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CONVERGENCE OF THE DISCONTINUOUS GALERKIN METHOD FOR DISCONTINUOUS SOLUTIONS*

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Abstract. We consider linear first order scalar equations of the form $\rho_t + \operatorname{div}(\rho v) + a\rho = f$ with appropriate initial and boundary conditions. It is shown that approximate solutions computed using the discontinuous Galerkin method will converge in $L^2[0,T;L^2(\Omega)]$ when the coefficients v and a and data f satisfy the minimal assumptions required to establish existence and uniqueness of solutions. In particular, v need not be Lipschitz, so characteristics of the equation may not be defined, and the solutions being approximated my not have bounded variation.

Key words. convergence, discontinuous Galerkin method, hyperbolic equations

AMS subject classifications. 65M60, 65M12

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1. Introduction.

1.1. Overview. In 1989 DiPerna and Lions [5] proved that weak solutions $\rho : [0,T) \times \Omega \to \mathbb{R}$ of the convection equation

$$\rho_t + \operatorname{div}(\rho v) + a\rho = f \quad \text{in } \Omega, \qquad \rho|_{t=0} = \rho_0, \tag{1.1}$$

are unique when the velocity field v is in certain Sobolev spaces; in particular, v need not be continuous. While weak solutions exist under much weaker hypotheses, the uniqueness result is subtle. They also proved the following remarkable stability result (specialized here to $L^2(\Omega)$).

THEOREM 1.1 (see DiPerna and Lions [5]). *Let $\{\rho_k\}_{k=0}^{\infty} \subset L^\infty[0,T;L^2(\Omega)]$ be weak solutions of*

$$\rho_{kt} + \operatorname{div}(\rho v_k) + a_k \rho_k = f_k \quad \text{in } \Omega$$

with initial data $\rho_k(0) = \rho_{k0}$. Assume that

- *$\{v_k\} \subset L^1[0,T;H_0^1(\Omega)]$, $v_k \to v$ in $L^1[0,T;L^2(\Omega)]$, and $\operatorname{div}(v_k) \to \operatorname{div}(v)$ in $L^1[0,T;L^\infty(\Omega)]$ with $v \in L^1[0,T;H_0^1(\Omega)]$, and*
- *$a_k \to a$ in $L^1[0,T;L^\infty(\Omega)]$, $f_k \to f$ in $L^1[0,T;L^2(\Omega)]$, and $\rho_{k0} \to \rho_0$ in $L^2(\Omega)$.*

Then $\rho_k \to \rho$ in $L^2[0,T;L^2(\Omega)]$, where ρ is the unique solution of (1.1).

In this paper we prove an analogue of this theorem for approximate solutions computed using the discontinuous method.

While the original motivation of DiPerna and Lions concerned the Bolzman equation, their results provided the tools needed to establish existence of solutions for the incompressible Navier–Stokes equations with variable density and viscosity. In modeling the flow of incompressible immiscible fluids the density is discontinuous, and both

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the density and viscosity, $\mu = \mu(\rho)$, appear as coefficients multiplying the principle terms of the momentum equation. To obtain existence of the coupled system the strong convergence guaranteed by Theorem 1.1 is used in an essential fashion [14].

Similarly, to prove convergence of numerical approximations to the equations modeling incompressible, immiscible fluids, or flows containing particles, strong convergence of the approximate densities ρ_h will be required when the velocity fields are also computed approximately, and this is what is addressed here. Below we show that approximate solutions of the density equation converge strongly in $L^2[0,T;L^2(\Omega)]$ when the coefficients and data satisfy the minimal hypotheses required to obtain existence and uniqueness of the continuous problem.

While many numerical schemes have been proposed for the solution of first order hyperbolic equations, the discontinuous Galerkin method stands out as one of the best schemes in practice. In his introductory text [7] Johnson states "the discontinuous Galerkin method performs remarkably well and we know of no (linear) finite difference method that is better." Another advantage is that discontinuous Galerkin approximations of the density equation couple correctly with natural approximations of the momentum equation so that the discrete system inherits estimates on the kinetic energy $\rho(|v|^2/2)$. To recover such energy estimates it is natural to multiply the density equation by $|v|^2/2$. Since stable approximations of the momentum equation typically approximate v with piecewise quadratic functions, this suggests that piecewise quartic functions are a natural choice of discrete spaces for the discontinuous Galerkin approximation of the balance of mass. Our results below show that the discontinuous Galerkin scheme will converge when piecewise polynomial approximations of arbitrary degree in the spatial variables are used; however, for technical reasons the degree of the piecewise polynomial temporal variation is restricted to be zero or one.

1.2. Discontinuous Galerkin method. The discontinuous Galerkin method was introduced to simulate neutron transport, and in this context the coefficients v and a are constant. Most of the analysis of this method concerns rates of convergence [8, 11, 12] and requires the solution to be smooth, so is not applicable to problems involving discontinuous solutions. One exception is the work of Lin and Zhou [13], who consider equations in the form $V.\nabla\rho + a\rho = f$, which is slightly more general than the evolution equation considered here (put $V = (1, v)$ and $\nabla = (\partial_t, \nabla_x)$ to recover the evolution form). Lin and Zhou require $V \in W^{1,1}(\Omega)$ and show that if the solution is in $H^{1/2}(\Omega)$, then piecewise constant solutions of the discontinuous Galerkin method will converge to certain weak solutions (the definition of a weak solution in [13] is not standard). Below we exploit the evolution structure of the equation in an essential fashion. This allows us to avoid any smoothness assumptions on v with respect to the time variable, $v \in L^1[0,T;H_0^1(\Omega)]$. Under this assumption ρ will not be of bounded variation and typically will not belong to the fractional Sobolev space $H^{1/2}$.

Since (1.1) is a conservation law in divergence form, it is natural to consider the substantial literature concerning numerical schemes developed for nonlinear conservation laws of the form $\rho_t + \mathrm{div}(F(\rho)) = f$. Essentially all the methods (including the discontinuous Galerkin method) and theory developed in this context assume that F is independent of x and t, the exception being Kruzkov's original work [10], which allows F to depend on x and t in a C^1 fashion. In this context solutions of the conservation laws are regular in the sense that they have bounded variation and frequently stability of numerical schemes is ensured by flux limiters, which limit the variation [3, 4, 6, 9]. The best regularity one can expect for the velocity field of immiscible fluids is $v \in L^2[0,T;H_0^1(\Omega)]$ and the variation of the corresponding density will fail

to be bounded, so any scheme that limits the variation of the discrete solution could not converge strongly.

The level set method has also been used to solve certain hyperbolic equations [15]. If $a = f = \mathrm{div}(v) = 0$, then (1.1) can be written as $\rho_t/|\nabla\rho| = -v.(\nabla\rho/|\nabla\rho|)$, showing that the level sets of ρ move with normal velocity $-v$. Frequently it is possible to find a smooth function ϕ_0 having the same level sets as ρ_0 with $\rho_0 = \beta(\phi_0)$, where $\beta : \mathbb{R} \to \mathbb{R}$ is a discontinuous function. Since $\rho = \beta(\phi)$ for all subsequent times, this eliminates the need to deal with discontinuous functions. This approach has been applied to problems in fluid mechanics [2].

1.3. Notation. In this paper, $\Omega \subset \mathbb{R}^d$ is a bounded domain with unit outward normal n. We consider a regular family of finite element meshes $\{\mathcal{T}_h\}_{h>0}$ each of which we assume triangulates Ω exactly. It is assumed that the finite elements have uniformly bounded aspect ratio. The parameter $h > 0$ represents the diameter of the largest element in $\mathcal{T}_h$ and $|K|$ denotes the area (two dimensions) or volume (three dimensions) of an element $K \in \mathcal{T}_h$. Similarly, if $e \subset K$ is an edge or face, then $|e|$ denotes the length or area of e, respectively. The discontinuous Galerkin method is constructed using space-time elements of the form $K \times (t^{m-1}, t^m)$ with $K \in \mathcal{T}_h$, where $\{t^m\}_{n=0}^M$ is a partition of $[0, T]$. The space of polynomials of degree k on an element K is denoted $\mathcal{P}_k(K)$. For simplicity we assume that for each $h > 0$ a uniform partition of $[0, T]$ used with $t^m = m\tau$ where $\tau = T/M$ is assumed to converge to zero as h tends to zero. We denote the approximate solutions by ρ_h; in particular, the dependence on τ is implicit. If $a \in \mathbb{R}$, the positive and negative parts are denoted by $a^\pm$ with $a^+ = \max(a, 0)$ and $a^- = \min(a, 0)$.

