Putnam E.12

Po-Shen Loh

19 November 2024

1 Problems

Putnam 2019/B1. Denote by $\mathbb{Z}^2$ the set of all points (x, y) in the plane with integer coordinates. For each integer $n \geq 0$, let P_n be the subset of $\mathbb{Z}^2$ consisting of the point $(0, 0)$ together with all points (x, y) such that $x^2 + y^2 = 2^k$ for some integer $k \leq n$. Determine, as a function of n, the number of four-point subsets of P_n whose elements are the vertices of a square.

Putnam 2019/B2. For all $n \geq 1$, let

$$a_n = \sum_{k=1}^{n-1} \frac{\sin\left(\frac{(2k-1)\pi}{2n}\right)}{\cos^2\left(\frac{(k-1)\pi}{2n}\right)\cos^2\left(\frac{k\pi}{2n}\right)}.$$

Determine

$$\lim_{n\to\infty} \frac{a_n}{n^3}.$$

Putnam 2019/B3. Let Q be an n-by-n real orthogonal matrix, and let $u \in \mathbb{R}^n$ be a unit column vector (that is, $u^T u = 1$). Let $P = I - 2uu^T$, where I is the n-by-n identity matrix. Show that if 1 is not an eigenvalue of Q, then 1 is an eigenvalue of PQ.