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COMPACTNESS PROPERTIES OF THE DG AND CG TIME STEPPING SCHEMES FOR PARABOLIC EQUATIONS*

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Abstract. For a broad class of parabolic equations it is shown that numerical solutions computed using the discontinuous Galerkin or the continuous Galerkin time stepping schemes of arbitrary order will inherit the compactness properties of the underlying equation. Convergence of numerical schemes for a phase field approximation of the flow of two fluids with surface tension is presented to illustrate these results.

Key words. discontinuous Galerkin, continuous Galerkin, parabolic equations

AMS subject classifications. 76M10, 76M12, 65M60

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1. Introduction.

We consider the numerical approximation of solutions, $u : [0,T) \to U$, of evolution equations: $u(0) = u_0$, and

$$u_t + A(u) = f(u). \tag{1.1}$$

Here U is a Banach space and each term of the equation takes values in U'. We allow both $A(u) = A(t,u)$ and $f = f(t,u)$ to be nonlinear and to depend explicitly upon t but do not express the latter dependence in the notation. Existence of solutions of such equations is typically established by first proving a priori bounds. These bounds, and the structure of the equation, are then used to establish sufficient compactness to control the nonlinear terms. In this paper we show that numerical schemes which use discontinuous Galerkin (DG) or continuous Galerkin (CG) time stepping schemes of arbitrary order will inherit the stability and compactness properties of a broad class of evolution equations. Under these circumstances Lax's meta-theorem is applicable:

> "A (linear) numerical scheme converges if and only if it is stable and consistent."

Of course the nonlinear analogue of this old adage requires a compactness hypothesis to guarantee convergence of the nonlinear terms, and this is the main focus of this work.

Stability. Prototypically, a priori bounds on solutions of (1.1) are established by multiplying it either by u or by u_t. In these two situations we show that the following rule of thumb will frequently give rise to stable and convergent numerical schemes:

- If multiplication by u gives a priori bounds, use a DG time stepping scheme.
- If multiplication by u_t gives a priori bounds, use a CG time stepping scheme.

Recall that the lowest order DG and CG schemes are the (first order) implicit Euler and (second order) Crank–Nicolson schemes, respectively. Since the amount of work required for each scheme is the same, this would suggest that CG time stepping schemes may be more desirable; however, their stability is more delicate.

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If the spatial operator is coercive in the sense that $(A(u), u) \geq c\|u\|_U^p$ for some $c > 0$ and $1 \leq p < \infty$, then multiplying (1.1) by u and integrating shows

$$(1/2)\|u(t)\|_H^2 + \int_0^t (A(u), u) \leq (1/2)\|u_0\|_H^2 + \int_0^t (f(u), u). \tag{1.2}$$

Here we assume a pivot space structure $U \hookrightarrow H \hookrightarrow U'$, where U is embedded in a Hilbert space H which is identified with its dual, and $(.,.)$ denotes the duality pairing. Under suitable structural and/or continuity hypotheses on $f(.)$, Gronwall's inequality can then be used to show

$$\|u\|_{L^\infty[0,T;H]}^2 + c\|u\|_{L^p[0,T;U]}^p \leq C\left(\|u_0\|_H^2 + C(f)\right).$$

Alternatively, if $A = \Phi'$ is the gradient of a potential, $\Phi : U \to \mathbb{R}$ (we assume Φ is independent of time), then multiplying (1.1) by u_t and integrating gives

$$\int_0^t \|u_t\|_H^2 + \Phi(u(t)) = \Phi(u_0) + \int_0^t (f(u), u_t). \tag{1.3}$$

If Φ is coercive in the sense that $\Phi(u) \geq c\|u\|_U^p$, and if f satisfies suitable structural and/or continuity hypotheses, it follows that

$$\|u_t\|_{L^2[0,T;H]} + c\|u\|_{L^\infty[0,T;U]}^p \leq C\left(\Phi(u_0) + C(f)\right). \tag{1.4}$$

Below we show that solutions computed using the DG and CG time stepping schemes inherit discrete analogues of the stability estimates (1.2) and (1.3), respectively. In this situation stability of the schemes is "automatically" inherited from the structural properties of the underlying equation.

Compactness. Compactness of solutions to the evolution equation (1.1) is frequently established using the Lions–Aubin theorem.

THEOREM 1.1 (Lions–Aubin). *Let B_0, B, and B_1 be Banach spaces satisfying $B_0 \hookrightarrow\!\!\!\!\rightarrow B \hookrightarrow B_1$, where the first inclusion is compact. If $0 < T < \infty$, $1 \leq p, q < \infty$, and*

$$W = W(0,T) = \{u \in L^p[0,T;B_0] \mid u' \in L^q[0,T;B_1]\},$$

then the inclusion $W \hookrightarrow\!\!\!\!\rightarrow L^p[0,T;B]$ is compact.

In the current context, a typical application of this theorem would set $(B_0, B, B_1) = (U, H, U')$. For parabolic equations the embedding $U \hookrightarrow\!\!\!\!\rightarrow H$ is typically compact and the a priori estimates bound solutions in $L^p[0,T;U]$. The crucial hypothesis on u_t is typically found by writing the equation as

$$\int_0^T (u_t, v) = \int_0^T (f(u), v) - (A(u), v). \tag{1.5}$$

The continuity and/or structural hypotheses of f and A, and the a priori estimates on u, are then used to bound u_t in $L^2[0,T;U']$. The Lions–Aubin theorem then establishes compactness of solutions in $L^p[0,T;H]$. This line of argument fails for numerical solutions constructed using the DG time stepping scheme; indeed, the solutions are discontinuous so their time derivatives are not integrable. This difficulty is circumvented in Theorem 3.1 below, where we show that solutions satisfying the discrete analogue of (1.5) are compact in $L^p[0,T;H]$.

When $A = \Phi'$ is the derivative of potential, $U \hookrightarrow\!\!\!\rightarrow H$, and solutions satisfy the bounds in (1.4), compactness of solutions in $L^p[0,T;H]$ follows directly from the Lions–Aubin theorem with $(B_0, B, B_1) = (U, H, H)$. In Theorem 4.1 it is shown that this compactness property is inherited by numerical approximations constructed using the CG scheme. However, typically more is true. Writing the equation as

$$A(u) = f(u) - u_t,$$

the bounds in (1.4) show $A(u)$ is bounded in $L^p[0,T;H]$. For parabolic equations the subspace $D(A) = \{u \in U \mid A(u) \in H\}$ is typically compact in U. Application of the Lions–Aubin theorem with $(B_0, B, B_1) = (D(A), U, H)$ then establishes compactness in $L^p[0,T;U]$. Typically numerical solutions, u_h, are in U, but are never in $D(A)$, so this argument is not applicable. When A is linear, Theorem 4.5 below circumvents this difficulty by establishing compactness in $L^2[0,T;U]$ of numerical solutions constructed using the CG scheme.

1.1. Flows of incompressible immiscible fluids. We will illustrate the applicability of our results by proving convergence of a class of numerical schemes for a phase field approximation of the equations modeling the flow of two incompressible Newtonian fluids where the surface tension between them is significant. To eliminate technical detail it is assumed that the density ρ of each fluid is the same. Using the ideas in [6] our analysis can be extended to include different densities.

The strong form of the problem considers $\Omega = \Omega_1(t) \cup \Omega_2(t) \cup S(t)$, where $\Omega_1(t) \cap \Omega_2(t) = \emptyset$ partitions Ω into the two regions occupied by each fluid and $S(t)$ is the surface separating them. Adopting the usual notation, equations modeling the balance of mass and momentum require $\operatorname{div}(u) = 0$, and

$$\rho u_t + (\rho u.\nabla)u + \nabla p - \operatorname{div}(\mu D(u)) = \rho f \qquad \text{in } \Omega_1(t) \cup \Omega_2(t),$$

and $[-pI + \mu D(u)]n = \gamma H n$ on $S(t)$, where H is the mean curvature of $S(.)$, n is a normal, and $[.]$ denotes the jump across the surface. We implicitly assume that the velocity is continuous across the surface, which is transported with the flow, and we assume Dirichlet boundary data for the velocity on $\partial\Omega$.

A weak statement of the momentum equation is

$$\int_\Omega \big(\rho(u_t + (u.\nabla)u).v - p \operatorname{div}(v) + \mu D(u) : D(v)\big) + \int_{S(t)} \gamma H v.n = \int_\Omega \rho f.v. \tag{1.6}$$

This is a very singular problem in the sense that the surface is not likely to remain smooth, so that H, n, and the surface measure may fail to exist. We consider a regularization based upon a phase field approximation of these quantities. Let $\phi : (0,T) \times \Omega \to \mathbb{R}$ be a smooth approximation of the phase function

$$\phi_0(t,x) = \begin{cases} -1, & x \in \Omega_1(t), \\ \;\;\,1, & x \in \Omega_2(t), \end{cases}$$

so that $S(t) \simeq \{x \in \Omega \mid \phi(t,x) = 0\}$. In section 6 formal asymptotic expansions are presented which motivate the following regularized approximation of the two-fluid problem:

$$\begin{aligned} \int_\Omega \Big\{ &\rho u_t.v + (\rho/2)\big((u.\nabla)u.v - (u.\nabla)v.u\big) \\ &- p \operatorname{div}(v) + \mu D(u) : D(v) + (\gamma/\lambda)\big(\phi_t + u.\nabla\phi\big)\nabla\phi.v \Big\} = \int_\Omega \rho f.v. \end{aligned} \tag{1.7}$$

In this equation the density is constant, $\rho \in \mathbb{R}^+$, and the viscosity is a function of ϕ; $\mu = \mu(\phi)$. The phase function ϕ satisfies the equation

$$\int_\Omega \left(\phi_t + u.\nabla\phi\right)\psi + \lambda\epsilon\nabla\phi.\nabla\psi + (\lambda/\epsilon)F'(\phi)\psi = 0, \tag{1.8}$$

where $F \in C^1(\mathbb{R})$ is a classical "double well potential" with quadratic growth at infinity,

$$F(\phi) = \begin{cases} (1/2)(\phi^2-1)^2, & |\phi| \le \sqrt{3}, \\ 2\phi^2 - 4, & |\phi| \ge \sqrt{3}, \end{cases}$$

and $\lambda > 0$ is a "relaxation" parameter.

Bounds for solutions of this system of equations are naturally obtained by setting the test functions $v = u$ and $\psi = (\gamma/\lambda)\phi_t$ in the weak statements of the momentum equation and phase equations,

$$\frac{d}{dt}\int_\Omega \left((\rho/2)|u|^2 + (\gamma\epsilon/2)|\nabla\phi|^2 + (\gamma/\epsilon)F(\phi)\right) + \int_\Omega (\gamma/\lambda)(\phi_t + u.\nabla\phi)^2 + \mu|D(u)|^2 = \int_\Omega \rho f.u.$$

In section 6 we illustrate our results by showing that numerical approximations constructed using classical finite element spaces for the spatial discretizations, and the DG and CG time stepping schemes for the momentum and phase equations, respectively, converge.

1.2. Related results. Thomee's book [27] presents a detailed analysis of the DG scheme for the heat equation. A majority of papers that address convergence of numerical schemes for nonlinear parabolic equations focuses exclusively upon the first order implicit Euler time stepping scheme, e.g., [2, 12, 13, 22], and occasionally the Crank–Nicolson scheme [16, 26]. The dearth of higher order schemes is frequently justified by the supposition that the regularity is low; however, in practice solutions are frequently piecewise smooth and the implicit Euler time stepping error swamps the spatial error if, for example, classical P_2/P_1 Taylor–Hood elements are used for the Navier–Stokes equations. A more likely reason is that for many equations the implicit Euler scheme trivially inherits the stability, monotonicity, and compactness properties of the underlying equation [9, 10, 26].

Frequently higher order time stepping schemes are analyzed using finite difference methodology, e.g., [1, 26], which is not well adapted to the variational setting where the equations are naturally posed. This typically leads to dense technical calculations to establish discrete versions of the stability and compactness estimates enjoyed by the equation. This contrasts with the DG and CG schemes analyzed herein which, being Galerkin schemes, inherit much of the underlying variational structure and functional analytic properties.

1.3. Notation. Spaces of Bochner integrable functions from a time interval $[0, T]$ to a Banach space U will be denoted by $L^p[0, T; U]$, and $\mathcal{P}_\ell[0, T; U]$ will indicate functions of the form $u(t) = \sum_{i=0}^{\ell} p_i(t)u_i$ with $u_i \in U$ and $p_i \in \mathcal{P}_\ell(0, T)$, the space of polynomials with degree less than or equal to ℓ. The derivative of $u \in L^p[0, T; U]$ is denoted by u' or u_t. The notation $U \hookrightarrow H$ is used to indicate that U is continuously embedded in H and $U \hookrightarrow\!\!\!\!\rightarrow H$ denotes a compact embedding. A bilinear mapping $a : U \times U \to \mathbb{R}$ is coercive if there exists $c > 0$ such that $a(u, u) \ge c\|u\|_U^2$.

In the examples, $\Omega \subset \mathbb{R}^d$ denotes a bounded open set with Lipschitz boundary, and standard notation is employed for the Lebesgue spaces, $L^p(\Omega)$, and Sobolev spaces, $W^{1,p}(\Omega)$, $H^1(\Omega)$.

Given a partition $0 = t^0 < t^1 < \cdots < t^N = T$ of $[0,T]$, a subspace $U_h \subset U$, and $u_h \in C[0,T;U_h]$, we frequently write $u^n = u_h(t^n)$, and if u_h is piecewise continuous we write $u^n_\pm$ for the left and right limits. If necessary, notation of the form u^0_h, or u^0_{h-}, is used for the sequence of initial values $u_h(0)$ or $u_h(0_-) \in U_h$. The duality pairing between $f \in U'$ and $u \in U$ is written as $f(u) = (f,u)$.

The following form of the Arzelà–Ascoli theorem will be used in several places.

THEOREM 1.2. *Let X and Y be metric spaces and let X be compact. Then $\mathcal{F} \subset C(X,Y)$ (with the sup metric) is precompact if and only if $\mathcal{F}_x \equiv \{f(x) : f \in \mathcal{F}\}$ is precompact in Y for each $x \in X$, and $\mathcal{F}$ is equicontinuous; i.e., for each $\epsilon > 0$ there exists $\delta > 0$ such that*

$$d_X(x_1,x_2) < \delta \qquad \Rightarrow \qquad d_Y(f(x_1),f(x_2)) < \epsilon \text{ for all } f \in \mathcal{F}.$$

We will also make frequent use of the property that if X, Y, and Z are Banach spaces, $X \hookrightarrow Y \hookrightarrow Z$, and $\|x\|_Y \le C\|x\|_X^\theta \|x\|_Z^{1-\theta}$ for some $\theta \in (0,1)$ and all $x \in X$, then bounded subsets of X which are compact in Z are also compact in Y.

1.4. Outline. In the next section we state the DG and CG schemes for parabolic equations in a general setting. Sections 3 and 4 establish compactness properties of the discrete DG and CG solutions, respectively. When the spatial operator $A : U \to U'$ is linear, self-adjoint, and elliptic, we show that the CG scheme inherits additional regularity properties in the second half of section 4. In these sections various bounds are assumed upon the solution in order to decouple stability issues from compactness properties. In this setting, once bounds are established for a particular equation one can then refer to the appropriate theorem for the compactness properties.