The traces of functions ρ_h play an important role in the discontinuous Galerkin method and give rise to a lot of notation. In general, ρ_h denotes a discontinuous piecewise polynomial approximate solution of the convection equation, and annotations of the form ρ^m, ρ_-, etc., refer to various traces of ρ_h (i.e., the subscript h is omitted). We write $\rho^m = \rho_h(t^m_-) = \lim_{t\nearrow t^m} \rho_h(t)$, and the trace from above is denoted by $\rho^m_+ = \lim_{t\searrow t^m} \rho_h(t)$. The jump in ρ_h at t^m is denoted by $[\rho^m] = \rho^m_+ - \rho^m$. Integrals of the form $\int_{\partial K}(\rho_h \ldots)$ compute the trace of ρ_h from within K: $\rho_h|_{\partial K}(t, x) = \lim_{\epsilon\searrow 0} \rho_h(t, x - \epsilon n)$, where n is the unit outward normal to ∂K. An orientation of each edge, e, (face in three dimensions) between two finite elements is selected by arbitrarily selecting one of its normals, which is denoted by N (see Figure 2.1). We write $e = K_+ \cap K_-$, where N points from K_- to K_+, and write $[\rho_e] = \rho_+ - \rho_-$, where $\rho_\pm$ are the traces taken from within $K_\pm$.

Standard notation is adopted for the Lebesgue spaces, $L^p(\Omega)$, and the Sobolev spaces, $W^{m,p}(\Omega)$ or $H^m(\Omega)$. The dual exponent to p is denoted by p', $1/p + 1/p' = 1$. Solutions of the evolution equation will be functions from $[0, T]$ into these spaces, and we adopt the usual notion, $L^2[0, T; H^1(\Omega)]$, $C[0, T; H^1(\Omega)]$, etc., to indicate the temporal regularity of such functions. The space of C^∞ test functions having compact support in Ω is denoted by $\mathcal{D}(\Omega)$. For vector valued quantities, such as the velocity v, we write $v \in L^2(\Omega)$, to indicate that each component lies in the specified space. The space $H(\mathrm{div}; \Omega)$ is the set of vector valued functions in $L^2[0, T; L^2(\Omega)]$ with divergence in $L^2[0, T; L^2(\Omega)]$. Strong convergence of a sequence is indicated as $\rho_h \to \rho$ and weak convergence by $\rho_h \rightharpoonup \rho$.

2. Background. In this section we recall the essential results developed by DiPerna and Lions [5] for (1.1) and recall the discontinuous Galerkin method for approximating solutions of (1.1). Our proof of convergence is essentially a verification of the old adage "a stable consistent scheme is convergent." To make this rigorous

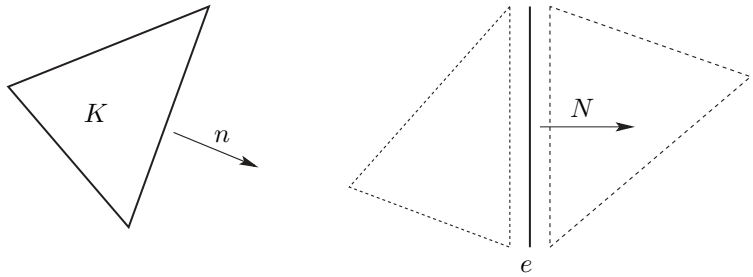

FIG. 2.1. (a) *The outward normal vector of a triangle (tetrahedra in three dimensions) is denoted by n.* (b) *A normal to each edge (face in three dimensions) is arbitrarily chosen and denoted by N.*

we use Theorem 2.2, taken from [14], in a crucial fashion. This theorem states that certain elementary relations that hold for classical solutions of (1.1) continue to hold for weak solutions. As stated above, our convergence results require the approximate solutions to be either piecewise constant or piecewise linear in time. This can be directly attributed to a lack of stability; in general, there are no estimates for the time derivative of the discrete solution. Lacking bounds on the time derivative we can't show that natural piecewise constant approximations converge weakly to the same limit as higher degree piecewise polynomial approximations; see Corollary 3.2.

2.1. DiPerna–Lions theory. For technical reasons DiPerna and Lions [5] considered velocity fields v which vanished on the boundary, and for this reason we always require $v(t) \in H_0^1(\Omega)$. In this situation no boundary conditions are required for ρ; otherwise, ρ would be specified on the inflow boundary, where $v.n < 0$. The following definition of a weak solution of (1.1) is standard and allows us to admit the possibility of discontinuous solutions.

DEFINITION 2.1. *Let $v|_{\partial\Omega} = 0$; then $\rho : [0,T) \times \Omega \to \mathbb{R}$ is a weak solution of* (1.1) *if*

$$-\int_0^T \int_\Omega \rho(\psi_t + v.\nabla\psi - a\psi) = \int_\Omega \rho_0 \psi(0) + \int_0^T \int_\Omega f\psi \tag{2.1}$$

for all $\psi \in \mathcal{D}([0,T) \times \Omega)$.

If $\beta : \mathbb{R} \to \mathbb{R}$ is differentiable, then multiplying (1.1) by $\beta'(\rho)$ and formally rearranging the derivatives shows that $\beta(\rho)$ satisfies

$$\beta(\rho)_t + \operatorname{div}(\beta(\rho)v) + \big(\rho\beta'(\rho) - \beta(\rho)\big)\operatorname{div}(v) + a\rho\beta'(\rho) = f\beta'(\rho). \tag{2.2}$$

The following theorem by DiPerna and Lions [5] states that weak solutions of (1.1) will also be weak solutions of (2.2) provided each term is integrable.

THEOREM 2.2. *Let $1 \le p \le \infty$ and suppose that*

$$v \in L^1[0,T,W_0^{1,p'}(\Omega)], \qquad a, \operatorname{div}(v) \in L^1[0,T;L^\infty(\Omega)], \qquad f \in L^1[0,T;L^p(\Omega)].$$

Then for each $\rho_0 \in L^p(\Omega)$ there exists a unique weak solution $\rho \in L^\infty[0,T,L^p(\Omega)]$ of (1.1). *This solution satisfies as follows:*

- $\rho \in C[0,T;L^p(\Omega)]$ *if* $p < \infty$.

- *If $\beta \in C^1(\mathbb{R})$ satisfies $\beta'(t) \le C(1+|t|^r)$ for $C > 0$, and $r = p-1$ if $p < d/(d-1)$, $r < p-1$ if $p = d/(d-1)$, and $r = p/d$ if $p > d/(d-1)$ (r arbitrary if $p = \infty$), then* (2.2) *is satisfied weakly.*
- *If $\beta \in C^1(\mathbb{R})$ satisfies $\beta'(t) \le C(1+|t|^r)$ for $C > 0$ and $r \le p-1$ (β arbitrary if $p = \infty$), then*

$$\frac{d}{dt}\int_\Omega \beta(\rho) + \int_\Omega \operatorname{div}(v)(\rho\beta'(\rho) - \beta(\rho)) + a\rho\beta'(\rho) = \int_\Omega f\beta'(\rho). \tag{2.3}$$

Remark. The restrictions on r in the second statement of the theorem and the Sobolev embedding theorem guarantee that the term $\beta(\rho)v.\nabla\psi$ is integrable. Similarly, the restriction $r \le p-1$ in the third statement guarantees that each term is integrable.

