B-I. Polynomials

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1 Warm-Ups

1. Consider the cubic equation $ax^3 + bx^2 + cx + d = 0$. The roots are

$$\begin{aligned} x &= \sqrt[3]{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right) + \sqrt{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right)^2 + \left(\frac{c}{3a} - \frac{b^2}{9a^2}\right)^3}} \\ &+ \sqrt[3]{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right) - \sqrt{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right)^2 + \left(\frac{c}{3a} - \frac{b^2}{9a^2}\right)^3}} \\ &- \frac{b}{3a}. \end{aligned}$$

Prove that no such general formula exists for a quintic equation.

2 Theory

Thanks to Elgin Johnston (1997) for these theorems.

Rational Root Theorem Let $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ be a polynomial with integer coefficients. Then any rational solution r/s (expressed in lowest terms) must have $r|a_0$ and $s|a_n$.

Descartes's Rule of Signs Let $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ be a polynomial with real coefficients. Then the number of positive roots is equal to $N - 2k$, where N is the number of sign changes in the coefficient list (ignoring zeros), and k is some nonnegative integer.

Eisenstein's Irreducibility Criterion Let $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ be a polynomial with integer coefficients and let q be a prime. If q is a factor of each of a_{n-1}, a_{n-2}, ..., a_0, but q is not a factor of a_n, and q^2 is not a factor of a_0, then $p(x)$ is irreducible over the rationals.

Einstein's Theory of Relativity Unfortunately, this topic is beyond the scope of this program.

Gauss's Theorem If $p(x)$ has integer coefficients and $p(x)$ can be factored over the rationals, then $p(x)$ can be factored over the integers.

Lagrange Interpolation Suppose we want a degree-n polynomial that passes through a set of $n+1$ points: $\{(x_i, y_i)\}_{i=0}^n$. Then the polynomial is:

$$p(x) = \sum_{i=0}^{n} \frac{y_i}{\text{normalization}}(x - x_0)(x - x_1)\cdots(x - x_n),$$

where the i-th "normalization" factor is the product of all the terms $(x_i - x_j)$ that have $j \neq i$.

3 Problems

Thanks to Elgin Johnston (1997) for most of these problems.

1. (Crux Math., June/July 1978) Show that $n^4 - 20n^2 + 4$ is composite when n is any integer.

   Solution: Factor as difference of two squares. Prove that neither factor can be ± 1.

2. (St. Petersburg City Math Olympiad 1998/14) Find all polynomials $P(x, y)$ in two variables such that for any x and y, $P(x + y, y - x) = P(x, y)$.

   Solution: Clearly constant polynomials work. Also, $P(x, y) = P(x + y, y - x) = P(2y, -2x) = P(16x, 16y)$. Suppose we have a nontrivial polynomial. Then on the unit circle, it is bounded because we can just look at the fixed coefficients. Yet along each ray $y = tx$, we get a polynomial whose translate has infinitely many zeros, so it must be constant. Hence P is constant along all rays, implying that P is bounded by its max on the unit circle, hence bounded everywhere. Now suppose maximum degree of y is N. Study the polynomial $P(z^{N+1}, z)$. The leading coeff of this is equal to the leading coeff of $P(x, y)$ when sorted with respect to x as more important. Since the z-poly is also bounded everywhere, it too must be constant, implying that the leading term is a constant.

3. (Putnam, May 1977) Determine all solutions of the system

$$\begin{aligned} x + y + z &= w \\ \frac{1}{x} + \frac{1}{y} + \frac{1}{z} &= \frac{1}{w}. \end{aligned}$$

   Solution: Given solutions x, y, z, construct 3-degree polynomial $P(t) = (t - x)(t - y)(t - z)$. Then $P(t) = t^3 - wt^2 + At - Aw = (t^2 + A)(t - w)$. In particular, roots are w and a pair of opposites.

4. (Crux Math., April 1979) Determine the triples of integers (x, y, z) satisfying the equation

$$x^3 + y^3 + z^3 = (x + y + z)^3.$$

   Solution: Move z^3 to RHS and factor as $x^3 \pm y^3$. We get $(x + y) = 0$ or $(y + z)(z + x) = 0$. So two are opposites.

5. (USSR Olympiad) Prove that the fraction $(n^3 + 2n)/(n^4 + 3n^2 + 1)$ is in lowest terms for every positive integer n.

   Solution: Use Euclidean algorithm for GCD. $(n^3 + 2n)n = n^4 + 2n^2$, so difference to denominator is $n^2 + 1$. Yet that's relatively prime to $n(n^2 + 2)$.