In section 5 we prove two lemmas used in sections 3 and 4 to establish equicontinuity for solutions of the DG and CG schemes, respectively. The arguments in this section exploit the structure of the piecewise polynomial spaces used to construct the DG and CG schemes and are the key ingredients of the compactness proofs.

In the final section we illustrate the utility of our results by showing that numerical schemes for the two-fluid problem will converge. The equations for the two-fluid problem have the prototypical structure of a "complex fluid" where the momentum equation is coupled to an equation governing the evolution of the "microstructure." Indeed, this work was motivated by the author's interest in the Ericksen–Leslie equations for the flow of nematic liquid crystals [17] which have this structure.

2. DG and CG schemes. In this section we review the DG and CG time stepping schemes and their basic properties. Throughout we assume that $U \hookrightarrow H \hookrightarrow U'$ are dense embeddings of Banach spaces and that the pivot space, H, is a Hilbert space. The numerical schemes we consider approximate the weak statement of (1.1):

$$(u_t, v) + a(u,v) = f(v), \qquad v \in U, \tag{2.1}$$

where $a : U \times U \to \mathbb{R}$ is characterized by $a(u,v) = (A(u),v)$. For each subspace $U_h \subset U$ and partition $0 = t^0 < t^1 < \cdots < t^N = T$ of $[0,T]$, the DG and CG schemes construct on each interval (t^{n-1},t^n) a function in $\mathcal{P}_\ell[t^{n-1},t^n;U_h]$ which approximates a discrete version of (2.1). Below it will be assumed that the temporal partitions are quasi-uniform in the following sense.

DEFINITION 2.1. *A family of partitions* $\{t_h^n\}_{n=0}^{N_h}$ *of* $[0,T]$ *indexed by a parameter* h *is* quasi-uniform *if there exists* $\vartheta \in (0,1]$ *independent of* h *such that*

$$\vartheta \max_{1\le n\le N}(t_h^n - t_h^{n-1}) \le \min_{1\le n\le N}(t_h^n - t_h^{n-1}). \tag{2.2}$$

2.1. DG scheme. Given $u_-^0 \in U_h$, the DG scheme seeks $u_h \in L^2[0,T;U_h]$ satisfying, on each interval (t^{n-1},t^n), $u_h \in \mathcal{P}_\ell[t^{n-1},t^n;U_h]$ and

$$\int_{t^{n-1}}^{t^n} \Big((u_{ht},v_h) + a(u_h,v_h)\Big) + (u_+^{n-1} - u_-^{n-1}, v_+^{n-1}) = \int_{t^{n-1}}^{t^n} (f(u_h),v_h) \tag{2.3}$$

for all $v_h \in \mathcal{P}_\ell[t^{n-1},t^n;U_h]$.

The distinguishing property of the DG scheme is that the test and trial spaces are the same, so it is possible to set $v_h = u_h$. With this choice we find

$$(1/2)\|u_-^n\|_H^2 + (1/2)\|[u_-^{n-1}]\|_H^2 + \int_{t^{n-1}}^{t^n} a(u_h,u_h) = (1/2)\|u_-^{n-1}\|_H^2 + \int_{t^{n-1}}^{t^n} (f(u_h),u_h).$$

When $a(u,u) \ge c\|u\|_U^p$, summing this equation shows

$$\|u_-^n\|_H^2 + \sum_{m=0}^{n-1} \|[u^m]\|_H^2 + c\int_0^{t^n} \|u_h\|_U^p \le \|u_-^0\|_H^2 + \int_0^{t^n} |(f(u_h),u_h)|. \tag{2.4}$$

If f satisfies appropriate growth and/or continuity hypotheses, the discrete solutions computed with the DG scheme will satisfy

$$\max_{1\le n\le N} \|u_-^n\|_H^2 + \sum_{n=0}^{N-1} \|[u^n]\|_H^2 + c\|u_h\|_{L^p[0,T;U]}^p \le C\left(\|u_-^0\|_H^2 + C(f)\right).$$

In section 3 we show that this bound is sufficient for the discrete solutions to be compact in $L^r[0,T;H]$ for $1 \le r < 2p$.

One flaw of the higher order ($\ell > 1$) DG schemes is that (2.4) only bounds $\|u_h\|_H$ at the partition points. In certain problems further estimates are needed in order to apply Gronwall's inequality to bound the solution [7, 8].

2.2. CG scheme. Given $u^0 \in U_h$, the CG scheme seeks $u_h \in C[0,T;U_h]$ satisfying, on each interval (t^{n-1},t^n), $u_h \in \mathcal{P}_\ell[t^{n-1},t^n;U_h]$ and

$$\int_{t^{n-1}}^{t^n} (u_{ht},v_h)_H + a(u_h,v_h) = \int_{t^{n-1}}^{t^n} (f(u_h),v_h)_H, \qquad v_h \in \mathcal{P}_{\ell-1}[t^{n-1},t^n;U_h]. \tag{2.5}$$

Since $u_{ht} \in \mathcal{P}_{\ell-1}[t^{n-1},t^n;U_h]$ when $u_h \in \mathcal{P}_\ell[t^{n-1},t^n;U_h]$ we may set $v_h = u_{ht}$. If $A(u) = \Phi'(u)$, where $\Phi : U \to \mathbb{R}$ is independent of time, this choice of test function gives

$$\int_{t^{n-1}}^{t^n} \|u_t\|_H^2 + \Phi(u^n) = \Phi(u^{n-1}) + \int_{t^{n-1}}^{t^n} (f(u_h),u_{ht}).$$

Summing this equation shows

$$\int_0^{t^n} \|u_{ht}\|_H^2 + \Phi(u^n) = \Phi(u^0) + \int_0^{t^n} (f(u_h),u_{ht}). \tag{2.6}$$

If $\Phi(u) \geq c\|u\|_U^p$ and f satisfies appropriate growth and/or continuity hypotheses, the discrete solutions will satisfy

$$\|u_{ht}\|^2_{L^2[0,T;H]} + c \max_{1 \leq n \leq N} \|u^n\|_U^p \leq C \left(\Phi(u_h^0) + C(f) \right).$$

In section 4 it is shown that this bound is sufficient for the discrete solutions to be compact in $C[0,T;H]$.

One problem with higher order CG schemes ($\ell > 1$) is that (2.6) only bounds $\|u_h\|_U$ at the partition points. This can be problematic if, for example, $\|f(u)\|_H \leq C\|u\|_U$ since Gronwall's inequality will no longer be applicable. In this situation one may wish to set $v = u$ to first obtain an estimate on $L^p[0,T;U]$; however, this is not possible with the CG scheme. One may also wish to set $v = u$ if the CG scheme is used for an equation for which $a(.,.)$ is coercive but not the gradient of a potential.

These issues with stability can sometimes be circumvented upon introducing a projection of u_h onto $\mathcal{P}_{\ell-1}[t^{n-1}, t^n, U_h]$.

Definition 2.2. *Let $\ell > 0$ be an integer, let $0 = t^0 < t^1 < \cdots < t^N = T$ be a partition of $[0,T]$, and let U be a Banach space with $U_h \subset U$ a closed subspace. Let $u \mapsto \bar{u}$ denote the projection of $L^1[0,T;U]$ onto $\{u_h \in L^1[0,T;U_h] \mid u_h|_{(t^{n-1},t^n)} \in \mathcal{P}_{\ell-1}[t^{n-1}, t^n; U_h]\}$ characterized by*

$$0 = \int_{t^{n-1}}^{t^n} p(t) \left(u(t) - \bar{u}(t) \right) dt, \qquad p \in \mathcal{P}_{\ell-1}(t^{n-1}, t^n).$$

Note that this projection depends upon the partition of $[0,T]$ and subspace U_h; however, this is not expressed in the notation.

If $b : U \times U \to \mathbb{R}$ is bilinear and continuous and if $u \in L^1[0,T;U]$, a calculation shows

$$\int_{t^{n-1}}^{t^n} b(u, \bar{v}) = \int_{t^{n-1}}^{t^n} b(\bar{u}, \bar{v}), \qquad \bar{v} \in \mathcal{P}_{\ell-1}[t^{n-1}, t^n, U_h],$$

so if $a(.,.)$ is bilinear and independent of time, the CG scheme may be written as

$$\int_{t^{n-1}}^{t^n} (u_{ht}, v_h)_H + a(\bar{u}_h, v_h) = \int_{t^{n-1}}^{t^n} (f(u_h), v_h), \qquad v_h \in \mathcal{P}_{\ell-1}(t^{n-1}, t^n; U_h).$$

If $a(.,.)$ also is coercive, setting $v_h = \bar{u}_h$ and summing shows

$$(1/2)\|u^n\|_H^2 + c\int_0^{t^n} \|\bar{u}_h\|_U^2 \leq (1/2)\|u^0\|_H^2 + \int_0^{t^n} (f(u_h), \bar{u}_h). \tag{2.7}$$

Under suitable hypotheses on f, this gives a bound of the form

$$\max_{1 \leq n \leq N} \|u^n\|_H^2 + \|\bar{u}_h\|_{L^2[0,T;U]} \leq C \left(\|u^0\|_H^2 + C(f) \right).$$

In section 4 we show that this estimate is sufficient for the projections $\{\bar{u}_h\}_{h>0}$ to be compact in $L^r[0,T;H]$ for $1 \leq r < 4$.

When $a(.,.)$ is symmetric, bilinear, coercive, and independent of time, solutions of the CG scheme will satisfy the bounds in (2.6) and (2.7). This is not so for nonlinear operators. Solutions of the CG scheme obtained upon replacing $a(u_h, v_h)$ with $a(\bar{u}_h, v_h)$ will satisfy (2.7) but not (2.6).

3. Compactness properties of the DG scheme. When solutions of the evolution equation $u_t + A(u) = f(u)$ are bounded, estimates on the time derivative may be obtained by writing $u_t = f(u) - A(u) \equiv F(u)$. This frequently provides sufficient temporal regularity to show that solutions lie in a compact set of $L^2[0,T;H]$. The next theorem shows that the same is true for discrete solutions computed using a DG time stepping scheme. The theorem does not explicitly require a bound on the initial data; this is subsumed by the bounds assumed upon the solution.

THEOREM 3.1. *Let H be a Hilbert space, let U be a Banach space, and let $U \hookrightarrow\!\!\!\rightarrow H \hookrightarrow U'$ be dense compact embeddings. Fix $\ell \geq 0$ to be an integer, and let $1 \leq p, q < \infty$. Let $h > 0$ be a (mesh) parameter, and for each h let $U_h \subset U$ be a subspace and let $\{t_h^i\}_{i=0}^{N_h}$ be a quasi-uniform family of partitions of $[0,T]$. Assume the following:*

1. *For each $h > 0$, $u_h \in \{u_h \in L^p[0,T;U] \mid u_h|_{(t_h^{n-1},t_h^n)} \in \mathcal{P}_\ell(t_h^{n-1}, t_h^n; U_h)\}$ and on each interval satisfies*

$$\int_{t_h^{n-1}}^{t_h^n} (u_{ht}, v_h)_H + (u_+^{n-1} - u_-^{n-1}, v_+^{n-1})_H = \int_{t_h^{n-1}}^{t_h^n} (F_h, v_h)$$

for each $v_h \in \mathcal{P}_\ell(t^{n-1}, t^n; U_h)$.

2. *$\{u_h\}_{h>0}$ is bounded in $L^p[0,T;U]$ and $\{\|F_h\|_{L^q[0,T;U_h']}\}_{h>0}$ is bounded.*

Then

1. *if $p > 1$, then $\{u_h\}_{h>0}$ is compact in $L^r[0,T;H]$ for $1 \leq r < 2p$;*
2. *if $1 \leq 1/p + 1/q < 2$ and $\sum_{n=1}^{N_h} \|[u_h^n]\|_H^2 < C$ is bounded independently of h, then $\{u_h\}_{h>0}$ is compact in $L^r[0,T;H]$ for $1 \leq r < 2/(1/p + 1/q - 1)$.*

Notice that the condition $1/p + 1/q < 2$ is just the requirement that one of p or q is strictly larger than 1. Note too that for linear equations with $F(t,v) = f(t) - a(u,v)$, where $a : U \times U \to \mathbb{R}$ is bilinear, continuous, and coercive, then $p = q = 2$, so $u_h \in L^r[0,T;H]$ for any $1 \leq r < \infty$ (cf. the discussion at the end of section 2.1).

We break the proof of this theorem into the following four steps:

1. A mild generalization of the Lions–Aubin theorem which replaces the differentiability hypothesis with an equicontinuity hypothesis is first established.
2. The key step is to show that approximations computed using the DG time stepping scheme are equicontinuous. Lemma 3.3 states a little more; namely, they are uniformly equicontinuous.
3. Uniform equicontinuity implies that the solutions are bounded in a Besov space. The additional integrability in time follows from embedding results for these spaces.
4. The theorem follows upon combining the above.

The following theorem generalizes the Lions–Aubin theorem and is similar in spirit to those found in [18, 19, 24]. This theorem does not require continuity in time and is closely related to Kolmogorov's characterization of compact subsets of $L^p(\Omega)$ [23, 29].

THEOREM 3.2. *Let B_0 and B be Banach spaces, and let $B_0 \hookrightarrow\!\!\!\rightarrow B$ be a compact embedding. Let $\mathcal{F} \subset L^1[0,T;B_0]$ be bounded, and suppose for some $1 \leq p < \infty$ that $\mathcal{F}$ is equicontinuous in $L^p[0,T;B]$ in the sense that for all $\epsilon > 0$ there exists $\delta > 0$ such that*

$$\int_{\delta'}^{T} \|u(t) - u(t-\delta')\|_B^p \, dt \leq \epsilon, \qquad u \in \mathcal{F}, \;\; \delta' < \delta.$$

Then for all $0 < \theta < T/2$ *the set* $\mathcal{F}|_{(\theta,T-\theta)} \equiv \{u|_{(\theta,T-\theta)} \mid u \in \mathcal{F}\}$ *is precompact in* $L^p[\theta, T-\theta; B]$.

Remark. Necessary and sufficient conditions for compactness in $L^p[0,T;B]$ are equicontinuity and uniform integrability: for all $\epsilon > 0$ there exists $\theta > 0$ such that

$$\int_0^\theta \|u(t)\|_B^p \, dt + \int_{T-\theta}^T \|u(t)\|_B^p \, dt \le \epsilon.$$

Uniform integrability is immediate if $\{u_h\}_{h>0}$ is bounded in $L^r[0,T;B]$ for some $r > p$.

Proof. Let $\phi : \mathbb{R} \to \mathbb{R}$ be smooth, nonnegative, with support in $(-1,1)$, and have unit integral. For $\delta > 0$ let $\mathcal{F}_\delta = \{\phi_\delta * u \mid u \in \mathcal{F}\}$ be the mollifications, where u is extended by zero off $[0,T]$ and $\phi_\delta(s) = (1/\delta)\phi(s/\delta)$.