2.2. Discontinuous Galerkin method. We allow for the possibility that the coefficients are computed only approximately on each mesh; $v \simeq v_h$ and $a \simeq a_h$. Since $\operatorname{div}(v_h)$ may not be bounded in $L^1[0,T;L^\infty(\Omega)]$, care is required to construct a stable approximation scheme. Let

$$\begin{aligned}\mathcal{R}_h = \{\rho_h \in L^2[0,T;L^2(\Omega)] \mid \rho_h|_{K\times(t^{m-1},t^m)} \in \mathcal{P}_k(K)\otimes\mathcal{P}_\ell(t^{m-1},t^m),\\ K \in \mathcal{T}_h,\ m = 1,2,\ldots\}.\end{aligned}$$

The discontinuous Galerkin method requires $\rho_h \in \mathcal{R}_h$ to satisfy

$$\begin{aligned}&\int_K \left(\rho^m\psi(t^m_-) - \rho^{m-1}\psi(t^{m-1}_+)\right)\\ &\quad - \int_{t^{m-1}}^{t^m}\int_K \rho_h\left(\psi_t + v_h.\nabla\psi + (1/2)(\operatorname{div}(v_h) - \operatorname{div}(v))\psi - a_h\psi\right)\\ &\quad + \int_{t^{m-1}}^{t^m}\int_{\partial K} \rho_{in}(v_h.n)\psi = \int_{t^{m-1}}^{t^m}\int_K f\psi\end{aligned} \tag{2.4}$$

for each $K \in \mathcal{T}_h$, $m = 1,2,\ldots$, and $\psi \in \mathcal{R}_h$. Since functions in $\mathcal{R}_h$ are discontinuous at the boundary of each space-time element, $K \times (t^{m-1},t^m)$, we specify how the traces are to be evaluated. In all instances traces of ρ_h are taken from the upwind direction, and traces of ψ are taken from within $K \times (t^{m-1},t^m)$. That is, ρ^m and ρ^{m-1} are the traces taken from below, $\rho^m = \lim_{s\nearrow t^m}\rho_h(s)$, ρ_{in} is the inflow trace, $\rho_{in}(x) = \lim_{\epsilon\searrow 0}\rho_h(x-\epsilon v_h)$, and the subscripts $t_\pm$ are used to indicate the traces of ψ at each end of the time interval.

If $(v.n) = (v.n)^+ + (v.n)^-$ is the decomposition of $v.n|_{\partial K}$ into positive and negative parts and $e = K \cap K_-$ is an edge (face in three dimensions) common to K and K_-, then the upwind term can be written as

$$\rho_{in}(v.n) = \rho_h|_{\partial K_-}(v.n)^- + \rho_h|_{\partial K}(v.n)^+.$$

If a global orientation of e is determined by (arbitrarily) selecting one of its normals N (see Figure 2.1), then the weak statement on each element can be summed to give

$$\begin{aligned}&\int_\Omega \rho^m\psi(t^m_-) - \sum_{k=0}^{m-1}\int_\Omega \rho^k[\psi^k] - \int_0^{t^m}\int_\Omega \rho_h\left(\psi_t + v_h.\nabla\psi + (1/2)(\operatorname{div}(v_h) - \operatorname{div}(v))\psi - a_h\psi\right)\\ &\quad - \sum_e \int_0^{t^m}\int_e \left(\rho_-(v_h.N)^+ + \rho_+(v_h.N)^-\right)[\psi_e] = \int_\Omega \rho^0\psi(t^0_-) + \int_0^{t^m}\int_\Omega f\psi.\end{aligned} \tag{2.5}$$

The jump in ψ on the element boundaries is denoted by $[\psi_e] = \psi_+ - \psi_-$ with $\psi_\pm$ determined by the orientation N on an edge and the positive time direction at a temporal interface. We abused the notation by writing $\sum_K \int_K(.) = \int_\Omega(.)$ for integrands involving gradients and similarly for temporal integrals. When ψ is continuous the above expression reduces to a standard weak statement of (1.1).

Remarks. The variant of the discontinuous Galerkin scheme presented here is formulated to be convergent when the velocity field v was known only approximately but the divergence of the v was known precisely. The canonical example of this would be when v_h is an approximation of the velocity of an incompressible fluid for which $\mathrm{div}(v) = 0$. Typically $v_h \to v$ in $L^2[0,T;L^2(\Omega)]$, but $\mathrm{div}(v_h) \neq 0$ since it is difficult to construct divergence free subspaces of $H_0^1(\Omega)$; in particular, $\mathrm{div}(v_h) \not\to 0$ in $L^1[0,T;L^2(\Omega)]$.

The assumptions $v \in L^1[0,T;H_0^1(\Omega)]$ and $\mathrm{div}(v) \in L^1[0,T;L^\infty(\Omega)]$ are required for uniqueness of the solution of (1.1); however, the approximation v_h of v used in the computations need not converge to v in these spaces. For the scheme to be well defined, traces of $v_h.n$ must exist on element boundaries and the traces from each side must agree. For these reasons we require v_h to lie in the space

$$V_h = \{v_h \in L^1[0,T;H(\mathrm{div};\Omega)] \mid v_h(t)|_K \in \mathcal{P}_k(K)\}$$

for some fixed integer $k \geq 0$. Note that uniqueness of the continuous problem requires $v \in L^2[0,T;H_0^1(\Omega)]$, but we do not require $v_h \in L^2[0,T;H_0^1(\Omega)]$.

3. Stability. The following stability result is standard.

THEOREM 3.1 (stability). *Let $\rho_h \in \mathcal{R}_h$ be the approximate solution of* (1.1) *obtained with the discontinuous scheme* (2.4) *and suppose that $\rho^0 \in L^2(\Omega)$, $v_h \in V_h$,*

$$v \in L^1[0,T;H_0^1(\Omega)], \qquad a_h, \mathrm{div}(v) \in L^1[0,T;L^\infty(\Omega)], \qquad f \in L^1[0,T;L^2(\Omega)];$$

then

$$\begin{aligned}(1/2)\|\rho^m\|^2_{L^2(\Omega)} + (1/2)\sum_{k=0}^{m-1}\|[\rho^k]\|^2_{L^2(\Omega)} + (1/2)\sum_e \int_0^{t^m}\int_e |v_h.n|[\rho_h]^2 \\ + \int_0^{t^m}\int_\Omega (\mathrm{div}(v)/2 + a_h)\rho_h^2 = (1/2)\|\rho^0\|^2_{L^2(\Omega)} + \int_0^{t^m}\int_\Omega f\rho_h.\end{aligned} \tag{3.1}$$

(1) *If* $(\mathrm{div}(v)/2 + a_h) \geq c > 0$ *and* $f \in L^2[0,T;L^2(\Omega)]$, *then*

$$\begin{aligned}\|\rho^m\|^2_{L^2(\Omega)} + \sum_{k=0}^{m-1}\|[\rho^k]\|^2_{L^2(\Omega)} + \sum_e \int_0^{t^m}\int_e |v_h.n|[\rho_h]^2 + c\int_0^{t^m}\|\rho_h(s)\|^2_{L^2(\Omega)}\,ds \\ \leq \|\rho^0\|^2_{L^2(\Omega)} + (1/c)\int_0^{t^m}\|f(s)\|^2_{L^2(\Omega)}\,ds.\end{aligned}$$

(2) *If ρ_h is piecewise constant in time or piecewise linear in time ($\ell = 0$ or 1 in the definition of $\mathcal{R}_h$), and τ is sufficiently small, then*

$$\|\rho^m\|^2_{L^2(\Omega)} + \sum_{k=0}^{m-1}\|[\rho^k]\|^2_{L^2(\Omega)} + \sum_e \int_0^{t^m}\int_e |v_h.n|[\rho_h]^2 \leq C_1\|\rho^0\|^2_{L^2(\Omega)}\exp(C_2 t^m),$$

where C_1 and C_2 depend on the coefficients v and a_h and the data f.

Remark. When $\operatorname{div}(v)$ and a are bounded it is possible to introduce a change of variables to guarantee that $\operatorname{div}(v)/2 + a \geq c > 0$. Specifically, if $\rho = re^{\alpha t}$, then r satisfies

$$r_t + \operatorname{div}(rv) + (\alpha + a)r = e^{-\alpha t} f.$$

Proof. Selecting $\psi = \rho_h$ in (2.4) gives

$$(1/2)\int_K (\rho^m)^2 + [\rho^{m-1}]^2 - (\rho^{m-1})^2 + \int_{t^{m-1}}^{t^m}\int_K (\operatorname{div}(v)/2 + a_h)\rho_h^2$$
$$+ (1/2)\int_{t^{m-1}}^{t^m}\int_{\partial K} \rho_h^2 (v_h.n)^+ + \rho_-^2 (v_h.n)^- - [\rho_h]^2 (v_h.n)^- = \int_{t^{m-1}}^{t^m}\int_K f\rho_h.$$

Summing this expression and collecting terms establishes (3.1), and if $(\operatorname{div}(v)/2 + a_h) \geq c > 0$, statement (1) follows immediately.

