Finite model theory, stability, and Ramsey's theorem

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August 4th, 1995

Abstract

We prove some cool theorems on the border of Ramsey theory (finite partition calculus) and model theory. Also a begining of classification theory for classes of finite models is attempted.

Introduction

Frank Ramsey in his fundamental paper (see [21] and page 18-27 of [10]) was interested in "a problem of formal logic" (- the title of [21]). The theorem known as "finite Ramsey's theorem": $\forall r, c < \omega, \forall k < \omega \ \exists n < \omega$ such that $n \to (k)^r_c$. Let $n(k, r, c)$ be the first such n. The theorem was used by him to construct finite models for finite universal theory such that the universe is canonical with respect to the relations in the language (model theorists call canonical sets Δ-indiscernibles (see Definition 1.1)).

Much is known about the order of magnitude of the function $n(k, r, c)$ and some of its generalizations (see [8], and TODORCEVIC 1993/4 ??). An upper bound is an $(r-1)$ - times iterated exponential of a polynomial in k and c. Many feel that the upper bound is tight. However especially for $r \geq 3$ the gap between the lower and upper bounds is huge.

In 1956 A. Ehernfeucht and A. Mostowski [7] rediscovered the usefullness of Ramsey's theorem in logic, and introduced the notion we now call indiscernibles. Several people continued exploiting the connections between partition theorems and logic (i.e. Model Theory), among them M. Morley (see [18] and [19]), and S. Shelah who has published virtually uncountable number of papers related to indiscernibles (see e.g. "The Bible" [27]). Morley [19] used indiscernibles to construct models of very large cardinality (relatively to the cardinality of the reals), namely he proved that the Hanf number of $L_{\omega_1,\omega}$ is $\beth_{\omega_1}$.

One of the most important developments in mathematical logic (certainly the most important in model theory) in the last 30 years is what is known as "classification theory" or also as "stability theory". There are several books dedicated entirely to some aspects of the subject, among them are books by: J. Baldwin [2], D. Lascar [16], S. Shelah [27], and A. Pillay [20].

Lately Shelah and others have done extensive work in extending classification theory from the context of first order logic, to classifying arbitrary classes of models usually for infinitary logics extending first order logic (for example see [3], [4], [5], [12], [14], [17] , [23] , [28] , [24]). [24] contains several philosophical and personal comments about why this research is interesting; [25] is video tape from the Shelah's planery talk at the International Congress of Mathematics at Berkeley in 1986.

This raises a question of fundamental importance: *Is there a classification theory for finite structures?* One of the basic problems in doing finite model theory for finite structures is in choosing an approperiate "sub model" relation, or in category-theoretic terminology introducing a natural morphism. In classification theory for elementary classes (- models of a first order (usually complete) theory) the right notion of morphism is "elementary embedding" which is defined using the relation $M \prec N$, which is a strenghtening of the notion of being submodel (denoted by $M \subseteq N$). Unfortunately for finite structures always $M \prec N$ implies $M = N$. Moreover, in many cases even $M \subseteq N$ implies $M = N$ (e.g. when N is a group of prime order). We need a substitute. One of the basic observations to make is that when we limit our attention to structures in a relational language only (no function symbols) then $M \subseteq N$ does not imply $M = N$. In general this seems to be insufficient (to force that the substructure will inherit some of the properties of the bigger structure (e.g. satisfying the same first order sentences)). It was observed already by Ramsey (in [21]) that if $M \subseteq N$ then for every universal sentence ϕ, $N \models \phi$ implies $M \models \phi$. So when studying the class of models of a universal first order theory the relation $M \subseteq N$ is reasonable (but not for more complicated theories, e.g. not every subfield of an algebraically closed field is algerbraically closed). Such a concept for classes of finite strucutres is introduced below (see Definition 4.9).

This paper has several goals:

1. *Study Ramsey numbers for definable coloring inside models for a stable theory.*

   This can be viewed as a direct extension of Ramsey's work, namely by taking into account the first order properties of the structures. A typical example is the field of complex numbers $\langle \mathbf{C}, +, \cdot \rangle$. It is well known that its first order theory $Th(\mathbf{C})$ has many nice properties: It is $\aleph_1$ - categorical, $\aleph_0$ - stable, and does not have the finite cover property. We will be interested in the following general situation: Given a first order (complete) theory T, and (an infinte) model $M \models T$. Let k, r, and c be natural numbers, and let F be a definable (by a first order formula in the language $L(T)$ maybe with parameters from M) coloring of a set of r- tuples from M by c colors. Let $n := n_F(k, r, c)$ be the minimal natural number such that for every $S \subseteq |M|$ of cardinbality n, if $F : [S]^r \to c$ then there exists $S^* \subseteq S$ of cardinality k such that F is constant on $[S^*]^r$. It turns out that for stable theories, (or even for theories without the independence property) we get better upper bounds than for the general Ramsey numbers. This indicates that one can not improve the lower bounds by looking at stable structures.

2. *Introduce stability-like properties (e.g. n-order property, k-independence property, d-cover property), as well as averages of finite sequences of indiscernibles. Some of the interconnections, and the effect on existence of indiscernibles are presented.*

3. *Develope classification theory for classes of finite structures, in particular introduce a notion that correspond to stable amalgamation, show that it is symmetric for many models.*

   See Example 4.8 below.

4. *Bringing down uncountable techniques into finite context.*

   We believe that much of the machinery developed (mainly by Shelah) to deal with problems concerning categoricity of infinitary logics, and the behaviour of the spectrum function at cardinalities $\geq \beth_{\omega_1}$ depends on some very powerfull combinatorial ideas. We try here to extract some of these ideas and present them in a finite context.

Shelah [27] proved that instability is equivalent to the presence of either the strict order property or the independence property. In a combinatorial setting, stability means that for arbitrarily large sets, the number of types over a set is polynomial in the cardinality of the set. We address the finite case here in which we restrict our attention to when the number of ϕ- types over a finite set is bounded by a polynomial in the size of the set of parameters.

First we find precisely the degree of the polynomial bound on the number of these types given to us by the absence of the strict order or independence properties. This is an example of something relevant in the finite case which is of no concern in the usual classification theory framework.

Once we have these sharper bounds we can find sequences of indiscernibles in the spirit of [27]. It should be noted here that everything we do is "local", involving just a single formula (or equivalently a finite set of formulas). We then work through the calculations for uniform hypergraphs as a case study. This raises questions about "stable" graphs and hypergraphs which we begin to answer.

In the second half of the paper, we examine classes of finite structures in the framework of Shelah's classification for non-elementary classes (see [26]). In particular, we make an analogy to Shelah's "abstract elementary classes" and prove results similar to his.

Notation: Everything is standard . Often x, y, and z will denote free variables, or finite sequences of variables, when x is a sequence $l(x)$ denote its length. It should be clear from the context whether we deal with variables of sequences of variables. L will denote a similarity type (aka - language or signature), Δ will stand for (usually a finite) set of L formulas. M and N will stand for L - structures, $|M|$ the universe of the structure M, $\|M\|$ the cardinality of the universe of M. Given a fixed structure M, subsets of its universe will be denoted by A, B, C, and D. So when we write $A \subseteq M$ we really mean that $A \subseteq |M|$, while $N \subseteq M$ stands for "N is a submodel of M". Let M be a structure. By $a \in M$ we mean $a \in |M|$, when a is a finite sequence of elemets then $a \in M$ stands for "all the elements of the sequence a are elements of $|M|$".

Since all our work will be inside a given structure M (with the exception of section 4), all the notions a relative to it. For example for $a \in M$ and $A \subseteq M$ we denote by $tp_\Delta(a, A)$ the type $tp_\Delta(a, A, M)$ which is $\{\phi(x; b) : M \models \phi[a; b], b \in A, \phi(x; y) \in \Delta\}$ and if $A \subseteq M$ then $S_\Delta(A, M) := tp_\Delta(a, A) : a \in M$. Note that in [27] $S_\Delta(A, M)$ stands for the set of all complete Δ -types (with parameters from A that are consistent with $Th(\langle M, c_a \rangle_{a \in A})$. It is important for us to limit attention to the types realized in M, in order to avoid dependence on the compactness theorem. When $\Delta = \{\phi\}$ then instead of writing $tp_\Delta(\cdots)$ and $S_\Delta(\cdots)$ will write $tp_\phi(\cdots)$ and $S_\phi(\cdots)$ respectively.

1 The effect of the order and independence properties on the number of local types

We first fix some notation and terms and then define the first important concepts, the first three are from [27] #4 is a generalization of a definition of Shelah, #5 is from Grossberg and Shelah [13].