Since $\mathcal{F}$ is bounded in $L^1[0,T;B_0]$ it follows that $\mathcal{F}_\delta$ is a bounded subset of $C[0,T;B_0]$; in particular, $\mathcal{F}_{\delta t} \equiv \{u_\delta(t) \mid u_\delta \in \mathcal{F}_\delta\}$ is a bounded subset of B_0 for each $t \in [0,T]$, and hence precompact in B. Also

$$\|u_\delta(t_2) - u_\delta(t_1)\|_B \le \int_{|s|<\delta} |\phi_\delta(t_2-s) - \phi_\delta(t_1-s)| \, \|u(s)\|_B \, ds \le |t_2-t_1| \, \|\phi_\delta\|_{Lip} \, \|u\|_{L^1[0,T;B]},$$

so $\mathcal{F}_\delta$ is uniformly Lipschitz. By the Arzelà–Ascoli theorem $\mathcal{F}_\delta$ is relatively compact in $C[0,T;B]$, so it can be covered by an ϵ net for any $\epsilon > 0$. Moreover, since $C[0,T;B] \hookrightarrow L^p[0,T;B]$, it follows that for each $\delta > 0$ there is an $\epsilon/2$ net covering $\mathcal{F}_\delta$ with the $L^p[0,T;B]$ metric.

Next, write

$$u(t) - u_\delta(t) = \int_{|s|<\delta} \phi_\delta(s)(u(t) - u(t-s)) \, ds,$$

and use Hölder's inequality (with $\phi_\delta = \phi_\delta^{1/p}\phi_\delta^{1/p'}$) to get

$$\|u(t) - u_\delta(t)\|_B^p \le \int_{|s|<\delta} \phi_\delta(s) \, \|u(t) - u(t-s)\|_B^p \, ds.$$

If $\delta < \theta$, it follows that

$$\int_\theta^{T-\theta} \|u(t) - u_\delta(t)\|_B^p \, dt \le \int_\theta^{T-\theta} \int_{|s|<\delta} \phi_\delta(s) \|u(t) - u(t-s)\|_B^p \, ds \, dt \le \sup_{\delta'<\delta} \int_\delta^T \|u(t) - u(t-\delta')\|_B^p \, dt.$$

Thus for any $\epsilon > 0$, equicontinuity of $\mathcal{F}$ in $L^p[0,T;B]$ enables us to select δ sufficiently small to guarantee

$$\|u - u_\delta\|_{L^p[\theta,T-\theta,B]} \le \epsilon/2.$$

Since $\mathcal{F}_\delta$ can be covered by an $\epsilon/2$ net, it follows that $\mathcal{F}|_{(\theta,T-\theta)}$ is contained in the corresponding ϵ net, and the theorem follows. ∎

The key hypothesis is equicontinuity of the translates. The next lemma shows that translates of solutions computed using the DG time stepping scheme are bounded by a power of δ.

LEMMA 3.3. *Let* H *be a Hilbert space, let* U *be a Banach space, and let* $U \hookrightarrow H \hookrightarrow U'$ *be dense embeddings. Let* $0 = t^0 < t^1 < \cdots < t^N = T$ *be a partition of*

$[0,T]$, let $U_h \subset U$ be a subspace, and let $\ell \geq 0$. Let $1 \leq p, q < \infty$ with $1/p + 1/q \geq 1$ and assume that $u_h \in \{u_h \in L^p[0,T;U] \mid u_h|_{(t^{n-1},t^n)} \in \mathcal{P}_\ell(t^{n-1},t^n;U_h)\}$ and on each interval satisfies

$$\int_{t^{n-1}}^{t^n} (u_{ht}, v_h)_H + (u_+^{n-1} - u_-^{n-1}, v_+^{n-1})_H = \int_{t^{n-1}}^{t^n} (F, v_h)$$

for each $v_h \in \mathcal{P}_\ell(t^{n-1},t^n;U_h)$ where $F \in L^q[0,T;U_h']$.

Then there exists a constant $C > 0$ of the form $C(\ell,\vartheta)\|F\|_{L^q[0,T;U_h']}\|u_h\|_{L^p[0,T;U]}$ such that for all $0 \leq \delta \leq T$

$$\int_\delta^T \|u_h(t) - u_h(t-\delta)\|_H^2 \, dt \leq C \max(\tau,\delta)^{1/q'} \delta^{1/p'}$$

and

$$\int_\delta^T \|u_h(t) - u_h(t-\delta)\|_H^2 \, dt \leq \delta \sum_{n=1}^{N-1} \|[u^n]\|_H^2 + C\delta^{1/p'+1/q'},$$

where p' and q' are the dual exponents to p and q, respectively, and $\tau = \max_{1 \leq n \leq N}(t^n - t^{n-1})$ and $\vartheta = \min_{1 \leq n \leq N}(t^n - t^{n-1})/\tau$.

The proof of this lemma is rather long so we postpone it until section 5 below. The final ingredient needed for the proof of Theorem 3.1 states that estimates on the modulus of continuity provide additional integrability.

LEMMA 3.4. *Let W be a Banach space and let $u \in L^p[0,T;W]$ for some $1 \leq p < \infty$. If $0 \leq \alpha < 1$ and*

$$\int_\delta^T \|u(t) - u(t-\delta)\|_W^p \, dt \leq C\delta^\alpha, \qquad 0 < \delta < T,$$

then $u \in L^q[0,T;W]$ for any $1 \leq q < p/(1-\alpha)$ with norm depending only upon $L^p[0,T;W]$ and C.

Notice that it suffices to prove this result for $W = \mathbb{R}$ since if $g(t) = \|u(t)\|_W$, then $\|u\|_{L^q[0,T;W]} = \|g\|_{L^q(0,T)}$, and

$$|g(t) - g(t-\delta)| = \big|\|u(t)\|_W - \|u(t-\delta)\|_W\big| \leq \|u(t) - u(t-\delta)\|_W.$$

The hypotheses in the lemma characterize functions in the Besov space $B_\infty^{\alpha/p,p}[0,T;W]$. Since $B_\infty^{\alpha/p,p}(\mathbb{R}) \subset W^{s,p}(\mathbb{R})$ for any $s < \alpha/p$ [14, 25], the Sobolev embedding theorem shows $B_\infty^{\alpha/p,p}(\mathbb{R}) \hookrightarrow L^q(\mathbb{R})$ for any $q < p/(1-sp) = p/(1-\alpha)$. We can now prove Theorem 3.1.

Proof of Theorem 3.1. By hypothesis $\mathcal{F} \equiv \{u_h\}_{h>0}$ is bounded in $L^p[0,T;U]$, and Lemma 3.3 shows that $\mathcal{F}$ is equicontinuous in $L^2[0,T;H]$. It follows that Theorem 3.2 is applicable with $B_0 = U$ and $B_1 = H$, which establishes compactness of $\mathcal{F}$ in $L^2[\theta, T-\theta; H]$ for all $0 < \theta < T/2$.

Lemma 3.3 establishes the modulus of the continuity hypothesis of Lemma 3.4, so $\mathcal{F}$ is bounded in $L^r[0,T;H]$ for any $1 \leq r < r^*$, where $r^* = 2p$ in the first case and $r^* = 2/(1/p + 1/q - 1)$ in the second. In either case $r^* > 2$, so $\mathcal{F}$ is compact in $L^2[0,T;H]$. If $r < r^*$, let $r < s < r^*$ and write $r = 2\theta + (1-\theta)s$ for some $\theta \in (0,1)$. Then

$$\|u\|_{L^r[0,T;H]} \leq \|u\|_{L^2[0,T;H]}^\theta \|u\|_{L^s[0,T;H]}^{1-\theta},$$

so $\mathcal{F}$ is compact in $L^r[0,T;H]$. ∎

4. Compactness for the CG scheme. When $A(u) = \Phi'(u)$ is the gradient of a coercive potential, the natural bound satisfied by solutions of the CG scheme is

$$\|u_{ht}\|_{L^2[0,T;H]} + \max_{1\le i\le N} \|u^n\|_U \le C\left(\|u^0\|_U + C(f)\right).$$

We begin by showing this is sufficient for compactness in $C[0,T;H]$.

THEOREM 4.1. *Let $U \hookrightarrow\!\!\!\!\rightarrow H$ be a compact embedding of Banach spaces and let $\mathcal{U} \equiv \{u_h\}_{h>0} \subset C[0,T;H]$ have derivatives bounded in $L^q[0,T;H]$ for some $1 \le q \le \infty$. Assume that there exists $C > 0$ such that for each $h > 0$ there exists a partition $0 = t_h^0 < t_h^1 < \cdots < t_h^{N_h} = T$ with*

$$\max_{0\le n\le N_h} \|u_h(t_h^n)\|_U \le C \qquad \text{and} \qquad \lim_{h\to 0} \max_{1\le n\le N_h} |t_h^n - t_h^{n-1}| = 0.$$

Then $\mathcal{U}$ is bounded in $C[0,T;H]$, and (i) *if $q > 1$, then $\mathcal{U}$ is precompact in $C[0,T;H]$, and* (ii) *if $q = 1$, then $\mathcal{U}$ is precompact in $L^r[0,T;H]$ for any $1 \le r < \infty$.*

Proof. For each $h > 0$ define $\hat{u}_h \in C[0,T;U]$ to be the piecewise linear interpolant of $\{u_h(t_h^n)\}_{n=0}^{N_h}$. If $\tau_h^n = t_h^n - t_h^{n-1}$ and $t \in (t_h^{n-1}, t_h^n)$, then

$$\|\hat{u}_{ht}(t)\|_H = (1/\tau_h^n)\|u_h(t_h^n) - u_h(t_h^{n-1})\|_H \le (1/\tau_h^n)\int_{t_h^{n-1}}^{t_h^n} \|u_{ht}(s)\|_H \, ds,$$

so $\|\hat{u}_{ht}\|_{L^q[0,T;H]} \le \|u_{ht}\|_{L^q[0,T;H]}$ is bounded. It follows that $\{\hat{u}_h\}_{h>0}$ is equicontinuous in $C[0,T;H]$, and since $\{\hat{u}_h\}_{h>0}$ is bounded in $C[0,T;U]$ the sections $\hat{\mathcal{U}}_t \equiv \{\hat{u}_h(t)\}_{h>0}$ are bounded in U, and thus precompact in H, for each $t \in [0,T]$. The Arzelà–Ascoli theorem then shows $\{\hat{u}_h\}_{h>0}$ is precompact in $C[0,T;H]$.

To show compactness of $\{u_h\}_{h>0}$, fix $t \in (t_h^{n-1}, t_h^n)$ and compute

$$\|u_h(t) - \hat{u}_h(t)\|_H \le \|u_h(t) - u_h(t_h^{n-1})\|_H + \|\hat{u}_h(t) - u_h(t_h^{n-1})\|_H \le 2\int_{t_h^{n-1}}^{t_h^n} \|u_{ht}(s)\| \, ds.$$

If $q > 1$, it follows that

$$\|u_h - \hat{u}_h\|_{C[0,T;H]} \le 2 \max_{1\le n<N_h} |t_h^n - t_h^{n-1}|^{1/q'} \|u_{ht}\|_{L^q[0,T;H]} \to 0,$$

so $\mathcal{U}$ is also compact in $C[0,T;H]$. If $q = 1$, then

$$\|u_h - \hat{u}_h\|_{L^1[0,T;H]} \le 2 \max_{1\le n<N_h} |t_h^n - t_h^{n-1}| \|u_{ht}\|_{L^1[0,T;H]} \to 0.$$

Since both $\{u_h\}$ and $\{\hat{u}_h\}_{h>0}$ are bounded in $C[0,T;H]$ it follows that $\|u_h - \hat{u}_h\|_{L^r[0,T;H]} \to 0$ for $1 \le r < \infty$. ∎

Setting the test function in the CG scheme to be the projection of the solution onto $\mathcal{P}_{\ell-1}[0,T;U_h]$ gives estimates of the form

$$\max_{1\le n\le N} \|u^n\|_H^2 + \|\bar{u}_h\|_{L^p[0,T;U]}^p \le C\left(\|u_h^0\|_H + C(f)\right).$$

The next theorem shows that this suffices to establish compactness of the projections $\{\bar{u}_h\}_{h>0}$ in $L^r[0,T;H]$ for $1 \le r < 2p$ under mild restrictions on the meshes.

ASSUMPTION 4.2. *The orthogonal projection $P_h : H \to U_h$ is stable when restricted to U; that is, there exists $C_P > 0$ such that $\|P_h u\|_U \le C_P \|u\|_U$ for all $u \in U$ and $h > 0$.*

When $U = H^1(\Omega) \hookrightarrow L^2(\Omega) = H$ and $U_h \subset U$ is a classical finite element space constructed over a triangulation of Ω, the orthogonal projections $P_h : H \to U_h$ are stable when restricted to U for a broad class of meshes [3, 5, 11].

THEOREM 4.3. *Let U and H be Hilbert spaces, and let $U \hookrightarrow\!\!\!\rightarrow H \hookrightarrow\!\!\!\rightarrow U'$ be dense compact embeddings. Fix $\ell \geq 0$ to be an integer, and let $1 \leq p, q < \infty$. Let $h > 0$ be a (mesh) parameter, and for each h let $U_h \subset U$ be a finite-dimensional subspace and let $\{t_h^i\}_{i=0}^{N_h}$ be a partition of $[0,T]$, and assume that $\lim_{h\to 0} \max_{1\leq n\leq N_h} |t_h^{n+1} - t_h^n| = 0$. Assume that the orthogonal projections $P_h : H \to U_h$ satisfy Assumption 4.2. If the following hold, then $\{\bar{u}_h\}_{h>0}$ is precompact in $L^r[0,T;H]$ for $1 \leq r < 2p$:*

1. *For each $h > 0$, $u_h \in \{u_h \in C[0,T;U] \mid u_h|_{(t_h^{n-1}, t_h^n)} \in \mathcal{P}_\ell[t_h^{n-1}, t_h^n; U_h]\}$ and on each interval satisfies*

$$\int_{t_h^{n-1}}^{t_h^n} (u_{ht}, v_h)_H = \int_{t_h^{n-1}}^{t_h^n} (F_h, v_h)$$

for each $v_h \in \mathcal{P}_{\ell-1}[t^{n-1}, t^n; U_h]$, where $F_h \in L^q[0,T;U_h']$.

2. *The projections $\{\bar{u}_h\}_{h>0}$ onto $\mathcal{P}_{\ell-1}[0,T;U_h]$ characterized in Definition 2.2 are bounded in $L^p[0,T;U]$, and $\{\|F_h\|_{L^q[0,T;U_h']}\}_{h>0}$ is bounded.*
3. *The initial data $\{u_h^0\}_{h>0}$ are bounded in H.*

The proof of Theorem 4.3 uses properties of the projections summarized in the following lemma.

LEMMA 4.4. *Let $\ell > 0$ be an integer, let U_h be a finite-dimensional subspace of the Hilbert space H, and let $P_h : H \to U_h$ denote the orthogonal projection. Let $(.,.)_W$ be a (semi-)inner product on U_h. Then there exists a constant $C_\ell > 0$ depending only upon ℓ, such that the projection $u \mapsto \bar{u}$ of Definition 2.2 satisfies*

$$\|\bar{u}\|_{L^p[t^{n-1},t^n;W]} \leq C_\ell \|P_h u\|_{L^p[t^{n-1},t^n;W]}$$

for all $1 \leq p \leq \infty$.