If ρ_h is piecewise constant in time, then

$$\int_0^{t^m}\int_\Omega f\rho_h - (\operatorname{div}(v)/2 + a_h)\rho_h^2 \leq F^m \max_{1\leq k\leq n} \|\rho^k\|_{L^2(\Omega)} + \sum_{k=1}^m \gamma_k \|\rho^k\|_{L^2(\Omega)}^2,$$

where

$$\gamma_k = \int_{t^{k-1}}^{t^k} \left((1/2)\|\operatorname{div}(v(s))\|_{L^\infty(\Omega)} + \|a_h(s)\|_{L^\infty(\Omega)}\right) ds, \qquad F^m = \int_0^{t^m} \|f(s)\|_{L^2(\Omega)}\, ds.$$

If ρ_h is piecewise linear in time, then for $s \in (t^{m-1}, t^m)$

$$\begin{aligned}\|\rho_h(s)\|_{L^2(\Omega)} &= \|\rho^m(s - t^{m-1})/\tau + \rho_+^{m-1}(t^m - s)/\tau\|_{L^2(\Omega)}\\ &= \|\rho^m(s - t^{m-1})/\tau + ([\rho^{m-1}] - \rho^{m-1})(t^m - s)/\tau\|_{L^2(\Omega)}\\ &\leq \|\rho^m\|_{L^2(\Omega)}(s - t^{m-1})/\tau + \|[\rho^{m-1}] - \rho^{m-1}\|_{L^2(\Omega)}(t^m - s)/\tau.\end{aligned}$$

Then

$$\|\rho_h\|_{L^\infty[0,t^m;L^2(\Omega)]} \leq \max_{0\leq k\leq m} \|\rho^k\|_{L^2(\Omega)} + \max_{0\leq k\leq m-1} \|[\rho^k]\|_{L^2(\Omega)}$$

and

$$\|\rho_h(s)\|_{L^2(\Omega)}^2 \leq \|\rho^m\|_{L^2(\Omega)}^2 + 2\|\rho^{m-1}\|_{L^2(\Omega)}^2 + 2\|[\rho^{m-1}]\|_{L^2(\Omega)}^2.$$

It follows that

$$\int_0^{t^m}\int_\Omega f\rho_h - (\operatorname{div}(v)/2 + a_h)\rho_h^2 \leq F^m \left(\max_{0\leq k\leq m} \|\rho^k\|_{L^2(\Omega)} + \max_{0\leq k\leq m-1} \|[\rho^k]\|_{L^2(\Omega)}\right)$$
$$+ \left(\gamma_m \|\rho^m\|_{L^2(\Omega)}^2 + 2\gamma_1 \|\rho^0\|_{L^2(\Omega)}^2\right) + \sum_{k=1}^{m-1} \left((\gamma_k + 2\gamma_{k+1})\|\rho^k\|_{L^2(\Omega)}^2 + 2\gamma_k \|[\rho^{k-1}]\|_{L^2(\Omega)}^2\right).$$

In each instance an estimate of the form

$$\|\rho^m\|_{L^2(\Omega)}^2 + (1 - \hat{\gamma}) \sum_{k=0}^{m-1} \|[\rho^k]\|_{L^2(\Omega)}^2 + \sum_e \int_0^{t^m}\int_e |v_h.n|[\rho_e]^2$$
$$\leq (1 + \hat{\gamma}_0)\|\rho^0\|_{L^2(\Omega)}^2 + (C/c)(F^m)^2 + \sum_{k=1}^m \left(\hat{\gamma}_k \|\rho^k\|_{L^2(\Omega)}^2\right)$$
$$+ c\left(\max_{0\leq k\leq m} \|\rho^k\|_{L^2(\Omega)}^2 + \max_{0\leq k\leq m-1} \|[\rho^k]\|_{L^2(\Omega)}^2\right)$$

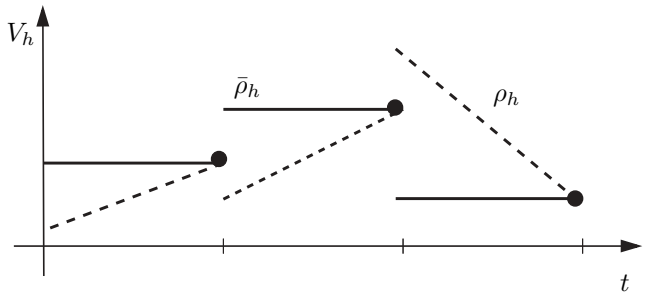

FIG. 3.1. *$\bar\rho_h$ is piecewise constant time and is equal to $\rho_h(t^m_-)$ on $(t^{m-1}, t^m]$.*

holds, where $c > 0$ is arbitrary and $\hat\gamma$ and $\hat\gamma_k$ are bounded by integrals of functions in $L^1[0,T]$ over intervals of length τ. If τ is sufficiently small, all these constants will be less than 1/2, and statement (2) follows from the discrete Gronwall inequality. ∎

When $\mathrm{div}(v)/2 + a_h = 0$ the stability estimate only bounds ρ_h at the discrete times $\{t^m\}_{n=0}^M$. If $\bar\rho_h(t) = \rho^m$ on $(t^{m-1}, t^m]$ (see Figure 3.1), the lemma shows that $\bar\rho_h$ can be bounded in $L^\infty[0,T;L^2(\Omega)]$. Clearly $\bar\rho_h = \rho_h$ when ρ_h is piecewise constant in time ($\ell = 0$); however, when $\ell > 1$ it may happen that $\bar\rho_h$ and ρ_h have different weak limits. The following corollary shows that this will not happen when $\ell = 1$.

COROLLARY 3.2. *Let $\rho_h \in \mathcal{R}_h$ be piecewise linear in time ($\ell = 1$) and let $\bar\rho_h \in \mathcal{R}_h$ be the function piecewise constant in time equal to $\rho^m = \rho_h(t^m_-)$ on $(t^{m-1}, t^m]$.*

- *If $\psi \in L^2[0,T;L^2(\Omega)]$, then*

$$\left|\int_0^T\int_\Omega(\bar\rho_h - \rho_h)\psi\right| \le \|\bar\rho_h\|_{L^2[0,T;L^2(\Omega)]}\|\psi - \psi(.+\tau)\|_{L^2[0,T-\tau;L^2(\Omega)]}$$
$$+\sqrt{(\tau/2)}\left(\|\rho^M\|_{L^2(\Omega)} + \|\rho^0\|_{L^2(\Omega)} + \left(\sum_{m=0}^{M-1}\|[\rho^m]\|^2_{L^2(\Omega)}\right)^{1/2}\right)\|\psi\|_{L^2[0,T;L^2(\Omega)]}.$$

- $\|\rho_h\|_{L^\infty[0,T;L^2(\Omega)]} \le \|\bar\rho_h\|_{L^\infty[0,T;L^2(\Omega)]} + \max_{0\le m\le M-1}\|[\rho_h^m]\|_{L^2(\Omega)}$ *and*

$$\|\bar\rho_h - \rho_h\|^2_{L^2[0,T;L^2(\Omega)]}$$
$$\le (2/3)\left(\|\bar\rho_h - \bar\rho_h(.+\tau)\|_{L^2[0,T-\tau;L^2(\Omega)]} + \tau\sum_{m=0}^{M-1}\|[\rho^m]\|^2_{L^2(\Omega)}\right).$$

Proof. On the interval $(t^{m-1}, t^m]$ we have $\bar\rho_h - \rho_h = (\rho^m - \rho^{m-1}_+)(t^m - t)/\tau$ (recall that $\rho^m = \rho^m_-$). Then

$$\int_0^T\int_\Omega(\bar\rho_h - \rho_h)\psi = \int_\Omega\sum_{m=1}^M\int_0^\tau(\rho^m - \rho^{m-1}_+)(1 - s/\tau)\psi(t^{m-1}+s)\,ds$$
$$= \int_\Omega\int_0^\tau\Bigg(\rho^M\psi(t^{M-1}+s) - \rho^0_+\psi(s)$$
$$+\sum_{m=1}^{M-1}(\rho^m\psi(t^{m-1}+s) - \rho^m_+\psi(t^m+s))\Bigg)(1 - s/\tau)\,ds$$

$$= \int_\Omega \int_0^\tau \Bigg(\rho^M \psi(t^{M-1}+s) - \rho^0 \psi(s) + \sum_{m=1}^{M-1} \rho^m (\psi(t^{m-1}+s) - \psi(t^m+s)) + \sum_{m=0}^{M-1} (\rho^m - \rho_+^m)\psi(t^m+s) \Bigg) (1-s/\tau)\, ds.$$

