

Multi-Component Multiphase Flow Through a Poroelastic Medium

Brian Seguin¹  · Noel J. Walkington²

In memory of Walter Noll

Received: 20 June 2018
© Springer Nature B.V. 2019

Abstract An axiomatic development for a continuum description of a multi-component multiphase porous flow in an elastic medium is developed. The Coleman–Noll procedure is used to derive constitutive restrictions which guarantee that the resulting model satisfies an appropriate statement of the second law of thermodynamics and a corresponding dissipation inequality. Many of the models and formulations appearing in the engineering literature are shown to be special cases of the model developed here.

Keywords Poroelasticity · Multiphase flow · Biot theory · Thermodynamics

Mathematics Subject Classification 76S05 · 74F10 · 80A17

1 Introduction

Coleman–Noll theory is utilized to develop a continuum description of flow through an elastic porous medium which satisfies the statement of the second law of thermodynamics developed in their pioneering paper [12]. Geological problems involve fluids and media for which a detailed description of their properties is not available and are too complex to simulate precisely. Models of the gross properties of these systems utilize mixture theory with substantial constitutive postulates to model the dissipation, phase formation, and interaction between the phases and medium. In another paper the authors developed an axiomatic

N.J. Walkington was supported in part by National Science Foundation Grants DMS-1418991 and DMREF-1729478. This work was also supported by the NSF through the Center for Nonlinear Analysis.

✉ B. Seguin
bseguin@luc.edu

N.J. Walkington
noelw@andrew.cmu.edu

¹ Department of Mathematics and Statistics, Loyola University Chicago, Chicago, IL 60660, USA

² Department of Mathematical Sciences, Carnegie Mellon University, Pittsburgh, PA 15213, USA

formulation for flows in a rigid media [26]; extension to include elastic media requires additional constitutive postulates to model how the fluid pressure(s) enter into the macroscopic stress–strain relation of the combination of medium and fluids. Classical poroelasticity models utilize Biot theory to develop the macroscopic stress–strain relation. Biot theory is typically developed in the context of linear elasticity with a single fluid pressure [10]; in particular, the reference and current configurations of the medium and fluid(s) are not distinct in this theory. The poroelastic model below is developed in the current configuration; while this is less convenient for linearization of the elastic response of the medium, it significantly simplifies the constitutive theory for the fluid(s) in the pores.

1.1 Porous Flow Models

Porous flows of interest typically involve multiple fluids in a tortuous medium with interactions occurring over multiple spatial and temporal scales. Moreover, only gross properties of the spatial and compositional makeup of the medium and fluids are available, so models based on first principles can only realize macroscopic balance laws for mass, force, and energy, along with an entropy inequality. Modeling the kinematics, dynamics, and thermodynamics, required to close the systems of conservation laws has a long history. Darcy’s experimental results of 1857 resulted in his eponymous law relating the mass flow and pressure gradient [13], and the series of papers by Biot and Willis [3–6] dating from the 1940’s integrated elasticity of the medium into the modeling. The monograph by Bear [2] presents a comprehensive review of the classical theory, and the more recent monograph by Chen, Huan, and Ma [9] includes a detailed discussion of numerical methods to simulate solutions. The monograph by Cheng [10] provides an extensive overview of poroelastic models.

Due to their importance in the oil industry, a majority of the work on porous flow focuses on incompressible one or two phase problems [1, 7, 8, 14, 16, 17, 19, 22, 27, 31]. When multiple components are present a separation of time scales is assumed so that classical thermodynamics can be used to determine the formation and evolution of the various phases [11, 15, 18, 20, 23]. Integrating classical statements of thermodynamics into a continuum description of flow in a rigid medium consistent with the second law formulated by Coleman and Noll [12] was undertaken by the authors in [26]. Classical elasticity theory assumes that stresses can be determined from the deformation, but this is not so for a poroelastic medium since the stress in the medium also depends upon the pressure in the pores. In addition, deformation of the medium results in changes of pore volume and hence the fluid pressure. Homogenization theory for a periodic linear elastic medium with pore pressure p [25] shows that there exists a homogenized elasticity tensor \mathbb{C} and two Biot parameters, a tensor \mathbf{B} and a scalar β , for which the homogenized stress \mathbf{T}_0 in the solid and a volumetric strain ξ take the form