Proof. For $u_h \in U_h$ let $L_h u_h \in U_h$ be the "discrete Laplacian," that is,

$$(L_h u_h, v_h)_H = (u_h, v_h)_W, \qquad v_h \in U_h.$$

The key step is to observe that for $u \in U$ and $\bar{u} \in U_h$,

$$(L_h \bar{u}, u)_H = (L_h \bar{u}, P_h u)_H = (\bar{u}, P_h u)_W.$$

Since L_h is linear it maps $\mathcal{P}_{\ell-1}[t^{n-1}, t^n; U_h]$ to itself, so

$$\|\bar{u}\|^2_{L^2[t^{n-1},t^n;W]} = \int_{t^{n-1}}^{t^n} (L_h \bar{u}, \bar{u})_H = \int_{t^{n-1}}^{t^n} (L_h \bar{u}, u)_H \leq \|\bar{u}\|_{L^{p'}[t^{n-1},t^n;W]} \|P_h u\|_{L^p[t^{n-1},t^n;W]}.$$

The finite dimensionality of $\mathcal{P}_\ell(t^{n-1}, t^n)$, and a scaling argument, shows there exists a constant $C_\ell > 0$ depending only upon ℓ, such that

$$\|\bar{u}\|_{L^p[t^{n-1},t^n;W]} \|\bar{u}\|_{L^{p'}[t^{n-1},t^n;W]} \leq C_\ell \|\bar{u}\|^2_{L^2[t^{n-1},t^n;W]}, \qquad \bar{u} \in \mathcal{P}_\ell[t^{n-1}, t^n; U_h].$$

Combining the estimates establishes the lemma. ∎

Proof of Theorem 4.3. We claim that it suffices to show that $\bar{\mathcal{U}} \equiv \{\bar{u}_h\}_{h>0}$ is precompact in $L^s[0,T;U']$ for all $1 \le s < \infty$. Granted this, note that $\|u\|_H^2 \le \|u\|_U \|u\|_{U'}$ for any $u \in U$, so

$$\|\bar{u}_h\|_{L^r[0,T;H]} \le \|\bar{u}_h\|^{1/2}_{L^{r/(2\theta)}[0,T;U]} \|\bar{u}_h\|^{1/2}_{L^{r/(2(1-\theta))}[0,T;U']}, \qquad \theta \in [0,1].$$

Setting $1 \le r < 2p$ and $\theta = r/2p$ shows

$$\|\bar{u}_h\|_{L^r[0,T;H]} \le \|\bar{u}_h\|^{1/2}_{L^p[0,T;U]} \|\bar{u}_h\|^{1/2}_{L^{rp/(2p-r)}[0,T;U']},$$

from which we conclude that $\bar{\mathcal{U}}$ is compact in $L^r[0,T;H]$.

If $\mathcal{U} \equiv \{u_h\}_{h>}$ is compact in $L^s[0,T;U']$, then the previous lemma shows that

$$\|\bar{u}_h\|_{L^s[t^{n-1},t^n;U']} \le C_\ell \|P_h u\|_{L^s[t^{n-1},t^n;U']}.$$

Since

$$\|P_h u\|_{U'} = \sup_{v\in U} \frac{(P_h u, v)_H}{\|v\|_U} = \sup_{v \in U} \frac{(u, P_h v)_H}{\|v\|_U} \le \sup_{v\in U} \frac{\|u\|_{U'} \|P_h v\|_U}{\|v\|_U} \le C_P \|u\|_{U'},$$

it follows that

$$\|\bar{u}_h\|_{L^s[t^{n-1},t^n;U']} \le C_\ell C_P \|u\|_{L^s[t^{n-1},t^n;U']}.$$

Then compactness of $\bar{\mathcal{U}}$ in $L^s[0,T;U']$ will follow from compactness of $\mathcal{U}$ in $L^s[0,T;U']$.

Compactness of $\mathcal{U}$ in $L^s[0,T;U']$ follows from Theorem 4.1 with spaces $H \hookrightarrow\!\!\!\rightarrow U'$. The only hypothesis to be verified is the differentiability. Given $v \in L^{q'}[t^{n-1}, t^n, U]$, we compute

$$\begin{aligned}
\int_{t^{n-1}}^{t^n} (u_{ht}, v)_H &= \int_{t^{n-1}}^{t^n} (u_{ht}, \bar{v})_H \\
&= \int_{t^{n-1}}^{t^n} (F_h, \bar{v}) \\
&\le \|F_h\|_{L^q[t^{n-1},t^n;U']} \|\bar{v}\|_{L^{q'}[t^{n-1},t^n;U]} \\
&\le \|F_h\|_{L^q[t^{n-1},t^n;U_h']} C_\ell C_P \|v\|_{L^{q'}[t^{n-1},t^n;U]},
\end{aligned}$$

where the previous lemma was used in the last line. It follows that $\{u_{ht}\}_{h>0}$ is bounded in $L^q[0,T;U']$, so $\mathcal{U}$ is precompact in $L^s[0,T;U']$ for any $1 \le s < \infty$. ∎

4.1. Compactness in $\boldsymbol{L^2[0,T;U]}$. When u_t and $f(u)$ are in $L^p[0,T;H]$ we may write (1.1) as $Au = f - u_t$ to conclude $Au \in L^p[0,T;H]$. Elliptic regularity may then be used to show that solutions are compact in $L^p[0,T;U]$. In this section analogous estimates are obtained for projections of solutions to the CG scheme when $A : U \to U'$ is linear.

THEOREM 4.5. *Let U be a Banach space, let H be a Hilbert space, let $U \hookrightarrow\!\!\!\rightarrow H$ be a compact embedding, and let $a : U \times U \to \mathbb{R}$ be bilinear, symmetric, continuous, and coercive. Fix $\ell \ge 0$ to be an integer, $1 \le q < \infty$ and $1 < p < \infty$. Let $h > 0$ be a (mesh) parameter, and for each h let $U_h \subset U$ be a closed subspace and let $\{t_h^i\}_{i=0}^{N_h}$ be a quasi-uniform family of partitions of $[0,T]$. Assume that $\{U_h\}_{h>0} \subset U$ is dense in the sense that for all $v \in U$ there exists $v_h \in U_h$ such that $\lim_{h\to 0} \|v - v_h\|_U \to 0$.*

For each $h > 0$ let $u_h \subset C[0,T;U_h]$ satisfy $u_h|_{(t_h^{n-1},t_h^n)} \in \mathcal{P}_\ell[t_h^{n-1},t_h^n;U_h]$ and

$$\int_{t_h^{n-1}}^{t_h^n} a(\bar{u}_h, v_h) = \int_{t_h^{n-1}}^{t_h^n} (F_h, v_h), \qquad v_h \in \mathcal{P}_{\ell-1}[t^{n-1}, t^n; U_h],$$

where $\bar{u}_h$ is the projection of u_h onto $\{\bar{u} \in L^1[0,T;U_h] \mid \bar{u}|_{(t_h^{n-1},t_h^n)} \in \mathcal{P}_{\ell-1}[t_h^{n-1},t_h^n;U_h]\}$ of Definition 2.2*. Assume that* (i) *$\{u_{ht}\}_{h>0}$ is bounded in $L^q[0,T;H]$,* (ii) *$\{F_h\}_{h>0}$ is bounded in $L^p[0,T;H]$, and* (iii) *$\{u_h^0\}_{h>0}$ is bounded in H. Then $\bar{\mathcal{U}} \equiv \{\bar{u}_h\}_{h>0}$ is bounded in $L^\infty[0,T;H]$ and precompact in $L^2[0,T;U]$.*

In the proof it will be convenient to work with the strong form of the equation, $u_t + Au = F$, which involves the operator $A : D(A) \to H$.

Definition 4.6. *Let U be a Banach space, let H be a Hilbert space, and let the embeddings $U \hookrightarrow H \hookrightarrow U'$ be dense. If $a : U \times U \to \mathbb{R}$ is linear in the second argument, let*

$$D(A) = \{u \in U \mid \text{ there exists } C_u > 0 \text{ such that } |a(u,v)| \le C_u \|v\|_H \text{ for all } v \in U\}.$$

Then $A : D(A) \to H$ is the map characterized by $(Au, v)_H = a(u, v)$ for all $v \in U$.

In general the projections $\bar{u}$ introduced in Definition 2.2 will not be continuous in time. The following lemma shows that if u is differentiable, it is possible to bound the translates of $\bar{u}$ needed to establish equicontinuity.

Lemma 4.7. *Let $\ell \ge 1$ be an integer, let $0 = t^0 < t^1 < \cdots < t^N = T$ be a partition of $[0,T]$, and let $\tau = \max_{1\le n\le N}(t^n - t^{n-1})$ and $\vartheta = \min_{1\le n\le N}(t^n - t^{n-1})/\tau$. Let W be a (semi-)inner product space, $1 \le q \le \infty$, and suppose that*

$$w \in \{C[0,T;W] \mid w|_{(t^{n-1},t^n)} \in \mathcal{P}_\ell(t^{n-1},t^n;W),\ \ 1 \le n \le N\}$$

and that $w' \in L^q[0,T;W]$. Then there exists a constant $C = C(\ell,\vartheta)$ depending only upon ℓ and ϑ such that

$$\left(\int_\delta^T \|\bar{w}(t) - \bar{w}(t-\delta)\|_W^q \, dt\right)^{1/q} \le C\|w'\|_{L^q[0,T;W]} \max(\delta,\tau)^{1/q'} \delta^{1/q},$$

where $\bar{w}$ is the projection of w introduced in Definition 2.2*.*

We postpone the proof of this lemma until section 5 and focus instead upon its application to the CG scheme.

Proof of Theorem 4.5. We first verify that $\bar{\mathcal{U}} \equiv \{\bar{u}_h\}_{h>0}$ is compact in $L^s[0,T;H]$ for any $1 \le s < \infty$. The coercivity of $a(.,.)$ and the bound on $\{F_h\}_{h>0}$ imply $\{\bar{u}_h\}_{h>0}$ is bounded in $L^p[0,T;U]$. Then Lemma 4.7 (with space $W = H$) establishes the equicontinuity hypotheses of Theorem 3.2, from which we conclude (pre)compactness of $\bar{\mathcal{U}}$ in $L^p[\theta, T-\theta;H]$ for any $0 < \theta < T$. Since $\{u_{ht}\}$ is bounded in $L^q[0,T;H]$, and $\{u_h^0\}_{h>0}$ is bounded in H, it follows that $\mathcal{U} \equiv \{u_h\}_{h>0}$ is bounded in $C[0,T;H]$. Lemma 4.4 then shows $\bar{\mathcal{U}}$ is bounded in $L^\infty[0,T;H]$, so $\bar{\mathcal{U}}$ is precompact in $L^s[0,T;H]$ for any $1 \le s < \infty$.

Let $A_h : U_h \to U_h$ be the discrete approximation of A characterized by

$$(A_h w_h, z_h)_H = a(w_h, z_h) \qquad \text{for all } z_h \in U_h.$$

Then $A_h \bar{u}_h \in \mathcal{P}_{\ell-1}[t_h^{n-1}, t_h^n; U_h]$ on each interval of the partition, and by hypothesis $\{A_h \bar{u}_h\}_{h>0}$ is bounded in $L^p[0,T;H]$. We may then pass to a subsequence for which $\bar{u}_h \rightharpoonup u$ in $L^p[0,T;U]$, $u_h \to u$ in $L^{p'}[0,T;H]$, and $A_h u_h \rightharpoonup \alpha$ in $L^p[0,T;H]$.

By hypothesis $a(.,.)$ is an inner product equivalent to $(.,.)_U$, so it suffices to show that

$$\int_0^T \|\bar{u}_h\|_U^2 = \int_0^T a(\bar{u}_h, \bar{u}_h) \to \int_0^T a(u,u) = \int_0^T \|u\|_U^2.$$

For $v \in C[0,T;U]$ fixed, the approximation properties assumed for the discrete spaces guarantee the existence of $\{\bar{v}_h\}_{h>0}$ with $\bar{v}_h \in \{\bar{v}_h \in C[0,T;U_h] \mid \bar{v}_h|_{(t_h^{n-1},t_h^n)} \in \mathcal{P}_{\ell-1}[t_h^{n-1}, t_h^n; U_h]\}$ for which $\bar{v}_h \to v$ in $C[0,T;U]$. Then

$$\int_0^T (\alpha, v)_H = \lim_{h\to 0} \int_0^T (A_h \bar{u}_h, \bar{v}_h)_H = \lim_{h\to 0} \int_0^T a(\bar{u}_h, \bar{v}_h) = \int_0^T a(u,v) = \int_0^T (Au, v)_H.$$

It follows that $\alpha = Au$. Then

$$\int_0^T a(\bar{u}_h, \bar{u}_h) = \int_0^T (A_h \bar{u}_h, \bar{u}_h)_H \to \int_0^T (Au, u)_H = \int_0^T a(u,u),$$

and $\|\bar{u}_h\|_{L^2[0,T;U]} \to \|u\|_{L^2[0,T;U]}$ so that $\bar{u}_h \to u$ in $L^2[0,T;U]$. ∎

Remarks. The linearity and symmetry assumptions were used to give a concise statement and proof of this theorem. In many situations these assumptions can be relaxed.

- Symmetry of $a(.,.)$ was used in an essential way to obtain the a priori bounds; however, it is not essential in order to obtain strong convergence in $L^p[0,T;U]$ under the hypotheses of the theorem. This may be useful when this equation is part of a system for which bounds are available independently. Briefly, with $\alpha = Au \in L^p[0,T;H]$ as above, let $\Pi_h u(t) \in U_h$ be the solution of $a(\Pi_h u, v_h) = a(u, v_h)$ for all $v_h \in U_h$. Classical finite element theory shows $\|u - \Pi_h u\|_U \le C\|u - v_h\|_U$ for all $v_h \in U_h$. The density assumption then implies $\Pi_h u \to u$ in $L^2[0,T;U] \cap C[0,T;H]$. Then

$$\begin{aligned} c\|\bar{u}_h - \Pi_h u\|_{L^2[0,T;U]}^2 &\le \int_0^T a(\bar{u}_h - \Pi_h u, \bar{u}_h - \Pi_h u) \\ &= \int_0^T (A_h \bar{u}_h - Au, \bar{u}_h - \Pi_h u)_H \to 0. \end{aligned}$$

- This line of argument also works for many nonlinear operators $A: U \to U'$. For example, for many monotone operators the map $u \mapsto (Au, u)$ is nonnegative and strictly convex. Convergence of such functions often implies strong convergence in U [4, 28].

4.1.1. Higher integrability in time. The natural energy estimate (2.6) of the CG scheme only provides a discrete $L^\infty[0,T;U]$ bound of the form

$$\max_{1\le i\le n} \|u^n\|_U \le C.$$

Under the hypotheses of Theorem 4.5 we show that some higher integrability of u_h in time is available. Observe that

$$c\|\bar{u}_h\|_U^2 \le a(\bar{u}_h, \bar{u}_h) = (A_h \bar{u}_h, \bar{u}_h)_H \le \|A_h \bar{u}_h\|_H \|\bar{u}_h\|_H.$$

Since $\bar{\mathcal{U}} = \{\bar{u}_h\}_{h>0}$ is bounded in $L^\infty[0,T;H]$ and $\{A_h\bar{u}_h\}_{h>0}$ is bounded in $L^p[0,T;H]$ it follows that

$$\sqrt{c}\|\bar{u}_h\|_{L^{2p}[0,T;U]} \leq \|\bar{u}_h\|^{1/2}_{L^\infty[0,T;H]}\|A_h\bar{u}_h\|^{1/2}_{L^p[0,T;H]}. \tag{4.1}$$

Under the hypotheses of Theorem 4.5 this shows $\bar{\mathcal{U}}$ is precompact in $L^q[0,T;U]$ for any $1 \leq q < 2p$.