The first statement then follows upon observing that the last term can be bounded by

$$\int_\Omega \int_0^\tau \sum_{m=0}^{M-1} (\rho^m - \rho_+^m)\psi(t^m+s)(1-s\tau) \le \sqrt{(\tau/2)} \left(\sum_{m=0}^{M-1} \|[\rho^m]\|^2_{L^2(\Omega)} \right)^{1/2} \|\psi\|_{L^2[0,T;L^2(\Omega)]}.$$

To establish the second statement we compute

$$\begin{aligned} \|\bar\rho_h - \rho_h\|^2_{L^2[0,T;L^2(\Omega)]} &= \sum_{m=1}^{M} \int_0^\tau \|\rho^m - \rho_+^{m-1}\|^2 (1-s/\tau)^2\, ds \\ &\le \sum_{m=1}^{M} (\tau/3) \|\rho^m - \rho^{m-1} + [\rho^{m-1}]\|^2 \\ &\le (2/3) \left(\|\bar\rho_h - \bar\rho_h(.+\tau)\|_{L^2[0,T-\tau;L^2(\Omega)]} + \tau \sum_{m=0}^{M-1} \|[\rho^m]\|^2_{L^2(\Omega)} \right). \qquad \square \end{aligned}$$

4. Consistency. The bounds established above show that subsequences of the approximate solutions converge weakly-star in $L^\infty[0,T;L^2(\Omega)]$. In this section we show that the limits of these sequences are weak solutions of (1.1). This is easy to show when the space $\mathcal{R}_h$ contains the continuous finite element spaces; however, when $k=0$ the only continuous functions are constants, which complicates the proof. We begin with a technical lemma.

LEMMA 4.1. *Let $K \subset \mathbb{R}^d$ be a simplex, $v \in H^1(K)^d$, and $\psi \in W^{1,p}(K)$ with $p \ge 4d/(d+4)$. Then there exists a constant C depending only on d and the aspect ratio of K such that*

$$\int_{\partial K} |v.n||\psi - \bar\psi|^2 \le C|K|^{(1/2-2/p)} h_K \|v\|_{H^1(K)} \|\psi\|^2_{W^{1,p}(K)},$$

where $\bar\psi = (1/|K|)\int_K \psi$ is the average of ψ on K and h_K is the diameter of K.

If each component of v is a polynomial of degree k, then

$$\int_{\partial K} |v.n||\psi - \bar\psi|^2 \le C|K|^{(1/2-2/p)} h_K \|v\|_{L^2(K)} \|\psi\|^2_{W^{1,p}(K)},$$

where C may now depend additionally on k.

Proof. Let $\hat K$ be the usual reference simplex and $\chi(\xi) = x_0 + B\xi$ be an affine mapping of $\hat K$ to K. We use a hat to denote the natural correspondence between functions defined on K and $\hat K$, $\hat\psi = \psi \circ \chi$. Writing the integral over the boundary as

the sum over the faces $e \subset \partial K$, the trace theorem with $p \geq 4d/(d+4)$ enables us to write

$$\begin{aligned}\int_{\partial K} |v.n||\psi - \bar{\psi}|^2 &= \sum_{e \subset \partial K} \int_e |v.n||\psi - \bar{\psi}|^2 \\ &= \sum_{\hat{e} \subset \partial \hat{K}} \frac{|e|}{|\hat{e}|} \int_{\hat{e}} |\hat{v}.n||\hat{\psi} - \bar{\psi}|^2 \\ &\leq C \sum_{\hat{e} \subset \partial \hat{K}} |e| \|\hat{v}\|_{H^1(\hat{K})} \|\hat{\psi} - \bar{\psi}\|^2_{W^{1,p}(\hat{K})} \\ &\leq C \sum_{\hat{e} \subset \partial \hat{K}} |e| \|\hat{v}\|_{H^1(\hat{K})} |\hat{\psi}|^2_{W^{1,p}(\hat{K})}.\end{aligned}$$

We used the fact that the average of ψ is the average of $\hat{\psi}$ and the Poincaré inequality to pass to the $W^{1,p}$ seminorm in the last line. Notice that if $v \in \mathcal{P}_k(K)^d$, then $\|\hat{v}\|_{H^1(\hat{K})}$ is equivalent to $\|\hat{v}\|_{L^2(\hat{K})}$ since $\mathcal{P}_k(\hat{K})$ is finite dimensional.

Recalling that $\|\hat{v}\|_{L^2(\hat{K})} = (|\hat{K}|/|K|)^{1/2} \|\hat{v}\|_{L^2(K))}$, $|B| \leq C h_K$,

$$|\hat{\psi}|_{W^{1,p}(\hat{K})} \leq (|\hat{K}|/|K|)^{1/p} |B| |\psi|_{W^{1,p}(K)} \leq C(|\hat{K}|/|K|)^{1/p} h_K |\psi|_{W^{1,p}(K)},$$

and $|\nabla \hat{v}|_{L^2(\hat{K})} \leq C(|\hat{K}|/|K|)^{1/2} h_K \|\nabla v\|_{L^2(K)}$, we deduce that

$$\int_{\partial K} |v.n||\psi - \bar{\psi}|^2 \leq C \sum_{e \subset \partial K} \frac{|e|}{|K|^{(1/2+2/p)}} h_K^2 \|v\|_{H^1(K)} |\psi|^2_{W^{1,p}(K)}$$

(with $\|v\|_{L^2(K)}$ replacing $\|v\|_{H^1(K)}$ if $v \in \mathcal{P}_k(K)^d$). Since

$$|K| = (1/d)|e| \times (\text{perpendicular height}) \geq c|e|h_K,$$

where c depends on the aspect ratio of K, it follows that $|e|h_K \leq C|K|$, and the proof follows. ∎

The following lemma provides sufficient conditions on the coefficients v_h and a_h that suffice to establish consistency of the discontinuous Galerkin method.

LEMMA 4.2. *Let $\{\rho_h\}$ be a (sub-) sequence of solutions of the discontinuous Galerkin scheme* (2.4) *computed on a sequence of quasi-regular meshes and suppose that $\rho_h \rightharpoonup^* \rho$ in $L^\infty[0,T;L^2(\Omega)]$. Assume that $f \in L^1[0,T;L^2(\Omega)]$, $v \in L^1[0,T;H^1_0(\Omega)]$, $v_h(t) \in V_h$, $\rho^0 \rightharpoonup \rho_0$ in $L^2(\Omega)$,*

$$v_h \to v, \quad \textit{and} \quad a_h \to a, \ \textit{in } L^1[0,T;L^2(\Omega)],$$

and either (1) $\operatorname{div}(v_h) \to \operatorname{div}(v)$ *in* $L^1[0,T;L^2(\Omega)]$ *or* (2) $(\operatorname{div}(v_h) - \operatorname{div}(v))|_K \perp \mathcal{P}_k(K)$ *in* $L^2(K)$ *for each* $K \in \mathcal{T}_h$. *Then ρ is a weak solution of* (1.1).

Proof. When k, $\ell > 0$ $\mathcal{R}_h$ contains the usual continuous finite element spaces, so if $\psi \in \mathcal{D}([0,T) \times \Omega)$, then the classical Lagrange interpolant $\psi_h \in \mathcal{R}_h \cap C([0,T] \times \bar{\Omega})$ converges to ψ in $W^{1,\infty}((0,T) \times \Omega)$. When ψ_h is substituted into (2.5) all the jump

terms vanish to give

$$-\int_0^T\int_\Omega \rho_h\big(\psi_{ht}+v_h.\nabla\psi_h+(1/2)(\mathrm{div}(v_h)-\mathrm{div}(v))\psi_h-a_h\psi_h\big)=\int_\Omega \rho^0\psi(0)+\int_0^T\int_\Omega f\psi_h.$$

If $\mathrm{div}(v_h)\to\mathrm{div}(v)$ in $L^1[0,T;L^2(\Omega)]$ the hypotheses suffice to pass to the limit term by term in the above equation to show that ρ is a weak solution of (1.1). If $(\mathrm{div}(v_h)-\mathrm{div}(v)|_K\perp\mathcal{P}_k(K)$ for $K\in\mathcal{T}_h$ the term involving $\mathrm{div}(v_h)-\mathrm{div}(v)$ still vanishes since

$$\begin{aligned}\int_0^T\int_\Omega \rho_h(\mathrm{div}(v_h)-\mathrm{div}(v))\psi_h &= \int_0^T\int_\Omega \rho_h(\mathrm{div}(v_h)-\mathrm{div}(v))(\psi_h-\bar\psi)\\ &\le \|\rho_h\|_{L^\infty[0,T;L^2(\Omega)]}\|\mathrm{div}(v_h)-\mathrm{div}(v)\|_{L^1[0,T;L^2(\Omega)]}Ch\\ &\to 0,\end{aligned}$$

where $\bar\psi(t,x)$ is the average of $\psi_h(t,.)$ over the element K containing x.