Definition 1.1 *1. For a set Δ of L - formulas and a natural number n, a $\underline{(\Delta, n) - \text{type over a set } A}$ is a set of formulas of the form $\phi(x; a)$ where $\phi(x; y) \in \Delta$ and $a \in A$ with $l(x) = n$. If $\Delta = L$, we omit it, and we just say "ϕ – type" for a $(\{\phi(x; a)\}, l(x))$ – type.*

2. *Given a (Δ, n) type p over A, define $dom(p) = \{a \in A \,:\, \text{for some } \phi \in \Delta,\ \phi(x; a) \in p\}$.*

3. *A type p $\underline{(\Delta_0, \Delta_1) - \text{splits over } B \subseteq dom(p)}$ if there is a $\phi(x; y) \in \Delta_0$ and $b, c \in dom(p)$ such that $tp_{\Delta_1}(b, B) = tp_{\Delta_1}(c, B)$ and $\phi(x; b), \neg\phi(x; c) \in p$. If p is a Δ – type and $\Delta_0 = \Delta_1 = \Delta$, then we just say $\underline{p \text{ splits over } B}$.*

4. *We say that $\underline{(M, \phi(x; y)) \text{ has the } k \text{ - independence property}}$ if there are $\{a_i : i < k\} \subseteq M$, and $\{b_w : w \subseteq k\} \subseteq M$, such that $M \models \phi[a_i; b_w]$ if and only if $i \in w$. We will say that $\underline{M \text{ has the } k - \text{independence property}}$ when there is a formula ϕ such that (M, ϕ) does.*

5. *$\underline{(M, \phi(x; y)) \text{ has the } n - \text{order property}}$ (where $l(x) = l(y) = k$) if there exists a set of k – tuples $\{a_i : i < n\} \subseteq M$ such that $i < j$ iff $M \models \phi[a_i, a_j]$ for all $i, j < n$. We will say that $\underline{M \text{ has the } n - \text{order property}}$ if there is a formula ϕ so that (M, ϕ) has the n – order property.*

WARNING: This use of "order property" corresponds to neither the order property nor the strict order property in [27]. The definition comes rather from [11].

Note that the following monotonicity properties are easy to prove:

Proposition 1.2 *1. let p be a complete Δ type, suppose $B \subseteq C \subseteq A$, and $Domp \subseteq A$ If p does not split over B then p does not split over C.*

2.

Fact 1.3 (Shelah see [27]) *Let T be a complete first order theory. The following conditions are equivalent:*

1. *T is unstable.*

2. *There are $\phi(x;y) \in L(M)$, $M \models T$, and $\{a_n : n < \omega\} \subseteq M$ such that $l(x) = l(y) = l(a_n)$, and for every $n, k < \omega$ we have*
   $n < k \Leftrightarrow M \models \phi[a_n; a_k]$.

Using the compactness theorem it is easy to prove the following-

Corollary 1.4 *Let T be a stable theory, suppose that $M \models T$ is an infinite model.*

1. *For every $\phi(x, y) \in L(M)$ there exists a natural number $n(\phi)$ such that (M, ϕ) does not have the $n(\phi)$ – order property.*

2. *For every $\phi(x, y) \in L(M)$ there exists a natural number $k(\phi)$ such that (M, ϕ) does not have the $k(\phi)$ – independence property.*

3. *If T is categorical in some cardinality greater than $|T|$ then for every $\phi(x, y) \in L(M)$ there exists a natural number $d(\phi)$ such that (M, ϕ) does not have the $d(\phi)$ – cover property (see Definition 1.14).*

We first establish that the failure of either the independence property or the order property for ϕ implies there is a polynomial bound on the number of ϕ – types. The more complicated of these to deal with is the failure of the order property. At the same time this is perhaps the more natural property to look for in a given structure. Shelah have proved that the failure of the independence property gives us a far better bound (i.e., smaller degree polynomial) with less work, this will be reproduced here in Theorem 1.13.

This first lemma is the finite version of Lemma 5 from [11].

Lemma 1.5 *Let $\phi(x;y)$ be a formula in L, n a positive integer, $s = l(y)$, $r = l(x)$, $\psi(y;x) = \phi(x;y)$. Suppose that $\{A_i \subseteq M_i \ : \ i \leq 2n\}$ is an increasing chain such that for every $B \subseteq A_i$ with $|B| \leq 3sn$, every type in $S_\phi(B,M)$ is realized in A_{i+1}. Then if there is a type $p \in S_\phi(A_{2n},M)$ such that $p|A_{i+1}$ (ψ,ϕ) – splits over every subset of A_i of size at most $3sn$, then (M,ρ) has the $n-$ order property, where*

$$\rho(x_0,x_1,x_2;y_0,y_1,y_2) \stackrel{\text{def}}{=} [\phi(x_0;y_1) \leftrightarrow \phi(x_0;y_2)]$$

Proof: Let d realize p. Define $\{a_i,b_i,c_i \in A_{2i+2} \ : \ i < n\}$ by induction on i. Assume for some $j < n$ that we have defined these for all $i < j$. Let $B_j = \bigcup\{a_i,b_i,c_i \ : \ i < j\}$. Notice that $|B_j| \leq 3sj < 3sn$, so by the assumption, $p|A_{2j+1}$ (ψ,ϕ) – splits over B_j. That is, there are a_j, $b_j \in A_{2j+1}$ such that

$$tp_\psi(a_j,B_j,M) = tp_\psi(b_j,B_j,M),$$

and

$$M \models \phi[d;a_j] \wedge \neg\phi[d;b_j].$$

Now choose $c_j \in A_{2j+1}$ realizing $tp(d,B_j \cup a_j \cup b_j,M)$ (which can be done since $|B_j \cup a_j \cup b_j| \leq 3sj + 2s < 3s(j+1) \leq 3sn$). This completes the inductive definition.

For each i, let $d_i = c_ia_ib_i$. We will check that the sequence of d_i and the formula

$$\rho(x_0,x_1,x_2;y_0,y_1,y_2) \stackrel{\text{def}}{=} [\phi(x_0;y_1) \leftrightarrow \phi(x_0;y_2)]$$

witness the n – order property for M.

If $i < j < n$, then $c_i \in B_j$. By choice of a_j and b_j, $tp_\psi(a_j,B_j,M) = tp_\psi(b_j,B_j,M)$, so in particular,

$$M \models \phi[c_i;a_j] \leftrightarrow \phi[c_i;b_j]$$

That is, $M \models \rho[d_i;d_j]$.

On the other hand, if $i \leq j < n$, then $\phi(x;a_i) \in tp_\phi(d,B_j \cup a_j \cup \ b_j,M)$ and $\phi(x;b_i) \notin tp_\phi(d,B_j \cup a_j \cup \ b_j,M)$, and so, by the choice of c_j, we have that

$$M \models \phi[c_j;a_i] \wedge \neg\phi[c_j;b_i].$$

That is, $M \models \neg\rho[d_j; d_i]$ in this case. ∎

In order to see the relationship between this definition of the order property and Shelah's, we mention Corollary 1.8.

Definition 1.6 *(M, ϕ) has the weak m – order property if there exist $\{d_i : i < m\} \subseteq M$ such that for each $j < m$,*

$$M \models \exists x \bigwedge_{i<m} \phi(x; d_i)^{if(i \geq j)}.$$

REMARK: This is what Shelah calls the m – order property.

Definition 1.7 *We write $x \to (y)^a_b$ if for every partition Π of the a - element subsets of $\{1, \ldots, x\}$ with b parts, there is a y - element subset of $\{1, \ldots, x\}$ with all of* its *a - element subsets in the same part of Π.*

Corollary 1.8

1. *If $(2n) \to (m+1)^2_2$ and the hypotheses for Lemma 1.5 hold, then ϕ has the weak m – order property in M.*
2. *If $n \geq \frac{2^{2m-1}}{\pi m}$ and the hypotheses for Lemma 1.5 hold, then ϕ has the weak m – order property in M.*

Proof: (This is essentially [27] I.2.10(2))

1. Let a_i, b_i, c_i for $i < n$ be as in the proof of Lemma 1.5. For each pair $i < j \leq n$, define

$$\chi(i,j) := \begin{cases} 1 & \text{if } M \models \phi[c_i; a_j] \\ 0 & \text{otherwise.} \end{cases}$$

Since $(2n) \to (m+1)^2_2$, we can find a subset I of $2n$ of cardinality $m+1$ on which χ is constant. Write $I = \{i_0, \ldots, i_m\}$.

