)\, dx \ : \ \gamma : [0,1] \to B_{g_{x_0}}(0, \rho),\ \gamma(0) = z_1,\ \gamma(1) = z_2 \right\}.$$

We plug in $\gamma(t) = t z_2 + (1-t) z_1$ as a test curve. By convexity, $\gamma(t) \in B_{g_{x_0}}(0, \rho)$ for all $t \in [0,1]$, and since $\rho < \rho_2$, we can apply Lemma 3.3 to find

$$\begin{aligned} d_{F_*}\big(\psi(z_1), \psi(z_2)\big) &\le \int_0^1 \psi^* F_*\big(t z_2 + (1-t) z_1,\ z_2 - z_1\big)\, dt \\ &\le (1 + C\rho) \int_0^1 \psi^* F_*\big(0,\ z_2 - z_1\big)\, dt = (1 + C\rho)\, \psi^* F_*\big(0, z_2 - z_1\big) = (1 + C\rho)\, F_*\big(x_0, z_2 - z_1\big). \end{aligned}$$

Here $C = C(\mathsf{C}, \mathfrak{m}) = C(g, F)$ is the constant from (3.3).

Now we prove the first inequality in (3.6); the proof is similar but slightly more involved. Let $\hat\gamma : [0,1] \to M$ be a curve with $\hat\gamma(0) = \psi(z_1)$ and $\hat\gamma(1) = \psi(z_2)$ such that

$$\int_0^1 F_*\big(\hat\gamma(t), \dot{\hat\gamma}(t)\big)\, dt \le (1 + \rho) d_{F_*}\big(\psi(z_1), \psi(z_2)\big).$$

Since, by the triangle inequality, we have $d_g(\psi(z_1), \psi(z_2)) \le 2\rho$, Lemma 3.4 guarantees that the image of $\hat\gamma$ is contained in $B_g(\psi(z_1), \frac{4\mathfrak{M}}{\mathfrak{m}}\rho)$, which in turn is contained in $B_g(x_0, (1 + \frac{4\mathfrak{M}}{\mathfrak{m}})\rho)$. So, $(1 + \frac{4\mathfrak{M}}{\mathfrak{m}})\rho < \rho_2 < \mathrm{inj}_g M/2$ we may consider the pulled-back curve $\gamma = \psi^{-1}\hat\gamma : [0,1] \to B_{g_{x_0}}(0, (1 + \frac{4\mathfrak{M}}{\mathfrak{m}})\rho) \subset T_{x_0}M$, which has $\gamma(0) = z_1$, and $\gamma(1) = z_2$. It is easy to check using duality that for any $z_1, z_2 \in T_{x_0}M$,

$$F_*(x_0, z_2 - z_1) = \inf \Big\{ \psi^* F_*(0, \dot\gamma(t))\, dt \ : \ \gamma : [0,1] \to T_x M,\ \gamma(0) = z_1,\ \gamma(1) = z_2 \Big\}. \tag{3.8}$$

Using γ as a competitor in (3.8) and applying the bound (3.3) and $\dot\gamma(t) = d\psi_{\hat\gamma(t)}(\dot{\hat\gamma}(t))$, we find

$$F_*(x_0, z_2 - z_1) \le \int_0^1 \psi^* F_*(0, \dot\gamma(t))\, dt \le (1 + C\rho) \int_0^1 \psi^* F_*(\gamma(t), \dot\gamma(t))\, dt$$
$$= (1 + C\rho) \int_0^1 F_*(\hat\gamma(t), \dot{\hat\gamma}(t))\, dt = (1 + C\rho) d_{F_*}(\psi(z_1), \psi(z_2)).$$

Here $C = C(\mathsf{C}, \mathfrak{m}, \mathfrak{M}) = C(F, g)$. This proves (3.6); (3.7) follows immediately from (3.6) up to decreasing ρ_1 depending on C (and thus on F, g) and doubling the constant C. □

We are now ready to prove Lemma 3.2.

Proof of Lemma 3.2. Let $\rho_1 = \rho_1(g, F)$ be chosen according to Lemma 3.5. Let $\rho_0 > 0$ be a fixed constant to be specified in the proof, small enough so that $(1 + \mathfrak{m}^{-1})\rho_0 \le \rho_1$. Fix $r \in (0, \rho_0]$ and $\rho \in [0, \rho_0]$ and choose $x_0, x_1 \in M$ with $d_g(x_0, x_1) \le r$. Let $\psi = \exp_{x_0} : T_{x_0}M \to M$. Provided we choose ρ_0 small enough in terms of $g, \mathfrak{m}, \mathfrak{M}$, we can pull back $B_{F_*}(x_1, 2\rho_0)$ by ψ and it suffices to show that

$$d_{H, g_{x_0}}\big(\partial r K_{x_0} + z_1,\, \psi^{-1}(\partial B_{F_*}(x_1, r))\big) \le C\rho\, r, \tag{3.9}$$
$$d_{H, g_{x_0}}\big(r K_{x_0} + z_1,\, \psi^{-1}(B_{F_*}(x_1, r))\big) \le C\rho\, r\,. \tag{3.10}$$

We prove (3.9), with the proof of (3.10) being analogous. Toward (3.9), fix $z_2 \in \partial r K_{x_0} + z_1 \subset T_{x_0}M$. So, $F_*(x_0, z_2 - z_1) = r$. By assumption $|z_1|_{g_{x_0}} < \rho$, and thus using (3.1), we also have $|z_2|_{g_{x_0}} < \rho + \mathfrak{m}^{-1} r \le \rho_1$. So, we can apply Lemma 3.5: recalling that $F_*(x_0, z_2 - z_1) = r$, (3.7) guarantees that

$$|d_{F_*}(x_1, \psi(z_2)) - r| \le C\rho\, r. \tag{3.11}$$

Note that (3.11) implies $z \in \psi^{-1}(B_{F_*}(x_1, \hat r))$ for $\hat r$ with $|\hat r - r| < C\rho\, r$, Together with (3.1) this implies

$$z_2 \in U_{g_{x_0}}(\psi^{-1}(\partial B_{F_*}(x_1, r)),\, C\rho\, r) \tag{3.12}$$

where $C = C(g, F)$.

The other direction is analogous. Take $z_2 \in \psi^{-1}(\partial B_{F_*}(x_1, r))$ so that $d_{F_*}(x_1, \psi(z_2)) = r$. Again we have assumed that $|z_1|_{g_{x_0}} < \rho$ and using (3.1) deduce that $|z_2|_{g_{x_0}} < \rho + \mathfrak{m}^{-1} r < \rho_1$ as well. Hence, we are in a position to apply Lemma 3.5, which guarantees that

$$|F_*(x_0, z_2 - z_1) - r| \le C\rho\, r\,.$$

So, we see that $z_2 \in \partial K \hat r + z_1$ for some $\hat r$ with $|\hat r - r| < C\rho\, r$. Together with (3.1), this proves that

$$z_2 \in U_{g_{x_0}}(\partial r K_{x_0} + z_1,\, C\rho\, r)$$

with $C = C(g, F)$ and completes the proof. □

4. The diameter bound

In this section we prove a uniform scale-invariant diameter bound for volume-constrained ϵ_0-local minimizers Ω_v of $\mathcal{F}$ with sufficiently small volume v: Ω_v is contained in the union of J_0 balls of radius $2v^{1/n}$. This estimate is uniform in v in the sense that, with respect to the rescaled metric $h = v^{-2/n} g$, $|\Omega_v|_h = 1$ and Ω_v is contained in J_0 balls of radius 2.

Theorem 4.1. *Let (M, g) be a closed Riemannian manifold of dimension $n \ge 2$ and let $\mathcal{F}$ be an anisotropic surface energy with integrand F. Fix $\epsilon_0 > 0$ and $\kappa > 0$. There exist $v_0 = v_0(n, g, F, \epsilon_0, \kappa) \in (0, |M|_g)$ and $J_0 = J_0(n, g, F, \epsilon_0, \kappa) \in \mathbb{N}$ such that for any volume-constrained ϵ_0-local minimizer Ω_v of $\mathcal{F}$ with volume $v < v_0$ and $\mathcal{F}(\Omega_v) \le \kappa v^{(n-1)/n}$, there is a collection of points $x_1, \ldots, x_{J_0} \in M$ such that*

$$\Omega_v \subset \bigcup_{i=1}^{J_0} B_g(x_i, 2v^{1/n}).$$

The proof of Theorem 4.1 has three steps. First, Lemma 4.2 uses the concavity of the function $t \mapsto t^{(n-1)/n}$ to imply that if a set of finite perimeter is the union of N disjoint sets, then a significant portion of its volume is contained in the union of J_0 of those sets. Next, combining Lemma 4.2 with a grid argument inspired by [FM13, MM16], we show in Proposition 4.3 that any set of finite perimeter with volume v and a uniform perimeter (or $\mathcal{F}$) bound has most of its volume contained in the union of J_0 balls of radius $v^{1/n}$. Finally, we prove a differential inequality that allows us to improve this measure bound to containment in J_0 balls in the case of a volume-constrained ϵ_0-local minimizer.

4.1. **A lemma about concavity and sequences of real numbers.** The following lemma says that if a nonnegative decreasing sequence $\{a_i\}$ sums to 1 and has $\|\{a_i\}\|_{\ell^\alpha}$ bounded for a concave power $\alpha \in (0,1)$, then the tail end of the sequence has small ℓ^1 norm. This lemma has been already used in concentration compactness arguments, see for instance [GNR22, Proposition 3.7], [CTG21, Proposition 3.1], [CGOS18, Lemma 5.6, Lemma 6.6], [NPST22, Theorem 3.3]. We will apply it with $\alpha = \frac{n-1}{n}$ as described above.

Lemma 4.2. *Fix $\alpha \in (0,1)$, $\kappa > 0$, and $\eta > 0$. There exists $J_0 = J_0(\alpha, \kappa, \eta) \in \mathbb{N}$ such that for any sequence $\{a_i\}_{i\in\mathbb{N}}$ of nonnegative real numbers with $a_1 \geq a_2 \geq \dots$ and such that*

$$\sum_{i\in\mathbb{N}} a_i = 1 \qquad \text{and} \qquad \sum_{i\in\mathbb{N}} a_i^\alpha \leq \kappa,$$

we have $\sum_{i=1}^{J_0} a_i \geq 1-\eta$.