When $\ell=0$ or $k=0$, functions in $\mathcal{R}_h$ are piecewise constant in time or space, respectively. In this situation the terms involving ψ_{ht} or $\nabla\psi_h$ vanish and it is necessary to show that the corresponding jump terms in (2.5) approximate the missing terms: $-\sum_{k=0}^{M-1}\int_\Omega\rho^k[\psi^k]\sim-\int_0^T\int_\Omega\rho\psi_t$ and

$$-\sum_e\int_0^T\int_e\left(\rho_-(v_h.N)^+ + \rho_+(v_h.N)^-\right)[\psi_h]\sim\int_0^T\int_\Omega\rho v.\nabla\psi.$$

If $\ell=0$ and $\psi\in\mathcal{D}([0,T)\times\Omega)$, let ψ^k be a projection of $\psi(t^k)$ onto the (spatial) finite element space $\{\psi_h\in L^2(\Omega)\mid\psi_h|_K\in\mathcal{P}_k(K),\,K\in\mathcal{T}_h\}$, and let $\hat\psi_h$ denote the piecewise linear interpolant of $\{\psi^k\}$ in time. Then $\hat\psi_h$ converges to ψ in $W^{1,\infty}[0,T;L^\infty(\Omega)]$ and temporal jump terms become

$$-\sum_{k=0}^{M-1}\int_\Omega\rho^k[\psi^k]=-\sum_{k=0}^{M-1}\int_\Omega\rho^k(\psi^{k+1}-\psi^k)=-\int_0^T\int_\Omega\rho_h\hat\psi_{ht}\to-\int_0^T\int_\Omega\rho\psi_t$$

as required.

Finally consider the spatial jump terms when $k=0$. Let ψ be piecewise polynomial of degree ℓ in time with values in $\mathcal{D}(\Omega)$. Selecting $\psi_h(t)|_K$ to be the spatial average of $\psi(t)$ over each element the spatial jump terms in (2.5) become

$$\begin{aligned}&-\sum_e\int_0^T\int_e\left(\rho_-(v_h.N)^+ + \rho_+(v_h.N)^-\right)[\psi_h]\\ &=-\sum_e\int_0^T\int_e\left(\rho_-(v_h.N)^+ + \rho_+(v_h.N)^-\right)[\psi_h-\psi]\\ &=-\int_0^T\sum_K\int_{\partial K}\left(\rho_{K_-}(v_h.n)^- + \rho_K(v_h.n)^+\right)(\psi_K-\psi),\end{aligned}$$

where ψ_K is the average value of ψ on K, ρ_K is the value of ρ_h on K, and ρ_{K_-} is the

value of ρ_h on the upwind element K_-. Then

$$
\begin{aligned}
&-\sum_e \int_0^T \int_e \left(\rho_-(v_h.N)^+ + \rho_+(v_h.N)^-\right)[\psi_h] \\
&= -\int_0^T \sum_K \int_{\partial K} \left((\rho_{K_-} - \rho_K)(v_h.n)^- + \rho_K(v_h.n)\right)(\psi_K - \psi) \\
&= -\int_0^T \sum_K \int_K \operatorname{div}(\rho_K v_h(\psi_K - \psi)) + \int_{\partial K} [\rho_h](v_h.n)^-(\psi_K - \psi) \\
&= -\int_0^T \int_\Omega \left(\rho_h v_h.\nabla\psi + \rho_h \operatorname{div}(v_h)(\psi_h - \psi)\right) + \int_0^T \sum_K \int_{\partial K} [\rho_h](v_h.n)^-(\psi_K - \psi).
\end{aligned}
$$

Clearly the first term converges to $\iint \rho v.\nabla\psi$; we need to show that the second term vanishes in the limit. Using Lemma 4.1, with $p = 4$, we obtain

$$
\begin{aligned}
&\int_0^T \sum_K \int_{\partial K} [\rho_h](v_h.n)^-(\psi_K - \psi) \\
&\le \left(\int_0^T \sum_K \int_{\partial K} |v_h.n|[\rho_h]^2\right)^{1/2} \left(\int_0^T \sum_K \int_{\partial K} |v_h.n|(\psi_K - \psi)^2\right)^{1/2} \\
&\le \left(\int_0^T 2\sum_e \int_e |v_h.N|[\rho_h]^2\right)^{1/2} \left(\sum_K Ch_K \int_0^T \|v_h\|_{L^2(K)} \|\psi\|^2_{W^{1,4}(K)}\right)^{1/2} \\
&\le \left(\int_0^T \sum_e \int_e |v_h.N|[\rho_h]^2\right)^{1/2} C\sqrt{h}\, \|v_h\|^{1/2}_{L^1[0,T;L^2(\Omega)]} \|\psi\|_{L^\infty[0,T;W^{1,4}(\Omega)]}.
\end{aligned}
$$

Theorem 3.1 shows that the first term is bounded so the expression above vanishes as $h \to 0$. $\square$

Remark. The second hypotheses on the divergence in Lemma 4.2 is useful in the context of finite element approximations of incompressible fluids. If $\tilde{v}_h \in L^1[0,T; H_0^1(\Omega)]$ is a classical finite element approximation of v, then typically $\tilde{v}_h \to v$ in $L^1[0,T;L^2(\Omega)]$, but $\operatorname{div}(\tilde{v}_h)$ will only converge weakly to $\operatorname{div}(v)$ in $L^2[0,T;L^2(\Omega)]$. However, we can construct v_h satisfying the hypothesis of the Lemma as follows. At each time t let $v_h(t)$ be the $L^2(\Omega)$ projection of $\tilde{v}_h(t)$ onto the space

$$
\bar{V}_h(v) = \left\{ v_h \in RT_h^k(\Omega) \;\middle|\; \int_K \operatorname{div}(v_h)p(x) = \int_K \operatorname{div}(v)p(x),\ p \in \mathcal{P}_k(K),\ K \in \mathcal{T}_h \right\}.
$$

Here $RT_h^k(\Omega)$ is the Raviart–Thomas subspace of $H(\operatorname{div};\Omega)$ constructed using piecewise polynomials of degree k on $\mathcal{T}_h$ [1].

By construction $\operatorname{div}(v_h) - \operatorname{div}(v)$ is orthogonal to $\mathcal{P}_k(K)$ for each element K. To see that v_h also converges to v first observe that

$$
(\tilde{v}_h - v_h, w_h)_{L^2(\Omega)} = 0 \qquad \forall\ w_h \in \bar{V}_h \equiv \bar{V}_h(0).
$$

Then

$$
\begin{aligned}
\|\tilde{v}_h - v_h\|^2_{L^2(\Omega)} &= (\tilde{v}_h - v_h, \tilde{v}_h - I_h v + I_h v - v_h)_{L^2(\Omega)} \\
&= (\tilde{v}_h - v_h, \tilde{v}_h - I_h v)_{L^2(\Omega)},
\end{aligned}
$$

where $I_h v$ is the interpolant of v onto the Raviart–Thomas space [1]. The degrees of freedom of I_h are constructed to guarantee that $I_h v - v_h \in \bar{V}_h$. Then

$$\begin{aligned}\|\tilde{v}_h - v_h\|_{L^2(\Omega)} &\le \|\tilde{v}_h - v\|_{L^2(\Omega)} + \|v - I_h v\|_{L^2(\Omega)} \\ &\le \|\tilde{v}_h - v\|_{L^2(\Omega)} + Ch\|v\|_{H_0^1(\Omega)},\end{aligned}$$

so $v_h \to v$ if $\tilde{v}_h \to v$ and v is bounded in $L^1[0,T;H_0^1(\Omega)]$.