If χ is 1 on I, then for every k with $1 \leq k \leq m+1$

$$\{\neg\phi(x; b_{i_j})^{if(j>k)} \ : \ 1 \leq j < m\}$$

is realized by $c_{i_{k-1}}$. Therefore, the sequence $\{b_{i_0}, \ldots, b_{i_m}\}$ witnesses the weak m – order property of ϕ in M.

On the other hand, if χ is 0 on I, then for every k with $1 \leq k \leq m+1$

$$\{\neg\phi(x; a_{i_j})^{if(j>k)} \ : \ 1 \leq j < m\}$$

is realized by $c_{i_{k-1}}$. Therefore, the sequence $\{a_{i_0}, \ldots, a_{i_m}\}$ witnesses the weak m – order property of $\neg\phi$ in M. To finish, we need only remark that it is equivalent for ϕ and $\neg\phi$ to have the weak m – order property in M.

2. By Stirling's formula, $n \geq \frac{2^{2m-1}}{\pi m}$ implies that $n \geq \frac{1}{2}\begin{pmatrix} 2m \\ m \end{pmatrix}$, and from [10], $n \geq \frac{1}{2}\begin{pmatrix} 2m \\ m \end{pmatrix}$ implies that $(2n) \rightarrow (m+1)^2_2$.

■

We can now establish the relationship between the number of types and the order property.

Theorem 1.9 *If $\phi(x; y) \in L(M)$ is such that*

$$\rho(x_0, x_1, x_2; y_0, y_1, y_2) \stackrel{\text{def}}{=} \phi[(x_0; y_1) \leftrightarrow \phi(x_0; y_2)]$$

does not have the n – order property in M, then for every set $A \subseteq M$ with $|A| \geq 2$, we have that $|S_\phi(A, M)| \leq 2n|A|^k$, where $k = 2^{(3ns)^{t+1}}$ for $r = l(x)$ and $s = l(y)$ and $t = \max\{r, s\}$.

Proof: Suppose that there is some $A \subseteq M$ with $|A| \geq 2$ so that $|S_\phi(A, M)| > (2n)|A|^k$. Let $\psi(y; x) = \phi(x; y)$, $m = |A|$, and let $\{a_i \ : \ i \leq (2n)m^k\} \subseteq M$ be witnesses to the fact that $|S_\phi(A, M)| > (2n)m^k$. (That is, each of these tuples realizes a different ϕ – type over A.) Define $\{A_i : i < 2n\}$, satisfying

1. $A \subseteq A_i \subseteq A_{i+1}$,

2. $|A_i| \leq c^{e(i)} m^{(3ns)^i}$,where $c = 2^{2+(3sn)^t}$ and $e(i) := \frac{(3ns)^i - 1}{3ns-1}$, and

3. for every $B \subseteq A_i$ with $|B| \leq 3sn$, every $p \in S_\phi(B, M) \cup S_\psi(B, M)$ is realized in A_{i+1}.

To see that this can be done, one need only check the cardinality constraints. There are at most $|A_i|^{3sn}$ subsets of A_i with cardinality at most $3sn$, and over each such subset B, there are at most $2^{(3sn)^r}$ and $2^{(3sn)^s}$ types in $S_\psi(B, M)$ and $S_\phi(B, M)$, respectively, so there are at most $2^{(3sn)^r} + 2^{(3sn)^s} \leq 2^{1+(3sn)^t}$ types in $S_\psi(B, M) \cup S_\phi(B, M)$ for each such B. Therefore, A_{i+1} can be defined so that

$$\begin{aligned} |A_{i+1}| &\leq |A_i| + (2^{1+(3sn)^t})|A_i|^{3sn} \\ &\leq c|A_i|^{3sn} \\ &\leq c(c^{e(i)} m^{(3ns)^i})^{3sn} \\ &= c^{1+e(i)(3sn)} m^{(3sn)^{i+1}} \\ &= c^{e(i+1)} m^{(3sn)^{i+1}}. \end{aligned}$$

Claim 1.10 *There is a $j < (2n)m^k$ such that for every $i < 2n$ and every $B \subseteq A_i$ with $|B| \leq 3sn$, $tp(a_j, A_{i+1})$ (ψ, ϕ) – splits over B.*

Proof: (Of Claim 1.10) Suppose not. That is, for every $j \leq (2n)m^k$, there is an $i(j) < 2n$ and a $B \subseteq A_{i(j)}$ with $|B| \leq 3sn$, so that $tp(a_j, A_{i(j)+1})$ does not (ψ, ϕ) – split over B. Since i is a function from $1 + (2n)m^k$ to $2n$, there must be a subset S of $1+(2n)m^k$ with $|S| > m^k$, and an integer $i_0 < 2n$ such that for all $j \in S$, $i(j) = i_0$. Now similarly, there are less than $|A_{i_0}|^{3sn}$ subsets of A_{i_0}, with cardinality at most $3sn$, so there is a $T \subseteq S$ with

$$|T| > \frac{m^k}{|A_{i_0}|^{3sn}},$$

and a $B_0 \subseteq A_{i_0}$, with $|B_0| \leq 3sn$ such that for all $j \in T$, $tp(a_j, A_{i_0+1})$ does not (ψ, ϕ) – split over B_0. Since $|A_{i_0}| \leq c^{e(i_0)} m^{(3ns)^{i_0}} \leq (cm)^{(3sn)^{2n}}$, then

$$|T| \geq \frac{m^k}{(cm)^{(3sn)^{2n}}}.$$

Let $C \subseteq A_{i_0+1}$ be obtained by adding to B_0, realizations of every type in $S_\phi(B_0, M) \cup S_\psi(B_0, M)$. This can clearly be done so that $|C| \leq 3ns + 2^{(3ns)^r} + 2^{(3ns)^s}$. The maximum number of ϕ – types over C is at most $2^{|C|^s} \leq 2^{c^s}$. ■

Claim 1.11 $m^{k-(3ns)^{2n}} > (2^{c^s})(c^{(3ns)^{2n}})$

Proof: (Of Claim 1.11) Since $c = 2^{2+(3ns)^t}$, we have $c^s + (3ns)^{2n}(2 + (3ns)^t)$ as the exponent on the right-hand side above. Since $m \geq 2$, it is enough to show that

$$\begin{aligned} k &> (c^s + (3ns)^{2n}(2 + (3ns)^t) + (3ns)^{2n} \\ &= 2^{s(2+(3ns)^t)} + (3ns)^{2n}(3 + (3ns)^t). \end{aligned}$$

This follows from the definition of k (recall that $k = 2^{(3ns)^{t+1}}$), so we have established Claim 1.11. ■

Therefore, $|T|$ is greater than the number of ϕ – types over C, so there must be $i \neq j \in T$ such that $tp_\phi(a_i, C) = tp_\phi(a_j, C)$. Since $tp_\phi(a_i, A) \neq tp_\phi(a_j, A)$, we may choose $a \in A$ so that $M \models \phi[a_i, a] \wedge \neg\phi[a_j, a]$. Now choose $a' \in C$ so that $tp_\psi(a, B_0) = tp_\psi(a', B_0)$ (this is how C is defined after all). Since $tp_\phi(a_i, A_{i_0+1})$ does not (ψ, ϕ) – split over B_0, we have that

$$\phi(x; a) \in tp_\phi(a_i, A_{i_0+1}) \text{ iff } \phi(x; a') \in tp_\phi(a_i, A_{i_0+1}),$$

so $M \models \phi[a_i, a'] \wedge \neg\phi[a_j, a']$, contradicting the fact that $tp_\phi(a_i, C) = tp_\phi(a_j, C)$ and thus completing the proof of Claim 1.10. Now letting j be as in Claim 1.10 above and applying Lemma 1.5 completes the proof of Theorem 1.9. ■

Theorem 1.13 below gives us a better result under different assumptions. The next lemma is II, 4.10, (4) in [27]. It is a question due to Erdős about the so-called "trace" of a set system which was answered by Shelah and Perles (see [22]) in 1972. Purely combinatorial proofs (i.e., proofs in the language of combinatorics) can also be found in most books on extremal set systems (e.g., Bollobas [6]).

Lemma 1.12 *If S is any family of subsets of the finite set I with*

$$|S| > \sum_{i<k} \binom{|I|}{i},$$

then there exist $\alpha_i \in I$ for $i < k$ such that for every $w \subseteq k$ there is an $A_w \in S$ so that $i \in w \Leftrightarrow \alpha_i \in A_w$. (The conclusion here is equivalent to trace$(I) \geq k$ in the language of [6].)