Proof. Since $\sum_{i=1}^{J} a_i \leq 1$ and the sequence is decreasing, we observe that $a_J \leq 1/J$ for every $J \in \mathbb{N}$. Hence we compute

$$1 - \sum_{i=1}^{J} a_i = \sum_{i>J} a_i = \sum_{i>J} a_i^\alpha a_i^{1-\alpha} \leq a_J^{1-\alpha} \sum_{i>J} a_i^\alpha \leq \frac{1}{J^{1-\alpha}}\kappa.$$

The proof follows choosing J large enough so that $\frac{1}{J^{1-\alpha}}\kappa \leq \eta$. □

4.2. **A diameter bound in measure.** In this section, we prove that a set of finite perimeter in (M,g) with small enough volume v has all but an η-fraction contained in J_0 balls of radius $v^{1/n}$, where J_0 depends only on dimension, the scale invariant anisotropic perimeter bound and η. The proposition applies to all sets of finite perimeter of sufficiently small volume and does not require minimality.

Proposition 4.3. *Fix $\kappa \geq 1$ and $\eta > 0$. Let (M,g) be a closed Riemannian manifold of dimension $n \geq 2$ and fix an anisotropic surface energy $\mathcal{F}$ with integrand F. There exist $v_0 = v_0(n,g,F,\eta) > 0$ and $J_0 = J_0(n,F,\kappa,\eta) \in \mathbb{N}$ such that the following holds. For any finite perimeter set Ω with volume $v \in (0, v_0]$ and $\mathcal{F}(\Omega) \leq \kappa v^{(n-1)/n}$, we may find points $x_1, \dots x_{J_0}$ in M such that*

$$\Big|\Omega \setminus \bigcup_{i=1}^{J_0} B_g\big(x_i, v^{1/n}\big)\Big|_g \leq \eta v\,. \tag{4.1}$$

The idea of the proof of Proposition 4.3 is the following. First, we intersect Ω with a collection of disjoint open sets $\{Q_i\}$ of diameter at most $v^{1/n}$ that cover Ω up to a null set, decomposing Ω into finitely many pairwise disjoint sets with the desired diameter bound. Importantly, Lemma 4.5 below ensures the collection $\{Q_i\}$ can be constructed in such a way that we quantitatively control the amount of surface energy that is added through taking intersections. Next, the Wulff inequality (2.8) yields a bound on the sum of a *concave power* of the volume of each component. Finally, by Lemma 4.2 we conclude that most of the volume of Ω must be contained in J_0 of the disjoint components.

Remark 4.4. The smallness of v_0 in the statement of Proposition 4.3 is used to apply the Wulff inequality in the form (2.8) on (M,g). In Euclidean space with a translation invariant $\mathcal{F}$ (in particular the perimeter), the Wulff inequality (2.8) holds for every volume: hence Proposition 4.3 holds, with the same proof, for sets of finite perimeter of any volume.

Lemma 4.5. *Let (M, g) be a closed Riemannian manifold of dimension $n \geq 2$. There exist $c = c(n, g) > 0$ and $r_0 = r_0(n, g) > 0$ such that the following holds. For every finite perimeter set $E \subset M$ and $r \in (0, r_0)$, there is a finite collection of pairwise disjoint open sets $\{Q_i\}_{i=1}^N$ in M with $\mathrm{diam}_g(Q_i) \leq r$ such that $|E \setminus \cup_{i=1}^N Q_i|_g = 0$,*

$$\mathcal{H}^{n-1}_g(\{x \in \partial^* E \cap \partial^* Q_i \,:\, \nu_E(x) = \nu_{Q_i}(x)\}) = 0, \tag{4.2}$$

and

$$\frac{|E|_g}{r} \geq c \sum_{i=1}^N \mathcal{H}^{n-1}_g\big(E^{(1)} \cap \partial Q_i\big). \tag{4.3}$$

In [FM13, Lemma 5.1], a statement analogous to Lemma 4.5 is shown in Euclidean space by decomposing $\mathbb{R}^n$ into cubes whose sides are parallel to a judiciously chosen orthonormal basis depending on the set E itself. To prove Lemma 4.5, we will obtain an initial collection of sets by applying [FM13, Lemma 5.1] in charts, and then refine the sets by hand to ensure they satisfy the properties of the lemma. Although [FM13, Lemma 5.1] is stated without using the set $E^{(1)}$ of points of density 1 and without the property (4.2), as observed in [MM16, Proof of Lemma 3.1, Step four] we will need to use $E^{(1)}$ and (4.2) in the proof of Proposition 4.3 in order to apply [Mag12, Theorem 16.3 (16.7)] to obtain (4.5).

Proof of Lemma 4.5. Step 1. First, we construct an initial collection of open sets $\{Q_i\}_{i=1}^N$, possibly not pairwise disjoint, with diameter at most r that cover E up to a null set and satisfy (4.3).

Choose $r_0 < (\mathrm{inj}_g M)/4$ small enough so that $\frac{1}{2} g_{euc} \leq \psi^* g \leq 2 g_{euc}$ on $B_{g_{euc}}(0, 4r_0) \subset \mathbb{R}^n$, where ψ is the normal coordinate map centered at any $x \in M$. Choose a finite collection of points $x_1, \ldots, x_K$ such that the balls $\{B_g(x_k, 2r_0)\}_{k=1}^K$ cover M and let $\psi_k : B_{g_{euc}}(0, 4r_0) \to M$ be the normal coordinate map centered at x_k.

Let $G_k = \psi_k^{-1}(E \cap B_g(x_k, 3r_0)) \subset \mathbb{R}^n$, so that $G_k \Subset B_{g_{euc}}(0, 4r_0)$ and $E = \cup_{k=1}^K \psi_k(G_k)$. Fix $k \in \{1, \ldots, K\}$ and $r < r_0$. Applying [FM13, Lemma 5.1] to the set G_k, we obtain a collection of disjoint open cubes $\{Q'_k\}$ of diameter $r/2$ with parallel sides that cover Lebesgue almost all of $\mathbb{R}^n$ such that $|G_k|_{g_{euc}} \geq \frac{r}{4n} \sum_{Q'_k} \mathcal{H}^{n-1}_{g_{euc}}(G_k^{(1)} \cap \partial Q'_k)$. Let $\{Q'_{k,a}\}_{a \in A'_k} \subset \{Q'_k\}$ be the finite collection of those cubes that intersect G_k nontrivially. Note that $Q'_{k,a} \subset B_{g_{euc}}(0, 4r_0)$ for each $a \in A'_k$, and

$$|G_k|_{g_{euc}} \geq \frac{r}{4n} \sum_{a \in A'_k} \mathcal{H}^{n-1}_{g_{euc}}\big(G_k^{(1)} \cap \partial Q'_{k,a}\big)\,. \tag{4.4}$$

For each $a \in A'_k$, let $Q_{k,a} = \psi_k(Q'_{k,a})$. Notice that $\mathrm{diam}_g(Q_{k,a}) \leq r$ for all $k \in \{1, \ldots, K\}$ and $a \in A'_k$. As an initial refinement of this collection, we let $A_1 = A'_1$ and for $k \geq 2$ let

$$A_k := \Big\{ a \in A'_k \ : \ Q_{k,a} \not\subset \bigcup_{j<k} \bigcup_{a \in A_j} Q_{j,a} \Big\}.$$

The collection $\{Q_{k,a}\}_{a \in A_k, 1 \leq k \leq K}$ consists of open sets with $\mathrm{diam}_g(Q_{k,a}) \leq r$ and covers E up to a set of measure zero. Moreover, applying (4.4) in charts, we find that

$$\begin{aligned} |E|_g \geq \frac{1}{K} \sum_{k=1}^K |E \cap V_k|_g \geq \frac{1}{2K} \sum_{k=1}^K |G_k|_{g_{euc}} &\geq \frac{r}{8nK} \sum_{k=1}^K \sum_{a \in A_k} \mathcal{H}^{n-1}_{g_{euc}}\big(G_k^{(1)} \cap \partial Q'_{a,k}\big) \\ &\geq \frac{r}{16nK} \sum_{k=1}^K \sum_{a \in A_k} \mathcal{H}^{n-1}_g\big(E^{(1)} \cap \partial Q_{a,k}\big), \end{aligned}$$

so the collection satisfies (4.3). However, the sets in this collection are not pairwise disjoint.

Step 2: We now slightly modify the collection of sets from Step 1 above so that they are pairwise disjoint and are still open with diameter at most r, cover E up to a null set, and satisfy the estimate (4.3). Fix $2 \leq k \leq K$ and $b \in A_k$. Let $I_{k,b} = \{a \in A_1 : Q_{1,a} \cap Q_{k,b} \neq \emptyset\}$ be the indices corresponding to cubes from the chart ψ_1 that intersect $Q_{k,b}$. By the construction from disjoint cubes in charts, the cardinality of $I_{k,b}$ is at most C_n. Let $\hat{Q}_{k,b} := Q_{k,b} \setminus \cup_{a\in I_{k,b}} \overline{Q}_{1,a}$. Then

$$\mathcal{H}^{n-1}_g\big(\partial\hat{Q}_{k,b} \cap E^{(1)}\big) \le \mathcal{H}^{n-1}_g\big(\partial Q_{k,b} \cap E^{(1)}\big) + \sum_{a\in I_{k,b}} \mathcal{H}^{n-1}_g\big(\partial Q_{1,a} \cap Q_{k,b} \cap E^{(1)}\big)\,.$$

Summing this up over all $b \in A_k$ and $2 \le k \le K$, we find that

$$\sum_{k=2}^{K}\sum_{b\in A_k} \mathcal{H}^{n-1}_g\big(\partial\hat{Q}_{k,b} \cap E^{(1)}\big) \le \sum_{k=2}^{K}\sum_{b\in A_k} \mathcal{H}^{n-1}_g\big(\partial Q_{k,b} \cap E^{(1)}\big) + C_nK \sum_{a\in A_1} \mathcal{H}^{n-1}_g\big(\partial Q_{1,a} \cap E^{(1)}\big)\,.$$

Here we have used the fact that any $x \in \partial Q_{1,a}$ is contained in $Q_{k,b}$ for at most C_nK cubes, thanks to the construction from disjoint cubes in charts. Adding $\sum_{a\in A_1} \mathcal{H}^{n-1}_g\big(\partial Q_{1,a}\cap E^{(1)}\big)$ to both sides and recalling (4.4), we see that

$$\sum_{a\in A_1} \mathcal{H}^{n-1}_g\big(\partial Q_{1,a} \cap E^{(1)}\big) + \sum_{k=2}^{K}\sum_{b\in A_k} \mathcal{H}^{n-1}_g\big(\partial\hat{Q}_{k,b} \cap E^{(1)}\big) \le C_nK \sum_{k=1}^{K}\sum_{a\in A_k} \mathcal{H}^{n-1}_g(E^{(1)} \cap \partial Q_{a,k}) \le \frac{|E|_g}{r}.$$

So, the collection of sets $\{Q_{1,a}\}_{a\in A_1} \cup \{\hat{Q}_{k,b}\}_{2\le k\le K, b\in A_k}$ satisfies (4.3), each set is open with diameter at most r, and the sets $Q_{1,a}$ are pairwise disjoint and also have trivial intersection with any $\hat{Q}_{k,b}$.