4.1.2. Higher spatial regularity. When $a(.,.)$ is the bilinear form corresponding to an elliptic operator, functions in $D(A) \subset U$ typically exhibit higher spatial regularity. For classical finite element spaces $U_h \cap D(A)$ consists of the constant functions, and bounds on $A_h u_h$ in H do not immediately imply higher regularity. In this section we recall how to recover some additional regularity and bounds.

Example 4.8. In the classical situation with $U = H^1_0(\Omega)$ and $a(u,v) = (\nabla u, \nabla v)_{L^2(\Omega)}$, the domain of the operator is $D(A) = H^2(\Omega) \cap H^1_0(\Omega)$, provided $\Omega \subset \mathbb{R}^d$ is sufficiently regular. In three dimensions

$$\|u\|_{W^{1,6}(\Omega)} \leq \|u\|_{H^2(\Omega)} \leq C\|Au\|_H$$

and

$$\|u\|_{L^\infty(\Omega)} \leq C\|u\|^{1/2}_{H^1(\Omega)}\|u\|^{1/2}_{H^2(\Omega)}. \leq C\|u\|^{1/2}_U\|Au\|^{1/2}_H.$$

Classical finite element spaces satisfy $U_h \subset L^\infty(\Omega) \cap W^{1,6}(\Omega)$, and analogous bounds often hold when $A_h u_h$ is substituted for Au.

Given $u_h \in U_h$ we write $\tilde{u}_h \in D(A)$ for the solution of $A\tilde{u}_h = A_h\bar{u}_h$; that is,

$$\tilde{u}_h \in U, \qquad a(\tilde{u}_h, v) = (A_h\bar{u}_h, v)_H, \qquad v \in U.$$

Restricting v to U_h shows

$$a(\bar{u}_h, v_h) = a(\tilde{u}_h, v_h), \qquad v_h \in U_h,$$

so $\bar{u}_h$ is the classical finite element approximation of $\tilde{u}_h$. In particular, when $a(.,.)$ is continuous and coercive,

$$\|\bar{u}_h - w_h\|_U \leq C \inf_{w_h \in U_h} \|\tilde{u}_h - w_h\|_U.$$

Also,

$$c\|\tilde{u}_h\|^2_U \leq a(\tilde{u}_h, \tilde{u}_h) = (A_h\bar{u}_h, \tilde{u}_h)_H = (A_h\bar{u}_h, P_h\tilde{u}_h)_H = a(\bar{u}_h, P_h\tilde{u}_h) \leq C\|\bar{u}_h\|_U\|P_h\tilde{u}_h\|_U.$$

Thus if the orthogonal projection $P_h : H \to U_h$ is stable when restricted to U, it follows that $\|\tilde{u}_h\|_U$ is bounded by $\|\bar{u}_h\|_U$.

By interpolating between U and $D(A)$ we can obtain estimates on $\tilde{u}_h$ in other spaces (such as $W^{1,6}(\Omega)$ in the previous example), and approximation and inverse estimates can often be used to bound $\bar{u}_h$ in the same spaces.

LEMMA 4.9. *Let U be a Banach space, let $a : U \times U \to \mathbb{R}$ be bilinear, continuous, and coercive, and let $U_h \subset U$ be finite-dimensional. Let W be a Banach space for which $U_h \subset W$ and $\|v\|_W \leq \|v\|^{1-\theta}_U\|Av\|^\theta_H$ for $v \in D(A)$ and some $\theta \in [0,1]$ fixed.*

Let $I_h : D(A) \to U_h$ be an interpolation operator satisfying

1. *$\|I_h v\|_W \leq C\|v\|_W$ for $v \in D(A)$, and*
2. *$\|v - I_h v\|_U \leq Ch^{\theta_1}\|v\|^{1-\theta}_U\|Av\|^\theta_H$ for some $\theta_1 \geq 0$,*

and let the following inverse inequality hold:

$$\sup_{0\neq v_h\in U_h} \frac{\|v_h\|_W}{\|v_h\|_U} \leq C/h^{\theta_1}.$$

Then $\|u_h\|_W \leq C\|\tilde{u}\|_U^{1-\theta}\|A\tilde{u}\|_H^{\theta}$ *for all* $u_h \in U_h$*, where* $\tilde{u} \in D(A)$ *is the solution of* $A\tilde{u} = A_h u_h$.

In particular, if Assumption 4.2 *holds and* $\{\bar{u}_h\}_{h>0}$ *is bounded in* $L^q[0,T;U]$ *and* $\{A_h\bar{u}_h\}_{h>0}$ *is bounded in* $L^p[0,T;H]$*, then* $\{\bar{u}_h\}_{h>0}$ *is bounded in* $L^r[0,T;W]$*, where* $1/r = (1-\theta)/q + \theta/p$:

$$\|\bar{u}_h\|_{L^r[0,T;W]} \leq C\|\bar{u}_h\|_{L^q[0,T;U]}^{1-\theta}\|A_h\bar{u}_h\|_{L^p[0,T;H]}^{\theta}.$$

Proof. The coercivity and continuity of $a(.,.)$ on U guarantee $\|u_h - w_h\|_U \leq C\|\tilde{u} - w_h\|_U$ for any $w_h \in U_h$. We then compute

$$\begin{aligned}\|u_h\|_W &\leq \|u_h - I_h\tilde{u}\|_W + \|I_h\tilde{u}\|_W \leq (C/h^{\theta_1})\|u_h - I_h\tilde{u}\|_U + \|I_h\tilde{u}\|_W \\ &\leq C\|\tilde{u}\|_U^{1-\theta}\|A\tilde{u}\|_H^{\theta} + C\|\tilde{u}\|_W,\end{aligned}$$

and the conclusion follows. □

Example 4.10 (after Heywood and Rannacher [15]). In the classical situation with $H = L^2(\Omega) \hookrightarrow H_0^1(\Omega) = U$ and $a(u,v) = (\nabla u, \nabla v)_{L^2(\Omega)}$, the domain of the operator is $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$ when $\Omega \subset \mathbb{R}^d$ is sufficiently regular.

Let $\{\bar{u}_h\}_{h>0}$ be (projections of) the solutions of the *CG* scheme bounded in $L^\infty[0,T;H]$ with $\{A_h\bar{u}_h\}_{h>0}$ bounded in $L^2[0,T;H]$. Then (4.1) shows

$$\|\bar{u}_h\|_{L^4[0,T;U]} \leq C\|\bar{u}_h\|_{L^\infty[0,T;H]}^{1/2}\|A_h\bar{u}_h\|_{L^2[0,T;H]}^{1/2}.$$

If $I_h : C(\bar{\Omega}) \to \mathbb{R}$ is the classical interpolation operator, then on regular meshes

$$\|v - I_h v\|_{H^1(\Omega)} \leq Ch^{\theta}\|v\|_{H^\theta(\Omega)} \leq Ch^{\theta}\|v\|_{H^1(\Omega)}^{1-\theta}\|v\|_{H^2(\Omega)}^{\theta},$$

so $\theta_1 = \theta$. Moreover, $\|I_h u\|_W \leq C\|u\|_W$ for all the classical Lebesgue and Sobolev spaces. Note that if, for example, the domain has reentrant corners, then $D(A) = H^{1+\mu}(\Omega)$ for some $0 < \mu < 1$ depending upon the reentrant angles. In this situation $\theta_1 = \mu\theta$.

(1) When $W = W^{1,s}(\Omega)$ and $\Omega \subset \mathbb{R}^d$, classical inverse inequalities for finite element function spaces constructed over quasi-uniform meshes state

$$\|v_h\|_{W^{1,s}(\Omega)} \leq Ch^{\min(0,d/s-d/2)}\|v_h\|_{H^1(\Omega)}, \qquad v_h \in U_h.$$

In three dimensions $\|u\|_{W^{1,6}(\Omega)} \leq C\|u\|_{H^2(\Omega)}$ and $\|u_h\|_{W^{1,6}(\Omega)} \leq (C/h)\|u_h\|_{H^1(\Omega)}$, so the theorem is applicable with $\theta = 1$. It follows that $\|\bar{u}_h\|_{L^2[0,T;W^{1,6}(\Omega)]} \leq C\|A_h\bar{u}_h\|_{L^2[0,T;L^2(\Omega)]}$.

Similarly, $\|u\|_{W^{1,3}(\Omega)} \leq C\|u\|_{H^1(\Omega)}^{1/2}\|u\|_{H^2(\Omega)}^{1/2}$ and $\|u_h\|_{W^{1,3}(\Omega)} \leq (C/h^{1/2})\|u_h\|_{H^1(\Omega)}$, so setting $s = 3$ and $\theta = 1/2$ shows

$$\begin{aligned}\|\bar{u}_h\|_{L^{8/3}[0,T;W^{1,3}(\Omega)]} &\leq C\|\bar{u}_h\|_{L^4[0,T;H^1(\Omega)]}^{1/2}\|A_h\bar{u}_h\|_{L^2[0,T;L^2(\Omega)]}^{1/2} \\ &\leq C\|\bar{u}_h\|_{L^\infty[0,T;L^2(\Omega)]}^{1/4}\|A_h\bar{u}_h\|_{L^2[0,T;L^2(\Omega)]}^{3/4}.\end{aligned}$$

(2) When $W = L^s(\Omega)$ the corresponding inverse estimate is

$$\|v_h\|_{L^s(\Omega)} \leq Ch^{1+\min(0,d/s-d/2)}\|v_h\|_{H^1(\Omega)}.$$

In three dimensions $\|v_h\|_{L^\infty(\Omega)} \leq (C/h^{1/2})\|v_h\|_{H^1(\Omega)}$, and we may apply the lemma with $\theta = 1/2$ to find

$$\|\bar{u}_h\|_{L^{8/3}[0,T;L^\infty(\Omega)]} \leq C\|\bar{u}_h\|^{1/4}_{L^\infty[0,T;L^2(\Omega)]}\|A_h\bar{u}_h\|^{3/4}_{L^2[0,T;L^2(\Omega)]}.$$

5. Proof of equicontinuity. In this section we prove the crucial equicontinuity estimates stated in Lemmas 3.3 and 4.7. In the proofs we use the discrete Young inequality: if $1/p + 1/q = 1 + 1/r$, then

$$\sum_{k,n} F_k G_n H_{k-n} \leq \|F\|_{\ell^q}\|G\|_{\ell^p}\|H\|_{\ell^{1/r'}} = \|F\|_{\ell^q}\|G\|_{\ell^p}\|H\|_{\ell^{1/p'+1/q'}}.$$

Precise limits on the sums will not be important since the sequences can always be extended by zero.

Proof of Lemma 3.3. Let $\tau^n = t^n - t^{n-1}$, $\tau = \max_{1\leq n\leq N}\tau^n$, and $\vartheta = (\min_{1\leq n\leq N}\tau^n)/\tau$. By hypothesis

$$\int_{t^{n-1}}^{t^n}(u_{ht}, v_h)_H + (u_+^{n-1} - u_-^{n-1}, v_+^{n-1})_H = \int_{t^{n-1}}^{t^n}\langle F_h(s), v_h\rangle\, ds, \tag{5.1}$$

with $\{F_h\}_{h>0}$ bounded in $L^q[0,T;U_h']$. Setting $v_h(s) = z_h \in U_h$ (independent of time) into the above and summing from m to n shows

$$(u_-^n - u_-^m, z_h)_H = \int_{t^m}^{t^n}\langle F_h(s), z_h\rangle\, ds, \qquad 0 \leq m < n \leq N. \tag{5.2}$$

In order to estimate $u_h(t)$ at intermediate times $t \in (t^{n-1}, t^n)$ we exploit certain properties of polynomials as in [7].

For $t \in (t^{n-1}, t^n)$ let $p = p(.;t) \in \mathcal{P}_\ell(t^{n-1}, t^n)$ satisfy

$$p(t^{n-1}) = 1, \qquad \int_{t^{n-1}}^{t^n} pq = \int_{t^{n-1}}^{t} q, \qquad q \in \mathcal{P}_{\ell-1}(t^{n-1}, t^n).$$

An explicit calculation shows

$$p(s;t) = 1 + \left(\frac{s - t^{n-1}}{\tau^n}\right)\sum_{k=0}^{\ell-1} c_k\hat{p}_k\left((s - t^{n-1})/\tau^n\right), \qquad \text{with } c_k = \int_{(t-t^{n-1})/\tau^n}^{1}\hat{p}_k(\xi)\, d\xi,$$

where $\{\hat{p}_k\}_{k=0}^{\ell-1}$ is an orthonormal basis of $\mathcal{P}_{\ell-1}(0,1)$ with weighted inner product

$$(\hat{p}, \hat{q}) = \int_0^1 \xi\hat{p}(\xi)\hat{q}(\xi)\, d\xi.$$

For $t \in (t^{n-1}, t^n)$ it follows that

$$\|p\|_{L^\infty(t^{n-1},t^n)} \leq 1 + ((t^n - t)/\tau^n)\sum_{k=0}^{\ell-1}\|\hat{p}_k\|^2_{L^\infty(0,1)} \leq C_\ell,$$

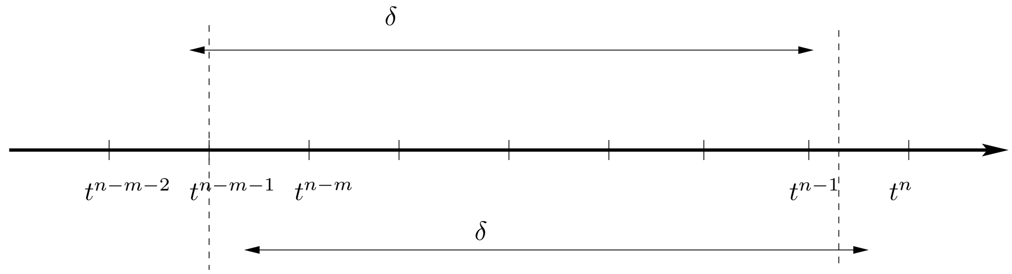

FIG. 5.1. *If $m\vartheta\tau \le \delta < (m+1)\vartheta\tau$ and $t \in (t^{n-1}, t^n)$, then $t - \delta > t^{n-m-2}$.*

so $p = p(.;t)$ is bounded independently of t.

Setting $v_h(s) = p(s;t) z_h$ into (5.1) shows

$$(u_h(t) - u_-^{n-1}, z_h)_H = \int_{t^{n-1}}^{t^n} \langle F_h(\eta), z_h \rangle p(\eta; t)\, d\eta. \tag{5.3}$$

Here we used the fact that $u_{ht} \in \mathcal{P}_{\ell-1}[t^{n-1}, t^n; U]$, so the integral of (u_{ht}, v_h) is $(u_h(t) - u_+^{n-1}, z_h)$. If $s < t$, $s \in (t^{m-1}, t^m)$, and $t \in (t^{n-1}, t^n)$, then setting $t = s$ and $n = m$ into this identity shows

$$(u_h(s) - u_-^{m-1}, z_h)_H = \int_{t^{m-1}}^{t^m} \langle F_h(\eta), z_h \rangle p(\eta; s)\, d\eta. \tag{5.4}$$

Combining (5.3), (5.2), and (5.4) shows

$$(u_h(t) - u_h(s), z_h)_H = \int_{t^{m-1}}^{t^n} \langle F_h(\eta), z_h \rangle p^{(t,s)}(\eta)\, d\eta$$

for any $z_h \in U_h$, where

$$p^{(t,s)}(\eta) = \begin{cases} 1 - p(\eta; s), & \eta \in (t^{m-1}, t^m), \\ 1, & \eta \in (t^m, t^{n-1}), \\ p(\eta; t), & \eta \in (t^{n-1}, t^n). \end{cases}$$

By setting $z_h = u_h(t) - u_h(s)$ we find

$$\|u_h(t) - u_h(s)\|_H^2 \le C \int_{t^{m-1}}^{t^n} \|F_h(\eta)\|_{U_h'}\, d\eta\, \|u_h(t) - u_h(s)\|_U, \quad 0 \le t^{m-1} < s \le t < t^n \le T, \tag{5.5}$$

where $\|p^{(s,t)}\|_{L^\infty} \le C$, a constant depending only upon ℓ.