Proof: See Theorem 1 in Section 17 of [6] or Ap.1.7(2) in [27]. ∎

Theorem 1.13 *If $\phi(x;y) \in L(M)$ $(r = l(x),\ s = l(y))$ does not have the k – independence property in M, then for every set $A \subseteq M$, if $|A| \geq 2$, then $|S_\phi(A, M)| \leq |A|^{s(k-1)}$.*

Proof: (Essentially [Sh], II.4.10(4)) Let F be the set of ϕ – formulas over A. Then

$$|F| < |A|^s.$$

So if $|S_\phi(A, M)| > |A|^{s(k-1)}$, then certainly

$$|S_\phi(A, M)| > \sum_{i<k} \binom{|F|}{i},$$

in which case Lemma 1.12 can be applied to F and $S_\phi(A, M)$ to get witnesses to the k – independence property in M, a contradiction. ∎

The "moral" of Theorem 1.9 and Theorem 1.13 is that there is a bound on the number of ϕ – types over a set A which is polynomial in $|A|$ in case ϕ has some nice properties. Note that the difference between the two properties is that the degree of the polynomial in the absence of the k – independence property is linear in k while in the absence of the n – order property the degree is exponential in n.

Another property discovered by Keisler (in order to study saturation of ultrapowers, see [15]), and studied extensively by Shelah is the "finite cover property" (see [27]) whose failure essentially provides us with a streghtening of the compactness theorem.

Definition 1.14 *We say that* <u>(M, ϕ) *does not have the* d *- cover property*</u> *if for every* $n \geq d$ *and* $\{b_i : i < n\} \subseteq M$ *such that*

$$\left(\forall w \subseteq n)[|w| < d \Rightarrow M \models \exists x \bigwedge_{i \in w} \phi(x; b_i)] \right) \Rightarrow M \models \neg \exists x \bigwedge_{i < n} \phi(x; b_i).$$

EXAMPLE **1.15** *If* $M = (M, R)$ *is the countable random graph, then* (M, R) *fails to have the 2 – cover property. If* M *is the countable universal homogeneous triangle - free graph, then* (M, R) *fails to have the 3 – cover property.*

The following is obvious:

Proposition 1.16 *Let* $d_1 \leq d_2$. *If* (M, ϕ) *fail to have the* d_1 *– cover property then* (M, ϕ) *does not have the* d_2 *– cover property.*

Before proceeding, we point out what happens in the "unstable" cases. In [27] II.4.10 (3) we find the following

Fact 1.17 *If for arbitrarily large* k, $(M, \phi(x; y))$ *(*$r = l(x)$, $s = l(y)$*) has the* k *– independence property, then there are arbitrarily large* $A \subseteq M$ *such that* $|S_\phi(A, M)| \geq 2^{|A|/s}$.

2 Indiscernible sequences in large finite sets

NOTE: The next defintion is an interpolant of Shelah's [27], I.2.3, and Ramsey's notion of canonical sequence.

Definition 2.1 *1. A sequence $I = \langle a_i : i < n\rangle \subseteq M$ is called an $\underline{(\Delta, m) \text{ – indiscernible sequence over } A \subseteq M}$ (where Δ is a set of $L(M)$ – formulas) if for every $i_0 < \ldots < i_{m-1} \in I$, $j_0 < \ldots < j_{m-1} \in I$ we have that $tp_\Delta(a_{i_0} \cdots a_{i_{m-1}}, A, M) = tp_\Delta(a_{j_0} \cdots a_{j_{m-1}}, A, M)$*

2. A sequence $I = \langle a_i : i < n\rangle \subseteq M$ is called an $\underline{(\Delta, m) \text{ – indiscernible set over } A \subseteq M}$ iff for every $\{i_0, \ldots, i_{m-1}\}$, $\{j_0, \ldots, j_{m-1}\} \subseteq I$ we have

$$tp_\Delta(a_{i_0} \cdots a_{i_{m-1}}, A, M) = tp_\Delta(a_{j_0} \cdots a_{j_{m-1}}, A, M).$$

EXAMPLE **2.2** *1. In the model $M_n = \langle m, 0, 1, \chi\rangle$ $(n \le m < \omega)$ where χ is function from the increasing n – tuples of m to $\{0, 1\}$, any increasing enumeration of a monochromatic set is an example of a (Δ, n) – indiscernible sequence over with $\Delta = \{\chi(x) = 0, \chi(x) = 1\}$.*

2. In a graph (G, R), cliques and independent sets are examples of $(R, 2)$ – indiscernible sets over $\emptyset$.

It turns out that similarly to the situation for first order theories (under the assumption that T is stable, every sequence of indiscernibles is a set of indiscernibles) also in our case this is true, however the failure of the n - *order property* is sufficient. Our argument follows closely that of Shelah [27].

Theorem 2.3 *If M does not have the n – order property, then any sequence $I = \langle a_i : i < n + r - 1\rangle \subseteq M$ which is $\phi(x; y)$ – indiscernible over $B \subseteq M$ is a set of ϕ – indiscernibles over B (where $r = l(x)$).*

Proof: Since any permutation of $\{1, \ldots, n\}$ is a product of transpositions $(k, k + 1)$, and since I is an indiscernible sequence over B, it is enough to show that for each $b \in B$ and $k < r$,

$$M \models \phi[a_0 \cdots a_{k-1} a_{k+1} a_k \cdots a_{r-1}; b] \leftrightarrow \phi[a_0 \cdots a_{k-1} a_k a_{k+1} \cdots a_{r-1}; b].$$

Suppose not. Then we may choose $b \in B$ and $k < r$ so that

$$M \models \neg\phi[a_0 \cdots a_{k-1} a_{k+1} a_k \cdots a_{r-1}; b] \wedge \phi[a_0 \cdots a_{k-1} a_k a_{k+1} \cdots a_{r-1}; b].$$

Let $c = a_0 \cdots a_{k-1}$ and $d = a_{n+k+1} \cdots a_{n+r-2}$ making $l(c) = k$ and $l(d) = r - k - 2)$. By the indiscernibility of I,

$$M \models \neg\phi[c a_{k+1} a_k d; b] \wedge \phi[c a_k a_{k+1} d; b].$$

For each i and j with $k \leq i < j < n+k$, we have (again by the indiscernibility of the sequence I) that

$$M \models \neg\phi[c a_j a_i d; b] \wedge \phi[c a_i a_j d; b].$$

Thus the formula $\psi(x, y; cdb) \overset{\text{def}}{=} \phi(c, x, y, d; b)$ defines an order on $\langle a_i : k \leq i < n + k \rangle$ in M, a contradiction. ■

The following definition is an generalization of the notion of *end homogenous* sets in combinatorics (see section 15 of [8]) to the context of Δ - indiscernible sequences.

Definition 2.4 *1. A sequence $I = \langle a_i : i < n \rangle \subseteq M$ is called an end - (Δ, m) – indiscernible sequence over $A \subseteq M$ (where Δ is a set of $L(M)$ – formulas) if for every $\{i_0, \ldots, i_{m-2}\}$ (distinct) and $j_0, j_1 < n$ both larger than $\max\{i_0, \ldots, i_{m-2}\}$, we have*

$$tp_\Delta(a_{i_0} \cdots a_{i_{m-2}} a_{j_0}, A, M) = tp_\Delta(a_{i_0} \cdots a_{i_{m-2}} a_{j_1}, A, M)$$

2. If every formula $\phi(x; y) \in \Delta$ has $l(x) = m$, then we will write just Δ instead of (Δ, m) above.

Definition 2.5 *For the following lemma, let $F : \omega \to \omega$ be given, and fix the parameters, α, r, and s. We define the function F^* for each $k \geq r$ as follows:*

- $F^*(0) = 1$,

- $F^*(j+1) = [1 + F^*(j)]F(\alpha + j)$ *for* $j < k - 2 - r$, *and*
- $F^*(j+1) = 1 + F^*(j)$ *for* $k - 2 - r \leq j < k - 2$.

We will not need $j \geq k - 2$.

Lemma 2.6 *Let* $\psi(x; y) = \phi(x_1, \ldots, x_{r-1}, x_0; y)$. *If for every* $B \subseteq M$, $|S_\psi(B, M)| < F(|B|)$, *and* $I = \{a_i : i \leq F^*(k-2)\} \subseteq M$ *(where* $l(x) = r$, $l(y) = s$, $\alpha = |A|$*), then there is a* $J \subseteq I$ *such that* $|J| \geq k$ *and* J *is a* ϕ *– end - indiscernible sequence over* A.