Setting aside the sets $\{Q_{1,a}\}_{a\in A_1}$, we apply the same procedure with the index $k = 2$ playing the role of 1 to refine the sets $\{\hat{Q}_{k,b}\}$ for $3 \le k \le K$, $b \in A_k$, to make them disjoint from $\hat{Q}_{2,a}$ for any $a \in A_2$ and satisfy the properties above. Proceeding inductively and applying the refinement procedure K times, we obtain a collection of sets satisfying the properties of the lemma. In particular, property (4.2) can be obtained by slightly tilting the initial collection of open sets $\{Q_i\}_{i=1}^N$. □

We now prove Proposition 4.3.

Proof of Proposition 4.3. Let $\bar{v}$ be as in (2.8), let r_0 be as in Lemma 4.5, and set $v_0 := \min\{\bar{v}, r_0^n\}$. Let $\{Q_i\}_{i=1}^N$ be the collection of sets obtained applying Lemma 4.5 to $E = \Omega$ with $r = v^{1/n}$. We first apply the isoperimetric inequality (2.8) and then, using (4.2), we apply [Mag12, Theorem 16.3 (16.7)] to compute

$$\sum_{i=1}^{N} |\Omega \cap Q_i|_g^{\frac{n-1}{n}} \le C \sum_{i=1}^{N} \mathcal{F}(\Omega \cap Q_i) \le C\Big(\mathcal{F}(\Omega) + \sum_{i=1}^{N} \mathcal{H}^{n-1}_g(\Omega^{(1)} \cap \partial Q_i)\Big). \tag{4.5}$$

Applying estimate (4.3) from Lemma 4.5 to (4.5), we obtain

$$\sum_{i=1}^{N} |\Omega \cap Q_i|_g^{\frac{n-1}{n}} \le C\left(\kappa v^{\frac{n-1}{n}} + \frac{v}{r}\right).$$

Dividing by $v^{\frac{n-1}{n}}$ and using the choice $r = v^{\frac{1}{n}}$, we deduce that

$$\sum_{i=1}^{N} \left(\frac{|\Omega \cap Q_i|_g}{v}\right)^{\frac{n-1}{n}} \le C\frac{v^{1/n}}{r} + C\kappa \le C\kappa. \tag{4.6}$$

The sets Q_i are pairwise disjoint and cover E up to a set of measure zero, so $\sum_{i=1}^N \frac{|\Omega\cap Q_i|_g}{v} = 1$. Up to relabeling the indices, we can suppose that the sequence $a_i := \frac{|\Omega\cap Q_i|_g}{v}$ is non-increasing, hence we can apply Lemma 4.2 to $\{a_i\}_{i=1}^N$, to deduce that there exists $J_0 = J_0(n, \kappa, \eta) \in \mathbb{N}$ such that

$$\sum_{i=1}^{J_0} \frac{|\Omega \cap Q_i|_g}{v} \ge 1 - \eta.$$

For each $i = 1, \dots, J_0$, since $\mathrm{diam}_g(Q_i) \le r = v^{\frac{1}{n}}$, we can find a point $x_i \in M$ such that $Q_i \subset B_g(x_i, v^{1/n})$. So, again using the pairwise disjointness of the Q_i,

$$\left|\Omega \setminus \bigcup_{i=1}^{J_0} B_g(x_i, v^{1/n})\right|_g \le \left|\Omega_v \setminus \bigcup_{i=1}^{J_0} Q_i\right|_g = |\Omega|_g - \sum_{i=1}^{J_0} |\Omega \cap Q_i|_g \le \eta v. \tag{4.7}$$

Thus (4.1) holds, as desired. □

4.3. **Proof of the diameter bound.** In this section, we prove the diameter bound of Theorem 4.1. In the proof, we will use (slight modifications of) sets of the form $\Omega_v' := \Omega_v \cap \bigcup_{i=1}^{J_0} B_g(x_i, R)$ as competitors for the ϵ_0-local minimality of Ω_v, where $x_i, \dots, x_{J_0}$ are the points obtained in Proposition 4.3. To this aim, we first prove that $\Omega_v \Delta \Omega_v' \subset U_g(\partial\Omega_v, \delta)$ in the following lemma. Recall that $U_g(E, \delta)$ is the tubular neighborhood defined in (1.1).

Lemma 4.6. *Fix a Riemannian manifold (M, g) of dimension $n \ge 2$. Let $r_0 > 0$ be small enough so that $|B_g(x, r)|_g \ge \omega_n r^n/2$ for any $x \in M$ and $r \in (0, r_0)$. Fix $\delta > 0$ and $J_0 \in \mathbb{N}$. Let $\gamma < \frac{\omega_n}{2}\min\{r_0^n, \delta^n\}$. If $\Omega \subset M$ is a measurable set with*

$$\left|\Omega \setminus \bigcup_{i=1}^{J_0} B_g(x_i, R)\right|_g \le \gamma$$

for some $x_1, \dots, x_{J_0} \in M$ and $R > 0$, then

$$\Omega \setminus \bigcup_{i=1}^{J_0} B_g(x_i, R + \delta) \subset U_g(\partial\Omega, \delta). \tag{4.8}$$

Proof. Suppose there is a point $x_0 \in \Omega \setminus \bigcup_{i=1}^{J_0} B_g(x_i, R+\delta)$ with $x_0 \notin U_g(\partial\Omega, \delta)$. Then by definition,

$$B_g(x_0, \delta) \subset \Omega \setminus \bigcup_{i=1}^{J_0} B_g(x_i, R)$$

and so in particular

$$|B_g(x_0, \delta)|_g \le \left|\Omega \setminus \bigcup_{i=1}^{J_0} B_g(x_i, R)\right|_g \le \gamma.$$

On the other hand, $|B_g(x_0, \delta)|_g \ge \frac{\omega_n}{2}\min\{r_0^n, \delta^n\}$, contradicting our choice of γ. We conclude that no such point exists and the containment (4.8) holds. □

We are now ready to prove the main result of the section, using Lemma 4.6 and Proposition 4.3 to establish a differential inequality for the volume of Ω_v outside J_0 balls of radius r.

Proof of Theorem 4.1. We begin by fixing parameters. Let $\eta = \eta(n, \epsilon_0) \in (0, 1/2)$ be a fixed number to be determined later in the proof. Let $J_0 = J_0(n, F, \kappa, \eta) = J_0(n, F, \kappa, \epsilon_0)$ be chosen according to Proposition 4.3. Choose $r_0 = r_0(g, \eta, J_0, F) = r_0(g, n, \kappa, F) < \mathrm{inj}_g M$ to be small enough according to the assumptions of Lemma 4.6 and such that

$$\|\psi^* g - g_{euc}\|_{C^1(B_{g_{euc}}(0, r_0))} \le \frac{\eta}{12 J_0} \qquad \text{and} \qquad \sup_{\nu \in S^{n-1}} \|\psi^* F(\cdot, \nu)\|_{C^1(B_{g_{euc}}(0, r_0))} \le 1, \tag{4.9}$$

in normal coordinates ψ at any $x \in M$. Let $v_0 = v_0(n, g, F, \kappa, \eta) < r_0^n \omega_n/2$ be small enough to apply Proposition 4.3 and such that $3v_0^{1/n} < r_0/100J_0$. Let $v < v_0$ be fixed. Throughout the proof, c_n denotes a dimensional constant whose value may change from line to line.

By Proposition 4.3, we can find a collection of points $x_1, \dots, x_{J_0} \in M$ such that

$$\left|\Omega_v \setminus \bigcup_{i=1}^{J_0} B_g(x_i, v^{1/n})\right|_g < \eta v\,. \tag{4.10}$$

Let $I = [v^{1/n}, 3v^{1/n}]$, and for $r \in I$ let

$$A(r) = \bigcup_{i=1}^{J_0} B_g(x_i, r) \qquad \text{and} \qquad u(r) = \frac{|\Omega_v \setminus A(r)|_g}{v}.$$

Note that u is decreasing in r, and $u(v^{1/n}) < \eta$ by (4.10). We claim there exists $c_n > 0$ such that

$$[(vu(r))^{1/n}]' \le -c_n \tag{4.11}$$

for all $r \in I$ with $u(r) > 0$. Before proving the differential inequality (4.11), let us see how it will allow us to conclude the proof of the proposition. Take $r \in I$ such that $u(r) > 0$. Since u is decreasing, (4.11) holds for all $s \in (v^{1/n}, r]$. Integrating this differential inequality and recalling that $u(v^{1/n}) < \eta$, we find

$$c_n\big(\hat{r} - v^{1/n}\big) \le (\eta v)^{1/n} - [vu(\hat{r})]^{1/n} < (\eta v)^{1/n}.$$

In particular, provided we choose $\eta < c_n/2$, we find that $r < 2v^{1/n}$. Thus u vanishes on $[2v^{1/n}, 3v^{1/n}]$ and $\Omega_v \subset A(2v^{1/n})$. This shows the claim in the proposition.

It therefore remains to prove (4.11). By the coarea formula, we find that

$$u'(r) = -\frac{1}{v}\,\mathcal{H}^{n-1}_g(\partial A(r) \cap \Omega_v) \tag{4.12}$$

for a.e. $r \in I$. To gain information about the right-hand side of (4.12), we would like to use the sets

$$E_r := A(r) \cap \Omega_v$$

for $r \in I$ as competitors for the local minimality of Ω_v. But, since we may have $|E_r| < v$, we must modify the sets using Lemma 2.1 to make them admissible competitors. To this end, note that $A(3v^{1/n})$ has $1 \le K \le J_0$ connected components $A_1, \dots, A_K$, and each connected component A_k has diameter at most $6J_0\, v^{1/n}$. Thus, we can find a collection of disjoint open sets $V_1, \dots, V_K$ in M such that $\mathrm{diam}_g(V_k) \le 12J_0\, v^{1/n}$ and $U_g(A_k, v^{1/n}) \subset V_k$. In particular, $U_g(E_r, v^{1/n}) \subset \bigcup_{k=1}^K V_k$ for each $r \in I$.

