Putnam Σ.12

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1 Problems

Putnam 2013/B4. For any continuous real-valued function f defined on the interval $[0,1]$, let

$$\mu(f) = \int_0^1 f(x)\,dx,\ \mathrm{Var}(f) = \int_0^1 (f(x) - \mu(f))^2\,dx,$$
$$M(f) = \max_{0\le x\le 1} |f(x)|\,.$$

Show that if f and g are continuous real-valued functions defined on the interval $[0,1]$, then

$$\mathrm{Var}(fg) \le 2\mathrm{Var}(f)M(g)^2 + 2\mathrm{Var}(g)M(f)^2.$$

Putnam 2013/B5. Let $X = \{1, 2, \ldots, n\}$, and let $k \in X$. Show that there are exactly $k \cdot n^{n-1}$ functions $f : X \to X$ such that for every $x \in X$ there is a $j \ge 0$ such that $f^{(j)}(x) \le k$. [Here $f^{(j)}$ denotes the j^{th} iterate of f, so that $f^{(0)}(x) = x$ and $f^{(j+1)}(x) = f(f^{(j)}(x))$.]

Putnam 2013/B6. Let $n \ge 1$ be an odd integer. Alice and Bob play the following game, taking alternating turns, with Alice playing first. The playing area consists of n spaces, arranged in a line. Initially all spaces are empty. At each turn, a player either

- places a stone in an empty space, or
- removes a stone from a nonempty space s, places a stone in the nearest empty space to the left of s (if such a space exists), and places a stone in the nearest empty space to the right of s (if such a space exists).

Furthermore, a move is permitted only if the resulting position has not occurred previously in the game. A player loses if he or she is unable to move. Assuming that both players play optimally throughout the game, what moves may Alice make on her first turn?