A Course in Model Theory I:

Introduction[1]

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[1]This **preliminary draft** is dated from August 14, 2019. The book will be published by Cambridge University Press. The book is approximately 98.52% complete, I expect that the final version will have about 800 pages, many sections of the current version will be revised and few will be added. I hope to have a stable version of this volume soon.
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Contents

Preface 5
1. About this book 5
2. A mathematical introduction to the book 14
Course outlines 21

Part 1. Definability 23

Chapter 1. Fundamentals 25
Introduction 25
1. Structures and languages 29
2. The basic concepts 45
3. On existence of models and elementary submodels 81
4. The Erdős-Rado Theorem 101
5. Applications of the compactness theorem 116
6. Joint embedding and the Amalgamation properties in first-order logic 131
7. Types and the diagram of T 138
8. Some extensions of first-order logic 151
9. Countable models and Henkin's omitting types theorem 172
10. Models of weak set theory 189
11. Absoluteness 197
12. Two cardinal theorems, by Vaught, Chang, Keisler and Morley 199
13. Model complete-theories 212
14. Skolemization 221

Chapter 2. Abstract Elementary Classes 227
Introduction 227
1. Abstract Classes 229
2. Abstract Elementary Classes 245
3. The major open questions concerning AEC 260
4. Shelah's presentation Theorem and 2-categories 265
5. Basic Examples 275
6. PC-classes and omitting types 280
7. $\mathrm{I}(\aleph_0, \mathcal{K}) = \mathrm{I}(\aleph_1, \mathcal{K}) = 1 \implies \mathcal{K}_{\aleph_2} \neq \emptyset$. 302
8. Categoricity in $\aleph_1$ for AECs is not absolute 309
9. Random power set in higher order 323
10. $\mathrm{Ext}^1_{\mathbf{Z}}(\mathrm{G}, \mathbf{Z})$ 324
11. Weak amalgamation 325
12. Few models imply the amalgamation property 328

Chapter 3. More Fundamentals 335
Introduction 335
1. The filter of closed unbounded sets 336
2. Ultraproducts 348

3. Ehrenfeucht-Fraissé games 362
4. Two applications to algebra 365
5. Non-standard analysis* 371
6. When does a class have a structure theory? Shelah's thesis 371

Part 2. Galois Theory 373

Chapter 4. Complete types and indiscernibles 375
Introduction 375
1. Saturated models 377
2. The monster model, homogenous and special models 390
3. Indiscernibles and Ehrenfeucht-Mostowski models 414

Chapter 5. Galois types and monster models in Abstract Elementary Classes 435
Introduction 435
1. Types in Abstract Elementary Classes 437
2. Galois saturation and model-homogenuity are the same 441
3. α-limit models a substitute to saturation 444
4. Existence and uniqueness of Galois saturated models 451

Chapter 6. More on Types 459
Introduction 459
1. Definability and the Lascar group 461
2. Using models of set theory to establish consistency of a first-order theory 473
3. Game theoretic characterization of elementary embedding and isomorphism 476
4. Saturation of ultraproducts 478
5. Keisler-Shelah's theorem* 478
6. More on model complete theories* 479
7. Shelah's Generalization of Eherenfeucht-Mostowski models 480
8. $\mathrm{D}(T)$ as a topological space* 483
9. The topology of Lascar's groups 490
10. More on existence, omitting types, and the completeness theorem 491
11. The Paris Harrington's theorem* 497
12. More on two cardinal theorems* 498
13. Chang's conjecture and Jónsson algebras* 502

Chapter 7. Morley's Theorem 509
Introduction 509
1. Dimension in model theory 512
2. A rank function 514
3. $\aleph_0$-stability 522
4. Existence of indiscernibles, non-splitting and cohiers 530
5. Prime, primary and atomic models 541
6. Every model is saturated 550
7. Chang's Conjecture is true for $\aleph_0$-stable theories 556
8. Quasi-minimal formulas and an omitting types Theorem 558
9. Strongly minimal sets and the Baldwin-Lachlan proof 565
10. Some properties of $|T|^+$-categorical theories 584
11. The Baldwin Lachlan proof 589
12. Keisler's rank-free proof of Morley's theorem 591
13. Morely's rank and the local rank 597
14. Some properties of $\aleph_0$-stable theories 599

Chapter 8. Basics of Stability 601

Introduction 601
1. Local Types 602
2. Infinitely many Rank Functions 607
3. Characterizations of stability by rank and φ-types 623
4. Definability of types is equivalent to stablity 626
5. The order dichotomy 631
6. Sequences and sets of indiscernibles 641
7. Towards Los conjecture for uncountable first-order theories 650
8. The independence and strict-order properties 652
9. Superstable theories 665
10. Simple Theories 666
11. Noetherian topological spaces 667

Chapter 9. Forking calculus 671
Introduction 671
1. Basics of Forking 672
2. Stability spectrum theorem 700
3. Forking in Simple theories is symmetric and transitive 701
4. Applications of forking 705
5. Forking is canonical 706

Chapter 10. Applications 707
Introduction 707
1. Harnik's theorem 707
2. Uniqueness and characterization of prime models 708
3. Uniqueness of prime models 709
4. Stability spectrum 709

Chapter 11. Survey 711
Introduction 711
1. The main gap (Shelah's great theorem) 711
2. Classification theory for non-elementary classes 713
3. Geometric stability (or the fine structure theory) 713
4. Lang-Mordell 713
5. Ax and Kochen 713
6. *o*-minimal theories 713
7. Abstract model theory 714
8. Finite model theory 715
9. Non standard analysis 715

Chapter 12. A miniguide to the literature 717

Chapter 13. Open Problems 719
Introduction 719
1. Classification theory for non-elementary classes 719
2. Shelah's categoricity conjecture 719
3. Main Gap for uncountable theories 719
4. Other problems 720

Chapter 14. Historical comments 723

APPENDIX 733

Chapter 15. Some set theory 735
Introduction 735

1. Sets, functions and relations 735
2. Cardinal numbers 737
3. The Axiom of Choice, Zorn's Lemma and the well-ordering theorem 743
4. Ordinals 745
5. Martin's Axiom 749
6. On weak diamonds 750
7. The small subsets of λ^+ is a normal ideal 760
8. Kuratowski's and Hajnal's free subset theorems 763
9. The building-stones of many models 767

Chapter 16. Combinatorial geometry 771
Introduction 771
1. Pregeometries (or Matroids) 771
2. Abstract dependence 781
3. Projective geometries 781

Chapter 17. Plato: The Allegory of the Cave, from book VII The Republic 783

Bibliography 787

Index 799