In terms of the rescaled metric $h = v^{-2/n} g$, this means that $\mathrm{diam}_h(V_k) \le 12J_0$, and $U_h(E_r, 1) \subset \bigcup_{k=1}^K V_k$, and $|E_r|_h = 1 - u(r) \in [1-\eta, 1]$. Moreover, since v_0 was chosen small enough that $v^{1/n} < r_0/12J_0$, the estimates (4.9) hold with h in place of g and $B(0, 12J_0)$ in place of $B(0, r_0)$. Thus, for each $r \in I$, we may apply Lemma 2.1 on (M, h) with $D = 12J_0$ and $E = E_r$. In terms of the metric g, the conclusion of the lemma tells us there is a set $\tilde{E}_r$ with $|\tilde{E}_r| = v$ such that

$$\tilde{E}_r \Delta E_r \Subset U_g(\partial E_r, c_n J_0 \eta v^{1/n}), \tag{4.13}$$

$$\mathcal{F}(\tilde{E}_r) \le \{1 + c_n J_0 u(r)\}\,\mathcal{F}(E_r). \tag{4.14}$$

We now claim that, if η is chosen to be small enough, we have

$$\tilde{E}_r \Delta \Omega_v \subset U_g(\partial\Omega_v, \epsilon_0 v^{1/n}), \tag{4.15}$$

thus $\tilde{E}_r$ is an admissible competitor for the local minimality of Ω_v. Thanks to the triangle inequality property of the symmetric difference (2.1), it suffices to show that $\Omega_v \Delta E_r$ and $E_r \Delta \tilde{E}_r$ are both contained in this neighborhood of $\partial\Omega_v$. To obtain the first containment, we apply Lemma 4.6 with $\delta = \frac{\epsilon_0}{2} v^{1/n}$ and $\gamma = \eta v$, with $\eta > 0$ chosen small enough depending on ϵ_0 so that $\eta v < \frac{\omega_n}{2}\min\{r_0^n, \epsilon_0^n v/2^n\}$. This implies

$$E_r \Delta \Omega_v = \Omega_v \setminus A(r) \subset U_g(\partial\Omega_v, \frac{\epsilon_0}{2} v^{1/n}). \tag{4.16}$$

Next, to show the containment of $E_r \Delta \tilde{E}_r$, thanks to (4.13), it suffices to show that

$$U_g(\partial E_r, c_n J_0 \eta v^{1/n}) \subset U_g(\partial \Omega_v, \epsilon_0 v^{1/n}).$$

Since $\partial E_r = (\partial\Omega_v \cap A(r)) \cup (\partial A(r) \cap \Omega_v)$ we thus find that

$$\begin{aligned} U_g(\partial E_r, c_n J_0 \eta r) &= U_g(\partial\Omega_v \cap A(r), c_n J_0 \eta r) \;\cup\; U_g(\partial A(r) \cap \Omega_v, c_n J_0 \eta r) \\ &\subset U_g(\partial\Omega_v, c_n J_0 \eta r) \;\cup\; U_g(\partial A(r) \cap \Omega_v, c_n J_0 \eta r)\,. \end{aligned}$$

Since $\partial A(r) \cap \Omega_v \subset \Omega_v \setminus A(r)$ and $r \le 3v^{1/n}$, if we take $c_n J_0 \eta r \le \epsilon_0 v^{1/n}/2$, we have $U_g(\partial\Omega_v \cap A(r), c_n J_0 \eta r) \subset U_g(\partial\Omega_v, \epsilon_0 v^{1/n})$ by (4.16) above. Thus (4.15) holds.

Hence, $\tilde{E}_r$ is an admissible competitor for the local minimality of Ω_v, and so we find that

$$\begin{aligned}\mathcal{F}(\Omega_v) \le \mathcal{F}(\tilde{E}_r) &\overset{(4.14)}{\le} (1+Cu(r))\mathcal{F}(E_r)\\ &= (1+Cu(r))\big(\mathcal{F}(\Omega_v;A(r)) + C\mathcal{H}^{n-1}(\partial A(r)\cap\Omega_v)\big)\\ &\overset{(4.12)}{\le} (1+Cu(r))(\mathcal{F}(\Omega_v;A(r)) + Cv|u'(r)|)\\ &\le \mathcal{F}(\Omega_v;A(r)) + C\kappa v(v^{-1/n}u(r) + |u'(r)|).\end{aligned}$$

Subtracting $\mathcal{F}(\Omega_v;A(r))$ from both sides and adding $\int_{\partial A(r)\cap\Omega_v} F(x,-\nu_{A(r)})\,d\mathcal{H}^{n-1}$ to both sides (and noting the latter term is bounded above by $Cv|u'(r)|$), we find

$$\mathcal{F}(\Omega_v\setminus A(r)) \le C\kappa v(v^{-1/n}u(r) + |u'(r)|)$$

from which we deduce from (2.8) that

$$(vu(r))^{(n-1)/n} \le C\kappa v(v^{-1/n}u(r) + |u'(r)|).$$

Choosing η small enough that $C\kappa v u(r) \le \frac{1}{2}u(r)^{(n-1)/n}$, we obtain (4.11) and conclude the proof. ☐

5. Uniform convergence to a Wulff shape, qualitatively

In this section, we prove Theorem 5.1, showing that for v sufficiently small, a volume-constrained ϵ_0-local minimizer Ω_v of $\mathcal{F}$ is uniformly close to a tangent Wulff shape of the appropriate volume at some point $x_0 \in M$. At this stage, the estimates are qualitative with respect to the volume parameter v.

Theorem 5.1. *Fix a closed Riemannian n-manifold (M,g) and an anisotropic surface energy $\mathcal{F}$ with integrand F. For every $\kappa > 0$ and $\epsilon_0 > 0$, there exists $v_0 = v_0(n,g,F,\kappa,\epsilon_0) \in (0,|M|_g)$ such that the following holds. Let Ω_v be a volume-constrained ϵ_0-local minimizer Ω_v of volume $v < v_0$ with $\mathcal{F}(\Omega_v) \le \kappa v^{(n-1)/n}$. Then Ω_v is connected and there is a point $x_0 \in M$ such that*

$$d_{H,g}\Big(\partial\Omega_v,\, \exp_{x_0}(\partial(v^{1/n}K_{x_0}))\Big) < \frac{\epsilon_0 v^{1/n}}{\beta_0} \qquad \text{and} \qquad \Big|\Omega_v \Delta \exp_{x_0}(v^{1/n}K_{x_0})\Big|_g < \frac{\epsilon_0 v}{\beta_0}. \tag{5.1}$$

Here K_{x_0} is the tangent Wulff shape at x_0 defined in (2.5), and

$$\beta_0(n,\kappa,F,\epsilon_0) := \frac{8\mathcal{C}\epsilon_0}{\min\Big\{\Big(\frac{\mathfrak{m}}{2\Lambda}n\omega_n^{1/n}\Big)^n, \frac{\epsilon_0^n\,\omega_n}{2^{n+1}}\Big\}}.$$

Remark 5.2. In Theorem 5.6 we show that estimate (5.1) actually holds for any $\beta_0 > 0$, provided v_0 is also taken to be sufficiently small depending on β_0. However, the explicit choice of β_0 in the theorem statement provides a volume threshold under which local minimizers are connected.

Remark 5.3. Since $K_{x_0} \subset B_{g_{x_0}}(0,R')$ for a constant $R' > 0$ depending only on F and g, Theorem 5.1 implies that Ω_v satisfies a diameter bound $\Omega_v \subset B_g(x_0, R\,v^{1/n})$ where $R = R' + 2\epsilon_0/\beta_0$ depends only on F, g, and n. Moreover, (5.1) implies that

$$\Omega_v \Delta \exp_{x_0}(v^{1/n}K_{x_0}) \subset U_g(\partial\Omega_v, \epsilon_0 v^{1/n}/\beta_0).$$

We prove Theorem 5.1 via a compactness argument, using Theorem 4.1 crucially at various points. To see the idea, take a sequence of volume-constrained ϵ_0-local minimizers Ω_k of volume $v_k \to 0$. Using Theorem 4.1, we can pull back Ω_k in charts and show that, after rescaling, the resulting sequence of sets E_k in $\mathbb{R}^n$ subsequentially converges in L^1 to a limit set E with unit volume. Again using Theorem 4.1, we show in Section 5.1 that the sets Ω_k satisfy scale-invariantly uniform density estimates. This upgrades the L^1 convergence of E_k to Hausdorff convergence of the boundaries and crucially allows us to deduce that the limit set E is itself a volume-constrained local minimizer of a translation invariant anisotropic surface energy. A scaling argument and the Alexandrov-type theorem [DRKS20, Corollary 6.8] show that E is a translation of the corresponding Wulff shape. The translation invariance leads to technical challenges bringing this statement back to (M, g), which we tackle by comparing tangent Wulff shapes at different points using Proposition 3.1.

5.1. **Uniform quasi-minimality.** In this section, we prove that volume-constrained ϵ_0-local minimizers of $\mathcal{F}$ with small volume satisfy a quasi-minimality property among non-volume-constrained competitors. In the language of [Mag12], after rescaling, they satisfy a local version of being (Λ, r_0)-minimizers of $\mathcal{F}$. Crucially, the parameters Λ and r_0 are independent of v. Theorem 4.1 is key in the proof, as it allows us to apply Lemma 2.1 to modify a local competitor into one with the prescribed volume while estimating the error in a uniform way.

Lemma 5.4. *Let (M, g) be a closed Riemannian manifold of dimension $n \geq 2$ and let $\mathcal{F}$ be an anisotropic surface energy on M. For each $\epsilon_0 > 0$ and $\kappa > 0$, there exist $v_0 = v_0(n, \epsilon_0, \kappa, g) > 0$ and $\Lambda = \Lambda(n, \kappa) > 1$ such that the following holds. If $\Omega_v \subset M$ is a volume-constrained ϵ_0-local minimizer of $\mathcal{F}$ with volume $v \in (0, v_0)$ and $\mathcal{F}(\Omega_v) \leq \kappa v^{(n-1)/n}$, then Ω_v is a local $(\Lambda v^{-1/n}, \frac{\epsilon_0}{2} v^{1/n})$-minimizer of $\mathcal{F}$, i.e.*

$$\mathcal{F}(\Omega_v) \leq \mathcal{F}(E) + \Lambda\, v^{-1/n} |\Omega_v \Delta E|_g.$$

for any set $E \subset M$ such that $\Omega_v \Delta E \Subset U_g(\partial\Omega_v, \frac{\epsilon_0}{2}\, v^{1/n})$.

