5. Functional equations

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1 Classical results

Cauchy. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function that satisfies $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. Show that there must be a real number c such that $f(x) = cx$ for all $x \in \mathbb{R}$.

Parallelogram Law. In a normed space, for every x and y:

$$2\|x\|^2 + 2\|y\|^2 = \|x+y\|^2 + \|x-y\|^2.$$

Group theory. Let $f(x, y)$ be a function from $\{1, \ldots, 29\}^2 \to \{1, \ldots, 29\}$, which satisfies $f(f(x, y), z) = f(x, f(y, z))$ for all $x, y, z \in \{1, \ldots, 29\}$. Suppose that there is an integer $a \in \{1, \ldots, 29\}$ such that for every integer $x \in \{1, \ldots, 29\}$, we have $f(x, a) = x$ and $f(a, x) = x$. Also suppose that for every integer $x \in \{1, \ldots, 29\}$, there is an integer $y \in \{1, \ldots, 29\}$ such that $f(x, y) = a$ and $f(y, x) = a$. Prove that for every integers $x, y \in \{1, \ldots, 29\}$, we must have $f(x, y) = f(y, x)$.

2 Problems

1. Determine all continuous functions from $\mathbb{R} \to \mathbb{R}$ which satisfy

$$f(x+y) + f(x-y) = 2\,[f(x) + f(y)]$$

   for all $x, y \in \mathbb{R}$.

2. Determine all continuous functions from $\mathbb{R}^+ \to \mathbb{R}$ which satisfy

$$f(xy) = f(x) + f(y)$$

   for all $x, y \in \mathbb{R}^+$. Here, the set $\mathbb{R}^+$ represents all positive real numbers.

3. Determine all continuous functions from $\mathbb{R}^+ \to \mathbb{R}$ which satisfy

$$f(xy) = f(x)f(y)$$

   for all $x, y \in \mathbb{R}^+$. Here, the set $\mathbb{R}^+$ represents all positive real numbers.

4. Let $f : \mathbb{C} \to \mathbb{C}$ be a continuous function which satisfies $f(z) + zf(1-z) = 1 + z$ for all $z \in \mathbb{C}$. Determine all possible such functions f.

5. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function which satisfies $f(\sqrt{x^2 + y^2}) = f(x)f(y)$ for all real x and y. Show that $f(x) = f(1)^{x^2}$.

6. Find all surjective functions $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ such that $f(n) \geq n + (-1)^n$ for all $n \in \mathbb{Z}^+$.

7. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that for all $x \in \mathbb{R}$:

$$f(1-x) = 1 - f(f(x)).$$

3 Homework

Please write up solutions to two of the problems, to turn in at next week's meeting. One of them may be a problem that we discussed in class.