Proof: We will construct $A_j = \{a_i : i \leq j\}$ and S_j by induction on $j < k - 1$ so that

1. $a_j = \min S_j$,
2. $S_{j+1} \subseteq S_j$,
3. $|S_j| > F^*(k - 2 - j)$, and
4. whenever $\{i_1, \ldots, i_{r-1}\} \subseteq j$, $b \in S_j$, and $1 \leq m \leq r$

$$tp_\phi(a_{i_1} \cdots a_{i_{r-1}} a_j, A, M) = tp_\phi(a_{i_1} \cdots a_{i_{r-1}} b, A, M).$$

The construction is completed by taking an arbitrary $a_{k-1} \in S_{k-2} - \{a_{k-2}\}$. (which is possible by (3) since $F^*(0) = 1$), and letting $J = \langle a_i : i < k \rangle$. We claim that J will be the desired ϕ – end - indiscernible sequence over A.

To see this, let $\{i_0, \ldots, i_{r-1}, j_0, j_1\} \subseteq k$ with $\max\{i_0, \ldots, i_{r-1}\} < j_0 < j_1 < k$ be given. Certainly then $\{i_0, \ldots, i_{r-1}\} \subseteq j_0$ and $a_{j_1} \in S_{j_0}$, so by (4) we have that

$$tp_\phi(a_{i_0} \cdots a_{i_{r-1}} a_{j_0}, A, M) = tp_\phi(a_{i_0} \cdots a_{i_{r-1}} a_{j_1}, A, M).$$

To carry out the construction, first set $a_j = j$ and $S_j = \{i : j \leq i \leq F^*(k-2)\}$ for $0 \leq j \leq r - 1$. Clearly, we have satisfied all conditions in this.

Now assume for some $j \geq r$ that A_{j-1} and S_{j-1} have been defined satisfying the conditions.

Define the equivalence relation $\sim$ on $S_{j-1} - \{a_{j-1}\}$ by $c \sim d$ iff for all $\{i_0, \ldots, i_{r-1}\}$,

$$tp_\phi(a_{i_1} \cdots a_{i_{r-1}} c, A, M) = tp_\phi(a_{i_1} \cdots a_{i_{r-1}} d, A, M)$$

The number of $\sim$ - classes then is at most $|S_\psi(A, A_j)| < F(\alpha+j)$. Therefore, at least one class S_j has cardinality at least $\frac{|S_{j-1}|-1}{F(\alpha+j)}$. Let $a_j = \min S_j$. By definition of F^*, $\frac{F^*(k-2-j+1)}{F(\alpha+j)} > F^*(k-2-j)$, so we have that $|S_j| > F^*(k-2-j)$. It is easy to see that condiiton (4) is satisfied. ■

For the following lemma, we once again need a function defined in terms of the parameters of the problem. We will need the parameter r and the function F^* defined for Lemma 2.6 (which depends on r, α, and s). Let f_i be the F^* that we get when $r = i$ (and α and s fixed) in Lemma 2.6.

For the following lemma define

$$g_i := \begin{cases} id & \text{if } i = 0 \\ f_{i-1} \circ (g_{i-1} - 2) & \text{otherwise.} \end{cases}$$

Lemma 2.7 *If $J = \{a_i : i \leq g_{r-1}(m-1)\} \subseteq M$ is a ϕ – end - indiscernible sequence over $A \subseteq M$, then there is a $J' \subseteq J$ such that $|J'| \geq m$ and J' is a ϕ – indiscernible sequence over A.*

Proof: (By induction on r.) Note that if $r = 1$, there is nothing to do since end - indiscernible *is* indiscernible in this case. Assume that the above result is true for all formulas with $l(x) < r$ ($r > 1$), and consider ϕ as given above. Let c be the last element in J. Define ψ so that

$$M \models \psi[a_1, \ldots, a_{r-1}; b] \text{ iff } M \models \phi[a_1 \cdots a_{r-1} c; b]$$

for all $a_0, \ldots, a_{r-1} \in J$, $b \in M$. Note then that $|S_\psi(B) \leq |S_\phi(B)|$ for all $B \subseteq M$, so we can use the same F_1 for ψ as for ϕ. (This result can be improved by using a sharper bound on the number of ψ – types.) By the definition of g_{r-1}, there must be a subset J'' of J with cardinality at least $g_{r-2}(m-1)$ which is ψ – end - indiscernible over A. By the inductive hypothesis, there is a subsequence of J'' with cardinality at least $m-1$ which is ψ – indiscernible over A. Form J' by adding c to the end of this sequence. It follows from the ϕ – end - indiscernibility of J and the ψ – indiscernibility of J'' that J' is ϕ – indiscernible over A. ■

Theorem 2.8 *For any $A \subseteq M$ and any sequence I from M with $|I| \geq g_r(m-1)$, there is a subsequence J of I with cardinality at least m which is ϕ – indiscernible over A.*

Proof: By Lemmas 2.6 and 2.7. ■

Our goal now is to apply this to theories with different properties to see how these properties affect the size of a sequence one must look in to be assured of finding an indiscernible sequence. First we will do a basic comparison between the cases when we do and do not have a polynomial bound on the number of types over a set. In each of these cases, we will give the bound to find a sequence indiscernible over $\emptyset$. We will use the notation $\log^{(i)}$ for

$$\log_2 \circ \log_2 \circ \cdots \circ \log_2 \ \ (i \text{ times}).$$

Corollary 2.9 *1. If $F(i) = 2^{i^r}$ (which is the worst possible case), then $\log^{(r)} g_r(m-1) \leq 4m$.*

2. If $F(i) = i^p$, then $\log^{(r)} g_r(m-1) \leq 2m + \log_2 m + \log_2 p$.

We now combine part (2) above with the results from the previous section to see what happens in the specific cases of structures without the n – order property and structures without the k – independence property. We define by induction on i the function $\beth(i,x)$ by $\beth(0,x) = x$ and $\beth(i+1,x) = 2^{\beth(i,x)}$, for all x. Recall that for the formula $\phi(x;y)$ we have defined the parameters $r = l(x)$, $s = l(y)$, and $t = \max\{r,s\}$.

Corollary 2.9 *3. If (M, ϕ) fails to have the k - independence property and $I = \{a_i : i < \beth(r, 2m + \log_2 m + \log_2 k + \log_2 r)\} \subseteq M$, then there is a $J \subseteq I$ so that $|J| \geq m$ and J is a ϕ – indiscernible sequence over $\emptyset$.*

4. If (M, ϕ) fails to have the n – order property and $I = \{a_i : i < \beth(r, 2m + \log_2 m + (3ns)^{t+1})\} \subseteq M$, then there is a $J \subseteq I$ so that $|J| \geq m$ and J is a ϕ – indiscernible sequence over $\emptyset$.

Finally, note that with the additional assumption of failure of the d – cover property, if d is smaller than k or n, then from the assumptions in (3) and (4) above, we could infer a failure of the d – independence propery or the d – order property improving the bounds even further.

3 Applications to Graph Theory

To illustrate some applications, we look to graph theory. The reader should be warned that the word "independent" has a graph - theoretic meaning, so care must be taken when reading "independent set" versus "independence property". A first question is "How much independence can one expect a random graph to have?" We will approach the answer to this question along the lines of Albert & Frieze [1]. There an analogy is made to the Coupon Collector Problem, and we will continue this here.

The Coupon Collector Problem (see Feller [9]) is essentially that if n distinct balls are independently and randomly distributed among m labelled boxes (so each distribution has the same probability m^{-n} of occurring), then what is the probability that no box is empty? Letting $q(n,m)$ be this probability, it is easy to compute that

$$q(n,m) = \sum_{i=0}^{m} (-1)^i \left(\begin{array}{c} m \\ i \end{array} \right) \left(1 - \frac{i}{m}\right)^n = \frac{m!\, S_{m,n}}{m^n}$$

where $S_{n,m}$ is the Stirling number of the second kind.

It is well - known that, for $\lambda = me^{-n/m}$, $q(n,m) - e^{-\lambda}$ tends to 0 as n and m get large with λ bounded.