6. (Po, 2004) Prove that $x^4 - x^3 - 3x^2 + 5x + 1$ is irreducible.

   Solution: Eisenstein with substitution $x \mapsto x + 1$.

7. (Canadian Olympiad, 1970) Let $P(x)$ be a polynomial with integral coefficients. Suppose there exist four distinct integers a, b, c, d with $P(a) = P(b) = P(c) = P(d) = 5$. Prove that there is no integer k with $P(k) = 8$.

   Solution: Drop it down to 4 zeros, and check whether one value can be 3. Factor as $P(x) = (x - a)(x - b)(x - c)(x - d)R(x)$; then substitute k. 3 is prime, but we'll get at most two ± 1 terms from the $(x - \alpha)$ product.

8. (Monthly, October 1962) Prove that every polynomial over the complex numbers has a nonzero polynomial multiple whose exponents are all divisible by 10^9.

   Solution: Factor polynomial as $a(x - r_1)(x - r_2) \cdots (x - r_n)$. Then the desired polynomial is $a(x^P - r_1^P) \cdots (x^P - r_n^P)$, where $P = 10^9$. Each factor divides the corresponding factor.

9. (Elgin, MOP 1997) For which n is the polynomial $1+x^2+x^4+\cdots+x^{2n-2}$ divisible by the polynomial $1+x+x^2+\cdots+x^{n-1}$?

Solution: Observe:

$$\begin{aligned}(x^2-1)(1+x^2+x^4+\cdots+x^{2n-2}) &= x^{2n}-1\\ (x-1)(1+x+x^2+\cdots+x^{n-1}) &= x^n-1\\ (x+1)(1+x^2+x^4+\cdots+x^{2n-2}) &= (x^n+1)(1+x+x^2+\cdots+x^{n-1}).\end{aligned}$$

So if the quotient is $Q(x)$, then $Q(x)(x+1)=x^n+1$. This happens iff -1 is a root of x^n+1, which is iff n is odd.

10. (Czech-Slovak Match, 1998/1) A polynomial $P(x)$ of degree $n\geq 5$ with integer coefficients and n distinct integer roots is given. Find all integer roots of $P(P(x))$ given that 0 is a root of $P(x)$.

Solution: Answer: just the roots of $P(x)$. Proof: write $P(x)=x(x-r_1)(x-r_2)\cdots(x-r_N)$. Suppose we have another integer root r; then $r(r-r_1)\cdots(r-r_N)=r_k$ for some k. Since degree is at least 5, this means that we have $2r(r-r_k)$ dividing r_k. Simple analysis shows that r is between 0 and r_k; more analysis shows that we just need to defuse the case of $2ab \mid a+b$. Assume $a\leq b$. Now if $a=1$, only solution is $b=1$, but then we already used ± 1 in the factors, so we actually have to have $12r(r-r_k)$ dividing r_k, no good. If $a>1$, then $2ab>2b\geq a+b$, contradiction.

11. (Hungarian Olympiad, 1899) Let r and s be the roots of

$$x^2-(a+d)x+(ad-bc)=0.$$

Prove that r^3 and s^3 are the roots of

$$y^2-(a^3+d^3+3abc+3bcd)y+(ad-bc)^3=0.$$

Hint: use Linear Algebra.

Solution: r and s are the eigenvalues of the matrix

$$\begin{pmatrix} a & b\\ c & d\end{pmatrix}$$

The y equation is the characteristic polynomial of the cube of that matrix.

12. (Hungarian Olympiad, 1981) Show that there is only one natural number n such that $2^8+2^{11}+2^n$ is a perfect square.

Solution: $2^8+2^{11}=48^2$. So, need to have 2^n as difference of squares N^2-48^2. Hence $(N+48)$, $(N-48)$ are both powers of 2. Their difference is 96. Difference between two powers of 2 is of the form $2^M(2^N-1)$. Uniquely set to 2^7-2^5.

13. (MOP 97/9/3) Let $S=\{s_1,s_2,\ldots,s_n\}$ be a set of n distinct complex numbers, for some $n\geq 9$, exactly $n-3$ of which are real. Prove that there are at most two quadratic polynomials $f(z)$ with complex coefficients such that $f(S)=S$ (that is, f permutes the elements of S).

14. (MOP 97/9/1) Let $P(x)=a_0x^n+a_1x^{n-1}+\cdots+a_n$ be a nonzero polynomial with integer coefficients such that $P(r)=P(s)=0$ for some integers r and s, with $0<r<s$. Prove that $a_k\leq -s$ for some k.