5. Convergence. When ρ_h is piecewise constant or linear in time ($\ell = 0$ or 1) the stability estimate guarantees that it is possible to pass to a subsequence for which

$$\bar{\rho}_h \rightharpoonup^* \bar{\rho} \ \text{ and } \ \rho_h \rightharpoonup^* \rho \qquad \text{in } L^\infty[0,T;L^2(\Omega)],$$

and Corollary 3.2 shows that the weak limits coincide, $\rho = \bar{\rho}$. (Recall that $\bar{\rho}_h$ is piecewise constant in time and assumes the values ρ^m in $(t^{m-1}, t^m]$ (see Figure 3.1).) If additionally $v \in L^1[0,T;H_0^1(\Omega)]$, then Theorem 2.2 shows that ρ is unique. In this situation the whole sequence $\{\rho_h\}$ converges weakly, and in the following theorem we establish strong convergence in $L^2[0,T;L^2(\Omega)]$.

THEOREM 5.1. *Let $\{\rho_h\}_{h>0}$ be the sequence of solutions of the discontinuous Galerkin scheme* (2.4) *with either $\ell = 0$ or $\ell = 1$ (ρ_h piecewise constant or linear in time) computed on a sequence of quasi-regular meshes. Assume $\rho_0 \in L^2(\Omega)$,*

$$v \in L^1[0,T;H_0^1(\Omega)], \qquad a,\ \mathrm{div}(v) \in L^1[0,T;L^\infty(\Omega)], \qquad f \in L^1[0,T;L^2(\Omega)],$$

and $\mathrm{div}(v)/2 + a \ge 0$. *If $\rho^0 \to \rho_0$ in $L^2(\Omega)$, $a_h \to a$ in $L^1[0,T;L^\infty(\Omega)]$, and $v_h \in V_h$ is an approximation of v for which $v_h \to v$ in $L^2[0,T;L^2(\Omega)]$ and either* (1) $\mathrm{div}(v_h) \to \mathrm{div}(v)$ *in $L^1[0,T;L^2(\Omega)]$ or* (2) $(\mathrm{div}(v_h) - \mathrm{div}(v))|_K \perp \mathcal{P}_k(K)$ *in $L^2(K)$ for each $K \in \mathcal{T}_h$, then ρ_h converges in $L^2[0,T;L^2(\Omega)]$ to ρ, the weak solution of* (1.1). *Moreover, the jump term*

$$J_h^M = (1/2)\sum_{k=0}^{M-1} \|[\rho^k]\|^2_{L^2(\Omega)} + (1/2)\sum_e \int_0^T \int_e |v_h.n|[\rho_e]^2$$

converges to zero.

Proof. The idea of the proof is to show that

$$\liminf_h \|\bar{\rho}_h\|_{L^2[0,T;L^2(\Omega)]} \le \|\rho\|_{L^2[0,T;L^2(\Omega)]}. \tag{5.1}$$

If $\ell = 0$, then $\bar{\rho}_h = \rho_h$ and strong convergence of ρ_h follows. When $\ell = 1$, the first estimate in Corollary 3.2 shows that $\bar{\rho}_h$ and ρ_h have the same weak limit so $\bar{\rho}_h$ converges strongly to ρ. The second estimate in Corollary 3.2 shows that strong convergence of $\bar{\rho}_h$ implies strong convergence of ρ_h.

To establish (5.1), the key step is to observe that (2.3) in Theorem 2.2 is an *equation* instead of the usual *inequality*. The hypotheses on v allow us to select $\beta(s) = (1/2)s^2$ in Theorem 2.2 to obtain

$$\begin{aligned}(1/2)\|\rho(t)\|^2_{L^2(\Omega)} + \int_0^t \int_\Omega ((1/2)\mathrm{div}(v) + a)\rho^2 &= (1/2)\|\rho_0\|^2_{L^2(\Omega)} + \int_0^t \int_\Omega f\rho \\ &= \liminf_{h\to 0}\left((1/2)\|\rho^0\|^2_{L^2(\Omega)} + \int_0^t \int_\Omega f\rho_h\right). \end{aligned} \tag{5.2}$$

Integrating both sides with respect to t and using the dominated convergence theorem to interchange the limit and integral gives

$$(1/2)\|\rho\|^2_{L^2[0,T;L^2(\Omega)]} + \int_0^T \gamma(t;\rho)\,dt = \liminf_{h\to 0} \int_0^T \left((1/2)\|\rho^0\|^2_{L^2(\Omega)} + \int_0^t \int_\Omega f\rho_h\right) dt,$$

where

$$\gamma(t;\rho) = \int_0^t \int_\Omega ((1/2)\mathrm{div}(v) + a)\rho^2.$$

Below we use (3.1) to show that

$$\begin{aligned} &\liminf_{h\to 0} \int_0^T \left((1/2)\|\rho^0\|^2_{L^2(\Omega)} + \int_0^t \int_\Omega f\rho_h\right) dt \\ &\geq \liminf_{h\to 0} \left((1/2)\|\bar{\rho}_h\|^2_{L^2[0,T;L^2(\Omega)]} + \int_0^T \gamma(t;\rho_h)\,dt\right), \end{aligned} \tag{5.3}$$

so that

$$(1/2)\|\rho\|^2_{L^2[0,T;L^2(\Omega)]} + \int_0^T \gamma(t;\rho)\,dt \geq \liminf_{h\to 0} \left((1/2)\|\bar{\rho}_h\|^2_{L^2[0,T;L^2(\Omega)]} + \int_0^T \gamma(t;\rho_h)\,dt\right).$$

Since $0 \leq (\mathrm{div}(v)/2 + a) \in L^1[0,T;L^\infty(\Omega)]$ it follows that $\gamma(t,.)$ is nonnegative and lower semicontinuous with respect to weak-star convergence in $L^\infty[0,T;L^2(\Omega)]$. Then applying Fatou's lemma to the right-hand side of the above expression shows that $\|\rho\|_{L^2[0,T;L^2(\Omega)]} \geq \liminf_h \|\bar{\rho}_h\|_{L^2[0,T;L^2(\Omega)]}$, which establishes (5.1).

To verify inequality (5.3), multiply (3.1) by τ and sum to obtain

$$\begin{aligned} \int_0^T \left((1/2)\|\rho^0\|^2_{L^2(\Omega)} + \int_0^t \int_\Omega f\rho_h\right) dt &= (1/2)\|\bar{\rho}_h\|^2_{L^2[0,T;L^2(\Omega)]} + \int_0^T \gamma(t,\rho_h) + \sum_{m=1}^M \tau J_h^m \\ &\quad + \int_0^T \int_0^t \int_\Omega f\rho_h - \sum_{m=1}^M \tau \int_0^{t^m} \int_\Omega f\rho_h \\ &\quad + \sum_{m=1}^M \tau\gamma(t^m;\rho_h) - \int_0^T \gamma(t,\rho_h)\,dt \\ &\quad + \sum_{m=1}^M \tau \int_0^{t^m} \int_\Omega (a - a_h)\rho_h^2. \end{aligned}$$

To complete the convergence proof we show that the terms in the last three lines vanish as $h \to 0$.

- The terms involving f can be combined as

$$\begin{aligned} \int_0^T \int_0^t \int_\Omega f\rho_h - \sum_{m=1}^M \tau \int_0^{t^m} \int_\Omega f\rho_h &= \sum_{m=1}^M \int_\Omega \left(\int_{t^{m-1}}^{t^m} \int_0^t f\rho_h - \tau \int_0^{t^m} f\rho_h\right) \\ &= \sum_{m=1}^M \int_\Omega \left(\int_{t^{m-1}}^{t^m} \int_{t^{m-1}}^t f\rho_h - \tau \int_{t^{m-1}}^{t^m} f\rho_h\right) \\ &= \sum_{m=1}^M \int_\Omega \int_{t^{m-1}}^{t^m} (t^m - s - \tau) f(s)\rho_h(s)\,ds, \end{aligned}$$

where the last line follows on interchanging the order of integration. It follows that

$$\left|\int_0^T\int_0^t\int_\Omega f\rho_h - \sum_{m=1}^M \tau\int_0^{t^m}\int_\Omega f\rho_h\right| \le 2\tau\|f\|_{L^1[0,T;L^2(\Omega)]}\|\rho_h\|_{L^\infty[0,T;L^2(\Omega)]}.$$

- A similar calculation is used to estimate the terms involving $\gamma(.;\rho_h)$,

$$\sum_{m=1}^M \tau\gamma(t^m;\rho_h) - \int_0^T \gamma(t,\rho_h)\,dt = \sum_{m=1}^M \int_\Omega\int_{t^{m-1}}^{t^m} (t^m - s - \tau)\big(\mathrm{div}(v)/2 + a\big)\,\rho_h^2\,ds,$$

so that

$$\left|\int_0^T\int_0^t \gamma(s;\rho_h)\,ds\,dt - \sum_{m=0}^M \tau\int_0^{t^m}\gamma(s;\rho_h)\,ds\right| \le 2\tau\|(1/2)\mathrm{div}(v) + a\|_{L^1[0,T;L^\infty(\Omega)]}\|\rho_h\|^2_{L^\infty[0,T;L^2(\Omega)]}.$$

- The last term is bounded by $T\|a - a_h\|_{L^1[0,T;L^\infty(\Omega)]}\|\rho_h\|^2_{L^\infty[0,T;L^2(\Omega)]}$.