The way that this will be applied in our context is as follows. We will say that a certain set $\{v_1, \ldots, v_k\}$ of vertices *witnesses the k - independence property in G* if (G, R) has the k – independence property with $a_i = v_i$ (see Definition 4). Notice that any k vertices $\{v_1, \ldots, v_k\}$ determine 2^k "boxes" defined by all possible Boolean combinations of formulas $\{R(x, v_1), \ldots, R(x, v_k)\}$ (a vertex being "in a box" meaning it witnesses the corresponding formula in G). The remaining $n - k$ vertices are then equally likely to fill each of the 2^k boxes, so the probability that these k vertices witness the k – independence property in G is just $q(n-k, 2^k)$. So for $\lambda = \lambda(n,k) = 2^k \exp(-(n-k)/2^k)$ bounded (as $n, k \to \infty$) we will have the probability that k particular vertices witness the k - independence property in a graph on n vertices tends to $e^{-\lambda}$, and the probability that a graph on n vertices has the k – independence property is at most

$$\binom{r}{k} e^{-\lambda} \leq n^k e^{-\lambda} \text{ as } n, k \to \infty.$$

If $n = k + k2^k$, then $q(n-k, 2^k) \to 1$, so the particular vertices $\{1, \ldots, k\}$ witness k – independence in a graph on $k + k2^k$ vertices almost surely. On the other hand,

Theorem 3.1 *A random graph on $n = k + 2^k(\log k)$ vertices has the failure of the k – independence property almost surely.*

Proof: For $n = k + 2^k \log k$, the λ from above is $\frac{2^k}{k}$, and $\log(n^k e^{-\lambda}) = k \log(k + 2^k \log k) - 2^k/k$ which clearly goes to $-\infty$ as $k \to \infty$, so the probability that a graph on n vertices has the k – independence property goes to 0.

3.1 Triangle - Free Graphs

We look at an example of how the independence property (and hence stability) can affect well - known problems in finite mathematics. We look at Ramsey numbers for hypergraphs and triangle - free graphs. Note that here $\lfloor x \rfloor$ refers to the greatest integer less than or equal to the real number x. ∎

Lemma 3.2 *Suppose G is a triangle - free graph on k vertices without the 2 – independence property. Then G has an independent set of size $k/2$.*

Proof: Choose a vertex v of G, and let $G_0 = G - \{v\}$. We claim that G is bipartite. Suppose this is not the case. Then there is an odd cycle C in G_0 which we may assume is as small as possible. As G is triangle - free, we know that C is at least a 5 - cycle, and since C is of minimal size, we know that it has no chords. Therefore, we can read off from C the vertices x_0, x_1,

x_2, x_3, and x_4 such that x_i and x_{i+1} are adjacent for $i < 5$, and these are the only adjacencies among these five vertices, except possibly for x_1 and x_5. We may also choose these vertices so that x_2 and x_4 are non-neighbors of v (because no two consecutive vertices in a path my be neighbors of v in the triangle - free G). Letting $a_0 = x_2$, $a_1 = x_4$, $b_\emptyset = v$, $b_{\{0\}} = x_1$, $b_{\{1\}} = x_5$, and $b_{\{1,2\}} = x_3$, we see that we have contradicted the assumption on the absence of the 2 – independence property, so G_0 must be bipartite. Therefore, G_0 has an independent set of cardinality at least $\lfloor k/2 \rfloor$, and this same set is an independent set in G. ■

QUESTION: What is the correct extension of this for k – independence?

3.2 Ramsey's theorem for finite hypergraphs

We can improve (for the case of hypergraphs without n - independence) the best known upper bounds for the Ramsey number $R_r(n, m)$. First we should say what this means.

Definition 3.3 1. *An r –* graph *is a set of* vertices V *along with a set of r – element subsets of V called* edges. *The edge set will be identified in the language by the r – ary predicate R.*

2. *A* complete r – graph *is one in which all r – element subsets of the vertices are edges. An* empty r – graph *is one in which none of the r – element subsets of the vertices are edges.*

3. *$R_r(n, m)$ denotes the smallest positive integer N so that in any r – hypergraph on N vertices there will be an induced subgraph which is either a complete r – graph on n vertices or an empty r - graph on m vertices.*

4. *We say that an r – graph G* has the k – independence property *if $(G, R(x))$ does (where $l(x) = r$).*

Note that the first suggested improvement of Lemma 2.6 applies in this situation – namely, the edge relation is symmetric. We can immediately make the following computations.

Lemma 3.4 *1. In an r – graph G, F is given by $F(i) = 2^q$ where $q = \begin{pmatrix} i \\ r-1 \end{pmatrix}$. Consequently, $F^*(m) \leq 2^{m^r}$ in this case.*

2. In an r – graph G which does not have the k – independence property, F is defined by

$$F(i) := \begin{cases} 1 & \textit{for } i < r \\ i^{(r-1)(k-1)} & \textit{otherwise.} \end{cases}$$

Consequently $F^*(m) \leq m^{(r-1)(k-1)m}$ in this case.

For a fixed natural number p, define the functions $E^{(j)}$ by

- $E^{(1)} = E = (\alpha \mapsto (\alpha + 1)^{p(\alpha+1)})$, and
- $E^{(i+1)} = E \circ E^{(i)}$ for $i \geq 1$.

Theorem 3.5 *If an r – graph G on at least $E^{(r-1)}(m-1)$ vertices does not have the k – independence property, then G has an induced subgraph on m vertices which is either complete or empty.*

Proof: (By induction on r)

For $r = 2$, the graph has at least $(m+1)^{p(m+1)} \geq (m+1)^{m+1} \geq 2^{2m}$ vertices, and it is well-known (see e.g. [10]) that $2^{2m} \to (m)^2_2$.

Assume $r \geq 3$, and let $G = (V, R)$ be an r – graph as described and set $L = E^{(r-2)}(m)$. Using $F(i) = i^{(r-1)(k-1)}$ for $i \geq 3$, $(F(0) = F(1) = 1)$ and computing F^* in Lemma 2.6, we first see that any r – graph on at least $(L+1)^{p(L+1)}$ vertices (where $p = (r-1)(k-1)$) will have an end - indiscernible sequence J over $\emptyset$ of cardinality L. Let v be the last vertex in J and define the relation R on the $(r-1)$ – sets from (the range of) J by "$R'(X)$ iff

$R(X \cup \{v\})$". Now (J, R) is an $(r-1)$ – graph of cardinality L, so by the inductive hypothesis there is an R' – indiscernible subsequence J_0 of J with cardinality m. Clearly $I = \{av : a \in J_0\}$ is an R – indiscernible sequence over $\emptyset$ of cardinality m. ■

Remark: No "cover property" is used, but could be.

3.3 Comparing upper bounds for $r = 3$

Note that for $r = 3$ we have $p = 2(k-1)$, and so we get $(2^{2m}+1)^{p(2^{2m}+1)}$ which is roughly $2^{km(2^{2m}+2)}$. The bound for $r = 3$ in [8] is roughly $2^{2^{4m}}$. So $\log_2 \log_2$ (their bound) $= 4m$ and

$$\log_2 \log_2 \text{(our bound)} = \log_2(km(2^{2m+2})) = \log_2 p + \log_2 m + (2m+2)$$

which is smaller than $4m$ as long as $2m - 2 - \log_2 m > \log_2 k$. This is true as long as $k < 2^{2m-2}/m$.

For example, for $m = 10$ our bound is about $2^{c(k-1)}$ where c is roughly 4×10^7 and theirs is about $2^{2^{40}}$. Since 2^{40} is roughly 10^{12}, this is a significant improvement.

3.4 Comparing upper bounds in general

Let a_r be the upper bound given in [8] and b_r be the upper bound as computed above (both as a function of m, the size of the desired indiscernible set). Since we have $b_{r+1} \le b_r^{(p)(b_r)}$, we get the relationship

$$\begin{aligned} \log^{(r)} b_{r+1} &\le \log^{(r-1)}[p\, b_r(\log b_r)] \\ &= \log^{(r-2)}(\log(r-1) + \log(k-1) + \log b_r + \log\log b_r) \end{aligned}$$

for $r \ge 3$, $\log\log b_3 = 2m + \log_2 m + \log_2 k + \log_2 r$, and $\log b_2 = 2m$. It follows that $\log^{(r)} b_{r+1}$ is less than (roughly) $2m + \log_2 m + \log_2 k$ for every r.

In [8], the bounds a_r satisfy $\log a_2 = 2m$, $\log\log a_3 = 4m$, and for $r \geq 3$,

$$\begin{aligned} \log^{(r)} a_{r+1} &= \log^{(r-1)}(a_r^r) = \\ \log^{(r-2)}(r \log a_r) &= \log^{(r-3)}[\log r + \log\log a_r]. \end{aligned}$$

We can then show that $\log^{(r-1)} a_r < 4m + 2$ for all r.

Clearly for each $r \geq 3$, $\frac{b_r}{a_r} \to 0$ as m gets large.

Final Remark: On a final note, the above comparison is only given for r - graphs with $r \geq 3$ because the technique enlisted does not give an improvement in the case of graphs. We have briefly mentioned what is known about the case of triangle free graphs, but certainly there is something to say in general (not to mention in the triangle - free case where so little is known.) This has not been pursued in this paper because it seems to be of no interest in the general study. However, the techniques may be of interest to the specialist.

