9. Linear Algebra

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1 Famous results

Inverse. The inverse of a matrix A is equal to the transpose of its cofactor matrix, divided by $\det A$.

Cramer's rule. Given an invertible matrix A and a column vector b, the solution to $Ax = b$ is given by

$$x_i = \frac{\det A_i}{\det A},$$

where x_i is the i-th entry of the solution vector x, and the matrix A_i is obtained by replacing the i-th column of A with b.

Trace. The trace $\operatorname{tr}(A)$ of a matrix is the sum of its diagonal entries, which is always equal to the sum of its eigenvalues (counting multiplicity). This can be used, for example, to estimate the largest absolute value of an eigenvalue.

Rank. The rank of a matrix is the size of its largest subset of rows which are linearly independent of each other. This is also equal to the size of its largest subset of columns with the same property. The rank of a product of matrices is always less than or equal to the rank of every matrix in the product.

Nilpotent. An $n \times n$ matrix A is called *nilpotent* if there is some positive integer d for which A^d is the zero matrix. This property is equivalent to having 0 as the only eigenvalue, and also equivalent to satisfying $A^n = 0$.

2 Problems

1. Alan and Barbara play a game in which they take turns filling entries of an initially empty 2008×2008 array. Alan plays first. At each turn, a player chooses a real number and places it in a vacant entry. The game ends when all the entries are filled. Alan wins if the determinant of the resulting matrix is nonzero; Barbara wins if it is zero. Which player has a winning strategy?

2. Let n be an integer which is at least 2, and let A_n be the $n \times n$ whose entries are $a_{ij} = |i - j|$. For example,

$$A_5 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 0 & 1 & 2 & 3 \\ 2 & 1 & 0 & 1 & 2 \\ 3 & 2 & 1 & 0 & 1 \\ 4 & 3 & 2 & 1 & 0 \end{pmatrix}.$$

   Prove that A_n has determinant $(-1)^{n-1}(n-1)2^{n-2}$.

3. Let A be an invertible $n \times n$ real matrix, all of whose elements are strictly positive. Prove that the number of nonzero elements in A^{-1} is always at least $2n$.

4. If $\mathbf{A}$ and $\mathbf{B}$ are $n \times n$ matrices such that $\mathbf{ABAB} = \mathbf{0}$, for which n does it follow that $\mathbf{BABA} = \mathbf{0}$?

5. Let A and B be real $n \times n$ matrices, and suppose that there are distinct real numbers $x_1, x_2, \ldots, x_{n+1}$ such that all of the matrices $A + x_i B$ are nilpotent (see definition above). Prove that both A and B must also be nilpotent.

6. Suppose that A is a 3×2 matrix and B is a 2×3 matrix. Prove that

$$AB = \begin{pmatrix} 8 & 2 & -2 \\ 2 & 5 & 4 \\ -2 & 4 & 5 \end{pmatrix} \qquad \Longrightarrow \qquad BA = \begin{pmatrix} 9 & 0 \\ 0 & 9 \end{pmatrix}.$$

3 Homework

Please write up solutions to two of the problems, to turn in at next week's meeting. One of them may be a problem that we discussed in class.