Case 1. $\delta \ge \vartheta\tau$: Let $m\vartheta\tau \le \delta < (m+1)\vartheta\tau$ and $t \in (t^{n-1}, t^n)$. Then (see Figure 5.1)

$$s \equiv t - \delta > t^{n-1} - (m+1)\vartheta\tau \ge t^{n-m-2},$$

and (5.5) shows

$$\int_{t^{n-1}}^{t^n} \|u_h(t) - u_h(t-\delta)\|_H^2\, dt \le C \int_{t^{n-m-2}}^{t^n} \|F_h(\eta)\|_{U_h'}\, d\eta \int_{t^{n-1}}^{t^n} \|u_h(t) - u_h(t-\delta)\|\, dt. \tag{5.6}$$

Note. We consider only times $t > \delta$. If $t^{n-1} < \delta < t^n$, then $u(t) - u(t-\delta)$ is neither needed nor defined for $t \in (t^{n-1}, \delta)$. However, (5.6) is well formed if we adopt the convention that $u_h(t) - u_h(t-\delta) \equiv 0$ when $t \le \delta$ and $t^k = 0$ if $k \le 0$. This convention will be used in the remainder of the proof.

Writing

$$\int_{t^{k-1}}^{t^k} \|F_h(s)\|_{U_h'}\, ds = F_k \qquad \text{and} \qquad \int_{t^{n-1}}^{t^n} \|u_h(t) - u_h(t-\delta)\|\, dt = G_n,$$

the right-hand side of (5.6) takes the from $\sum_{k=n-m-1}^{n} F_k G_n$, which we write as

$$\sum_{k=n-m-1}^{n} F_k G_n = \sum_{k=-m-1}^{N} F_k G_n H_{k-n},$$

where $H_\ell = 1$ if $\ell \in \{-m-1, \dots, 0\}$ and is zero otherwise. Summing with respect to n and using Young's inequality shows

$$\sum_{n=0}^{N} \sum_{k=n-m-1}^{n} F_k G_n \le \left(\sum_{k=1}^{N} F_k^q\right)^{1/q} \left(\sum_{n=1}^{N} G_n^p\right)^{1/p} (m+2)^{1/p'+1/q'}$$

(recall that $F_k \equiv 0$ for $k \le 0$). Hölder's inequality shows

$$\begin{aligned}
\left(\sum_{k=0}^{N} F_k^q\right)^{1/q} &= \left(\sum_{k=0}^{N} \left(\int_{t^{k-1}}^{t^k} \|F_h\|_{U_h'}\right)^q\right)^{1/q} \\
&\le \left(\sum_{k=0}^{N} \int_{t^{k-1}}^{t^k} \|F_h\|_{U_h'}^q \tau^{q/q'}\right)^{1/q} = \|F_h\|_{L^q[0,T;U_h']} \tau^{1/q'},
\end{aligned}$$

and similarly $(\sum_n G_n^p)^{1/p} \le \|u_h(.) - u_h(.-\delta)\|_{L^p[\delta,T;U]} \tau^{1/p'} \le 2\|u_h\|_{L^p[0,T;U]} \tau^{1/p'}$. Then

$$\begin{aligned}
\int_\delta^T \|u_h(t) - u_h(t-\delta)\|_H^2\, dt &\le C\|F_h\|_{L^q[0,T;U_h']} \|u\|_{L^p[0,T;U]} \left((m+2)\tau\right)^{1/p'+1/q'} \\
&\le C\|F_h\|_{L^q[0,T;U_h']} \|u\|_{L^p[0,T;U]} (3\delta/\vartheta)^{1/p'+1/q'},
\end{aligned}$$

where the conditions $m\vartheta\tau \le \delta$ and $m \ge 1$ were used for the final inequality.

Case 2. $\delta < \vartheta\tau$: Consider first the situation where $t^{n-1} < t - \delta < t \le t^n$, shown at the top of Figure 5.2. Equation (5.3) is now applicable for both times t and $t - \delta$, so

$$(u_h(t) - u_h(t-\delta), z_h) = \int_{t^{n-1}}^{t^n} \langle F_h(\eta), z_h\rangle \left(p(\eta;t) - p(\eta;t-\delta)\right) ds. \tag{5.7}$$

With the notation introduced at the beginning of the proof, we have the explicit formula

$$p(\eta;t) - p(\eta;t-\delta)) = \left(\frac{\eta - t^{n-1}}{\tau^n}\right) \sum_{k=0}^{\ell-1} c_k \hat{p}_k \left((\eta - t^{n-1})/\tau^n\right),$$

$$\text{with } \; c_k = \int_{(t-\delta-t^{n-1})/\tau^n}^{(t-t^{n-1})/\tau^n} \hat{p}_k(\xi)\, d\xi,$$

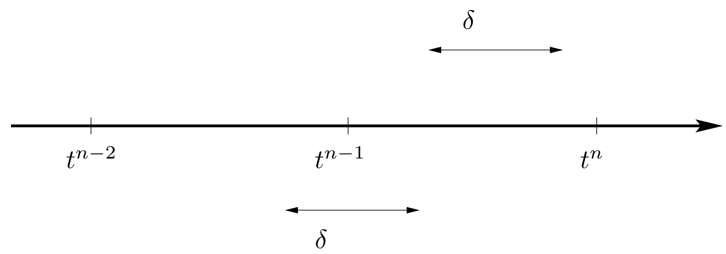

FIG. 5.2. *Two cases when* $\delta < \tau$.

so

$$\|p(.;t) - p(.;t-\delta)\|_{L^\infty(t^{n-1},t^n)} \le (\delta/\vartheta\tau) \sum_{k=0}^{\ell-1} \|\hat{p}_k\|^2_{L^\infty(0,1)} = C(\ell,\vartheta)(\delta/\tau).$$

Setting $z_h = u_h(t) - u_h(t-\delta)$ in (5.7) and using this bound shows

$$\|u_h(t) - u_h(t-\delta)\|^2_H \le (C\delta/\tau) \int_{t^{n-1}}^{t^n} \|F_h(\eta)\|_{U'_h}\, d\eta\, \|u_h(t) - u_h(t-\delta)\|_U,$$

so that

$$\int_{t^{n-1}+\delta}^{t^n} \|u_h(t) - u_h(t-\delta)\|^2_H\, dt \le (C\delta/\tau) \int_{t^{n-1}}^{t^n} \|F_h(\eta)\|_{U'_h}\, d\eta \int_{t^{n-1}}^{t^n} \|u_h(t) - u_h(t-\delta)\|_U\, dt.$$

With the notation from Case 1 above we can write the right-hand side as $\sum_k F_k G_n H_{k-n}$, where $H_i = \delta_{i0}$. Then summing and applying the Young and Hölder inequalities as before shows

$$\begin{aligned} \sum_{n=1}^N \int_{t^{n-1}+\delta}^{t^n} \|u_h(t) - u_h(t-\delta)\|^2_H\, dt &\le (C\delta/\tau)\tau^{1/p'+1/q'} \|F_h\|_{L^q[0,T;U'_h]} \|u\|_{L^p[0,T;U]} \\ &\le C\delta^{1/p'+1/q'} \|F_h\|_{L^q[0,T;U'_h]} \|u\|_{L^p[0,T;U]}, \end{aligned} \tag{5.8}$$

where the conditions $\delta < \tau$ and $1/p + 1/q \ge 1$ (so $1 \ge 1/p' + 1/q'$) were used in the last step.

Consider next the situation where $t - \delta < t^{n-1} < t \le t^n$, shown at the bottom of Figure 5.2. Note that $t \in (\delta, T)$, so $n \ge 2$.

To establish the first statement of the theorem substitute $s = t - \delta$ and $m = n-1$ into (5.5) and integrate over $t \in (t^{n-1}, t^{n-1} + \delta)$ to get

$$\int_{t^{n-1}}^{t^{n-1}+\delta} \|u_h(t) - u_h(t-\delta)\|^2_H\, dt \le C \int_{t^{n-2}}^{t^n} \|F_h(\eta)\|_{U'_h}\, d\eta \int_{t^{n-1}}^{t^{n-1}+\delta} \|u_h(t) - u_h(t-\delta)\|_U\, dt.$$

In this situation

$$\begin{aligned} \left[\sum_{n=1}^N \left(\int_{t^{n-1}}^{t^{n-1}+\delta} \|u_h(t) - u_h(t-\delta)\|_U\, dt\right)^p\right]^{1/p} &\le \|u_h(.) - u_h(.-\delta)\|_{L^p[0,T;U]}\delta^{1/p'} \\ &\le 2\|u_h\|_{L^p[0,T;U]}\delta^{1/p'}, \end{aligned}$$

and integrating F_h over the interval (t^{n-2}, t^n) gives a factor of $\tau^{1/q'}$ instead of $\delta^{1/q'}$. It follows that application of the Young and Hölder inequalities gives the estimate

$$\sum_{n=1}^{N} \int_{t^{n-1}}^{t^{n-1}+\delta} \|u_h(t) - u_h(t-\delta)\|_H^2 \, dt \leq C\|F_h\|_{L^q[0,T;U_h']} \|u\|_{L^p[0,T;U]} \tau^{1/q'} \delta^{1/p'}. \tag{5.9}$$

Combining this with the estimate in (5.8) and Case 1 above establishes the first statement of the theorem.

To obtain the second statement of the theorem we need to sharpen the estimate in (5.9). Using the triangle inequality

$$\|u_h(t) - u_h(t-\delta)\|_H \leq \|u_h(t) - u_h(t_+^{n-1})\|_H + \|[u^{n-1}]\|_H + \|u_h(t_-^{n-1}) - u_h(t-\delta)\|_H,$$

thus

$$\int_{t^{n-1}}^{t^{n-1}+\delta} \|u_h(t) - u_h(t-\delta)\|_H^2 \leq 3\delta\|[u^{n-1}]\|_H^2 + 3 \int_{t^{n-1}}^{t^{n-1}+\delta} \|u_h(t) - u_h(t_+^{n-1})\|_H^2 + \|u_h(t_-^{n-1}) - u_h(t-\delta)\|_H^2.$$

Since $(t_+^{n-1}, t) \subset (t^{n-1}, t^n)$ and $(t-\delta, t_-^{n-1}) \subset (t^{n-2}, t^{n-1})$, application of (5.3) shows

$$(u_h(t) - u_h(t_+^{n-1}), z_h)_H = \int_{t^{n-1}}^{t^n} \langle F_h(\eta), z_h\rangle \left(p(\eta; t) - p(\eta; t_+^{n-1})\right) d\eta,$$

$$(u_h(t_-^{n-1}) - u_h(t-\delta), z_h)_H = \int_{t^{n-2}}^{t^{n-1}} \langle F_h(\eta), z_h\rangle \left(p(\eta; t_-^{n-1}) - p(\eta; t-\delta)\right) d\eta.$$

As in the calculations following (5.7), explicit formulae for the polynomials $p(.,.)$ show

$$\|p(.;t) - p(.;t_+^{n-1})\|_{L^\infty(t^{n-2},t^{n-1})} \leq C(t - t^{n-1})/\tau^n \leq C(\ell,\vartheta)\delta/\tau$$

and

$$\|p(.;t_-^{n-1}) - p(.;t-\delta)\|_{L^\infty(t^{n-2},t^{n-1})} \leq C(t^{n-1} - (t-\delta))/\tau^n \leq C(\ell,\vartheta)\delta/\tau.$$

Proceeding as before, we now obtain an additional factor of $(\delta/\tau)^{1/q'}$ on the right-hand side of (5.9), which gives the second statement of the theorem. □

The proof of equicontinuity for the projections of the CG scheme has a similar form. We begin with an elementary lemma which, on each interval, characterizes $\bar{p}'$ as a projection of p', where $\bar{p}$ is the projection defined in Definition 2.2.

LEMMA 5.1. *Let $\ell \geq 1$ be an integer and let $0 = t^0 < t^1 < \cdots < t^N = T$ be a partition of $[0,T]$. If $p \in \{C(0,T) \mid p|_{(t^{n-1},t^n)} \in \mathcal{P}_\ell(t^{n-1},t^n),\ 1 \leq n \leq N\}$, let $\bar{p}$ be the projection of p of Definition 2.2. Then there exists a constant $C = C(\ell) > 0$ depending only upon ℓ such that*

$$\|\bar{p}'\|_{L^r(t^{n-1},t^n)} \leq C\|p'\|_{L^r(t^{n-1},t^n)} \qquad \text{and} \qquad |[\bar{p}^n]| \leq C\|p'\|_{L^r(t^{n-1},t^{n+1})}(t^{n+1}-t^{n-1})^{1/r'}$$

for any $1 \leq r \leq \infty$, where $[\bar{p}^n]$ denotes the jump in $\bar{p}$ at t^n. Similarly, if W is a (semi-)inner product space and

$$w \in \{C[0,T;W] \mid w|_{(t^{n-1},t^n)} \in \mathcal{P}_\ell(t^{n-1},t^n;W)\ \ 1 \leq n \leq N\},$$

there exists $C = C(\ell) > 0$ depending only upon ℓ such that

$$\|\bar{w}'\|_{L^r[t^{n-1},t^n;W]} \le C\|w'\|_{L^r[t^{n-1},t^n;W]}$$

and

$$\|[\bar{w}^n]\|_W \le C\|w'\|_{L^r[t^{n-1},t^{n+1};W]}(t^{n+1}-t^{n-1})^{1/r'}.$$

Proof. If $q \in \mathcal{P}_\ell(t^{n-1},t^n)$, then $q'(t) \in \mathcal{P}_{\ell-1}(t^{n-1},t^n)$, so using q as a test function in the definition of $\bar{p}$ and integrating by parts, one obtains

$$\int_{t^{n-1}}^{t^n} -\bar{p}'q + \bar{p}q|_{t^{n-1}}^{t^n} = \int_{t^{n-1}}^{t^n} -p'q + pq|_{t^{n-1}}^{t^n}, \qquad q \in \mathcal{P}_\ell(t^{n-1},t^n).$$

Selecting $q(t) = w(t)\hat{q}$ with $\hat{q} \in \mathcal{P}_{\ell-2}(t^{n-1},t^n)$ and $w(t) = (t-t^{n-1})(t^n-t)/\tau^2$, where $\tau = t^n - t^{n-1}$, shows

$$\bar{p}' \in \mathcal{P}_{\ell-2}(t^{n-1},t^n), \qquad \int_{t^{n-1}}^{t^n} \bar{p}'\hat{q}w = \int_{t^{n-1}}^{t^n} p'\hat{q}w, \qquad \hat{q} \in \mathcal{P}_{\ell-2}(t^{n-1},t^n).$$

A change of variables $t \mapsto (t-t^{n-1})/\tau$ and the equivalence of all norms on $\mathcal{P}_{\ell-2}(0,1)$ shows there exists a constant $C_\ell > 0$ such that $\|\bar{p}'\|_{L^r(t^{n-1},t^n)} \le C(\ell)\|p'\|_{L^r(t^{n-1},t^n)}$.