To show that the jump terms converge to zero we combine (5.2) and (3.1) to get

$$\|\rho(t)\|^2_{L^2(\Omega)} + \gamma(t,\rho) = \lim_{h\to 0}\left(\|\rho^m\|_{L^2(\Omega)} + \gamma(t^m,\rho_h) + J_h^m\right),$$

where $m = m(h)$ is chosen so that $t \in (t^{m-1}, t^m]$. With this choice $\rho^m = \bar\rho_h(t)$, and since $\|\bar\rho_h(t)\|_{L^2(\Omega)}$ converges to $\|\rho(t)\|_{L^2(\Omega)}$ in $L^2[0,T]$, selecting t to be a Lebesgue point and noting that $\gamma(.,.)$ is continuous on $\mathbb{R}\times L^2[0,T;L^2(\Omega)]$ shows that $\lim_{h\to 0} J_h^m = 0$. Since $J_h^m \ge J_h^M$ for $m \ge M$, and observing that the final time can be chosen arbitrarily, we can choose a Lebesgue point $t \ge T$ to conclude that $J_h^M \to 0$. ∎

6. Monotonicity of the piecewise constant scheme. As stated above, the form of the discontinuous Galerkin method proposed in section 2 was chosen to guarantee convergence when the velocity field was known only approximately. In particular, the factor of 1/2 in the term $(1/2)(\mathrm{div}(v_h) - \mathrm{div}(v))$ guarantees stability in $L^2(\Omega)$; however, $L^p(\Omega)$ estimates would require a different weight. In particular, the piecewise constant scheme, $k = \ell = 0$, may fail to be monotone.

When $k = \ell = 0$, the discontinuous Galerkin scheme (2.4) can be written as

$$|K|(\rho_K^m - \rho_K^{m-1}) + \int_{t^{m-1}}^{t^m}\int_K \big(a_h + (1/2)(\mathrm{div}(v) + \mathrm{div}(v_h))\big)\,\rho_K^m + \int_{t^{m-1}}^{t^m}\int_{\partial K}(\rho_-^m - \rho^m)(v_h.n)^- = \int_{t^{m-1}}^{t^m}\int_K f.$$

Notice that if K is selected so that ρ_K^m is maximal/minimal, then the boundary term in nonnegative/positive so

$$\max(\rho^m) \le \frac{\max(\rho^{m-1}) + F^m}{1 - C^m} \qquad \text{and} \qquad \min(\rho^m) \ge \frac{\min(\rho^{m-1})}{1 + C^m} - \frac{F^m}{1 - C^m},$$

where

$$F^m = \int_{t^{m-1}}^{t^m}\|f\|_{L^\infty(\Omega)} \quad \text{and} \quad C^m = \int_{t^{m-1}}^{t^m}\|a_h + (1/2)(\mathrm{div}(v) + \mathrm{div}(v_h))\|_{L^\infty(\Omega)},$$

and we have assumed that $\min(\rho^{m-1}) \geq 0$ and τ is sufficiently small to guarantee that $C^m < 1$. Defining

$$\gamma(t) = \int_0^t \|a_h + (1/2)(\operatorname{div}(v) + \operatorname{div}(v_h))\|_{L^\infty(\Omega)}$$

then, assuming $\max(\rho^0) \geq 0$, we compute

$$(1 - o(\tau)) \max(\rho^m) \leq e^{\gamma(t^m)} \max(\rho^0) + \int_0^{t^m} e^{\gamma(t^m)-\gamma(s)} \|f(s)\|_{L^\infty(\Omega)}\, ds,$$

and if $\min(\rho^0) \geq 0$

$$(1 + o(\tau)) \min(\rho^m) \geq e^{-\gamma(t^m)} \min(\rho^0) - \int_0^{t^m} e^{\gamma(t^m)-\gamma(s)} \|f(s)\|_{L^\infty(\Omega)}\, ds,$$

whenever the right-hand side is nonnegative. The terms $1 \pm o(\tau)$ arise when products of the form $\prod_{i=1}^m 1/(1 - C^i)$ are approximated by exponentials of the form $\exp(\sum_{i=1}^m C^i) = \exp(\gamma(t^m))$. Specifically, if $0 \leq C^i \leq 1$, then

$$1 - \sum_{i=1}^m (C^i)^2 \leq \left(\prod_{i=1}^m (1 - C^i)\right) \exp\left(\sum_{i=1}^m C^i\right).$$

Since $\sum_{i=1}^m C^i = \gamma(t^m)$ is bounded and $C^i \to 0$ as $\tau \to 0$, it follows that

$$1 - \sum_{i=1}^m (C^i)^2 \geq 1 - \left(\max_{1\leq i\leq m} C^i\right) \sum_{i=1}^m C^i = 1 - o(\tau).$$

Since it is unlikely that $\operatorname{div}(v_h)$ is bounded in $L^1[0,T;L^\infty(\Omega)]$, this estimate is not particularly useful as it stands. For example, typically it will not be possible to choose τ sufficiently small to guarantee that $C^m < 1$. However, if the construction at the end of section 4 is used to guarantee that $\operatorname{div}(v_h)$ and $\operatorname{div}(v)$ have the same average on each element, then the piecewise constant discontinuous Galerkin scheme becomes

$$|K|(\rho_K^m - \rho_K^{m-1}) + \int_{t^{m-1}}^{t^m} \int_K (a_h + \operatorname{div}(v))\rho_K^m + \int_{t^{m-1}}^{t^m} \int_{\partial K} (\rho_-^m - \rho^m)(v_h.n)^- = \int_{t^{m-1}}^{t^m} \int_K f.$$

In this situation the scheme will be monotone and convergent. The following theorem summarizes these observations.

THEOREM 6.1. *Let $\{\rho_h\}_{h>0}$ be the sequence of solutions of the piecewise constant ($k = \ell = 0$) discontinuous Galerkin scheme* (2.4). *Assume that $v \in L^1[0,T;H_0^1(\Omega)]$, $v_h \in V_h$ and that the averages of* $\operatorname{div}(v_h)$ *and* $\operatorname{div}(v)$ *agree on each element $K \in \mathcal{T}_h$. If a_h,* $\operatorname{div}(v)$, *and f are bounded in $L^1[0,T;L^\infty(\Omega)]$, $0 \leq \alpha \leq \rho^0 \leq \beta$, and τ is sufficiently small, then*

$$(1 - o(\tau)) \max(\rho^m) \leq e^{\gamma(t^m)} \max(\rho^0) + \int_0^{t^m} e^{\gamma(t^m)-\gamma(s)} \|f(s)\|_{L^\infty(\Omega)}\, ds$$

and

$$(1 + o(\tau)) \min(\rho^m) \geq e^{-\gamma(t^m)} \min(\rho^0) - \int_0^{t^m} e^{\gamma(t^m)-\gamma(s)} \|f(s)\|_{L^\infty(\Omega)}\, ds,$$

provided the right-hand side is nonnegative. Here

$$\gamma(t) = \int_0^t \|a_h + \operatorname{div}(v)\|_{L^\infty(\Omega)},$$

and τ *sufficiently small is interpreted to mean* $\gamma(t^m+\tau)-\gamma(t^m) < 1$ *for* $m = 0, 1, \ldots$.

Remark. The monotonicity estimates above can be improved slightly if one-sided bounds are used instead of absolute values. For example, in the upper bound $\|a_h + \operatorname{div}(v))\|_{L^\infty(\Omega)}$ can be replaced by $\|(a_h + \operatorname{div}(v))^-\|_{L^\infty(\Omega)}$ and $\|f\|_{L^\infty(\Omega)}$ by $\|f^+\|_{L^\infty(\Omega)}$.

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