Proof. Let v_0 and Λ be fixed constants to be specified in the proof and fix $v \in (0, v_0)$. Let $h = v^{-2/n} g$, so that Ω_v is a volume-constrained ϵ_0-local minimizer of $\mathcal{F}_h$ with $|\Omega_v|_h = 1$. In terms of the rescaled metric, we will prove that for any $E \subset M$ with $\Omega_v \Delta E \subset U_h(\partial\Omega_v, \frac{\epsilon_0}{2})$, we have

$$\mathcal{E}_h(\Omega_v) \leq \mathcal{E}_h(E) \qquad \text{where} \qquad \mathcal{E}_h(E) := \mathcal{F}_h(E) + \Lambda\, ||E|_h - 1|\,. \tag{5.2}$$

Then, noting that $||E|_h - 1| \leq |E\Delta\Omega_v|_h$ and scaling back to the original metric, this implies the lemma.

To show (5.2), it suffices to show that for any $E \subset M$ with $E\Delta\Omega_v \subset U_h(\partial\Omega_v, \frac{\epsilon_0}{2})$ and $\mathcal{E}_v(E) \leq 2\mathcal{E}_h(\Omega_v)$, we may find a set $\tilde{E}$ with

$$\mathcal{E}_h(\tilde{E}) \leq \mathcal{E}_h(E), \qquad |\tilde{E}|_h = 1, \qquad \tilde{E}\Delta\Omega_v \subset U_h(\partial\Omega_v, \epsilon_0),$$

since then taking $\tilde{E}$ as a competitor for the local minimality of Ω_v, directly implies (5.2).

So, fix $E \subset M$ with $E\Delta\Omega_v \Subset U_h(\partial\Omega_v, \frac{\epsilon_0}{2})$ and $\mathcal{E}_h(E) \leq 2\mathcal{E}_h(\Omega_v)$. Notice that

$$||E|_h - 1| \leq \frac{\mathcal{E}_h(E)}{\Lambda} \leq \frac{2\mathcal{E}_h(\Omega_v)}{\Lambda} = \frac{2\mathcal{F}_h(\Omega_v)}{\Lambda} \leq \frac{2\kappa}{\Lambda}$$

Recall the dimensional constants η_0, c_n in Lemma 2.1. Choose $\eta < \min\{\eta_0, \frac{\epsilon_0}{8c_n}\}$ and $\Lambda > 2\kappa/\eta$. According to Theorem 4.1, $\Omega_v \subset \cup_{j=1}^{J_0} B_h(x_i, 2)$ for points $x_1, \ldots x_{J_0} \in M$, and thus we also have $U_h(E, \frac{\epsilon_0}{2}) \subset U_h(\Omega_v, \epsilon_0) \subset \cup_{j=1}^{J_0} B_h(x_i, 4)$. Then, provided we choose v_0 small enough so (2.9) holds with $D = 4$, we can apply Lemma 2.1 to obtain a set $\tilde{E}$ that, thanks to our choice of η, satisfies

$$\tilde{E}\Delta E \subset U_h(\partial E, \frac{\epsilon_0}{2}) \subset U_h(\partial\Omega_v, \epsilon_0) \qquad \text{and} \qquad |\tilde{E}|_h = 1,$$

and

$$\begin{aligned} \mathcal{F}_h(\tilde{E}) &\leq \mathcal{F}_h(E)(1 + C_n\, ||E|_h - 1|) \\ &\leq \mathcal{F}_h(E) + 2\mathcal{F}(\Omega_v) C_n\, ||E|_h - 1| \leq \mathcal{F}_h(E) + C_n\kappa\, ||E|_h - 1|\,. \end{aligned}$$

Therefore, $\mathcal{E}_h(\tilde{E}) = \mathcal{F}_h(\tilde{E}) \leq \mathcal{F}_h(E) + C_n\kappa\, ||E|_h - 1| \leq \mathcal{E}_h(E)$ so long as $\Lambda > C_n\kappa$. ☐

5.2. **An intermediate form of Theorem 5.1.** Next, we prove a slightly weaker version of Theorem 5.1: In Theorem 5.5 below, a volume-constrained ϵ_0-local minimizer Ω_v is shown to be close to a (projected via $\exp_{x_0}$) Wulff shape translated by some $y \in T_{x_0}M$. The modulus of this translation, while tending to zero as $v \to 0$, could be very large relative to the natural length scale $v^{1/n}$. In Section 5.4 we will center to correct this translation error and prove that Ω_v is connected to complete the proof of Theorem 5.1.

Theorem 5.5. *Fix a closed Riemannian n-manifold (M, g) and an anisotropic surface energy $\mathcal{F}$ with integrand F. For every $\kappa > 0$, $\beta > 0$, $\epsilon_0 > 0$, and $\rho > 0$, there exists $v_0 = v_0(g, F, \kappa, \epsilon_0, \beta, \rho) > 0$ such that the following holds. Let Ω_v be a volume-constrained ϵ_0-local minimizer of volume $v < v_0$ with $\mathcal{F}(\Omega_v) \le \kappa v^{(n-1)/n}$. There are points $x \in M$ and $y \in B_{g_x}(0, \rho) \subset T_x M$ such that*

$$d_{H,g}\big(\partial\Omega_v, \exp_x(\partial v^{1/n} K_x + y)\big) < \frac{\epsilon_0 v^{1/n}}{\beta} \qquad \text{and} \qquad \big|\Omega_v \Delta \exp_x(v^{1/n} K_x + y)\big|_g < \frac{\epsilon_0 v}{\beta}\,. \tag{5.3}$$

Proof. We divide the proof in several steps:

Step 0: Setup. Supposing for the sake of contradiction that the statement is false, we find $r_0 \le \frac{\mathrm{inj}_g M}{2}$ and a sequence of numbers $v_i \to 0$ and of volume-constrained ϵ_0-local minimizers Ω_i of $\mathcal{F}$ with volume v_i such that

$$d_{H,g}\big(\partial\Omega_i, \exp_x(\partial v_i^{1/n} K_x + y)\big) \ge \frac{\epsilon_0 v_i^{1/n}}{\beta} \qquad \text{or} \qquad \big|\Omega_i \Delta \exp_x(v_i^{1/n} K_x + y)\big|_g \ge \frac{\epsilon_0 v_i}{\beta} \tag{5.4}$$

for every $x \in M$ and $y \in B_{g_x}(0, r_0) \subset T_x M$. Let v_0 be chosen according to Theorem 4.1. Since $v_i < v_0$ for i large enough, we may apply Theorem 4.1 to find a sequence of finite families of points $\{x_{i,j} : i \in \mathbb{N}, j = 1, \dots, J_0\} \subset M$ such that $\Omega_i \subset \bigcup_{j=1}^{J_0} B_g(x_{i,j}, 2v_i^{1/n})$. For fixed $i \in \mathbb{N}$ and for each $j = 1, \dots, J_0$, let $\Omega_{i,j}$ be the union of all the connected components of Ω_i that intersect $B_g(x_{i,j}, 2v_i^{1/n})$ and do not intersect any of the previous balls $\{B_g(x_{i,k}, 2v_i^{1/n})\}_{k=1}^{j-1}$. In particular, we observe that

$$\Omega_{i,j} \subset B_g\big(x_{i,j}, 4J_0 v_i^{1/n}\big) \quad \text{ for every } j = 1, \dots, J_0, \tag{5.5}$$

and $\Omega_{i,j}$ are pairwise disjoint in j. Since M is compact, $x_{i,j} \to x_j \in M$ for every $j = 1, \dots, J_0$ after passing to a subsequence in i. For i large enough, $B_g(x_{i,j}, 8J_0 v_i^{1/n}) \subset B_g(x_j, r_0)$ for every $j = 1, \dots, J_0$.

Step 1: Pulling back and rescaling the problem. Fix an orthonormal basis $\{e_1, \dots, e_n\}$ for Euclidean space, and for each $j = 1, \dots, J_0$, let $\psi_j : B_{g_{euc}}(0, r_0) \to B_g(x_j, r_0)$ be a normal coordinate map at x_j and let $z_j = 17J_0(j-1)e_1 \in \mathbb{R}^n$. For large enough i, we can define the map

$$\phi_{i,j} : B_{g_{euc}}\big(z_j,\, 8J_0\big) \subset \mathbb{R}^n \to M, \qquad \phi_{i,j}(x) := \psi_j\left(v_i^{1/n}(x - z_j) + \psi_j^{-1}(x_{i,j})\right),$$

which first maps its domain to $B_{g_{euc}}(\psi_j^{-1}(x_{i,j}), 8J_0 v_i^{1/n})$ homothetically, then maps this small ball to M by the normal coordinate map. Identifying the J_0 (a priori distinct) copies of Euclidean space via the basis $\{e_1, \dots, e_n\}$, we view the balls $B_{g_{euc}}\big(z_j,\, 8J_0\big)$ as disjoint subsets of the same Euclidean space.

In particular, by (5.5) we may define the pulled-back sets

$$E_{i,j} := \phi_{i,j}^{-1}(\Omega_{i,j}) \Subset B_{g_{euc}}\big(z_j, 6J_0\big) \subset \mathbb{R}^n\,. \tag{5.6}$$

We then let

$$E_i := \bigcup_{j=1}^{J_0} E_{i,j}, \qquad \text{with} \qquad E_i \Subset X := \bigcup_{j=1}^{J_0} B_{g_{euc}}\big(z_j, 6J_0\big) \subset \mathbb{R}^n\,.$$

Observe that $X \subset B_{g_{euc}}(0, 24J_0)$.

