# LIMIT MODELS IN CLASSES WITH AMALGAMATION

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Abstract. In abstract elementary classes limit models are sometimes the appropriate substitute for saturated models. For Galois-stable abstract elementary classes which satisfy the amalgamation property, we prove under the assumption that there is a mildly behaved dependence relation, that for any model $M$, any two limit models over $M$ of the same cardinality are isomorphic. This is useful in dealing with Shelah's categoricity conjecture.

## 1. introduction

In 1977, Shelah, building on the work of Jónsson and Fraïssé, identified a non-elementary context in which a model theoretic analysis could be carried out. Shelah began to study classes of models equipped with a partial order which exhibit many of the properties that the models of a first order theory have with respect to the elementary submodel relation. Such classes were named abstract elementary classes. They are broad enough to generalize $L_{\omega_1,\omega}(\mathbf{Q})$. We reproduce the definition here.

**Definition 1.1.** Let $\mathcal{K}$ be a class of structures all in the same similarity type $\mathrm{L}(\mathcal{K})$, and let $\prec_{\mathcal{K}}$ be a partial order on $\mathcal{K}$. The ordered pair $\langle \mathcal{K}, \prec_{\mathcal{K}}\rangle$ is an *abstract elementary class, AEC for short* iff

- A0 (Closure under isomorphism)
  - (a) For every $M \in \mathcal{K}$ and every $\mathrm{L}(\mathcal{K})$-structure $N$ if $M \cong N$ then $N \in \mathcal{K}$.
  - (b) Let $N_1, N_2 \in \mathcal{K}$ and $M_1, M_2 \in \mathcal{K}$ such that there exist $f_l : N_l \cong M_l$ (for $l = 1, 2$) satisfying $f_1 \subseteq f_2$ then $N_1 \prec_{\mathcal{K}} N_2$ implies that $M_1 \prec_{\mathcal{K}} M_2$.
- A1 For all $M, N \in \mathcal{K}$ if $M \prec_{\mathcal{K}} N$ then $M \subseteq N$.
- A2 Let $M, N, M^*$ be $\mathrm{L}(\mathcal{K})$-structures. If $M \subseteq N$, $M \prec_{\mathcal{K}} M^*$ and $N \prec_{\mathcal{K}} M^*$ then $M \prec_{\mathcal{K}} N$.
- A3 (Downward Löwenheim-Skolem) There exists a cardinal $\mathrm{LS}(\mathcal{K}) \geq \aleph_0 + |\,\mathrm{L}(\mathcal{K})|$ such that for every

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$M \in \mathcal{K}$ and for every $A \subseteq |M|$ there exists $N \in \mathcal{K}$ such that $N \prec_{\mathcal{K}} M$, $|N| \supseteq A$ and $\|N\| \leq |A| + \mathrm{LS}(\mathcal{K})$.

A4 (Tarski-Vaught Chain)
(a) For every regular cardinal $\mu$ and every $N \in \mathcal{K}$ if $\{M_i \prec_{\mathcal{K}} N \ : \ i < \mu\} \subseteq \mathcal{K}$ is $\prec_{\mathcal{K}}$-increasing (i.e. $i < j \Longrightarrow M_i \prec_{\mathcal{K}} M_j$) then $\bigcup_{i<\mu} M_i \in \mathcal{K}$ and $\bigcup_{i<\mu} M_i \prec_{\mathcal{K}} N$.
(b) For every regular $\mu$, if $\{M_i \ : \ i < \mu\} \subseteq \mathcal{K}$ is $\prec_{\mathcal{K}}$-increasing then $\bigcup_{i<\mu} M_i \in \mathcal{K}$ and $M_0 \prec_{\mathcal{K}} \bigcup_{i<\mu} M_i$.

For $M$ and $N \in \mathcal{K}$ a monomorphism $f : M \to N$ is called an $\mathcal{K}$-*embedding* iff $f[M] \prec_{\mathcal{K}} N$. Thus, $M \prec_{\mathcal{K}} N$ is equivalent to "$\mathrm{id}_M$ is a $\mathcal{K}$-embedding from $M$ into $N$".

For $M_0 \prec_{\mathcal{K}} M_1$ and $N \in \mathcal{K}$, the formula $f : M_1 \underset{M_0}{\to} N$ stands for $f$ is a $\mathcal{K}$-embedding such that $f \restriction M_0 = \mathrm{id}_{M_0}$.

For a class $\mathcal{K}$ and a cardinal $\mu \geq \mathrm{LS}(\mathcal{K})$ let

$$\mathcal{K}_\mu := \{M \in \mathcal{K} \ : \ \|M\| = \mu\}.$$

In reality, abstract elementary classes were not as approachable as one would expect and much work in non-elementary model theory takes place in contexts which additionally satisfy the amalgamation property:

**Definition 1.2.** Let $\mu \geq \mathrm{LS}(\mathcal{K})$. We say that $\mathcal{K}$ has the *$\mu$-amalgamation property* (*$\mu$-AP*) iff for any $M_\ell \in \mathcal{K}_\mu$ (for $\ell \in \{0, 1, 2\}$) such that $M_0 \prec_{\mathcal{K}} M_1$ and $M_0 \prec_{\mathcal{K}} M_2$ there are $N \in \mathcal{K}_\mu$ and $\mathcal{K}$-embeddings $f_\ell : M_\ell \to N$ such that $f_\ell \restriction M_0 = \mathrm{id}_{M_0}$ for $\ell = 1, 2$.

A model $M_0 \in \mathcal{K}_\mu$ satisfying the above requirement is called an *amalgamation base.*

We say that $\mathcal{K}$ has the *amalgamation property* (*AP*) iff any triple of models from $\mathcal{K}_{\geq \mathrm{LS}(\mathcal{K})}$ can be amalgamated.

**Remark 1.3.**
(1) Using the isomorphism axioms we can see that $\mathcal{K}$ has the $\lambda$-AP iff for any $M_\ell \in \mathcal{K}_\lambda$ (for $\ell \in \{0, 1, 2\}$) such that $M_0 \prec_{\mathcal{K}} M_\ell$ (for $\ell \in \{1, 2\}$) there are $N \in \mathcal{K}_\lambda$ and $f : M_1 \underset{M_0}{\to} N$ such that $N \succ_{\mathcal{K}} M_2$.
(2) Using the axioms of AECs it is not difficult to prove that if $\mathcal{K}$ has the $\lambda$-AP for every $\lambda \geq \mathrm{LS}(\mathcal{K})$ then $\mathcal{K}$ has the AP.

A stronger version of the amalgamation property is

**Definition 1.4.** Let $\mathcal{K}$ be an abstract class. $\mathcal{K}$ has the $\lambda$-*Disjoint Amalgamation Property* iff for every $M_\ell \in \mathcal{K}_\lambda$ (for $\ell = 0, 1, 2$) such that $M_0 \prec_{\mathcal{K}} M_\ell$ (for $\ell = 1, 2$) there are $N \in \mathcal{K}_\lambda$ which is a $\mathcal{K}$-extension of $M_2$ and a $\mathcal{K}$-embedding $f : M_1 \underset{M_0}{\to} N$ such that $f[M_1] \cap M_2 = M_0$.

We say that a class has the *disjoint amalgamation property* iff it has the $\lambda$-disjoint amalgamation property for every $\lambda \geq \mathrm{LS}(\mathcal{K}) + \aleph_0$. We write DAP for short.

An application of the compactness theorem establishes:

**Fact 1.5.** *If $T$ is a complete first-order theory then $\langle \mathrm{Mod}(T), \prec\rangle$ has the $\lambda$-DAP for all $\lambda \geq |\,\mathrm{L}(T)| + \aleph_0$*

The roots of the following fact can be traced back to Jónsson's 1960 paper [Jo], the present formulation is from [Gr1]:

**Fact 1.6.** *Let $\langle \mathcal{K}, \prec_{\mathcal{K}}\rangle$ be an AEC and $\lambda \geq \kappa > \mathrm{LS}(\mathcal{K})$ such that $K_{<\lambda}$ has the AP. Suppose $M \in \mathcal{K}$.*

*If $\lambda^{<\kappa} = \lambda \geq \|M\|$ then there exists $N \succ M$ of cardinality $\lambda$ which is $\kappa$-model-homogeneous.*

Thus if an AEC $\mathcal{K}$ has the amalgamation property then like in first-order stability theory we may assume that there is a large model-homogeneous $\mathfrak{C} \in \mathcal{K}$, that acts like a monster model.

We will refer to the model $\mathfrak{C}$ as the *monster model*. All models considered will be of size less than $|\mathfrak{C}|$, and we will find realizations of types we will construct inside this monster model.

From now on, we assume that the monster model $\mathfrak{C}$ has been fixed.

The notion of type as a set of formulas, does not seem to be as nicely behaved as in first-order logic. Thus we need a replacement which was introduced by Shelah in [Sh 394], in order to avoid confusion with the classical notion following [Gr2] we call this new notion *Galois-type*.

Since in this paper we deal only with AECs with the AP property, the notion of Galois type has a simpler definition than in the general case.

**Definition 1.7** (Galois types)**.** Suppose that $\mathcal{K}$ has the AP.

(1) Given $M \in \mathcal{K}$ consider the action of $\mathrm{Aut}_M(\mathfrak{C})$ on $\mathfrak{C}$, for an element $a \in |\mathfrak{C}|$ let $\text{ga-tp}(a/M)$ denote the *Galois type of $a$ over $M$* which is defined as the orbit of $a$ under $\mathrm{Aut}_M(\mathfrak{C})$.