To estimate the jumps, let $q(t) = (t-t^{n-1})/(t^n-t^{n-1})$ on (t^{n-1},t^n) and $q(t) = (t^{n+1}-t)/(t^{n+1}-t^n)$ for $t \in (t^n,t^{n+1})$. Then $|q| \le 1$ and

$$\bar{p}^n_- - \bar{p}^n_+ = \int_{t^{n-1}}^{t^{n+1}} (p'-\bar{p}')q, \qquad \text{so} \qquad |[\bar{p}^n]| \le C\|p'\|_{L^r(t^{n-1},t^{n+1})}(t^{n+1}-t^{n-1})^{1/r'}.$$

Next, if $w|_{(t^{n-1},t^n)} = \sum_{i=0}^{\ell} p_i(t)w_i \in \mathcal{P}_\ell[t^{n-1},t^n;W]$, we may assume without loss of generality that $\{w_i\}_{i=0}^{\ell}$ are orthonormal. If $r \le 2$, then

$$\|\bar{w}'(t)\|_W = \left(\sum_{i=0}^{\ell} |\bar{p}_i'(t)|^2\right)^{1/2} \le \left(\sum_{i=0}^{\ell} |\bar{p}_i'(t)|^r\right)^{1/r},$$

since $\|a\|_{\ell^s} \le \|a\|_{\ell^r}$ for $r \le s$. Then

$$\begin{aligned}\int_{t^{n-1}}^{t^n} \|\bar{w}'(t)\|_W^r\,dt &\le \int_{t^{n-1}}^{t^n} \sum_{i=0}^{\ell} |\bar{p}_i'(t)|^r\,dt \le C^r \int_{t^{n-1}}^{t^n} \sum_{i=0}^{\ell} |p_i'(t)|^r\,dt \\ &\le C^r \int_{t^{n-1}}^{t^n} \left(\sum_{i=0}^{\ell} |p'(t)|^2\right)^{r/2} (\ell+1)^{1-r/2}\,dt \\ &\le C^r \int_{t^{n-1}}^{t^n} \|w'(t)\|_W^r\,dt.\end{aligned}$$

Note that $(\ell+1)^{1/r-1/2} \le (\ell+1)^{1/2}$ when $1 \le r \le 2$, so $\|\bar{w}'\|_{L^r[0,T;W]} \le C\|w'\|_{L^r[0,T;W]}$ with constant independent of r. If $r \ge 2$, then Hölder's inequality is used in the first step and the nesting of the ℓ^s norms is used in the second step.

The estimate for the jump then follows as for the polynomials. ∎

Proof of Lemma 4.7. Let $\tau^n = t^n - t^{n-1}$, $\tau = \max_{1\le n\le N}\tau^n$, and $\vartheta = (\min_{1\le n\le N}\tau^n)/\tau$. Fix $0 \le s < t \le T$ with $s \in (t^{n-m-1}, t^{n-m})$ and $t \in (t^{n-1}, t^n)$ and, with a mild abuse of notation for the sums of integrals of $\bar{w}'$ on each interval, write

$$\bar{w}(t) - \bar{w}(s) = \int_s^t \bar{w}'(\xi)\,d\xi + \sum_{i=n-m}^{n-1} [\bar{w}^i].$$

Using Lemma 5.1 and Hölder's inequality we obtain

$$\|\bar{w}(t) - \bar{w}(s)\|_W \le \int_s^t \|\bar{w}'(\xi)\|_W\,d\xi + C\sum_{i=n-m}^{n-1}\int_{t^{i-1}}^{t^{i+1}} \|w'(\xi)\|_W\,d\xi \le C\int_{t^{m-n-1}}^{t^n} \|w'(\xi)\|_W\,d\xi.$$

If $0 \le t^{n-m-1} < s \le t < t^n \le T$, it follows that

$$\|\bar{w}(t) - \bar{w}(s)\|_W^q = C\int_{t^{n-m-1}}^{t^n} \|w'(\xi)\|_W^q\,d\xi\ ((m+1)\tau)^{q/q'}. \tag{5.10}$$

Case 1. $\delta \ge \vartheta\tau$: Let $m\vartheta\tau \le \delta < (m+1)\vartheta\tau$ and $t \in (t^{n-1}, t^n)$. Then

$$s \equiv t - \delta > t^{n-1} - (m+1)\vartheta\tau \ge t^{n-m-2},$$

and (5.10) shows

$$\int_{t^{n-1}}^{t^n} \|\bar{w}_h(t) - \bar{w}_h(t-\delta)\|_W^q\,dt \le C\tau^n \int_{t^{n-m-2}}^{t^n} \|w'(\xi)\|_W^q\,d\xi\ ((m+2)\tau)^{q/q'}.$$

Note. We consider only times $t > \delta$. If $t^{n-1} < \delta < t^n$, then $\bar{w}(t) - \bar{w}(t-\delta)$ is neither needed nor defined on (t^{n-1}, δ). However, the estimate above is meaningful if we adopt the convention that $\bar{w}_h(t) - \bar{w}_h(t-\delta) \equiv 0$ when $t \le \delta$ and $t^k = 0$ for $k \le 0$. This convention will be used in the remainder of the proof.

Summing the above, we obtain

$$\begin{aligned}\int_\delta^T \|\bar{w}_h(t) - \bar{w}_h(t-\delta)\|_W^q\,dt &\le C\tau\sum_{n=1}^N \sum_{i=n-m-1}^{n} \int_{t^{i-1}}^{t^i} \|w'(\xi)\|_W^q\,d\xi\,dt\ ((m+2)\tau)^{q/q'}\\ &\le C\int_0^T \|w'(\xi)\|_W^q\,d\xi\ ((m+2)\tau)^{1+q/q'}.\end{aligned}$$

For the case under consideration, $(m+2)\tau \le 3\delta/\vartheta$, so

$$\left(\int_\delta^T \|\bar{w}_h(t) - \bar{w}_h(t-\delta)\|_W^q\,dt\right)^{1/q} \le C\|w'\|_{L^q[0,T;W]}\delta.$$

Case 2. $\delta < \vartheta\tau$: Consider first the situation where $t - \delta < t^{n-1} < t \le t^n$, shown in the bottom of Figure 5.2. Substituting $s = t - \delta$ and $m = 1$ into (5.10) shows

$$\|\bar{w}(t) - \bar{w}(t-\delta)\|_W^q \le C\int_{t^{n-2}}^{t^n} \|w'(\xi)\|_W^q\,d\xi\,\tau^{q/q'},$$

so

$$\int_{t^{n-1}}^{t^{n-1}+\delta} \|\bar{w}(t) - \bar{w}(t-\delta)\|^q \le C\int_{t^{n-2}}^{t^n} \|w'(\xi)\|_W^q\,d\xi\,\tau^{q/q'}\delta.$$

Consider next the situation where $t^{n-1} < t - \delta < t \le t^n$, shown at the top of Figure 5.2. Then

$$\|\bar{w}(t) - \bar{w}(t-\delta)\|_W^q \le \int_{t-\delta}^t \|\bar{w}'(\xi)\|_W^q \, d\xi \, \delta^{q/q'}.$$

Integrating over $t \in (t^{n-1} + \delta, t^n)$ and changing the order of integration on the right gives

$$\begin{aligned}\int_{t^{n-1}+\delta}^{t^n} \|\bar{w}(t) - \bar{w}(t-\delta)\|_W^q \, dt &\le C \int_{t^{n-1}}^{t^n} \int_{\xi \vee (t^{n-1}+\delta)}^{(\xi+\delta)\wedge t^n} \|\bar{w}'(\xi)\|_W^q \, dt \, d\xi \, \delta^{q/q'} \\ &\le C \int_{t^{n-1}}^{t^n} \|\bar{w}'(\xi)\|_W^q \, d\xi \, \delta^{1+q/q'} \\ &\le C \int_{t^{n-1}}^{t^n} \|w'(\xi)\|_W^q \, d\xi \, \delta^q.\end{aligned}$$

Combining the above and summing shows that, in the case $\delta < \vartheta\tau$,

$$\left(\int_\delta^T \|\bar{w}_h(t) - \bar{w}_h(t-\delta)\|_W^q \, dt \right)^{1/q} \le C \|w'\|_{L^q[0,T;W]} \tau^{1/q'} \delta^{1/q}. \qquad \square$$

6. Application to the model problem. We recall certain asymptotic expansions that motivate the phase field approximation of the two-fluid problem introduced in section 1.1. If $\phi \simeq \chi_{\Omega_2} - \chi_{\Omega_1}$ is the phase function introduced in section 1.1 and ψ is a smooth test function, then formal asymptotic expansions show

$$\int_\Omega (\epsilon \Delta\phi - (1/\epsilon) F'(\phi))\, \psi \simeq \epsilon \int_S 2H\psi, \tag{6.1}$$

where $S = \bar{\Omega}_2 \cap \bar{\Omega}_1$ is the surface separating the two fluids and H is the mean curvature of S. Similarly, if v is a smooth vector-valued test function, then

$$\int_\Omega (\epsilon \Delta\phi - (1/\epsilon) F'(\phi))\, \nabla\phi . v \simeq \int_S (8/3) H n . v, \tag{6.2}$$

where n is the normal to S.

A regularized skew symmetric statement of the momentum equation (1.6) is obtained upon using (6.2) to approximate the surface integral and absorbing the constant of 8/3 into the definition of γ,

$$\begin{aligned}\int_\Omega \Big\{ &\rho u_t . v + (\rho/2) \big((u.\nabla) u . v - (u.\nabla) v . u \big) \\ &- p \operatorname{div}(v) + \mu D(u) : D(v) + \gamma \left(\epsilon \Delta\phi - (1/\epsilon) F'(\phi) \right) \nabla\phi . v \Big\} = \int_\Omega \rho f . v.\end{aligned} \tag{6.3}$$

This approximation of the momentum equation contains the Laplacian of the phase function, so a conforming finite element approximation would require Hermite elements for ϕ [20].

Since the surface separating the fluid is transported by the flow, the phase function should satisfy $\phi_t + u.\nabla\phi = 0$. However, solutions of this equation will not have sufficient regularity for the phase field approximation to be meaningful, so we admit approximations of the form $\phi_t + u.\nabla\phi = O(\epsilon)$. The perturbation we utilize is the one given in (6.1),

$$\phi_t + u.\nabla\phi = \lambda\left(\epsilon\Delta\phi - (1/\epsilon)F'(\phi)\right), \tag{6.4}$$

where $\lambda > 0$ is a "relaxation" constant. Equation (1.8) is the natural weak statement of this equation, and (1.7) is obtained upon using (6.4) to eliminate $\Delta\phi$ from the weak statement of the momentum equation (6.3). This regularization of the two-fluid problem was introduced in [21].

6.1. Finite element approximation. We consider discrete solutions of the model problem which use finite elements in space and DG time stepping for the momentum equation (1.7), and which use CG time stepping for the equation for the phase (1.8). We assume that $\Omega \subset \mathbb{R}^d$, $d = 2$ or 3, is a bounded Lipschitz domain and, for each $h > 0$, that $\mathcal{T}_h$ is a quasi-uniform triangulation of Ω with maximal element diameter h and $\{t_h^n\}_{n=0}^{N_h}$ is a quasi-uniform family of partitions of $[0,T]$. For definiteness, assume $(U_h, P_h) \subset H_0^1(\Omega)^d \times L^2(\Omega)/\mathbb{R}$ is a classical Taylor–Hood subspace constructed over $\mathcal{T}_h$, and let $D_h \subset \{\phi \in H^1(\Omega) \mid \phi|_{\partial\Omega} = 1\}$ be a classical Lagrange finite element subspace constructed over $\mathcal{T}_h$. Then set

$$\begin{aligned}
\mathbb{U}_h &= \{u_h \in L^2[0,T;U_h] \mid u_h|_{(t_h^{n-1},t_h^n)} \in \mathcal{P}_\ell[t_h^{n-1}, t_h^n; U_h]\},\\
\mathbb{P}_h &= \{p_h \in L^2[0,T;P_h] \mid p_h|_{(t_h^{n-1},t_h^n)} \in \mathcal{P}_\ell[t_h^{n-1}, t_h^n; P_h]\},\\
\mathbb{D}_h &= \{\phi_h \in C[0,T;D_h] \mid (\phi_h - 1)|_{(t_h^{n-1},t_h^n)} \in \mathcal{P}_\ell[t_h^{n-1}, t_h^n; D_h]\},\\
\mathbb{E}_h &= \{\psi_h \in L^2[0,T;D_h] \mid \psi_h|_{(t_h^{n-1},t_h^n)} \in \mathcal{P}_{\ell-1}[t_h^{n-1}, t_h^n; D_h]\}.
\end{aligned}$$

On each time interval the discrete solutions satisfy $(u_h, p_h, \phi_h) \in \mathbb{U}_h \times \mathbb{P}_h \times \mathbb{D}_h$ and

$$\begin{aligned}
&\int_{t_h^{n-1}}^{t_h^n} \Big\{(\rho\, u_{ht}, v_h) + (\rho/2)\big((u_h.\nabla)u_h, v_h\big) - (\rho/2)\big(u_h, (u_h.\nabla)v_h\big) - (p_h, \operatorname{div}(v_h))\\
&\qquad + (\mu_h D(u_h), D(v_h)) + (\gamma/\lambda)(\dot{\phi}_h, \nabla\bar{\phi}_h.v_h)\Big\}\\
&\qquad + \big(\rho(u_+^{n-1} - u_-^{n-1}), v_+^{n-1}\big) = \int_{t_h^{n-1}}^{t_h^n} (\rho f, v_h),\\
&\int_{t_h^{n-1}}^{t_h^n} (\operatorname{div}(u_h), q_h) = 0,\\
&\int_{t_h^{n-1}}^{t_h^n} (\dot{\phi}_h, \psi_h) + (\lambda\epsilon/2)(\nabla\phi_h, \nabla\psi_h) + (\lambda/\epsilon)(F'(\phi_h), \psi_h) = 0
\end{aligned}$$

for all $(v_h, q_h, \psi_h) \in \mathbb{U}_h \times \mathbb{P}_h \times \mathbb{E}_h$. In the above $\mu_h = \mu(\phi_h)$, $\dot{\phi}_h = \phi_{ht} + u_h.\nabla\bar{\phi}_h$, and $(.\,,.)$ denotes the natural $L^2(\Omega)$ pairing. For the analysis below it is important that the projection of the phase variable, $\bar{\phi}_h$, be used in the terms coupling the phase and momentum equations.