Let us define the rescaled metrics $h_i = v_i^{-2/n} g$, so that $|\Omega_i|_{h_i} = 1$ and $\mathcal{F}_{h_i}(\Omega_i) \le \kappa$ for all i. Up to passing to a subsequence with respect to i, we have

$$(1 - 1/i) g_{euc} \le \phi_{i,j}^* h_i \le (1 + 1/i) g_{euc} \quad \text{ on } B_{g_{euc}}\big(z_j, 8J_0\big) \qquad \forall i \in \mathbb{N},\, j = 1 \dots, J_0\,. \tag{5.7}$$

Hence, by (5.7),

$$\big||E_i|_{g_{euc}} - 1\big| \to 0. \tag{5.8}$$

Recall that the restriction $F_{x_j}(\cdot) := F(x_j, \cdot)$ of the anisotropic integrand defines a translation invariant surface energy $\bar{\mathcal{F}}_{x_j}$ on $T_{x_j} M$. The normal coordinate map $\psi_j : B_{g_{euc}}(0, r_0) \to B_g(x_j, r_0) \subset M$ induces the linear map $L = (d\psi_j)_{|0} : \mathbb{R}^n \to T_{x_j}M$, which has $\det L = 1$. Let us denote by $\mathcal{F}^*_{x_j}$ the pulled-back tangent surface energy at x_j by L, i.e. for a set $E \subset \mathbb{R}^n$ we let

$$\mathcal{F}^*_{x_j}(E) = \bar{\mathcal{F}}_{x_j}(L(E)) = \int_{\partial^* E} F(x_j, L(\nu_E(y)))\, d\mathcal{H}^{n-1}_{g_{euc}}(y).$$

Furthermore, for every set $E \subset X$ we define

$$\mathcal{F}^*(E) = \sum_{j=1}^{J_0} \mathcal{F}^*_{x_j}\big(E \cap B_{g_{euc}}(z_j, 8J_0)\big).$$

It is easy to see that $\mathcal{F}^*$ can be extended to every subset of $\mathbb{R}^n$ and that $\mathcal{F}^* \equiv \mathcal{F}^*_{x_j}$ for subsets of $B_{g_{euc}}(z_j, 8J_0)$ as these balls are disjoint. Let $\mathcal{F}^*_{i,j}(E) = \mathcal{F}_{h_i}(\phi_{i,j}(E))$ be the pulled-back h_i-surface energy of a set $E \subset B_{g_{euc}}(z_j, 8J_0)$ and let $\mathcal{F}^*_i(E) = \sum_{j=1}^{J_0} \mathcal{F}^*_{i,j}(E \cap B_{g_{euc}}(z_j, 8J_0))$ for every set $E \subset X$. By assumption, $\mathcal{F}^*_i(E_i) \le \kappa$. Moreover, by (5.7) and the continuity of F with respect to x, we have

$$|\mathcal{F}^*_{i,j}(E) - \mathcal{F}^*_{x_j}(E)| \le \omega(i)\mathcal{F}^*_{i,j}(E) \tag{5.9}$$

for a modulus of continuity ω depending on F and g.

Step 2: Compactness in L^1. Since $F(x,\cdot) \ge \mathfrak{m}$, thanks to (5.9) and the assumption $\mathcal{F}^*_i(E_i) \le \kappa$, we see that $P(E_i) \le 2\mathfrak{m}^{-1}\kappa$ for all i sufficiently large. Moreover, using (5.6), we see that up to a subsequence,

$$E_{i,j} \to E_{0,j} \text{ in } L^1 \text{ for sets of finite perimeter } E_{0,j} \subset B_{g_{euc}}(z_j, 7J_0) \tag{5.10}$$

and we define

$$\bigcup_{j=1}^{J_0} E_{0,j} =: E \Subset X.$$

By (5.8), $|E|_{g_{euc}} = 1$, and by (5.9) and the lower semi-continuity of $\mathcal{F}^*_{x_j}$ with respect to L^1 convergence,

$$\mathcal{F}^*_{x_j}(E_{0,j}) \le \liminf_{i\to\infty} \mathcal{F}^*_{x_j}(E_{i,j}) = \liminf_{i\to\infty} \mathcal{F}^*_{i,j}(E_{i,j}). \tag{5.11}$$

Step 3: Hausdorff convergence of the boundaries. For i sufficiently large and thus v_i sufficiently small, we apply Lemma 5.4; after rescaling the metric and using (5.7), we see that E_i is a local $(2\Lambda, \frac{\epsilon_0}{4})$-minimizer of the energy $\mathcal{F}^*_i$ on X. A standard adaptation of the classical argument (see for instance [Mag12, Theorem 21.11]) shows that the sets E_i enjoy uniform volume density estimates: there exist constants c_0 and r_0 depending only on $g, \mathfrak{m}, \mathfrak{M}, n, \Lambda$ and ϵ_0 such that for any $r < r_0$ and $x \in \partial E_i$,

$$c_0 \le \frac{|E_i \cap B(x,r)|_{g_{euc}}}{\omega_n r^n} \le 1 - c_0\,. \tag{5.12}$$

The density estimates (5.12) let us improve L^1 convergence to Hausdorff convergence of the boundaries:

$$d_{H,g_{euc}}(\partial E_i, \partial E) \to 0. \tag{5.13}$$

Indeed, if (5.13) does not hold, then for some $\mathsf{r} > 0$ and along an unrelabeled subsequence we have either:
(a) a sequence of points $x_i \in \partial E_i$ such that $B_{g_{euc}}(x_i, \mathsf{r}) \cap \partial E = \emptyset$ for all i, or else
(b) a sequence of points $x_i \in \partial E$ such that $B_{g_{euc}}(x_i, \mathsf{r}) \cap \partial E_i = \emptyset$.

In case (a), first suppose $B_{g_{euc}}(x_i, \mathsf{r}) \subset E$ for all i. The lower density estimate in (5.12) implies that

$$|E \Delta E_i|_{g_{euc}} \ge |E \setminus E_i|_{g_{euc}} \ge |B(x_i, \mathsf{r}) \setminus E_i|_{g_{euc}} \ge c_0 \mathsf{r}^n,$$

contradicting the L^1 convergence. If instead $B_{g_{euc}}(x_i, \mathsf{r}) \subset E^c$ for all i, the same argument using the upper density estimate in (5.12) we again reach a contradiction.

In case (b) we argue differently since we do not yet know that E satisfies density estimates. First suppose $B_{g_{euc}}(x_i, \mathsf{r}) \subset E_i^c$ for all i. By compactness, up to a further subsequence, $x_i \to x \in \partial E$, and thus $B_{g_{euc}}(x, \mathsf{r}/2) \subset E_i^c$ for all i sufficiently large. So, $1_{E_i}(x) = 0$ for all $y \in B_{g_{euc}}(x, \mathsf{r}/2)$. Since $1_{E_i} \to 1_E$ in

$L^1(\mathbb{R}^n)$ and thus pointwise a.e., we see that $|E \cap B_{g_{euc}}(x, \mathsf{r}/2)|_{g_{euc}} = 0$, contradicting (2.2). The analogous argument leads to the same contradiction when instead $B_{g_{euc}}(x_i, \mathsf{r}) \subset E_i$ for all i. This proves (5.13).

Step 4: E is a local minimizer of $\mathcal{F}^*$. Next, we claim that E is a volume-constrained $\frac{\epsilon_0}{4}$-local minimizer of the energy $\mathcal{F}^*$ in $\mathbb{R}^n$. To this end, take a set $G \subset \mathbb{R}^n$ with $|G|_{g_{euc}} = 1$ and $E\Delta G \subset U_{g_{euc}}(\partial E, \frac{\epsilon_0}{4})$. Thanks to (5.13), we also have $E\Delta E_i \subset U_{g_{euc}}(\partial E_i, \frac{\epsilon_0}{4})$ for all i sufficiently large. In turn, by the triangle inequality property of the symmetric difference (2.1), we have

$$E_i\Delta G \subset (E_i\Delta E) \cup (E\Delta G) \subset U_{g_{euc}}(\partial E_i, \frac{\epsilon_0}{4}) \cup U_{g_{euc}}(\partial E, \frac{\epsilon_0}{4}) \subset U_{g_{euc}}(\partial E_i, \frac{\epsilon_0}{2}).$$

Letting $G_j := G \cap B(z_j, 8J_0)$, note that $G_j \subset B(z_j, 7J_0)$ for every $j = 1, \dots, J_0$. Hence, we can define $\hat{G}_{i,j} = \phi_{i,j}(G_j) \subset M$ and $\hat{G}_i := \cup_{j=1}^{J_0} \hat{G}_{i,j}$. Letting $h_i = v_i^{-2/n} g$, we see that $\Omega_i \Delta \hat{G}_i \subset U_{h_i}(\partial\Omega_i, \epsilon_0)$ and thanks to (5.7), up to passing to a subsequence, $||\hat{G}_i|_{h_i} - 1| < 1/i$. Thus, applying Lemma 2.1, we obtain sets $\tilde{G}_i \subset M$ with $|\tilde{G}_i|_{h_i} = 1$, i.e. $|\tilde{G}_i|_g = v_i$ and $\mathcal{F}(\tilde{G}_i) \le (1 + 1/i)\mathcal{F}(\hat{G}_i)$. In particular, $\tilde{G}_i$ is an admissible competitor for the minimality of Ω_i, i.e. $\mathcal{F}(\Omega_i) \le \mathcal{F}(\tilde{G}_i) \le (1 + 1/i)\mathcal{F}(\hat{G}_i)$. Pulling the sets and energies back in charts and applying (5.9), we find that

$$\mathcal{F}_i^*(E_i) \le (1 + 1/i)\mathcal{F}_i^*(G) \le (1 + 2/i)\mathcal{F}^*(G).$$

Taking the limit infimum and recalling (5.11), we conclude that $\mathcal{F}^*(E) \le \mathcal{F}^*(G)$, proving the claim.

Step 5: E has one connected component. We claim that E has just one connected component. We will prove this by contradiction. Assume without loss of generality that E has two connected components E_1 and E_2, and that $1 = |E|_{g_{euc}} = |E_1|_{g_{euc}} + |E_2|_{g_{euc}}$.

First we observe that E_1 and E_2 cannot be tangent to each other, as otherwise a simple neck at the tangent point would decrease the energy $\mathcal{F}^*$ and increase the volume.

Denote by $\alpha_i := \mathcal{F}^*(E_i)$ and $v_i := |E_i|$ for $i = 1, 2$. For t small enough, we define

$$E^t := (1 + t)E_1 \cup g(t)E_2 \subset X,$$

where $g(t)$ is defined by the constraint of volume $(1 + t)^n v_1 + g(t)^n v_2 = v_1 + v_2$. Simple algebraic manipulations of this volume constraint shows that

$$g(t) = \Big(1 - ((1 + t)^n - 1)\, \frac{v_1}{v_2}\Big)^{1/n},$$

from which we compute

$$g'(0) = -\frac{v_1}{v_2}, \qquad g''(0) = (1 - n)\frac{v_1}{v_2}\Big(1 + \frac{v_1}{v_2}\Big).$$

We use these values to compute the derivatives of

$$f(t) := \mathcal{F}^*(E^t) = (1 + t)^{n-1}\alpha_1 + g(t)^{n-1}\alpha_2.$$

Via simple calculus and the local minimality of E, we compute

$$0 = f'(0) = (n - 1)\Big(\alpha_1 - \frac{v_1}{v_2}\alpha_2\Big),$$

which is equivalent to $\alpha_1/\alpha_2 = v_1/v_2$. Using this equality in computing $f''(0)$, and again by local minimality of E, we obtain the following contradiction

$$0 \le f''(0) = -\alpha_1(n - 1)\Big(1 + \frac{v_1}{v_2}\Big) < 0.$$

We conclude that E has only one connected component. In particular, E must be contained in only one of the balls $B_{g_{euc}}(z_j, 8J_0)$. We will assume, without loss of generality, that $E \subset B_{g_{euc}}(0, 8J_0)$ and in particular that E is a volume-constrained ϵ_0-local minimizer of the energy $\mathcal{F}^*_{x_1}$ in $\mathbb{R}^n$.