(2) For $M \in \mathcal{K}$, we let
$$\text{ga-S}(M) = \{\ \text{ga-tp}(a/M)\ :\ a \in |\mathfrak{C}|\}.$$

(3) $\mathcal{K}$ is $\lambda$-*Galois-stable* iff
$$N \in \mathcal{K}_\lambda \implies |\,\text{ga-S}(N)| \leq \lambda.$$

(4) Given $p \in \text{ga-S}(M)$ and $N \in \mathcal{K}$, we say that $p$ is *realized* by $a \in N$, iff $\text{ga-tp}(a/M) = p$.

Like in the first-order case we will write $a \models p$ when $a$ is a realization of $p$.

For a more detailed discussion of Galois types, extensions and restrictions, equivalent and more general formulations, the reader may consult [Gr2].

The main concept of this paper is Shelah's *limit model* which we will show serves as a substitute for saturation. Why do we need substitutes for saturation? When stability theory has been ported to contexts more general than first order logic, many situations have appeared when saturated models do not fulfill the main roles they play in elementary classes, for the transfer of categoricity (under reasonable stability conditions): existence and uniqueness. For example, homogeneous abstract elementary classes (see, for example, [BuLe]) study classes of models omitting given sets of types. Many other classes have this behavior. Thus, looking for notions that may appropriately substitute these roles of saturated models is crucial.

We first need to define universal extensions as they are the building blocks of limit models:

**Definition 1.8.** (1) Let $\kappa$ be a cardinal $\geq \mathrm{LS}(\mathcal{K})$. We say $M^*$ is *$\kappa$-universal over $N$* iff for every $N' \in \mathcal{K}_\kappa$ with $N \prec_{\mathcal{K}} N'$ there exists a $\mathcal{K}$-embedding $g : N' \underset{N}{\to} M^*$ such that:

$$\begin{tikzcd} N' \arrow[dr, dotted, "g"] & \\ N \arrow[u, "\mathrm{id}"] \arrow[r, "\mathrm{id}"'] & M^* \end{tikzcd}$$

(2) We say $M^*$ is *universal over $N$* or $M^*$ is *a universal extension of $N$* iff $M^*$ is $\|N\|$-universal over $N$.

**Theorem 1.9** (Existence)**.** *Let $\mathcal{K}$ be an AEC without maximal models and suppose it is Galois-stable in $\mu$. If $\mathcal{K}$ has the amalgamation property then for every $N \in \mathcal{K}_\mu$ there exists $M^* \gneq_{\mathcal{K}} N$, universal over $N$ of cardinality $\mu$.*

This theorem was stated without proof in [Sh 600], for a proof see [GrVa1] or [Gr1].

In [KoSh] and in [Sh 576] Shelah introduced a substitute for saturated models under the name of *$(\mu, \alpha)$-saturated models*. As Shelah himself in later papers e.g. [ShVi] and [Sh 600][1] refers to these models as *limit* models we use the more recent terminology.

**Definition 1.10.** [Limit models] Let $\mu \geq \mathrm{LS}(\mathcal{K})$ and $\alpha \leq \mu^+$ a limit ordinal and $N \in \mathcal{K}_\mu$. We say that $M$ is *$(\mu, \alpha)$-limit over $N$* iff there exists an increasing and continuous chain $\{M_i \mid i < \alpha\} \subseteq \mathcal{K}_\mu$ such that $M_0 = N$, $M = \bigcup_{i<\alpha} M_i$ and $M_{i+1}$ is universal over $M_i$ for all $i < \alpha$.

From Theorem 1.9 we get that for $\alpha \leq \mu^+$ there always exists a $(\mu, \alpha)$-limit model provided $\mathcal{K}$ has the AP, has no maximal models and is $\mu$-Galois-stable.

The following theorem partially clarifies the analogy with saturated models:

**Theorem 1.11.** *Let $T$ be a complete first-order theory and let $\mathcal{K}$ be the elementary class* $\mathrm{Mod}(T)$ *with the usual notion of elementary submodels.*

(1) *Suppose $T$ is superstable. If $M$ is $(\mu, \delta)$-limit model for $\delta$ a limit ordinal, then $M$ is saturated.*

[1]Baldwin informed us that in a version from Spring 2003 of that paper Shelah changed this name to *Brimmed*.

(2) *Suppose $T$ is stable. If $M$ is $(\mu,\delta)$-limit model for $\delta$ a limit ordinal with $\delta \geq |T|^+$, then $M$ is saturated.*

Thus under mild model-theoretic assumptions in elementary classes limit models are unique. This raises the following natural question for the situation in AECs:

**Question 1.12** (Uniqueness problem). *Let $\mathcal{K}$ be an AEC, $\mu \geq \mathrm{LS}(\mathcal{K})$, $\sigma_1, \sigma_2 < \mu^+$, $M \in \mathcal{K}_\mu$ and suppose that $N_\ell$ $(\mu, \sigma_\ell)$-limit models over $M$. What "reasonable" assumptions on $\mathcal{K}$ will imply that $\exists f : N_1 \cong_M N_2$?*

Using back and forth arguments one can show that when $\sigma_1 = \sigma_2$ then we get uniqueness without any assumptions on $\mathcal{K}$, in fact $\mathrm{cf}\,\sigma_1 = \mathrm{cf}\,\sigma_2$ suffices. More precisely:

**Fact 1.13.** *Let $\mu \geq \mathrm{LS}(\mathcal{K})$ and $\sigma < \mu^+$. If $M_1$ and $M_2$ are $(\mu,\sigma)$-limits over $M$, then there exists an isomorphism $g : M_1 \to M_2$ such that $g \restriction M = \mathrm{id}_M$. Moreover if $M_1$ is a $(\mu,\sigma)$-limit over $M_0$; $N_1$ is a $(\mu,\sigma)$-limit over $N_0$ and $g : M_0 \cong N_0$, then there exists a $\prec_{\mathcal{K}}$-mapping, $\hat{g}$, extending $g$ such that $\hat{g} : M_1 \cong N_1$.*

**Fact 1.14.** *Let $\mu$ be a cardinal and $\sigma$ a limit ordinal with $\sigma < \mu^+$. If $M$ is a $(\mu,\sigma)$-limit model, then $M$ is a $(\mu, \mathrm{cf}(\sigma))$-limit model.*

Thus Question 1.12 is meaningful for the case where $\mathrm{cf}\,\sigma_1 \neq \mathrm{cf}\,\sigma_2$. The main result of this paper is:

**Theorem 1.15** (Main Theorem). *Let $\mathcal{K}$ be an AEC and $\mu > \mathrm{LS}(\mathcal{K})$. Suppose $\mathcal{K}$ is satisfying the $\mu$-DAP. If $\mathcal{K}$ is $\mu$-Galois-stable, does not have long splitting chains, and satisfies locality of splitting*[2]*, then any two $(\mu,\sigma_\ell)$-limits over $M$ for ($\ell \in \{1,2\}$) are isomorphic over $M$.*

Notice that $\mu$-DAP is occasionally a property we get for free if the class $\mathcal{K}$ has an axiomatization in a logic with sufficient compactness; essentially, Robinson's consistency property is enough. In other occasions DAP is a known corollary of categoricity, even when AP is not assumed (see [ShVi] and [Va]).

Approximations to Theorem 1.15 and its relatives were considered by several authors:

Kolman and Shelah in [KoSh] prove the uniqueness of limit models in $\lambda$-categorical AECs that are axiomatized by a $L_{\kappa,\omega}$-sentence where $\lambda > \mu$ and $\kappa$ is a measurable cardinal. Both the measurability of $\kappa$ and the categoricity are used integrally in their proof.

Shelah in Theorem 6.3 of [Sh 394] gets uniqueness of limit models for classes with the amalgamation property under little more than categoricity in some $\lambda > \mu > \mathrm{LS}(\mathcal{K})$ together with existence of arbitrarily large models. An analysis of Ehrenfeucht-Mostowski models possible only when there are categoricity assumptions, gives uniqueness. Our assumptions in this paper

[2]See Assumption 2.4 for the precise description of long splitting chains and locality

may all be derived from the assumptions in [Sh 394], but our arguments do not require the Ehrenfeucht-Mostowski machinery.

Shelah in [Sh 576] (see Claim 7.8) proved a special case of the uniqueness of limit models under the assumption of $\mu$-AP, categoricity in $\mu$ and in $\mu^+$ as well as assuming $K_{\mu^{++}} \neq \emptyset$. In that paper Shelah needs to produce *reduced types* and use some of their special properties.

[ShVi] attempted to prove a uniqueness theorem without assuming any form of amalgamation; however, they assumed that $\mathcal{K}$ is categorical in some $\lambda > \mathrm{Hanf}(\mathcal{K}) + \mu$ and that every model in $\mathcal{K}$ has a proper extension. VanDieren in [Va] managed to prove the above uniqueness statement under the assumptions of [ShVi] together with the additional assumption that $\mathcal{K}^{am} := \{M \in \mathcal{K}_\mu \mid M \text{ is an amalgamation base}\}$ is closed under unions of increasing $\prec_{\mathcal{K}}$ chains.