4 Abstract properties

We begin to look at some of the abstract properties of a class K of finite L - structures with an appropriate partial ordering denoted by $\prec_K$. These properties come from Shelah's list of axioms in §1 of [26].

Definition 4.1 *1. Let L be a given similarity type, let Δ be a set of L - formulas, and let $n < \omega$, by Δ^*_n we denote the the minimal set of L - formulas containing the following set and all its subformulas:*

$$\{\exists x[\bigwedge_{i \in w} \phi(x; y_i) \wedge \bigwedge_{i \in n-w} \neg\phi(x; y_i)] : \phi(x; y) \in \Delta, w \subseteq n, l(y_1) = \cdots = l(y_n) = l(y)\}.$$

2. When $\Delta = \{\phi\}$ then $\{\phi\}^*_n$ will stand for Δ^*_n.

We will now look into natural values of k for the previous section.

Theorem 4.2 *If the formula ϕ fails to have the weak n – order property (and hence fails to have the n - independence property in K), then for every $\Delta^* \supseteq \{\phi\}^*_n$ – indiscernible sequence I of length at least $2n$ (which is in fact an indiscernible set by Theorem 2.3), either $|\{\phi[c; a] \ : \ a \in I\}| < n$ or $|\{\neg\phi[c; a] \ : \ a \in I\}| < n$.*

Proof: Suppose not. Then there is a Δ^* – indiscernible set $I = \{a_i : i < b\} \subseteq M$ with $b \geq 2n$ and a $c \in M$ such that both $|\{\phi[c; a] \ : \ a \in I\}| \geq n$ and $|\{\neg\phi[c; a] \ : \ a \in I\}| \geq n$. Let $\{a_0, \ldots, a_{2n-1}\} \subseteq I$ be such that

$$M \models \bigwedge_{i<n} \phi[c, a_i] \ \wedge \bigwedge_{i=n}^{i=2n-1} \neg\phi[c, a_i].$$

We complete the proof by showing that $\{a_0, \ldots, a_{n-1}\}$ exemplify the independence property: Let $w \subseteq n$ be given. Consider the formula

$$\psi_w(y_0, \ldots, y_{n-1}) := (\exists x)[\bigwedge_{i \in w} \phi(x; y_i) \ \wedge \bigwedge_{i \notin w} \neg\phi(x; y_i)].$$

Let $\{i_0, \ldots, i_{k-1}\}$, be an increasing enumeration of w. Clearly the following holds

$$M \models \psi_w[a_{i_0}, \ldots, a_{i_{k-1}}, a_n, \ldots, a_{2n-1-k}].$$

Since $\psi_w \in \Delta^*$ by indiscerniblity of I we have that also $M \models \psi_w[a_0, \ldots, a_{n-1}]$. For every $w \subseteq n$, let $b_w \in M$ be such that

$$M \models \bigwedge_{i \in w} \phi[b_w, a_i] \ \wedge \bigwedge_{i \in n-w} \neg\phi[b_w, a_i].$$

We are done since $\{a_0, \ldots, a_{n-1}\}$ and $\{b_w : w \subseteq n\}$ demonstrate that the pair (M, ϕ) has the n – independence property.

∎

The following definition is inspired by $\kappa(T)$, see Chapter III of [27].

Definition 4.3 *Let $n < \omega$ and let Δ be a finite set of formulas. $\kappa_{\Delta,n}(K)$ is the least positive integer so that for every $M \in K$, every sequence of Δ^*_n – indiscernibles $I = \langle a_i \ : \ i < \alpha < \omega\rangle \in M$ has either $M \models \phi[c; a_i]$ or $M \models \neg\phi[c; a_i]$ for less than $\kappa_\Delta(K)$ elements of I for every $\phi \in \Delta$ and $c \in M$.*

So the previous theorem states that if the formula ϕ fails to have the n – independence property in K, then $\kappa_{\phi,n}(K) \leq n$. When ϕ does not have the n – independence property κ_ϕ will stand for $\kappa_{\phi,n}$. In this case, the following definition makes sense.

Definition 4.4 *Let Δ and n be as above. Suppose I is a sequence of Δ^*_n-indiscernibles over $\emptyset$, define*

$$Av_\Delta(I, A, M) = \{\phi(x; a) \ : \ a \in A, \phi(x; y) \in \Delta \text{ and } |\{c \in I : M \models \phi[c; a]\}| \geq \kappa_\Delta(K)\}.$$

If M is understood, it will typically be omitted.

Theorem 4.5 *If ϕ has neither the m - independence property nor the n - cover property, let $\Delta \supseteq \{\phi\}^*_m$ be finite and I is a set of Δ – indiscernibles over $\emptyset$ of length greater than $\max\{n\,\kappa_\phi(K), 2m\}$, then $Av_\phi(I, A, M)$ is a complete ϕ – type over A.*

Proof: That $Av_\phi(I, A)$ is complete follows from the previous theorem. To see that it is consistent, we need only establish that every n formulas from it are consistent (by failure of n – cover property), and this follows from the size of I and the pigeonhole principle. ∎

Theorem 4.6 *Suppose that ϕ fail to have the m- independence property. If $I \subseteq M$ is a sequence of $\{\phi\}^*_m$ – indiscernibles over $\emptyset$ and $A \subseteq_{finite} M$, then there exists $J \subseteq I$ with $|J| \leq \kappa_\phi(K)|A|^{r-1}$ such that $I - J$ is ϕ – indiscernible over A. (Here $r = l(x)$.)*

Proof: I being ϕ – indiscernible over $\emptyset$ means simply that for every $i_1 < \ldots < i_r$ and $j_1 < \ldots < j_r \in I$,

$$M \models \phi[a_{i_1} \cdots a_{i_r}] \leftrightarrow \phi[a_{j_1} \cdots a_{j_r}].$$

By the definition, at most $\kappa_\phi(K)$ elements of I can give different truth values for each sequence of $r-1$ parameters $c \in A$ (in place of $x_2 \cdots x_r$ in ϕ), so by deleting at most

$$\kappa_\phi(K) \times (\text{the number of } (r-1) \text{ – sequences from } A) \leq \kappa_\phi(K)|A|^{r-1}$$

we must get a sequence indiscernible over A. ∎

So we will use the following term to denote when we are in a model in which this notion of average type exists.

Definition 4.7 *We will say that <u>M is (ϕ, m, n) – good</u> if (M, ϕ) has neither the m – independence property nor the n – cover property. In this case, we will define $\lambda_\phi(M) = \max\{n\kappa_{\{\phi^*_m\}}(K), 2m\}$.*

Example **4.8**

1. Let T be an $\aleph_1$ – categorical theory in a relational language (no function symbols), and let $M \models T$ be an uncountable model (e.g. M is an uncountable algebraically closed field of positive characteristic). Let $\phi \in L(T)$. By $\aleph_1$ – categoricity there exist integers n, m such that M is (ϕ, n, m) – good. (see Corollary 1.4).
   Let $K := \{N \subseteq M : \|N\| < \aleph_0\}$.

2. ??????.

Definition 4.9 *For a fixed (finite) relational language L and an L – formula ϕ, let K be a class of finite (ϕ, m, n) – good L – structures. Fix a positive integer k and an L – formula $\psi(x; y)$ and define $\prec_{K,\psi}$, as follows: $N \prec_{K,\psi} M$ if*

1. *$N \subseteq M$,*
2. *for every $a_0, \ldots, a_{k-1} \in N$ with $l(y) = l(a_i)$, if $M \models \exists x \bigwedge_{i<k} \psi(x; a_i)$, then $M \models \bigwedge_{i<k} \psi[b; a_i]$ for some $b \in N$.*
3. *for every $a \in M$, there is a sequence of indiscernibles $I \subseteq N$ of length at least $\lambda_\phi(K)$ so that $tp_\phi(a, N, M) = Av_\phi(I, N, M)$.*

We define the same relation for a set Δ of formulas simply by requiring that the above holds for each $\phi \in \Delta$. When K and ψ are understood we will just write $\prec$ for $\prec_{K,\psi}$.

Condition (2) is like $(\{\psi\}, \leq k)$ – relative saturation (i.e. every $\{\psi\}$ – type with $\leq k$ parameters that is realized in M is realized in N).

Before proceeding, let's see that this notion of substructure corresponds to a weak notion of elementary substructure from first order logic.