Step 6: E is a Wulff shape for $\mathcal{F}^*_{x_1}$. Through the choice of basis for $T_{x_1}M$ via the normal coordinate map, we have identified $T_{x_1}M$ with $\mathbb{R}^n$ and therefore may identify the volume-1 tangent Wulff shape $K_{x_1} \subset T_{x_1}M$ with a subset of $\mathbb{R}^n$, which we again denote by $K_{x_1} \subset \mathbb{R}^n$, that is the (Euclidean) unit-volume Wulff shape for the translation invariant surface energy $\mathcal{F}^*_{x_1}$ on $\mathbb{R}^n$ defined above.

The set $E \subset B_{g_{euc}}(0, 8J_0)$ is a set of finite perimeter that is a local minimizer of the smooth, uniformly elliptic, translation invariant anisotropic surface energy $\mathcal{F}^*_{x_1}$. According to the Alexandrov-type theorem of the first author, Kolasiński, and Santilli [DRKS20, Corollary 6.8], we deduce that E is the union of finitely many Wulff shapes with equal volume. In the previous step we showed that E is connected, so E comprises exactly one Wulff shape, i.e.

$$E = K_{x_1} + y \qquad \text{for some } y \in \mathbb{R}^n. \tag{5.14}$$

Moreover, keeping in mind that $0 \in K_{x_1}$ and $E \subset B_{g_{euc}}(0, 8J_0)$, we see that $|y|_{g_{euc}} \le 8J_0$.

Step 7: Contradiction to the initial claim. Together, (5.10), (5.13) and (5.14) show that for i sufficiently large,

$$d_{H,g_{euc}}\big(\partial E_i, \partial K_{x_1} + y\big) < \epsilon_0/\beta \qquad \text{and} \qquad \big|E_i \Delta (K_{x_1} + y)\big|_{g_{euc}} < \epsilon_0/\beta. \tag{5.15}$$

Mapping these sets onto M by $\phi_{i,1}$, (5.15) implies that

$$d_{H,g}\big(\partial\Omega_i, \exp_{x_1}\big(\partial(v_i^{1/n} K_{x_1}) + y_i\big)\big) < \frac{\epsilon_0\, v_i^{1/n}}{\beta}, \qquad \text{and} \qquad \big|\Omega_i \Delta \exp_{x_1}(v_i^{1/n} K_{x_1} + y_i)\big|_g < \frac{\epsilon_0 v_i}{\beta}.$$

By Step 1, $y_i = v_i^{1/n} y + \psi_1^{-1}(x_{i,1}) \in B_{g_{x_1}}(0, r_0)$. This contradicts (5.4) and completes the proof. $\square$

5.3. **Recentering.** We now improve Theorem 5.5, simply recentering our parametrization to correct the translation y, by means of Proposition 3.1, to obtain the following:

Theorem 5.6. *Fix a closed Riemannian n-manifold (M, g) and an anisotropic surface energy $\mathcal{F}$ with integrand F. For every $\kappa > 0$, $\epsilon_0 > 0$, and $\beta > 0$, there exists $v_0 = v_0(g, F, \kappa, \epsilon_0, \beta) > 0$ such that the following holds. Let Ω_v be a volume-constrained ϵ_0-local minimizer of volume $v < v_0$ with $\mathcal{F}(\Omega_v) \le \kappa v^{(n-1)/n}$. There is a point $x \in M$ such that*

$$d_{H,g}\Big(\partial\Omega_v, \exp_x(\partial(v^{1/n} K_x)\Big) < \frac{\epsilon_0 v^{1/n}}{\beta} \qquad \textit{and} \qquad \Big|\Omega_v \Delta \exp_x(v^{1/n} K_x)\Big|_g < \frac{\epsilon_0 v}{\beta}\,. \tag{5.16}$$

Proof of Theorem 5.6. We apply Theorem 5.5 with $\kappa = \kappa$, $\epsilon_0 = \epsilon_0$, $\beta = 2\beta$ and $\rho = \frac{\epsilon_0}{2C\beta}$. Notice that v_0 will now depend only on $g, F, \kappa, \epsilon_0, \beta$, as ρ depends just on ϵ_0 and β. We deduce the validity of (5.3). We now apply Proposition 3.1 choosing $x_0 = x$, $x_1 = \exp_x(y)$, $z_1 = y$, $\rho = \rho$, $r = v^{1/n}$, to deduce (5.16). $\square$

5.4. **Conclusion of the proof.** Finally, we can conclude the proof of Theorem 5.1. We need only to show that a volume-constrained ϵ_0-local minimizer is connected.

Proof of Theorem 5.1. Choose $\beta = \beta_0$ and let v_0 be as in Theorem 5.6. Since β_0 depends just on n, κ, F, ϵ_0, we deduce that v_0 depends only on $n, g, \kappa, F, \epsilon_0$. Applying Theorem 5.6, we deduce the validity of (5.1). We are now left to prove that Ω_v has one connected component. To prove this, we first claim that the volume of every connected component U of Ω_v satisfies the following lower bound:

$$|U|_g \ge \min\left\{ \left(\frac{\mathfrak{m}}{2\Lambda} n \omega_n^{1/n}\right)^n, \frac{\epsilon_0^n\, \omega_n}{2^{n+1}} \right\} v. \tag{5.17}$$

Otherwise, denote by U a connected component such that

$$|U|_g < \min\left\{ \left(\frac{\mathfrak{m}}{2\Lambda} n \omega_n^{1/n}\right)^n, \frac{\epsilon_0^n\, \omega_n}{2^{n+1}} \right\} v. \tag{5.18}$$

We wish to apply the local quasi-minimality (without volume constraint) from Lemma 5.4 with the competitor $\Omega_v \setminus U$. To this end, we argue along the lines of Lemma 4.6 to see that $U \subset U_g(\partial U, \frac{\epsilon_0}{2} v^{1/n})$: if

$x \in U \setminus U_g(\partial U, \frac{\epsilon_0}{2} v^{1/n})$, then $B_g(x, \frac{\epsilon_0}{2} v^{1/n}) \subset U$ and $|U|_g \geq {\epsilon_0}^n \omega_n v / 2^{n+1}$ (provided v_0 is sufficiently small in terms of g and n), contradicting (5.18). So, by Lemma 5.4 we have that

$$\mathcal{F}(\Omega_v \setminus U) + \Lambda\, v^{-1/n} |\Omega_v \Delta (\Omega_v \setminus U)|_g \geq \mathcal{F}(\Omega_v) = \mathcal{F}(U) + \mathcal{F}(\Omega_v \setminus U) \overset{(2.8)}{\geq} \Big(\frac{\mathfrak{m}}{2} n \omega_n^{1/n}\Big) |U|_g^{(n-1)/n} + \mathcal{F}(\Omega_v \setminus U),$$

from which we deduce the following contradiction:

$$\frac{\mathfrak{m}}{2} n \omega_n^{1/n} \overset{(5.18)}{>} \Lambda\, v^{-1/n} |U|_g^{1/n} \geq \frac{\mathfrak{m}}{2} n \omega_n^{1/n}.$$

Since (5.18) leads to contradiction, we deduce the validity of (5.17).

We also observe that, by (5.1), for every connected component U of Ω_v:

$$\partial U \subset \partial \Omega_v \subset U_g(\exp_x(\partial(v^{1/n} K_x) + y), \epsilon_0 v^{1/n} / \beta_0).$$

We deduce that either

(Case 1) $\exp_x(v^{1/n} K_x + y) \setminus U_g(\exp_x(\partial(v^{1/n} K_x) + y, \frac{\epsilon_0 v^{1/n}}{\beta_0}) \subset U$, or

(Case 2) $U \subset U_g(\exp_x(\partial(v^{1/n} K_x) + y), \frac{\epsilon_0 v^{1/n}}{\beta_0})$.

Given the L^1 estimate in (5.3), there can be only one connected component satisfying (Case 1). Moreover, (5.17) implies that no connected component can satisfy (Case 2), because (Case 2) provides the following volume upper bound for U:

$$\min\left\{ \Big(\frac{\mathfrak{m}}{2\Lambda} n \omega_n^{1/n}\Big)^n, \frac{\epsilon_0^n\, \omega_n}{2^{n+1}} \right\} v \leq |U|_g \leq 4 \frac{\epsilon_0 v^{1/n}}{\beta_0} P(v^{1/n} K_x + y) \overset{(2.7)}{\leq} 4 \frac{\epsilon_0 v^{1/n}}{\beta_0} \mathcal{C} v^{(n-1)/n} = \frac{4 \mathcal{C} \epsilon_0}{\beta_0} v$$

which contradicts the definition of

$$\beta_0(n, \kappa, F, \epsilon_0) := \frac{8 \mathcal{C} \epsilon_0}{\min\left\{ \Big(\frac{\mathfrak{m}}{2\Lambda} n \omega_n^{1/n}\Big)^n, \frac{\epsilon_0^n\, \omega_n}{2^{n+1}} \right\}}.$$

We conclude that Ω_v has just one connected component. □

6. Quantitative closeness to a Wulff shape

In this section we complete the proof of Theorem 1.1. To begin, in Corollary 6.1, we prove a quantitative version of Theorem 5.1 through an application of Figalli-Maggi-Pratelli's quantitative Wulff inequality [FMP10]. This application originates with [FM11] in the context of *global* minimizers of a related anisotropic variational problem on Euclidean space. A fundamental difference in the present setting of local minimizers is that it was essential to first prove the qualitative Theorem 5.1 in order to use a (projected) tangent Wulff shape as a competitor for local minimality.