The referee requested us to clarify the connection between our main Theorem and [Sh 600]. In [Sh 600] the basic context is that of a *good frame*, which is an axiomatization of the notion of superstability. Its full definition is more than a page long. Shelah's assumptions on the AEC include, among other things, the amalgamation property, the existence of a forking like dependence relation and of a family of types playing a role akin to that of regular types in first order superstable theories – Shelah calls them *bs*-types – and several requirements on the interaction of these types and the dependence relation. One of the axioms of a good frame is the existence of a non-maximal super-limit model. This axiom along with $\mu$ stability implies the uniqueness of limit models of cardinality $\mu$. In Claim 4.8 of [Sh 600] he states that in a good frame limit models are unique (i.e. the same conclusion of our Main Theorem). (While we don't claim that we understand Shelah's proof or believe in its correctness, he explicitly uses the interplay beween *bs*-types and the forking notion as well as no long forking chains and continuity of forking.) Thus, the main differences are two: first, our assumptions on $\mathcal{K}$ are weaker than what Shelah is using (and our use of various versions of superstability is different from that of [Sh 600], as we do not require the full power of good frames) and second, our methods are quite different from his.

The formal differences between our approches can be summarized as follows:

(a) Suppose that $\mathcal{K}$ is an AEC satisfying the disjoint amalgamation property and is categorical in $\lambda^+$ for some $\lambda > \mathrm{LS}(\mathcal{K})$; we then get uniqueness of limit models and no splitting chains of length $\omega$. This result is used in [GrVa2] to conclude that $\mathcal{K}$ is categorical in all $\mu > \mathrm{LS}(\mathcal{K})^+$. In this case DAP follows from the other assumptions. By way of comparison, in order to get a good frame, Shelah needs results of [Sh 576] (a 99 pages-long paper) and [Sh 705] (220 pages) to conclude that good frames exist from the assumption of categoricity in several consecutive cardinals + several weak-diamonds. All our results are in ZFC.

(b) If one takes $\mathcal{K}$ to be the class of models of a complete first-order theory $T$ then what Shelah is using in his uniqueness proof amounts to requiring (the full power of assuming) that $T$ is *super*stable. However our uniqueness theorem just needs, in addition to the stability of $T$, no splitting chains of length $\omega$. As we don't claim that our theorem is of great interest for first-order theories, the difference between this paper and Claim 4.8 perhaps can be made clearer when one considers the bigger picture as in (a) above.

The reason for these differences is that Shelah's papers [Sh 576], [Sh 600] and [Sh 705] focus on a problem entirely different from [GrVa2]'s. Grossberg and VanDieren's [GrVa2] (as well as [GrVa0]) were written with Shelah's categoricity conjecture in mind. The basic assumption is categoricity in a cardinal above $\mathrm{Hanf}(\mathcal{K})$ while Shelah's above mentioned work goal, motivated by questions asked by Grossberg in fall 1994 aimed to generalize [Sh87b] to AECs which are not $\mathrm{PC}_{\aleph_0,\aleph_0}$. As the problems are quite different also the methods used to solve them are different, however occasionally the same concepts appear in both.

We are particularly interested in Theorem 1.15 not only for the sake of generalizing Shelah's result from [Sh 576] but due to the fact that the first and second author use this uniqueness theorem in a crucial step to prove:

**Theorem 1.16** (Upward categoricity theorem). *Suppose that $\mathcal{K}$ has arbitrarily large models, is $\chi$-tame and satisfies the amalgamation and joint embedding properties. Let $\lambda$ be such that $\lambda > \mathrm{LS}(\mathcal{K})$ and $\lambda \geq \chi$. If $\mathcal{K}$ is categorical in $\lambda^+$ then $\mathcal{K}$ is categorical in all $\mu \geq \lambda^+$.*

The proof of this categoricity theorem is in [GrVa2].

We thank John Baldwin for his help to read the referee's report.

## 2. The Setting

For the remainder of the paper we assume that $\mathcal{K}$ is an AEC satisyfing both the $\mu$-amalgamation property and the $\mu$-disjoint amalgamation property. We will prove the uniqueness of limit models in classes which are equipped with a moderately well-behaved dependence relation. Thus we will additionally assume that $\mathcal{K}$ is stable in $\mu$. We will use $\mu$-splitting as the dependence relation, but any dependence relation which is local and has existence, uniqueness and extension properties suffices.

**Definition 2.1.** A type $p \in \text{ga-S}(M)$ *$\mu$-splits* over $N$ if and only if $N$ is a $\prec_{\mathcal{K}}$-submodel of $M$ of cardinality $\mu$ and there there exist $N_1, N_2 \in \mathcal{K}_\mu$ and a $\mathcal{K}$-mapping $h$ such that $N \prec_{\mathcal{K}} N_l \prec_{\mathcal{K}} M$ for $l = 1, 2$ and $h : N_1 \to N_2$ with $h \restriction N = \mathrm{id}_N$ and $p \restriction N_2 \neq h(p \restriction N_1)$.

The existence property for non-$\mu$-splitting types follows from Galois stability in $\mu$:

**Fact 2.2** (Claim 3.3 of [Sh 394]). *Assume $\mathcal{K}$ is an abstract elementary class and is Galois-stable in $\mu$. For every $M \in \mathcal{K}_{\geq\mu}$ and $p \in \text{ga-S}(M)$, there exists $N \in \mathcal{K}_\mu$ such that $p$ does not $\mu$-split over $N$.*

The uniqueness and extension property of non-$\mu$-splitting types holds for types over limit models:

**Fact 2.3** (Theorem I.4.15 of [Va]). *Suppose that $\mathcal{K}$ is an AEC. Let $N, M, M' \in \mathcal{K}_\mu$ be such that $M'$ is universal over $M$ and $M$ is universal over $N$. If $p \in$ ga-S$(M)$ does not $\mu$-split over $N$, then there is a unique $p' \in$ ga-S$(M')$ such that $p'$ extends $p$ and $p'$ does not $\mu$-split over $N$.*

Here are the assumptions of the paper:

**Assumption 2.4.**
1. $\mathcal{K}$ is an AEC with the $\mu$-amalgamation property.
2. $\mathcal{K}$ satisfies the $\mu$-disjoint amalgamation property.
3. $\mathcal{K}$ is stable in $\mu$.
4. $\mu$-splitting in $\mathcal{K}$ satisfies the following locality and existence properties.
   For every $\alpha \geq LS(\mathcal{K})$, for every sequence $\langle M_i \mid i < \alpha \rangle$ of limit models of cardinality $\mu$ and for every $p \in$ ga-S$(M_\alpha)$ we have that
   (a) If for every $i < \alpha$ we have that $p \upharpoonright M_i$ does not $\mu$-split over $M_0$, then $p$ does not $\mu$-split over $M_0$.
   (b) There exists $i < \alpha$ such that $p$ does not $\mu$-split over $M_i$.

**Remark 2.5.** Categoricity in a cardinal $\lambda > \mu$ implies all parts of Assumption 2.4. This is important to questions where categoricity in a large enough cardinal is guaranteed.

The Disjoint Amalgamation Property (DAP) comes for free in First Order Contexts, in Homogeneous Classes and in Local AECs. It also holds for cats consisting of existentially closed models of positive Robinson theories ([Za]). In each of these contexts dependence relations satisfying Assumption 2.4 have been developed. Finally, the locality and existence of non-$\mu$-splitting extensions are akin to consequences of superstability in first order logic.

## 3. Strong Types

Under the assumption of $\mu$-stability, we can define *strong types* as in [ShVi]. These strong types will allow us to achieve a better control of extensions of towers of models than what we obtain using just Galois types.

**Definition 3.1** (Definition 3.2.1 of [ShVi]). For $M$ a $(\mu, \theta)$-limit model (see definition 1.10),

1. Let
$$\mathfrak{St}(M) := \left\{ (p, N) \;\middle|\; \begin{array}{l} N \prec_{\mathcal{K}} M; \\ N \text{ is a } (\mu,\theta)\text{-limit model}; \\ M \text{ is universal over } N; \\ p \in \text{ga-S}(M) \text{ is non-algebraic} \\ \text{and } p \text{ does not } \mu\text{-split over } N. \end{array} \right\}$$
2. For types $(p_l, N_l) \in \mathfrak{St}(M)$ $(l = 1, 2)$, we say $(p_1, N_1) \sim (p_2, N_2)$ iff for every $M' \in \mathcal{K}_\mu$ extending $M$ there is a $q \in$ ga-S$(M')$ extending

both $p_1$ and $p_2$ such that $q$ does not $\mu$-split over $N_1$ and $q$ does not $\mu$-split over $N_2$.

(3) Two strong types $(p_1, N_1) \in \mathfrak{St}(M_1)$ and $(p_2, N_2) \in \mathfrak{St}(M_2)$ are *parallel* iff for every $M'$ of cardinality $\mu$ extending $M_1$ and $M_2$ there exists $q \in \text{ga-S}(M')$ such that $q$ extends both $p_1$ and $p_2$ and $q$ does not $\mu$-split over $N_1$ and $N_2$.