$$\mathbf{T}_0 = \mathbb{C}(\nabla \mathbf{u}) + p(\mathbf{B} - s_0 \mathbf{1}) \quad \text{and} \quad \xi = \mathbf{B} \cdot \nabla \mathbf{u} + \beta p, \quad (1)$$

where \mathbf{u} is the macroscopic displacement. If $V_s \subset V$ is the solid portion of a representative volume of the porous medium and $|\tilde{V}_s|$ is the volume of the deformed solid, then setting $s_0 = |V_s|/|V|$ and $s = |\tilde{V}_s|/|V|$ the volumetric strain is characterized by

$$s = s_0 + \xi + O(\epsilon),$$

where ϵ is the ratio of the pore size (period) of the medium to a characteristic length scale of the domain. These constitutive laws are comparable to those appearing in [10, equations (5.5) and (5.6)], except there the total stress is used, while in (1) only the stress in the solid phase is present. In the absence of detailed geometric information on the pores required to determine the homogenized coefficients (\mathbb{C} , \mathbf{B} , β), the Biot theory appearing in the engi-

neering literature typically assumes isotropic response so that $\mathbb{C}(\mathbf{E}) = 2\mu\mathbf{E} + \lambda \text{tr}(\mathbf{E})\mathbf{1}$ and \mathbf{B} is a multiple of the identity [24]. In Sect. 4.1 we show that formulae for the Biot parameters are developed from a linearization of the formulation presented here.

1.2 Overview

Currently it is accepted that computationally tractable simulations of porous flows through an elastic medium with acceptable predictive capability can be realized with a system of partial differential equations consisting of (i) a scalar equation representing the balance of mass for each conserved component (chemical species), (ii) a vector valued equation representing a macroscopic momentum balance, and (iii) a scalar equation representing the balance of energy. The model presented below fits into this paradigm, which results in a system of coupled partial differential equations for the mass densities $\{m_c\}_{c=1}^{N_c}$ of the conserved components,¹ the displacement of the solid \mathbf{u} , and the internal energy e . All other physical processes are modeled with algebraic equations. These include Darcy laws which enter as force balances for the fluid phases, and constitutive equations describing the formation of the phases.

The balances of mass take the form,

$$\frac{\partial}{\partial t} m_c + \text{div} \left(m_c \mathbf{v}_0 - \mathbf{K}_c \nabla \mu_c + \sum_{\pi=1}^{N_p} M_{c\pi} \mathbf{v}_\pi^r \right) = h_c,$$

where the quantities inside the divergence represent the mass flux of the component arising from the motion of the medium and each phase, and the right-hand side the mass sinks and sources. Denoting the deformation gradient of the solid by $\mathbf{F} := \mathbf{1} + \nabla \mathbf{u}$, the balance of forces for the solid/fluid system takes the form

$$\sum_{\pi=0}^{N_p} s_\pi \mathbf{b}_\pi + \text{div} \left[\left(\frac{\partial e_I}{\partial \mathbf{F}} - \theta \frac{\partial \eta_0}{\partial \mathbf{F}} \right) \mathbf{F}^\top + \left(e_0 - \theta \eta_0 - \sum_{\pi=1}^{N_p} s_\pi P_\pi \right) \mathbf{1} \right] = \mathbf{0}.$$

This equation neglects the inertial term since the motion is slow in relation to the time scales of interest. The term inside the divergence is the (total) stress, and the remaining term is the (total) body force. The balance of energy becomes

$$\begin{aligned} \dot{e} = r + \dot{\mathbf{m}} \cdot \boldsymbol{\mu} + \text{div}(\mathbf{K}_\theta \nabla \theta) + \sum_{c=1}^{N_c} \nabla \mu_c \cdot \mathbf{K}_c \nabla \mu_c + \sum_{\pi=1}^{N_p} \left(\mathbf{v}_\pi^r \cdot \sum_{\substack{\pi'=0 \\ \pi' \neq \pi}}^{N_p} \mathbf{A}_{\pi\pi'} (\mathbf{v}_\pi^r - \mathbf{v}_{\pi'}^r) \right) \\ + \nabla \mathbf{v}_0 \cdot \left[\left(\frac{\partial e_I}{\partial \mathbf{F}} - \theta \frac{\partial \eta_0}{\partial \mathbf{F}} \right) \mathbf{F}^\top - \theta \eta \mathbf{1} \right] - \sum_{\pi=1}^{N_p} \theta \text{div}(\eta_\pi \mathbf{v}_\pi^r). \end{aligned}$$

The left-hand side of this equation is the rate of change of the internal energy and the right-hand side represents the change due to external supply, chemical energy carried by the mass, diffusion of heat and mass, viscous dissipation due to the relative motion of the fluids, elastic energy, and transport due to the motion of the fluids relative to the solid.