6.2. Estimates. Selecting $v_h = u_h$, $q_h = p_h$, and $\psi_h = (\gamma/\lambda)\phi_{ht}$ gives us the discrete energy estimate

$$\begin{aligned}
&\int_\Omega (\rho/2)\|u_-^n\|_{L^2(\Omega)}^2 + (\gamma\epsilon/2)\|\nabla\phi^n\|_{L^2(\Omega)}^2 + (\gamma/\epsilon)\|F(\phi^n)\|_{L^1(\Omega)}\Big|_{t^{n-1}}^{t^n} + (\rho/2)\|[u^{n-1}]\|_{L^2(\Omega)}^2\\
&\qquad + \int_{t_h^{n-1}}^{t_h^n} \|\sqrt{\mu_h}D(u_h)\|_{L^2(\Omega)}^2 + (\gamma/\lambda)\|\dot{\phi}_h\|_{L^2(\Omega)}^2 = \int_{t_h^{n-1}}^{t_h^n} (\rho f, u_h).
\end{aligned}$$

Bounding the right-hand side as $\|f\|_{H^{-1}(\Omega)}\|u_h\|_{H^1(\Omega)}$, we find

$$\max_{1\le n\le N} E(u_-^n,\phi^n) \le E(u_-^0,\phi^0) + C\int_0^T \|f\|^2_{H^{-1}(\Omega)}$$

and

(6.5)

$$\|\sqrt{\mu_h}\,\nabla u_h\|^2_{L^2[0,T;L^2(\Omega)]} + (\gamma/\lambda)\|\dot{\phi}_h\|^2_{L^2[0,T;L^2(\Omega)]} \le E(u_-^0,\phi^0) + C\int_0^T \|f\|^2_{H^{-1}(\Omega)},$$

where

$$E(u,\phi) = (\rho/2)\|u\|^2_{L^2(\Omega)} + (\gamma\epsilon/2)\|\nabla\phi\|^2_{L^2(\Omega)} + (\gamma/\epsilon)\|F(\phi)\|_{L^1(\Omega)},$$

and C depends upon the Korn and Poincaré constants for Ω. In the above we have assumed that $\mu_h = \mu(\phi_h)$ is bounded above and below by positive constants. Below C will denote a constant which may depend upon the constants λ, ϵ, γ, etc., and the initial data and right-hand side f, but not upon the mesh parameter h.

Beginning with the energy estimate, it is possible to bootstrap higher order estimates and sufficient compactness to establish convergence:

1. Using the equation for ϕ we compute

$$\int_{t_h^{n-1}}^{t_h^n} (\epsilon/2)(\nabla\bar{\phi}_h,\nabla\psi_h) + (4/\epsilon)(\bar{\phi}_h,\psi_h) = \int_{t_h^{n-1}}^{t_h^n} (\epsilon/2)(\nabla\phi_h,\nabla\psi_h) + (4/\epsilon)(\phi_h,\psi_h)$$
$$= \int_{t_h^{n-1}}^{t_h^n} \Big((-1/\lambda)\dot{\phi}_h + (1/\epsilon)(4\phi_h - F'(\phi_h)),\psi_h\Big).$$

Note that $4\phi - F'(\phi) = 2\phi(3-\phi^2)$ if $|\phi|\le\sqrt{3}$ and zero otherwise, so $|4\phi - F'(\phi)| \le 4$. Thus if $A_h\phi_h \in D_h$ is characterized by

$$(A_h\bar{\phi}_h,\psi_h) = (\epsilon/2)(\nabla\bar{\phi}_h,\nabla\psi_h) + (4/\epsilon)(\bar{\phi}_h,\psi_h), \qquad \psi_h\in D_h,$$

then $A_h\bar{\phi}_h \in L^2[0,T;L^2(\Omega)]$; specifically,

$$\|A_h\bar{\phi}_h\|_{L^2[0,T;L^2(\Omega)]} \le (1/\lambda)\|\dot{\phi}_h\|_{L^2[0,T;L^2(\Omega)]} + (4/\epsilon)\sqrt{|\Omega|T} \le C(f,u^0_{h-},\phi^0,T),$$

and since $(A_h\bar{\phi}_h,\bar{\phi}_h) = \epsilon\|\nabla\bar{\phi}_h\|^2_{L^2(\Omega)} + (1/\epsilon)\|\bar{\phi}\|^2_{L^2(\Omega)}$ it follows that

$$\|\nabla\bar{\phi}_h\|_{L^2[0,T;L^2(\Omega)]} + \|\bar{\phi}\|_{L^2[0,T;L^2(\Omega)]} \le C(f,u_h^0,\phi^0,T).$$

In order to establish compactness of the solutions it is necessary to bound $\phi_{ht} = \dot{\phi}_h - u_h.\nabla\bar{\phi}_h$ in $L^2(\Omega)$. Higher order spatial estimates on u_h and $\bar{\phi}_h$ are needed for this.

2. Consider next the momentum equation. To verify the hypotheses of Theorem 3.1 the pressure is eliminated from the momentum equation. Define the discretely divergence-free subspace Z_h by

$$Z_h = \{u_h \in U_h \mid (\mathrm{div}(u_h),q_h) = 0 \text{ for all } q_h\in P_h\}.$$

Then on each interval, (t_h^{n-1}, t_h^n), the DG approximation of the momentum equation has $u_h \in \mathcal{P}_\ell(t^{n-1}, t^n; Z_h)$ and

$$\int_{t_h^{n-1}}^{t_h^n} (\rho\, u_{ht}.v_h) + \big(\rho(u_+^{n-1} - u_-^{n-1}), v_+^{n-1}\big)$$
$$= \int_{t_h^{n-1}}^{t_h^n} \Big\{(\rho f, v_h) - (\rho/2)\big(((u_h.\nabla)u_h, v_h) - (u_h, (u_h.\nabla)v_h)\big)$$
$$- (\mu_h D(u_h), D(v_h)) - (\gamma/\lambda)(\dot{\phi}_h, \nabla\bar{\phi}_h.v_h)\Big\} \tag{6.6}$$

for each $v_h \in \mathcal{P}_\ell(t_h^{n-1}, t_h^n; Z_h)$. Selecting the subspace in Theorem 3.1 to be Z_h and recalling that u_h is bounded in $L^2[0,T;H_0^1(\Omega)]$, the remaining hypothesis is to verify that the right-hand side of (6.6) is in $L^1[0,T;Z_h']$, which follows from the energy estimate (6.5) and the bound upon $\|\bar{\phi}\|_{L^2[0,T;H^1(\Omega)]}$. Theorem 3.1 (with $p=2$) then establishes compactness of $\{u_h\}_{h>0}$ in $L^r[0,T;L^2(\Omega)]$ for $1 \le r < 4$.

3. We next estimate ϕ_{ht} using the identity $\phi_{ht} = \dot{\phi}_h - u_h.\nabla\bar{\phi}_h$. As in Example 4.10 we have $\|\bar{\phi}_h\|_{L^2[0,T;W^{1,6}(\Omega)]}$ bounded, so

$$\begin{aligned}\|u_h.\nabla\bar{\phi}_h\|_{L^2(\Omega)} &\le \|u_h\|_{L^3(\Omega)}\|\bar{\phi}_h\|_{W^{1,6}(\Omega)}\\ &\le \|u_h\|_{L^2(\Omega)}^{1/2}\|u_h\|_{L^6(\Omega)}^{1/2}\|\bar{\phi}_h\|_{W^{1,6}(\Omega)}\\ &\le C\|u_h\|_{L^2(\Omega)}^{1/2}\|u_h\|_{H^1(\Omega)}^{1/2}\|\bar{\phi}_h\|_{W^{1,6}(\Omega)}.\end{aligned}$$

If $1 \le q < 8/7$, then $1/q > 1/8 + 1/4 + 1/2$ and Hölder's inequality shows that $\|u_h.\nabla\bar{\phi}_h\|_{L^q[0,T;L^2(\Omega)]}$, and hence $\|\phi_{ht}\|_{L^q[0,T;L^2(\Omega)]}$, is bounded. It follows that ϕ_h, and hence $\bar{\phi}_h$, is bounded in $L^\infty[0,T;L^2(\Omega)]$.

4. Direct application of Theorem 4.5 and Lemma 4.9 then shows $\{\bar{\phi}_h\}_{h>0}$ is compact in $L^q[0,T;H^1(\Omega)]$ for any $1 \le q < 4$ and $\{\bar{\phi}_h\}_{h>0}$ is bounded in $L^{8/3}[0,T;W^{1,3}(\Omega)]$.

5. We now have sharper estimates for the nonlinear terms in the momentum equation:

$$\begin{aligned}|(\dot{\phi}_h, v_h.\nabla\bar{\phi}_h)| &\le \|\dot{\phi}_h\|_{L^2(\Omega)}\|\bar{\phi}_h\|_{W^{1,3}(\Omega)}\|v_h\|_{L^6(\Omega)}\\ &\le C\|\dot{\phi}_h\|_{L^2(\Omega)}\|\bar{\phi}_h\|_{W^{1,3}(\Omega)}\|v_h\|_{H^1(\Omega)}.\end{aligned}$$

If $q = 8/7$, then $1/q = 1/2 + 1/(8/3)$, so

$$v_h \mapsto (\dot{\phi}_h, v_h.\nabla\bar{\phi}_h) \in L^{8/7}[0,T;H^{-1}(\Omega)].$$

The convection terms are bounded as

$$|(u_h.\nabla u_h, v_h)| \le \|u_h\|_{L^3(\Omega)}\|u_h\|_{H^1(\Omega)}\|v_h\|_{L^6(\Omega)} \le \|u_h\|_{L^2(\Omega)}^{1/2}\|u_h\|_{H^1(\Omega)}^{3/2}\|v_h\|_{H^1(\Omega)}$$

and

$$|(u_h.\nabla v_h, u_h)| \le \|u_h\|_{L^4(\Omega)}^2\|v_h\|_{H^1(\Omega)} \le \|u_h\|_{L^2(\Omega)}^{1/2}\|u_h\|_{H^1(\Omega)}^{3/2}\|v_h\|_{H^1(\Omega)}.$$

If $q < 8/7$, then $1/q > 1/8 + 1/(4/3)$, so

$$v_h \mapsto ((u_h.\nabla)u_h, v_h) - (u_h, (u_h.\nabla)v_h) \in L^q[0,T;H^{-1}(\Omega)].$$

We can then apply the second statement of Theorem 3.1 to deduce that $u_h \in L^r[0,T;L^2(\Omega)]$ for any $1 \le r < 2/(1/2+7/8-1) = 16/3$. Using this estimate to improve the above bound on the convection term shows that we may set $q = 8/7$ to obtain $u_h \in L^{16/3}[0,T;L^2(\Omega)]$.

6.3. Passage to the limit. Upon passing to a subsequence we establish convergence to a weak solution of the phase field approximation of the two-fluid problem. Granted the bounds above, we may assume that

$$\begin{aligned}
\phi_h &\rightharpoonup^* \phi && \text{in } L^\infty[0,T;L^2(\Omega)],\\
\phi_{ht} &\rightharpoonup \phi_t && \text{in } L^{8/7}[0,T;L^2(\Omega)],\\
\dot\phi_h &\rightharpoonup \eta && \text{in } L^2[0,T;L^2(\Omega)],\\
\bar\phi_h &\to \bar\phi && \text{in } L^2[0,T;H^1(\Omega)] \cap L^s[0,T;L^2(\Omega)], \qquad 1 \le s < \infty,\\
u_h &\rightharpoonup u && \text{in } L^2[0,T;H^1(\Omega)],\\
u_h &\to u && \text{in } L^2[0,T;L^2(\Omega)] \cap L^s[0,T;L^2(\Omega)], \qquad 1 \le s < 16/3.
\end{aligned}$$

To show that $\bar\phi = \phi$, let $\psi \in \mathcal{D}((0,T)\times\Omega)$ and let $\bar\psi_h \in C[0,T;D_h] \cap \mathbb{E}_h$ converge to ψ in $W^{1,\infty}[0,T;W^{1,\infty}(\Omega)]$. Then using the definition of the projection $\phi_h \mapsto \bar\phi_h$ we find

$$\int_0^T (\phi,\psi) = \lim_{h\to 0}\int_0^T (\phi_h,\bar\psi_h) = \lim_{h\to 0}\int_0^T (\bar\phi_h,\bar\psi_h) = \int_0^T (\bar\phi,\psi).$$

Next, to show that $\eta = \dot\phi \equiv \phi_t + u.\nabla\phi$, recall that $\dot\phi_h = \phi_{ht} + u_h.\nabla\bar\phi_h$ and compute

$$\begin{aligned}
\int_0^T (\phi_t + u.\nabla\phi, \psi) &= \int_0^T (\phi_t,\psi) - (\phi, \operatorname{div}(\psi u))\\
&= \lim_{h\to 0}\int_0^T (\phi_{ht},\psi) - (\bar\phi_h, \operatorname{div}(\psi u_h))\\
&= \lim_{h\to 0}\int_0^T (\dot\phi_{ht},\psi) = \int_0^T (\eta,\psi),
\end{aligned}$$

so $\eta = \dot\phi$.

If $v_h \in C[0,T;Z_h] \cap \mathbb{U}_h$ and $v_h(T) = 0$, the discrete momentum equation shows

$$\int_0^T \Big\{ -(\rho\, u_h, v_{ht}) + (\rho/2)\big(((u_h.\nabla)u_h, v_h) - (u_h, (u_h.\nabla)v_h)\big) \\ + (\mu_h D(u_h), D(v_h)) + \gamma(\dot\phi_h, \nabla\bar\phi_h.v_h)\Big\} = (\rho u_-^0, v_h(0)) + \int_0^T (\rho f, v_h).$$

Since the pair (U_h, P_h) satisfy the inf-sup condition it follows that for any $v \in C_0^\infty([0,T)\times\Omega)$ there exists a sequence $v_h \in C[0,T;Z_h] \cap \mathbb{U}_h$ with $v_h(T) = 0$ such that $v_h \to v$ in $W^{1,\infty}[0,T;W^{1,\infty}(\Omega)]$. Upon passing to a subsequence we may assume

$$\begin{aligned}
&u_h \to u \text{ in } L^2[0,T;L^2(\Omega)], && u_h \rightharpoonup u \text{ in } L^2[0,T;H^1(\Omega)],\\
&\dot\phi_h \rightharpoonup \dot\phi \text{ in } L^2[0,T;L^2(\Omega)], && \bar\phi_h \to \phi \text{ in } L^2[0,T;H^1(\Omega)].
\end{aligned}$$

The viscosity, $\mu_h = \mu(\phi_h)$, converges in $L^2[0,T;L^2(\Omega)]$ since $\mu : \mathbb{R} \to \mathbb{R}$ is continuous. It follows that we may pass to the limit in the momentum equation term by term.

Similarly, if $\psi \in \mathcal{D}((0,T) \times \Omega)$, there exists $\bar{\psi}_h \in C[0,T;D_h] \cap \mathbb{E}_h$ converging to ψ in $W^{1,\infty}[0,T;W^{1,\infty}(\Omega)]$. Since $F' : \mathbb{R} \to \mathbb{R}$ is continuous and has linear growth, and since Theorem 4.1 shows $\{\phi_h\}_{h>0}$ is compact in $C[0,T;L^2(\Omega)]$, it follows that $F'(\phi_h) \to F'(\phi)$ in $C[0,T;L^2(\Omega)]$. We may then pass to the limit term by term in the equation

$$\int_0^T (\dot{\phi}_h, \psi_h) + (\epsilon/2)(\nabla \phi_h, \nabla \psi_h) + (1/\epsilon)(F'(\phi_h), \psi_h) = 0$$

to conclude that (u, ϕ) are a weak solution of the phase field approximation of the two-fluid problem.

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