**Lemma 3.2** (Monotonicity of parallel types)**.** *Suppose $M_0, M_1 \in \mathcal{K}_\mu$ and $M_0 \prec_{\mathcal{K}} M_1$ and $(p, N) \in \mathfrak{St}(M_1)$. If $M_0$ is universal over $N$, then for any $(q, N_q)$ parallel to $(p \restriction M_0, N)$, we have that $(q, N_q)$ is also parallel to $(p, N)$. Additionally, if $(q, N_q)$ is parallel to $(p, N)$ and $q_1$ is a non-$\mu$-splitting extension of $q$, then $(q_1, N_q)$ is also parallel to $(p, N)$.*

*Proof.* Straightforward using the uniqueness of non-$\mu$-splitting extensions. $\dashv$

**Notation 3.3.** Let $M, M' \in \mathcal{K}_\mu$, Suppose that $M$ is a $\prec_{\mathcal{K}}$-submodel of $M'$. For $(p, N) \in \mathfrak{St}(M')$, if $M$ is universal over $N$, we define the restriction $(p, N) \restriction M \in \mathfrak{St}(M')$ to be $(p \restriction M, N)$.

If we write $(p, N) \restriction M$, we mean that $p$ does not $\mu$-split over $N$ and $M$ is universal over $N$.

Notice that $\sim$ is an equivalence relation on $\mathfrak{St}(M)$ (see [Va]). Stability in $\mu$ implies that there are few strong types over any model of cardinality $\mu$:

**Fact 3.4** (Claim 3.2.2 (3) of [ShVi])**.** *If $\mathcal{K}$ is Galois-stable in $\mu$, then for any $M \in \mathcal{K}$ of cardinality $\mu$, $|\mathfrak{St}(M)/\sim| \leq \mu$.*

## 4. TOWERS

To each $(\mu, \theta)$-limit model $M$ we can naturally associate a continuous tower $\bar{M} = \langle M_i \in \mathcal{K}_\mu \mid i < \theta\rangle$ witnessing that $M$ is a $(\mu, \theta)$-limit model (that is, $\bigcup_{i<\theta} M_i = M$ and $M_{i+1}$ is universal over $M_i$). Further, by Facts 1.13 and 1.14 we can require that this tower satisfy additional requirements such as $M_{i+1}$ is a limit model over $M_i$ .

To prove the uniqueness of limit models we will construct a model which is simultaneously a $(\mu, \theta_1)$-limit model over some fixed model $M$ and a $(\mu, \theta_2)$-limit model over $M$. Notice that, by Fact 1.13, it is enough to construct a model $M^*$ that is simultaneously a $(\mu, \omega)$-limit model and a $(\mu, \theta)$-limit model for arbitrary $\theta$. By Fact 1.14 we may assume that $\theta$ is a limit ordinal $< \mu^+$ such that $\theta = \mu \cdot \theta$.

So, we actually construct an array of models with $\omega + 1$ rows, such that the bottom corner of the array $(M^*)$ will be a $(\mu, \theta)$-limit model witnessed by a tower of models as described in the first paragraph of this section. This tower will appear in the last column of the array. We will see that $M^*$ is a $(\mu, \theta)$-limit model by examining the last (the $\omega$th) row of the array. This last row will be an $\prec_{\mathcal{K}}$-increasing sequence of models, $\bar{M}^*$ of length $\theta_2$. However we will not be able to guarantee that $M^*_{i+1}$ is universal over $M^*_i$. Thus we need another method to conclude that $M^*$ is a $(\mu, \theta_2)$-limit model. This involves attaching more information to our tower $\bar{M}^*$.

Under the assumption of Galois-stability, given any sequence $\langle a_i \mid i < \theta\rangle$ of elements with $a_i \in M_{i+1}\backslash M_i$, we can identify $N_i \prec_{\mathcal{K}} M_i$ such that ga-tp$(a_i/M_i)$ does not $\mu$-split over $N_i$. Furthermore, by Assumption 2.4, we may choose this $N_i$ such that $M_i$ is a limit model over $N_i$. We abbreviate this situation by a tower $(\bar{M}, \bar{a}, \bar{N})$:

**Definition 4.1.** We denote by $\mathcal{K}^*_{\mu,\theta}$ the set of towers $(\bar{M}, \bar{a}, \bar{N})$ where $\bar{M} = \langle M_i \mid i < \theta\rangle$ is a $\prec_{\mathcal{K}}$-increasing sequence of limit models of cardinality $\mu$; $\bar{a} = \langle a_i \mid i+1 < \theta\rangle$ and $\bar{N} = \langle N_i \mid i+1 < \theta\rangle$ satisfy $a_i \in M_{i+1}\backslash M_i$; ga-tp$(a_i/M_i)$ does not $\mu$-split over $N_i$; and $M_i$ is universal over $N_i$.

**Notation 4.2.** Similarly define $\mathcal{K}^*_{\mu,I}$ where $I$ is a well-ordered set. We use the notation $i+1$ for the successor of $i$ in $I$ when it is clear which index set $I$ we are using. At times there may be more than one index set and we will write $\mathrm{succ}_I(i)$ for the successor of $i$ in $I$ to distinguish it from the successor of $i$ in another index set. Finally when $I$ is a sub-order of $I'$ for any $(\bar{M}, \bar{a}, \bar{N}) \in \mathcal{K}^*_{\mu,I'}$ we write $(\bar{M}, \bar{a}, \bar{N}) \restriction I$ for the tower $\mathcal{K}^*_{\mu,I}$ given by $\bar{M} = \langle M_i \mid i \in I\rangle$, $\bar{N} = \langle N_i \mid i \in I\rangle$ and $\bar{a} = \langle a_i \mid i \in I\rangle$.

For a tower $(\bar{M}, \bar{a}, \bar{N})$, it was shown in [ShVi] and [Va], that even if $M_{i+1}$ is not universal over $M_i$, one can conclude that $\bigcup_{i<\theta} M_i$ is a $(\mu,\theta)$-limit model provided $\theta = \mu \cdot \theta$ and for every $i < \theta$, and every strong type $(p, N)$ over $M_i$, there is $j < i + \mu$ such that (ga-tp$(a_j/M_i), N_i)$ and (ga-tp$(a_j/M_i), N)$ are parallel. In fact slightly less is required:

**Definition 4.3.** Suppose that $I$ is a well-ordered set such that there exists a cofinal sequence $\langle i_\alpha \mid \alpha < \theta\rangle$ of $I$ of order type $\theta$ such that there are $\mu \cdot \omega$ many element between $i_\alpha$ and $i_{\alpha+1}$.

Let $(\bar{M}, \bar{a}, \bar{N})$ be a tower indexed by $I$ such that each $M_i$ is a $(\mu, \sigma)$-limit model. For each $i$, let $\langle M^\gamma_i \mid \gamma < \sigma\rangle$ witness that $M_i$ is a $(\mu, \sigma)$-limit model.

A tower $(\bar{M}, \bar{a}, \bar{N})$ is *full relative to* $(M^\gamma_i)_{\gamma<\sigma, i\in I}$ iff for every $\gamma < \sigma$ and every $(p, M^\gamma_i) \in \mathfrak{St}(M_i)$ with $i_\alpha \leq i < i_{\alpha+1}$, there exists $j \in I$ with $i \leq j < i_{\alpha+1}$ such that (ga-tp$(a_j/M_j), N_j)$ and $(p, M^\gamma_i)$ are parallel.

**Fact 4.4.** *Let $\theta$ be a limit ordinal $< \mu^+$ satisfying $\theta = \mu \cdot \theta$. Suppose that $I$ is a well-ordered set such that there exists a cofinal sequence $\langle i_\alpha \mid \alpha < \theta\rangle$ of $I$ of order type $\theta$ such that there are $\mu \cdot \omega$ many element between $i_\alpha$ and $i_{\alpha+1}$.*

*Let $(\bar{M}, \bar{a}, \bar{N}) \in \mathcal{K}_{\mu,I}$ be a tower made up of $(\mu, \sigma)$-limit models,. If $(\bar{M}, \bar{a}, \bar{N}) \in \mathcal{K}_{\mu,I}$ is full relative to $(M^\gamma_i)_{i\in I,\gamma<\sigma}$, then $M := \bigcup_{i\in I} M_i$ is a $(\mu, \theta)$-limit model.*

*Proof.* Without loss of generality we may assume that $\bar{M}$ is continuous. Let $M'$ be a $(\mu, \theta)$-limit model over $M_{i_0}$ witnessed by $\langle M'_\alpha \mid \alpha < \theta\rangle$. By Disjoint Amalgamation, we may assume that $M' \cap M = M_{i_0}$. Since $\theta = \mu \cdot \theta$, we may also arrange things so that the universe of $M'_\alpha$ is $\mu \cdot \alpha$ and $\alpha \in M'_{\alpha+1}$.

We will construct an isomorphism between $M$ and $M'$ by induction on $\alpha < \theta$. Define an increasing and continuous sequence of $\prec_{\mathcal{K}}$-mappings $\langle h_\alpha \mid \alpha < \theta\rangle$ such that

(1) $h_\alpha : M_{i_\alpha + j} \to M'_{\alpha+1}$ for some $j < \mu \cdot \omega$
(2) $h_0 = \mathrm{id}_{M_{0,0}}$ and
(3) $\alpha \in \mathrm{rg}(h_{\alpha+1})$.

For $\alpha = 0$ take $h_0 = \mathrm{id}_{M_{0,0}}$. For $\alpha$ a limit ordinal let $h_\alpha = \bigcup_{\beta<\alpha} h_\beta$. Since $\bar{M}$ is continuous, the induction hypothesis gives us that $h_\alpha$ is a $\prec_{\mathcal{K}}$-mapping from $M_{i_\alpha}$ into $M'_\alpha$ allowing us to satisfy condition (1) of the construction.