Corollary 6.1. *Fix a Riemannian n-manifold (M, g) and an anisotropic surface energy $\mathcal{F}$. For every $\kappa > 0$ and $\epsilon_0 > 0$, there exists $v_0 = v_0(g, F, \kappa, \epsilon_0)$ and $C(g, F, \kappa, \epsilon_0)$ such that the following holds. Let Ω_v be a volume-constrained ϵ_0-local minimizer Ω_v of volume $v < v_0$ with $\mathcal{F}(\Omega_v) \leq \kappa v^{(n-1)/n}$. Then Ω_v has one connected component and there is a point $x_0 \in M$ such that*

$$\frac{d_{H,g}\Big(\partial \Omega_v, \exp_{x_0}(\partial v^{1/n} K_{x_0})\Big)}{v^{1/n}} < C v^{1/2n^2} \qquad \textit{and} \qquad \frac{\Big|\Omega_v \Delta \exp_{x_0}(v^{1/n} K_{x_0})\Big|_g}{v} < C v^{1/2n}. \tag{6.1}$$

Proof. Let v_0, β_0 and $x_0 \in M$ be as in Theorem 5.1, and let $\psi = \exp_{x_0}$. To lighten notation, we let $K = K_{x_0}$ and $\bar{\mathcal{F}} = \bar{\mathcal{F}}_{x_0}$. By Remark 5.3 we may define $G_v = \psi^{-1}(\Omega_v) \subset B_{g_{x_0}}(0, R v^{1/n})$ where $R = R(n, g, F) > 0$. On $B_{g_{x_0}}(0, 10R)$ we have

$$(1 - c v^{2/n}) g_{x_0} \leq \psi^* g \leq (1 + c v^{2/n}) g_{x_0} \tag{6.2}$$

for a constant $c = c(g)$, and so (6.2)

$$\big||G_v|_{g_{x_0}} - v\big| \leq cv^{1+2/n} \qquad \text{and} \qquad \big||\psi(v^{1/n}K)|_g - v\big| \leq cv^{1+2/n}\,. \tag{6.3}$$

So, just as in the proof of Lemma 2.1 we can choose a dilation factor $\lambda > 0$ with $|\lambda - 1| \leq cv^{2/n}$ such that the set $E := \psi(\lambda v^{1/n}K)$ has $|E|_g = |\Omega_v|_g$. Moreover, up to further decreasing v_0 depending on g, F, n and applying (2.1) and Theorem 5.1, we have $\Omega_v \Delta E \subset U_g(\partial\Omega_v, \epsilon_0 v^{1/n})$. So, E is an admissible competitor for the local minimality of Ω_v and thus $\mathcal{F}(\Omega_v) \leq \mathcal{F}(E)$. On the other hand, thanks to (6.2) and (2.4),

$$\begin{aligned}\mathcal{F}(E) &\leq \big(1 + cv^{1/n}\big)\bar{\mathcal{F}}(\lambda v^{1/n}K)\\ &= \big(1 + cv^{1/n}\big)\lambda^{n-1}v^{(n-1)/n}\bar{\mathcal{F}}(K) \leq \big(1 + cv^{1/n}\big)v^{(n-1)/n}\bar{\mathcal{F}}(K),\end{aligned}$$

and $\mathcal{F}(\Omega_v) \geq (1 - cv^{1/n})\bar{\mathcal{F}}(G_v)$ for $c = c(F, g)$. Together these yield $\bar{\mathcal{F}}(G_v) \leq (1 + cv^{1/n})v^{(n-1)/n}\bar{\mathcal{F}}(K)$. In particular, additionally using (6.3), we estimates the scale-invariant deficit in the Wulff inequality (2.6):

$$\delta(G_v) = \frac{\bar{\mathcal{F}}(G_v)}{|G_v|_{g_{x_0}}^{\frac{n-1}{n}}\bar{\mathcal{F}}(K)} - 1 \leq Cv^{1/n}$$

for $C = C(n, g, F)$. By the quantitative Wulff inequality [FMP10], there exists $y \in \mathbb{R}^n$ such that

$$\frac{|G_v \Delta (y + v^{1/n}K)|_{g_{x_0}}^2}{v^2} \leq C\delta(G_v) \leq Cv^{1/n}.$$

Moreover, $|y| \leq Cv^{1/n}$ (5.1). Using the scale-invariantly uniform density estimates for Ω_v as in Step 3 of Theorem 5.5, we obtain the estimate

$$d_{H,g}(\partial\Omega_v, \psi(\partial(y + v^{1/n}K))^n \leq C|\Omega_v \Delta \psi(\partial(y + v^{1/n}K))|_g < Cv^{1+1/2n}.$$

It remains to eliminate that translation y. To this end we apply Proposition 3.1 choosing $x_0 = x_0$, $x_1 = \exp_{x_0}(y)$, $z_1 = y$, $\rho = Cv^{1/n}$, $r = v^{1/n}$. Since $\rho r \leq Cv^{1/n}v^{1/n} \leq Cv^{1/n}v^{1/2n^2}$, we obtain:

$$\frac{|\Omega_v \Delta \exp_{x_1}(v^{1/n}K_{x_1})|_g^2}{v^2} \leq Cv^{1/n}, \qquad \text{and} \qquad \frac{d_{H,g}(\partial\Omega_v, \exp_{x_1}(\partial(v^{1/n}K_{x_1}))}{v^{1/n}} < Cv^{1/2n^2}.$$

Up to relabelling the point x_1, this is exactly the desired claim (6.1). ☐

Finally, we conclude the proof of Theorem 1.1.

Proof of Theorem 1.1. Let v_0 and x_0 be as in Corollary 6.1. The quantitative estimate (6.1) and the assumed regularity of F can be used as in the proof of [FM11, Theorem 2] to show that Ω_v is of class $C^{2,\alpha}$ and to obtain the following almost anisotropic umbilicality estimate on the anisotropic second fundamental form of $\partial \exp_{x_0}^{-1}(\Omega_v)$:

$$\|D^2F_{x_0}(\nu_{\exp_{x_0}^{-1}(\Omega_v)})D\nu_{\exp_{x_0}^{-1}(\Omega_v)} - \mathrm{Id}\|_{C^0(\partial\exp_{x_0}^{-1}(\Omega_v))} \leq C(g, F, \kappa, \epsilon_0, \alpha)v^{\frac{2\alpha}{n+2\alpha}}. \tag{6.4}$$

Anisotropic almost umbilical surfaces enjoy higher order quantitative closeness to the Wulff shapes, see for instance [DRG19, DRG21].

Since the proof of (6.4) is obtained repeating the arguments that are laid out in detail in [FM11], we simply outline the steps and highlight the differences:

- The first step—analogous to [FM11, Theorem 8]—is to show that $\exp_{x_0}^{-1}(\partial\Omega_v) \subset T_{x_0}M$ is locally the graph of a $C^{1,\alpha}$ function over an affine hyperplane in $T_{x_0}M$. The idea is to show the hypotheses of the ϵ-regularity theorem ([Alm68, Bom82, SSA77, SS82, DS02]) are satisfied at each point on $\exp_{x_0}^{-1}(\partial\Omega_v)$ and at a small enough scale by constructing a competitor from the projection of $\exp_{x_0}^{-1}(\partial\Omega_v)$ locally onto a disk.

- The proof of [FM11, Theorem 8] goes through in a nearly identical fashion in the present context with the following two substituted ingredients: use Corollary 6.1 in place of [FM11, Corollary 1] to show that Ω_v is uniformly close to a Wulff shape, and use Lemma 5.4 and (2.8) in place of [FM11, Lemma 9] to obtain the non-volume-constrained quasi-minimality property akin to [FM11, Eqn. (C.162)] (which is applied only to local competitors as in Lemma 5.4).
- The next step is to used Schauder estimates to improve the local estimates to $C^{2,\alpha}$ estimates for the functions locally parametrizing $\exp_{x_0}^{-1}(\partial\Omega_v)$, as in [FM11, Theorem 12]. Here we use the assumption that F is $C^{2,\alpha}$ in x and ν.
- Finally (6.4) follows from interpolating between the Hausdorff estimates of Corollary 6.1 and the $C^{2,\alpha}$ estimates arguing just as in [FM11, Theorem 13].

We are left to prove that Ω_v is geodesically convex. Let ψ be the normal coordinate map at x_0 and observe that by (6.1) there exists $\eta(g, F, \kappa, \epsilon_0) > 0$ such that

$$B_{g_{euc}}(0, \eta v^{\frac{1}{n}}) \Subset E_v := \psi^{-1}(\Omega_v) \subset \mathbb{R}^n \tag{6.5}$$

and that by (6.4) E_v is uniformly convex. In particular, we have the following lower bound on the smallest eigenvalue λ_1 of the second fundamental form of E_v:

$$\lambda_1 \geq C(F)v^{-\frac{1}{n}}. \tag{6.6}$$

This implies that, given two points $a, b \in \partial E_v$, any curve connecting a, b that is contained in $\mathbb{R}^n \setminus E_v$ has length at least $|b-a| + C(F)v^{\frac{1}{n}}\frac{|b-a|^2}{2}$.

Assume by way of contradiction that Ω_v is not geodesically convex and so there exists a minimizing geodesic $\tilde{\gamma} : [0, \ell] \to M$ parametrized by arclength with end points $\tilde{a}, \tilde{b} \in \partial\Omega_v$ and such that $\tilde{\gamma}(0, \ell) \subset M \setminus \overline{\Omega}_v$. Let $a := \psi^{-1}(\tilde{a})$, $b := \psi^{-1}(\tilde{b})$, and $\gamma = \psi^{-1} \circ \tilde{\gamma} : [0, \ell] \to \mathbb{R}^n$.

If $|a-b| \geq \eta v^{1/n}/4$, then the observation above, together with (6.5), shows that the image through ψ of the segment $[a, b]$ has smaller length than any curve contained in $M \setminus \Omega_v$ joining $\tilde{a}$ with $\tilde{b}$ provided v_0 (and hence v) is chosen sufficiently small depending only on g, F, ϵ_0, and κ.

Thus we must have $|a-b| < \eta v^{1/n}/4$. Let $z = (a+b)/2$ be the midpoint and for $t \in [0, 1]$ set $a_t := a - tz$, $b_t := b - tz$ and $\Gamma_t = Im(\gamma) - tz$. We have $\Gamma_0 \subset B_{g_{euc}}(z, \eta v^{1/n})$ provided again v_0 is chosen sufficiently small depending only on g, F, ϵ_0, and κ, otherwise the straight-line competitor violates the minimality of γ. So, $\Gamma_1 \subset B_{g_{euc}}(z, \eta v^{1/n}) \Subset E_v$. Furthermore, by strict convexity, $a_t, b_t \in E_v$ for all $t \in (0, 1]$. So, $s_0 := \inf\{s \in [0, 1] : \Gamma_t \subset E_v\}$ lies in $(0, 1)$ and there exists $t_0 \in (0, \ell)$ such that $\gamma(t_0) - s_0 z \in \Gamma_{s_0} \cap \partial E_v$. This implies that at corresponding point $\gamma(t_0)$, the curvature of γ with respect to g_{euc} is at least $C(F)v^{-1/n}$ by (6.6). On the other hand, estimating the metric coefficients g_{ij} and Christoffel symbols Γ_{ij}^k, we see that for v_0 sufficiently small this contradicts the fact that γ satisfies the geodesic equation $\frac{d^2}{dt^2}\gamma^k + \Gamma_{ij}^k \frac{d}{dt}\gamma^i \frac{d}{dt}\gamma^j = 0$. Hence Ω_v is geodesically convex. ☐

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