Lemma 4.10 *$N \prec_{K,\psi} M$ if and only if for every $\psi(z) \in \Delta$ and every $a \in N$ with $l(a) = l(z) < l(x)\kappa_\phi(K)$ we have $M \models \psi[a]$ if and only if $N \models \psi[a]$, where Δ is a suitable set of combinations of instances of ϕ.*

4.1 Properties of K

We list the following as facts. The Roman numerals in parentheses indicate the corresponding Axioms in [26]. Again recall that k and Δ are fixed throughout.

A. (I) If $N \prec_K M$, then $N \subseteq M$, and $M \prec_K M$.

B. (V) If $N_0 \subseteq N_1 \prec_K M$ and $N_0 \prec_K M$, then $N_0 \prec_K N_1$.

C. (VI) There is a function $F : \omega \to \omega$ such that if $M_0 \subseteq A \subseteq N$ and $M_0 \prec_K N$, then there is an M_1 with $M_0 \prec_K M_1 \prec_K N$, $A \subseteq M_1$, and $||M_1|| < F(|A|)$.

Remark: The first two are just as in the case of elementary substructures in first order theories given the previous lemma above, so there is nothing surprising about them. The last property is the finite analog to the Downward Lowenheim – Skolem Theorem. Using the results of section 1 one can improve the obvious upper bounds on F under stability – like assumptions. It is not clear yet what to do with this.

The following definition is due to Shelah ([23], p. 290).

Definition 4.11 *Given (ϕ, m, n) – good models M, M_0, M_1, and M_2 with $M_l \prec_K M$, $M_0 \prec_K M_1$, and $M_0 \prec_K M_2$, we say that $\underline{(M_0, M_1, M_2) \text{ is in } \phi(x;y) \text{ – stable amalgamation inside } M}$ if for every $c \in M_2$ with $l(c) = l(x)$ there is a $\{\phi\}^*_m$ – indiscernible sequence $I \subseteq M_0$, of length at least $\lambda_\phi(K)$ such that $Av_\phi(I, M_1, M) = tp_\phi(c, M_1, M)$.*

To prove symmetry of stable amalgamation (with the assumption of nonorder), we must first establish the following lemma (corresponding to I.3.1 in [23]):

Lemma 4.12 *Assume M is $(\phi(x;y;z), m, n)$ – good, and I_l is a $\{\phi_l\}^*_m$ – indiscernible sequences in M each of length greater than $\max\{\lambda_{\phi_l}(M), \kappa_{\phi_l}(M) + \kappa_{\phi_0}(M)\kappa_{\phi_1}(M)\}$ for each $l < 2$ (where $I_l = \{a^l_k : k < m_l\}$ for $l < 2$, $\phi_0 = \phi$, and $\phi_1(y;x;z) = \phi(x;y;z)$.) Then for each $b \in M$, the following are equivalent.*

i. There exists $i_k < m_0$ for $k < m_0 - \kappa_{\phi_0}(M)$ such that for each k,

$$\phi(a^0_{i_k}, y, b) \in Av_{\phi_1}(I_1, |M|, M).$$

ii. There exists $j_l < m_1$ for $l < m_1 - \kappa_{\phi_1}(M)$ such that for each l,

$$\phi(x, a^1_{j_l}, b) \in Av_{\phi_1}(I_0, |M|, M).$$

Proof: Assume (i) holds. Choose $i_k < m_0$ for $k < m_0 - \kappa_{\phi_0}(M)$ and $j_{k,l} < m_1$ for $l < m_1 - \kappa_{\phi_1}(M)$ witnessing (i). Since $m_0 > \kappa_{\phi_l}(M) + \kappa_{\phi_0}(M)\kappa_{\phi_1}(M)$, we can find κ_{ϕ_1} – many of the $j_{k,l}$ each of which occurs for at least $\kappa_\phi(M)$ different i_k. Thus for each of these, $\phi(x; a^1_{j_{k,l}}; b) \in Av_\phi(I_0, |M|, M)$.

Now assume (ii) does not hold. That is, there are $j_l < m_1$ for each $l < m_1 - \kappa_{\phi_1}(M)$ such that $\neg\phi(x; a^1_{j_l}; b) \in Av_\phi(I_0, |M|, M)$. Clearly one of these j_l must correspond to one of the $j_{k,l}$ from before that occurs at least $\kappa_{\phi_1}(M)$ times. But as we noted above $\phi(x; a^1_{j_{k,l}}; b) \in Av_\phi(I_0, |M|, M)$, a contradiction.

Note that (ii) implies (i) by the symmetric argument. ∎

Theorem 4.13 (Symmetry) *Suppose M_0, M_1, $M_2 \prec_K M$, $M_0 \prec_K M_1$, and $M_0 \prec_K M_2$, each $\prec_K$ with respect to $\Delta = \{\phi, \psi\}$ where $\psi(y; x) = \phi(x; y)$. Then (M_0, M_1, M_2) is in ϕ – stable amalgamation inside M if and only if (M_0, M_2, M_1) is in ψ – stable amalgamation inside M.*

Proof: Assume that (M_0, M_1, M_2) is in ϕ – stable amalgamation in M. Let $c \in M$ with $l(c) = l(x)$ be given. (We need to find a ψ – indiscernible sequence $I \subseteq M$, with $Av_\psi(I, M_2, M) = tp_\psi(c, M_2, M)$.) By the definition of $M_0 \prec_K M_1$, we may choose a ψ – indiscernible $I \subseteq M_0$ of length at least $\lambda_\psi(K)$ so that $tp_\psi(c, M_0, M_1) = Av_\psi(I, M_0, M_1)$ (and so $tp_\psi(c, M_0, M) = Av_\psi(I, M_0, M)$).

We claim that $Av_\psi(I, M_2, M) = tp_\psi(c, M_2 M_2, M)$. (Note that the first type is defined since I is long enough.) To see this, let $b \in M_2$ be given such that $M \models \psi[c; b]$, and we will show that $\psi(x; b) \in Av_\psi(I, M_2, M)$.

Since (M_0, M_1, M_2) is in ϕ – stable amalgamation in M, we can choose a ϕ – indiscernible set $J \subseteq M$ of length at least $\lambda_\phi(K)$ so that $tp_\phi(b, M_0, M_2) = Av_\phi(J, M_0, M_2)$, and so $tp_\phi(b, M_0, M) = Av_\phi(J, M_0, M)$. Since $M \models \phi[b; c]$, we have $\phi(x;c) \in Av_\phi(J, M_0, M)$, so a large number of b_i from J satisfy $M \models \phi[b_i; c]$, or rather $\psi(y; b_i) \in tp_\psi(c, M_0, M) = Av_\psi(I, M_0, M)$ for each i. So $\psi(y; b_i) \in Av_\psi(I, M_0, M)$ for each i.

But then by the previous Lemma, we may choose a large number of c_j from I so that $\phi(x; c_j) \in Av(J, M, M)$ for each j. That is, $M \models \phi[b; c_j]$ for each j, and so $\psi(y; b) \in Av_\psi(I, M_2, M)$ as desired. ■

5 n - saturated graphs

We will say that a graph G is n - saturated if every sentence of the form

$$\phi_w \stackrel{\text{def}}{=} \exists x \ \wedge_{i<n} R(x, a_i)^{\text{ if } [i \in w]}$$

(where $w \subseteq n$ and $\{a_i \ : \ i < n\} \subseteq V_G$) is realized in G. The first question to answer is, "Do these graphs exist and if so how large are they?" The answer to this question can be given using a simple probabilistic argument. It turns out that for a fixed k, almost all graphs are k – saturated.

The probability that for a fixed set A of k vertices, a particular sentence ψ_w ($w \notin A$) is *not* realized in a graph on n vertices is $(1 - 1/2^k)^{n-k}$, so the probability that some set $w \subseteq A$ yields a sentence ψ_w which is not realized is at most $2^k(1 - 1/2^k)^{n-k}$, and thus the probability that some set of k vertices yields a subset w for which ψ_w is not realized is at most $\left(\begin{array}{c} n \\ k \end{array} \right) 2^k(1 - 1/2^k)^{n-k}$. As this is bounded above by $(2n)^k(1-1/2^k)^n$ which clearly goes to 0 as n gets large, the probability that a random graph on n vertices fails to be k – saturated goes to 0. This does not however tell us how large a graph we might need to have $n-$saturation in the first place.

To answer this question, one approach is to find n so that the expression (most loosely, $(2n)^k(1 - 1/2^k)^n$) above is less than 1. From this we can conclude that the complementary event (i.e., being k – saturated) occurs with positive probability. Taking $n = 2^k k^3$ accomplishes this, in fact for all $k \geq 2$.

So we have proved

Theorem 5.1 *For each positive integer $n \geq 2$, there exists an n - saturated graph G with less than $n^3 2^n$ vertices.*

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