Suppose that $h_\alpha$ has been defined. Let $j < \mu\cdot\omega$ be such that $h_\alpha : M_{i_\alpha+j} \to M'_{\alpha+1}$. There are two cases: either $\alpha \in \mathrm{rg}(h_\alpha)$ or $\alpha \notin \mathrm{rg}(h_\alpha)$. First suppose that $\alpha \in \mathrm{rg}(h_\alpha)$. Since $M'_{\alpha+2}$ is universal over $M'_{\alpha+1}$, it is also universal over $h_\alpha(M_{i_\alpha+j})$. This allows us to extend $h_\alpha$ to $h_{\alpha+1} : M_{i_{\alpha+1}} \to M'_{\alpha+2}$.

Now consider the case when $\alpha \notin \mathrm{rg}(h_\alpha)$. Since $\langle M^\gamma_{i_\alpha+j} \mid \gamma < \sigma\rangle$ witnesses that $M_{i_\alpha+j}$ is a $(\mu,\sigma)$-limit model, by Assumption 2.4, there exists $\gamma < \sigma$ such that ga-tp$(\alpha/M_{i_\alpha+j})$ does not $\mu$-split over $M^\gamma_{i_\alpha+j}$. By our choice of $\bar{M}'$ disjoint from $\bar{M}$ outside of $M_{i_0}$, we know that $\alpha \notin M_{i_\alpha+j}$. Thus ga-tp$(\alpha/M_{i_\alpha+j})$ is non-algebraic. By relative fullness of $(\bar{M},\bar{a},\bar{N})$, there exists $j'$ with $j \le j' < i_{\alpha+1}$ such that (ga-tp$(\alpha/M_{i_\alpha+j'}), M^\gamma_{i_\alpha+j})$ is parallel to (ga-tp$(a_{i_{\alpha+1}+j'}/M_{i_{\alpha+1}+j'}), N_{i_{\alpha+1}+j'})$. In particular we have that

$$(*)\quad \text{ga-tp}(a_{i_{\alpha+1}+j'}/M_{i_\alpha+j}) = \text{ga-tp}(\alpha/M_{i_\alpha+j}).$$

We can extend $h_\alpha$ to an automorphism $h'$ of $\mathfrak{C}$. An application of $h'$ to $(*)$ gives us

$$(**)\quad \text{ga-tp}(h'(a_{i_{\alpha+1}+j'})/h_\alpha(M_{i_\alpha+j})) = \text{ga-tp}(\alpha/h_\alpha(M_{i_\alpha+j})).$$

Since $M'_{\alpha+2}$ is universal over $h_\alpha(M_{i_\alpha})$, we may extend $h_\alpha$ to a $\mathcal{K}$-mapping $h_{\alpha+1} : M_{i_{\alpha+1}+j'} \to M'_{\alpha+2}$ such that $h_{\alpha+1}(a_{i_{\alpha+1}+j'}) = \alpha$.

Let $h := \bigcup_{\alpha<\theta} h_\alpha$. Clearly $h : M \to M'$. To see that $h$ is an isomorphism, notice that condition (3) of the construction forces $h$ to be surjective. $\dashv$

## 5. Uniqueness of Limit Models

By Fact 1.13 it is enough to construct a model $M^*$ that is simultaneously a $(\mu,\omega)$-limit model and a $(\mu,\theta)$-limit model. By Fact 1.14 we may assume that $\theta$ is a limit ordinal $< \mu^+$ such that $\theta = \mu\cdot\theta$.

We now begin the construction of $M^*$. The goal will be to build a $\theta$ by $\omega$ array of models so that the bottom row of the array is a relatively full tower. We also need to be able to guarantee that the last row of the tower witnesses that $M^*$ is a $(\mu,\omega)$-limit model. This will be done by imposing the following ordering on rows of the array:

**Definition 5.1.** For towers $(\bar{M},\bar{a},\bar{N}) \in \mathcal{K}_{\mu,I}$ and $(\bar{M}',\bar{a}',\bar{N}') \in \mathcal{K}_{\mu,I'}$ with $I \subseteq I'$, we write $(\bar{M},\bar{a},\bar{N}) < (\bar{M}',\bar{a}',\bar{N}')$ if and only if for every $i \in I$, $a_i = a'_i$, $N_i = N'_i$ and $M'_i$ is universal over $M_i$.

**Remark 5.2.** The ordering $<$ on towers is identical to the ordering $<^c_\mu$ defined in [ShVi]. The superscript was used by Shelah and Villaveces to distinguish this ordering from others. We only use one ordering on towers, so we omit the superscripts and subscripts here.

To get a relatively full tower at the end of the construction, we will require that at stage $n$ of our construction the tower that we build is indexed by $I_n$ described here:

**Notation 5.3.** Fix an increasing and continuous chain of well-ordered sets $\langle I_n \mid n \leq \omega\rangle$ and an increasing and continuous sequence of elements $\langle i_\alpha \mid \alpha \leq \theta\rangle$ such that each $I_n$ has a supremum $i_\theta$ and $\langle i_\alpha \mid \alpha < \theta\rangle$ is cofinal in each $I_n\backslash\{i_\theta\}$ We additionally require that $\mathrm{otp}(\{j \in I_n \mid i_\alpha < j < i_{\alpha+1}\})$ is $\mu \cdot n$ for each $\alpha < \theta$ and each $n \leq \omega$. An example of such $\langle I_n \mid n \leq \omega\rangle$ is $I_n = \theta \times (\mu \cdot n) \bigcup\{i_\theta\}$ ordered lexicographically, where $i_\theta$ is an element $\geq$ each $i \in \bigcup_{n<\omega} I_n$.

We verify that it is possible to carry out the induction step of the construction. This is a particular case of Theorem II.7.1 of [Va]. But since our context is somewhat easier, we do not encounter so many obstacles as in [Va] and we provide a different, more direct proof here:

**Theorem 5.4** (Dense $<$-extension property)**.** *Given $(\bar{M}, \bar{a}, \bar{N})^n \in \mathcal{K}^*_{\mu,I_n}$ there exists $(\bar{M}, \bar{a}, \bar{N})^{n+1} \in \mathcal{K}^*_{\mu,I_{n+1}}$ such that $(\bar{M}, \bar{a}, \bar{N})^n < (\bar{M}, \bar{a}, \bar{N})^{n+1}$ and for each $(p, N) \in \mathfrak{St}(M_i)$ with $i_\alpha \leq i < i_{\alpha+1}$, there exists $j \in I_{n+1}$ with $i < j < i_{\alpha+1}$ such that $(\text{ga-tp}(a_j/M_j), N_j)$ and $(p, N)$ are parallel.*

Before we prove Theorem 5.4, we prove a slightly weaker extension property:

**Lemma 5.5** ($<$-extension property)**.** *Given $(\bar{M}, \bar{a}, \bar{N}) \in \mathcal{K}^*_{\mu,I}$ for any $n < \mu^+$, there exists a $<$-extension $(\bar{M}', \bar{a}, \bar{N}) \in \mathcal{K}^*_{\mu,I}$ of $(\bar{M}, \bar{a}, \bar{N})$ such that for each $i \in I$, $M'_{\mathrm{succ}_I(i)}$ is a $(\mu, \mu \cdot n)$-limit model over $M'_i$.*

*Proof of Lemma 5.5.* Given $(\bar{M}, \bar{a}, \bar{N}) \in \mathcal{K}^*_{\mu,I}$ we will define a $<$ extension $(\bar{M}', \bar{a}, \bar{N})$ by a directed system by induction on $i \in I$. We define $M^+_i$ and a directed system of $\prec_{\mathcal{K}}$-embeddings $\langle f_{i,j} \mid i < j \in I\rangle$ such that for $i \in I$, $M_i \prec_{\mathcal{K}} M^+_i$ for $i \leq j$, $f_{i,j} : M^+_i \to M^+_j$ and $f_{i,j} \restriction M_i = \mathrm{id}_{M_i}$. We further require that $M^+_{\mathrm{succ}_I(i)}$ is a $(\mu, \mu \cdot n)$-limit model over $f_{i,\mathrm{succ}_I(i)}(M^+_i)$ and $\text{ga-tp}(a_i/f_{i,\mathrm{succ}_I(i)}(M^+_i))$ does not $\mu$-split over $N_i$.

This construction is done by induction on $i \in I$ using both the disjoint amalgamation property and the existence of non-$\mu$-splitting extensions. At limit stages we take direct limits so that $f_{j,i} \restriction M_j = \mathrm{id}_{M_j}$. This is possible by Subclaims II.7.10 and II.7.11 of [Va] or see Claim 2.17 of [GrVa2]. If the direct limit $M'$ is not universal over $M_i$, simply take an extension of both $M'$ and $M_i$ which is universal over $M_i$ and call this $M^+_i$. We forfeit continuity of the tower at this point, but it will be recovered later using reduced towers.

Let $f_{j,\sup\{I\}}$ and $M_{\sup\{I\}}$ be the direct limit of this system such that $f_{j,\sup\{I\}} \restriction M_j = \mathrm{id}_{M_j}$. We can now define $M'_j := f_{j,\sup\{I\}}(M^+_j)$ for each $j \in I$. The details of the verification that $(\bar{M}', \bar{a}, \bar{N})$ are as required are left to the reader, but can also be found in [Va]. $\dashv$

*Proof of Theorem 5.4.* Given $(\bar{M}, \bar{a}, \bar{N})^n \in \mathcal{K}^*_{\mu,I_n}$, let $(\bar{M}', \bar{a}, \bar{N}) \in \mathcal{K}^*_{\mu,I_n}$ be an extension of $(\bar{M}, \bar{a}, \bar{N})$ as in Lemma 5.5.

For each $i_\alpha$, let $\langle M'_j \mid j \in I_{n+1},\ i_\alpha < j < i_{\alpha+1}\rangle$ witness that $M'_{\mathrm{succ}_{I_n}(i)}$ is a $(\mu, \mu \cdot n)$-limit model over $M'_i$. Without loss of generality we may assume that each $M'_j$ is a limit model over its predecessor.

Fix $\{(p, N)^j_{i_\alpha} \mid i_\alpha + \mu \cdot n < j < i_{\alpha+1}\}$ an enumeration of $\bigcup\{\mathfrak{S}\mathrm{t}(M_i) : i \in I_n, i_\alpha \leq i < i_{\alpha+1}\}$. By our choice of $I_{n+1}$ and stability in $\mu$, such an enumeration is possible. Since $M'_{\mathrm{succ}_{I_{n+1}}(j)}$ is universal over $M'_j$, there exists a realization in $M'_{\mathrm{succ}_{I_{n+1}}}(j)$ of the non-$\mu$-splitting extension of $p^j_{i_\alpha}$ to $M'_j$. Let $a_j$ be this realization and take $N'_j := N^j_{i_\alpha}$.

Notice that $(\langle M'_j \mid j \in I_{n+1}\rangle, \langle a_j \mid j \in I_{n+1}\rangle, \langle N_j \mid j \in I_{n+1}\rangle)$ provide the desired extension of $(\bar{M}, \bar{a}, \bar{N})$ in $\mathcal{K}^*_{\mu,I_{n+1}}$. $\dashv$

We are almost ready to carry out the complete construction. However, notice that Theorem 5.4 does not provide us with a continuous extension. Therefore the bottom row of our array may not be continuous at $\theta$ which would prevent us from applying Fact 4.4 to conclude that $M^*$ is a $(\mu, \theta)$-limit model. So we will further require that the towers that occur in our array are all continuous. This can be guaranteed by restricting ourselves to reduced towers as in [ShVi] and [Va].

**Definition 5.6.** A tower $(\bar{M}, \bar{a}, \bar{N}) \in \mathcal{K}^*_{\mu,I}$ is said to be *reduced* provided that for every $(\bar{M}', \bar{a}, \bar{N}) \in \mathcal{K}^*_{\mu,I}$ with $(\bar{M}, \bar{a}, \bar{N}) \leq_{\mu,I} (\bar{M}', \bar{a}, \bar{N})$ we have that for every $i \in I$,

$$(*)_i \quad M'_i \cap \bigcup_{j \in I} M_j = M_i.$$

If we take a $<$-increasing chain of reduced towers, the union will be reduced. The following fact appears as Theorem 3.1.14 of [ShVi]. We provide the proof for completeness.

**Fact 5.7.** *If $\langle (\bar{M}, \bar{a}, \bar{N})^\gamma \in \mathcal{K}^*_{\mu,I_\gamma} \mid \gamma < \beta\rangle$ is $<$-increasing and continuous sequence of reduced towers, then the union of this sequence of towers is a reduced tower.*

*Proof.* Denote by $(\bar{M}, \bar{a}, \bar{N})^\beta$ the union of the sequence of towers and $I_\beta := \bigcup_{\gamma<\beta} I_\gamma$. Then $\bar{a}^\beta = \bar{a}^0$, $\bar{N}^\beta = \bar{N}^0$ and $\bar{M}^\beta = \langle M^\beta_i \mid i \in \bigcup_{\gamma<\beta} I_\gamma\rangle$ where $M^\beta_i = \bigcup_{\gamma<\beta} M^\gamma_i$.

Suppose that $(\bar{M}, \bar{a}, \bar{N})^\beta$ is not reduced. Let $(\bar{M}', \bar{a}, \bar{N}) \in \mathcal{K}^*_{\mu,I_\beta}$ witness this. Then there exists an $i \in I_\beta$ and an element $b$ such that $b \in (M'_i \cap$

$\bigcup_{j\in I_\beta} M^\beta_j)\backslash M^\beta_i$. There exists $\gamma<\beta$ such that $b\in\bigcup_{j\in I_\gamma} M^\gamma_j\backslash M^\gamma_i$. Notice that $(\bar M', \bar a, \bar N)\restriction I_\gamma$ witnesses that $(\bar M,\bar a,\bar N)^\gamma$ is not reduced. $\dashv$

The following appears in [ShVi] (Theorem 3.1.13).

**Fact 5.8** (Density of reduced towers)**.** *There exists a reduced $<$-extension of every tower in $\mathcal{K}^*_{\mu,I}$.*

*Proof.* Suppose for the sake of contradiction that no $<$-extension of $(\bar M,\bar a,\bar N)$ is reduced. This allows us to construct a $\leq$-increasing and continuous sequence of towers $\langle(\bar M,\bar a,\bar N)^\zeta\in\mathcal{K}^*_{\mu,I}\mid\zeta<\mu^+\rangle$ such that $(\bar M,\bar a,\bar N)^{\zeta+1}$ witnesses that $(\bar M,\bar a,\bar N)^\zeta$ is not reduced.

The construction is done inductively in the obvious way:

For each $b\in\bigcup_{\zeta<\mu^+,i\in I} M^\zeta_i$ define

$$i(b):=\min\big\{i\in I\mid b\in\bigcup_{\zeta<\mu^+}\bigcup_{j<i} M^\zeta_j\big\}\text{ and}$$

$$\zeta(b):=\min\big\{\zeta<\mu^+\mid b\in M^\zeta_{i(b)}\big\}.$$

$\zeta(\cdot)$ can be viewed as a function from $\mu^+$ to $\mu^+$. Thus there exists a club $E=\{\delta<\mu^+\mid\forall b\in\bigcup_{i<\in I} M^\delta_i,\ \zeta(b)<\delta\}$. Actually, all we need is for $E$ to be non-empty.

Fix $\delta\in E$. By construction $(\bar M,\bar a,\bar N)^{\delta+1}$ witnesses the fact that $(\bar M,\bar a,\bar N)^\delta$ is not reduced. So we may fix $i\in I$ and $b\in M^{\delta+1}_i\cap\bigcup_{j\in I} M^\delta_j$ such that $b\notin M^\delta_i$. Since $b\in M^{\delta+1}_i$, we have that $i(b)\leq i$. Since $\delta\in E$, we know that there exists $\zeta<\delta$ such that $b\in M^\zeta_{i(b)}$. Because $\zeta<\delta$ and $i(b)<i$, this implies that $b\in M^\delta_i$ as well. This contradicts our choice of $i$ and $b$ witnessing the failure of $(\bar M,\bar a,\bar N)^\delta$ to be reduced. $\dashv$

**Theorem 5.9** (Reduced towers are continuous)**.** *If $(\bar M,\bar a,\bar N)\in\mathcal{K}^*_{\mu,I}$ is reduced, then it is continuous.*

By a slight revision to Lemma 5.5, we can conclude:

**Lemma 5.10.** *Suppose that $(\bar M,\bar a,\bar N)\in\mathcal{K}^*_{\mu,I}$ is reduced, then for every $I_0\subseteq I$, $(\bar M,\bar a,\bar N)\restriction I_0$ is reduced.*

*Proof of Theorem 5.9.* Suppose the claim fails for $\mu$. Let $\delta$ be the minimal limit ordinal such that there exists an $I$ of order type $\alpha$ and $(\bar M,\bar a,\bar N)\in\mathcal{K}^*_{\mu,I}$ a reduced tower discontinuous at $\delta$. Without loss of generality $I=\alpha$. We are assuming that $M_\delta\succneqq_{\mathcal{K}}\bigcup_{i<\delta} M_i$.

We can apply Lemma 5.10, to assume that $\alpha=\delta+1$. Fix $(\bar M,\bar a,\bar N)\in\mathcal{K}^*_{\mu,\delta+1}$ reduced and discontinuous at $\delta$. Let $b\in M_\delta\backslash\bigcup_{i<\delta} M_i$ be given. Let $M^b$ be a model of cardinality $\mu$ inside $\mathfrak{C}$ containing $\bigcup_{i<\delta} M^i_i\cup\{b\}$.

**Claim 5.11.** *There exists a $<$-extension of $(\bar M,\bar a,\bar N)\restriction\delta$, say $(\bar M',\bar a\restriction\delta,\bar N\restriction\delta)\in\mathcal{K}^*_{\mu,\delta}$ containing $b$.*

Let $M'_\delta \prec_{\mathcal{K}} \mathfrak{C}$ be a limit model universal over $M_\delta$ containing $\bigcup_{i<\delta} M'_i$. Notice that $(\bar{M}', \bar{a}, \bar{N}) \in \mathcal{K}^*_{\mu,\delta+1}$ is an extension of $(\bar{M}, \bar{a}, \bar{N})$ witnessing that $(\bar{M}, \bar{a}, \bar{N})$ is not reduced.

*Proof of Claim 5.11.* We use the minimality of $\delta$ and the density of reduced towers to build a $<$-increasing and continuous sequence of reduced (and continuous) towers $\langle (\bar{M}, \bar{a}, \bar{N})^\zeta \mid \zeta < \delta \rangle$ such that $(\bar{M}, \bar{a}, \bar{N})^0 := (\bar{M}, \bar{a}, \bar{N}) \restriction \delta$. This gives us a $\delta$ by $\delta$ array of models. If $b$ appears in this array, we are done. So let us suppose that ga-tp$(b/\bigcup_{i<\delta} M^i_i)$ is non-algebraic. Since $\bigcup_{i<\delta} M^i_i$ is a $(\mu, \delta)$-limit model (witnessed by the diagonal of this array), there exists $\xi < \delta$ such that ga-tp$(b/\bigcup_{i<\delta} M^i_i)$ does not $\mu$-split over $M^\xi_\xi$.

We will find a $<$-extension of $(\bar{M}, \bar{a}, \bar{N})$ by defining an $\prec_{\mathcal{K}}$-increasing chain of models $\langle N^*_i \mid i < \alpha \rangle$ and an increasing chain of $\prec_{\mathcal{K}}$-mappings $\langle h_i \mid i < \delta \rangle$ with the intention that the pre-image of $N^*_i$ under an extension of $\bigcup_{i<\delta} h_i$ will form a sequence $\bar{M}^*$ such that $(\bar{M}, \bar{a}, \bar{N}) < (\bar{M}^*, \bar{a}, \bar{N})$, $b \in M^*_{\xi+1}$ and $M^*_i = M^i_i$ for all $i < \xi$. We choose by induction on $i < \delta$ a $\prec_{\mathcal{K}}$-increasing and continuous chain of limit models $\langle N^*_i \in \mathcal{K}_\mu \mid i < \delta \rangle$ and an increasing and continuous sequence of $\prec_{\mathcal{K}}$-mappings $\langle h_i \mid i < \delta \rangle$ satisfying

1. $N^*_{i+1}$ is a limit model and is universal over $N^*_i$
2. $h_i : M^i_i \to N^*_i$
3. $h_i(M^i_i) \prec_{\mathcal{K}} M^{i+1}_i$
4. ga-tp$(h_{i+1}(a_i)/N^*_i)$ does not $\mu$-split over $h_i(N_i)$
5. $M^b \prec_{\mathcal{K}} N^*_{\xi+1}$ and
6. for $i \leq \xi$, $N^*_i = M^i_i$ with $h_i = \mathrm{id}_{M^i_i}$.

The requirements determine the definition of $N^*_i$ for $i \leq \xi$. We proceed with the rest of the construction by induction on $i < \delta$. If $i$ is a limit ordinal $\geq \xi$, let $N^*_i = \bigcup_{j<i} N^*_j$ and $h_i = \bigcup_{j<i} h_j$.

Suppose that we have defined $h_i$ and $N^*_i$ satisfying the conditions of the construction. We now describe how to define $N^*_{i+1}$. First, we extend $h_i$ to $\bar{h}_i \in \mathrm{Aut}(\mathfrak{C})$. We can assume that $\bar{h}_i(a_i) \in M^{i+2}_{i+1}$. This is possible since $M^{i+2}_{i+1}$ is universal over $h_i(M^i_i)$ by construction (as $M^{i+2}_{i+1}$ is universal over $M^{i+2}_i \succ_{\mathcal{K}} h_i(M^i_i)$).

Since ga-tp$(a_i/M^i_i)$ does not $\mu$-split over $N_i$, we have that ga-tp$(\bar{h}_i(a_i)/h_i(M^i_i))$ does not $\mu$-split over $h_i(N_i)$. We now adjust the proof of the existence property for non-splitting extensions.

**Claim 5.12.** *We can find $g \in \mathrm{Aut}(\mathfrak{C})$ such that* ga-tp$(g(\bar{h}_i(a_i))/N^*_i)$ *does not $\mu$-split over $h_i(N_i)$, $g(\bar{h}_i(M^{i+1}_{i+1})) \prec_{\mathcal{K}} M^{i+2}_{i+1}$ and such that $g \restriction (h_i(M^i_i)) = \mathrm{id}_{h_i(M^i_i)}$.*

*Proof of Claim 5.12.* First we find a $\prec_{\mathcal{K}}$-mapping $f$ such that $f : N^*_i \to h_i(M^i_i)$ such that $f \restriction h_i(N_i) = \mathrm{id}_{h_i(N_i)}$ which is possible since $h_i(M^i_i)$ is universal over $h_i(N_i)$. Notice that ga-tp$(f^{-1}(\bar{h}_i(a_i))/N^*_i)$ does not $\mu$-split over $h_i(N_i)$ and

$$(+) \quad \text{ga-tp}(f^{-1}(\bar{h}_i(a_i))/h_i(M^i_i)) = \text{ga-tp}(\bar{h}_i(a_i)/h_i(M^i_i))$$

by a non-splitting argument as in the proof of Theorem 2.3.

Let $N^+$ be a limit model of cardinality $\mu$ containing $f^{-1}(\bar{h}_i(a_i))$ with $f^{-1}(\bar{h}_i(M^{i+1}_{i+1}) \prec_{\mathcal{K}} N^+$. Now using the equality of types (+) and the fact that $M^{i+2}_{i+1}$ is universal over $h_i(M^i_i)$ with $\bar{h}_i(a_i) \in M^{i+2}_{i+1}$, we can find a $\prec_{\mathcal{K}}$-mapping $f^+ : N^+ \to M^{i+2}_{i+1}$ such that $f^+ \restriction h_i(M^i_i) = \text{id}_{h_i(M^i_i)}$ and $f^+(f^{-1}(\bar{h}_i(a_i))) = \bar{h}_i(a_i)$. Now set $g := f^+ \circ f^{-1} : \bar{h}(M^{i+1}_{i+1}) \to M^{i+2}_{i+1}$.

$\dashv$

Fix such a $g$ as in the claim and set $h_{i+1} := g \circ \bar{h}_i \restriction M^{i+1}_{i+1}$. Let $N^*_{i+1}$ be a $\prec_{\mathcal{K}}$ extension of $N^*_i$, $M^b$ and $h_{i+1}(M^{i+1}_{i+1})$ of cardinality $\mu$ inside $\mathfrak{C}$. Furthermore, choose $N^*_{i+1}$ to be a limit model and universal over $N^*_i$.

This completes the construction.

We now argue that the construction of these sequences is enough to find a $<$-extension, $(\bar{M}^*, \bar{a}, \bar{N})$, of $(\bar{M}, \bar{a}, \bar{N})$ such that $b \in M^*_\zeta$ for some $\zeta < \delta$.

Let $h_\delta := \bigcup_{i<\delta} h_i$. We will be defining for $i < \delta$, $M^*_i$ to be pre-image of $N^*_i$ under some extension of $h_\delta$. The following claim allows us to choose the pre-image so that $M^*_\zeta$ contains $b$ for some $\zeta < \delta$.

**Claim 5.13.** *There exists $h \in \text{Aut}(\mathfrak{C})$ extending $\bigcup_{i<\delta} h_i$ such that $h(b) = b$.*

*Proof of Claim 5.13.* Let $h_\delta := \bigcup_{i<\delta} h_i$. Consider the increasing and continuous sequence $\langle h_\delta(M^i_i) \mid i < \delta\rangle$. (This sequence is increasing by our choice of $g$ in 5.12: if $d \in M^i_i$, then $h_{i+1}(d) = g(\bar{h}_i(d)) = g(h_i(d)) = h_i(d)$, by the definitions of $h_{i+1}$, $\bar{h}_i$ and $g$.) By invariance, $h_\delta(M^{i+1}_{i+1})$ is universal over $h_\delta(M^i_i)$ and each $h_\delta(M^i_i)$ is a limit model.

Furthermore, from our choice of $\xi$, we know that ga-tp$(b/M^\delta_i)$ does not $\mu$-split over $M^\xi_\xi$. Since $h_i(M^i_i) \prec_{\mathcal{K}} M^{i+1}_i \prec_{\mathcal{K}} \bigcup_{j<\delta} M^\delta_j$, monotonicity of non-splitting allows us to conclude that

$$\text{ga-tp}(b/h_\delta(M^i_i)) \text{ does not } \mu\text{-split over } M^\xi_\xi.$$

This allows us to apply Assumption 2.4, to ga-tp$(b/\bigcup_{i<\delta} h_\delta(M^i_i))$ yielding

$$(**) \quad \text{ga-tp}(b/\bigcup_{i<\delta} h_\delta(M^i_i)) \text{ does not } \mu\text{-split over } M^\xi_\xi.$$

We can extend $\bigcup_{i<\delta} h_i$ to an automorphism $h^*$ of $\mathfrak{C}$. We will first show that

$$(***) \quad \text{ga-tp}(b/h^*(\bigcup_{i<\delta} M^i_i), \mathfrak{C}) = \text{ga-tp}(h^*(b)/h^*(\bigcup_{i<\delta} M^i_i), \mathfrak{C}).$$

By invariance and our choice of $\xi$ in $(*)$,

$$\text{ga-tp}(h^*(b)/h^*(\bigcup_{i<\delta} M^i_i), \mathfrak{C}) \text{ does not } \mu\text{-split over } M^\xi_\xi.$$

We will use non-splitting to derive $(*{*}*)$. In accordance with the definition of splitting, let $N^1 = \bigcup_{i<\delta} M_i^i$, $N^2 = h^*(\bigcup_{i<\delta} M_i^i)$ and $p = \text{ga-tp}(b/N^2)$. By $(**)$, we have that $p \restriction N^2 = h^*(p \restriction N^1)$. In other words,

$$\text{ga-tp}(b/h^*(\bigcup_{i<\delta} M_i^i), \mathfrak{C}) = \text{ga-tp}(h^*(b)/h^*(\bigcup_{i<\delta} M_i^i), \mathfrak{C}),$$

as desired.

From $(* * *)$, we can find an automorphism $f$ of $\mathfrak{C}$ such that $f(h^*(b)) = b$ and $f \restriction h^*(\bigcup_{i<\delta} M_i^i) = \text{id}_{h^*(\bigcup_{i<\delta} M_i^i)}$. Notice that $h := f \circ h^*$ satisfies the conditions of the claim.

$\dashv$

Now that we have a automorphism $h$ fixing $b$ and $\bigcup_{i<\delta} M_i$, we can define for each $i < \delta$, $M_i^* := h^{-1}(N_i^*)$.

**Claim 5.14.** *$(\bar{M}^*, \bar{a}, \bar{N})$ is a $<$-extension of $(\bar{M}, \bar{a}, \bar{N})$ such that $b \in M^*_{\xi+1}$.*

*Proof of Claim 5.14.* By construction $b \in M_\delta^\delta \subseteq N^*_{\xi+1}$. Since $h(b) = b$, this implies $b \in M^*_{\xi+1}$. To verify that we have a $\leq$-extension we need to show for $i < \delta$:

i. $M_i^*$ is universal over $M_i$
ii. $a_i \in M^*_{i+1} \backslash M_i$ for $i + 1 < \delta$ and
iii. ga-tp$(a_i/M_i^*)$ does not $\mu$-split over $N_i$ whenever $i, i+1 \leq \delta$.

Item i. follows from the fact that $M_i^i$ is universal over $M_i$ and $M_i^i \prec_{\mathcal{K}} M_i^*$. Item iii. follows from invariance and our construction of the $N_i^*$'s. Finally, recalling that a non-splitting extension of a non-algebraic type is also non-algebraic, we see that Item iii implies $a_i \notin M_i^*$. By our choice of $h_{i+1}(a_i) \in M^{i+2}_{i+1} \prec_{\mathcal{K}} N^*_{i+1}$, we have that $a_i \in M^*_{i+1}$. Thus Item ii is satisfied as well.

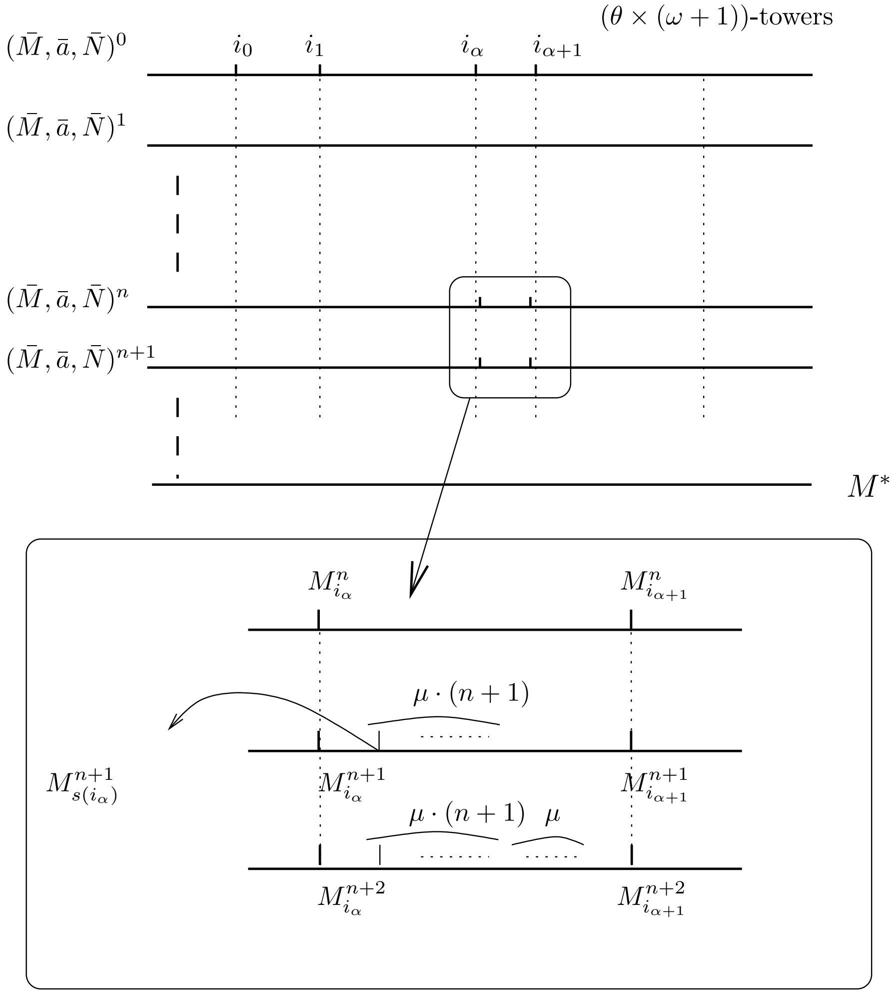
$\dashv$

$\dashv$

$\dashv$

**Corollary 5.15.** *In Theorem 5.4, we can choose $(\bar{M}, \bar{a}, \bar{N})^{\alpha+1}$ to be continuous.*

The construction:




Define by induction on $n \leq \omega$ a $<$-increasing and continuous chain of reduced towers $(\bar{M}, \bar{a}, \bar{N})^n \in \mathcal{K}^*_{\mu, I_n}$ such that $(\bar{M}, \bar{a}, \bar{N})^n$ is "dense in" $(\bar{M}, \bar{a}, \bar{N})^{n+1}$.

Corollary 5.15 tells us that the construction is possible in successor cases. In the limit case, let $I_\omega = \bigcup_{m<\omega} I_m$, and simply define, for $i \in I_\omega$,

$$M^\omega_i = \bigcup_{n \geq m} M^n_i,$$

where (say) $i \in I_m$. To see that the construction suffices for what we need, first notice that the last column of the array, $\langle \bigcup_{i \in I_n} M^n_i \mid n < \omega \rangle$, witnesses that $M^*$ is a $(\mu, \omega)$-limit model. In light of Fact 4.4 we need only verify that the last row of the array is a relatively full tower of cofinality $\theta$.

**Claim 5.16.** $(\bar{M}, \bar{a}, \bar{N})^\omega$ *is full relative to* $(M^n_i)_{n<\omega, i \in I_\omega}$.

*Proof.* Given $i$ with $i_\alpha \leq i < i_{\alpha+1}$, let $(p, M^n_i)$ be some strong type in $\mathfrak{St}(M^\omega_i)$. Notice that $(p \restriction M^{n+1}_i, M^n_i) \in \mathfrak{St}(M^{n+1}_i)$. By construction there is a $j \in I^{n+1}_\omega$ with $i \leq j < i_{\alpha+1}$ such that $(\text{ga-tp}(a_{i+j}/M^{n+2}_{i+j}), N^{n+2}_{i+j})$ is parallel to $p \restriction M_i^{n+1}$. We will show that $(\text{ga-tp}(a_{i+j}/M_{i+j}^\omega), N_{i+j}^\omega)$ is parallel to $(p, N)$.

First notice that ga-tp$(a_{i+j}/M_{i+j}^\omega)$ does not $\mu$-split over $N_{i+j}^\omega = N_{i+j}^{n+2}$. Since $(\text{ga-tp}(a_{i+j}/M_{i+j}^{n+2}), N_{i+j}^{n+2})$ is parallel to $(p \restriction M_i^{n+1}, M_i^n)$ there is $q \in \text{ga-S}(M_{i+j}^\omega)$ such that $q$ extends both $p \restriction M_i^{n+1}$ and ga-tp$(a_{i+j}/M_{i+j}^{n+2})$. By two separate applications of the uniqueness of non-$\mu$-splitting extensions we know that $q \restriction M_i^\omega = p$ and $q = \text{ga-tp}(a_{i+j}/M_{i+j}^\omega)$. To see that $(q, N_{i+j}^\omega)$ is parallel to $(p, M_i^n)$, let $M'$ be an extension of $M_{i+j}^\omega$ of cardinality $\mu$. Since $(p \restriction M_i^{n+1}, M_i^n)$ and $(q \restriction M_{i+j}^{n+2}, N_{i+j}^{n+2})$ are parallel, there is $q' \in \text{ga-S}(M')$ extending both $p \restriction M_i^{n+1}$ and $q \restriction M_{i+j}^{n+2}$ and not $\mu$-splitting over both $M_i^n$ and $N_{i+j}^{n+2}$. By the uniqueness of non-$\mu$-splitting extensions, we have that $q'$ is also an extension of $q$ and $p$. Thus $q$ witnesses that $(q, N_{i+j}^\omega)$ and $(p, M_i^n)$ are parallel. $\dashv$

This completes the proof of Theorem 1.15.

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