¹In porous flow contexts $\{m_c\}_{c=1}^{N_c}$ denote mass densities of components and $\{\rho_\pi\}_{\pi=1}^{N_p}$ are the mass densities of the phases.

Constitutive relations are required to close this system of partial differential equations. The development of constitutive relations to model phase formation and corresponding energy balance is non-trivial and is one of the main novelties of this work. Below we utilize the methodology introduced by Coleman and Noll [12], whereby the second law of thermodynamics is exploited to develop constitutive relations which guarantee a dissipation inequality for the resulting model. These relations naturally divide into two distinct types. The first type characterize the state of the fluids and phases at the pore scale and are introduced in Sect. 3.1. It is assumed that these processes evolve to an equilibrium at much shorter time scales than the macroscopic flow processes so that classical thermodynamics can be used to characterize the state of the fluids at the pore scale. This principle is referred to as local thermodynamic equilibrium, and while equilibrium has been assumed in poroelastic theories before [10], the novel formulation of this assumption we present provides a seamless integration of classical and continuum statements of the second law for multi-phase multicomponent fluids. In Sect. 3.2 these constitutive assumptions are introduced into the balance laws and the Coleman–Noll procedure is utilized to identify constitutive restrictions on the remaining variables which guarantee satisfaction of the second law of thermodynamics. The short Sect. 3.3 discusses the consequences of these restrictions. The second set of constitutive assumptions determine the dissipative processes, such as the Darcy law(s), and are discussed in Sect. 3.4. Section 3.5 discusses the simplifications that result when the temperature is assumed to be constant or is determined by that of the solid medium.

The next section introduces the notation required to describe systems with multiple components and phases, and develops the statements of the physical laws these quantities obey. The continuum model developed in Sect. 3 models a broad class of flows in poroelastic media. In Sect. 4 connections with other theories and the present theory are made. Finally, Sect. 5 summaries the proposed model by recapping the constitutive assumptions and balance laws.

2 Balance Laws

Balance laws will be formulated for a porous medium in which multiple phases of fluids flow through a porous solid phase. The fluid phases are formed by different components; that is, species which are conserved. There will be a total of N_c components that make up N_p phases. The formulation of the balance laws will be motivated by mixture theory and results from homogenization involving poroelastic materials.

\mathcal{B}_R will denote a reference configuration of the body (medium and fluids) in a three-dimensional Euclidean space \mathcal{E} with associated vector space \mathcal{V} , and \mathcal{I} an interval of time. Unless stated otherwise, *all densities are taken per unit volume of space in the present configuration.*

2.1 Introduction of Relevant Quantities

A large number of quantities are required to characterize the evolution of the porous medium under consideration. The variables used below are gathered here for easy reference.

The motion and volume fraction of each phase are described by the following.

- $\chi : \mathcal{B}_R \times \mathcal{I} \rightarrow \mathcal{E}$ is the motion of the medium, and is identified as the motion of the solid phase. Let

$$\mathcal{T} := \{(\chi(X, t), t) \in \mathcal{E} \times \mathcal{I} \mid (X, t) \in \mathcal{B}_R \times \mathcal{I}\}$$

be the trajectory of the motion.

- The deformation gradient of this motion is denoted by $\mathbf{F} = \nabla \chi$.
- A superscript dot will denote the material time-derivative relative to the motion of the solid; if ϕ is a spatial field, then

$$\dot{\phi} := \frac{\partial \phi}{\partial t} + \mathbf{v}_0 \cdot \nabla \phi. \tag{2}$$

- $\mathbf{v}_\pi : \mathcal{T} \rightarrow \mathcal{V}$ is the velocity of phase π , for $0 \leq \pi \leq N_p$, where phase $\pi = 0$ corresponds to the solid phase. We write $\mathbf{v}_\pi^r := \mathbf{v}_\pi - \mathbf{v}_0$ for the velocity of phase π relative to the solid phase.
- $s_\pi : \mathcal{T} \rightarrow [0, 1]$ represents the volume fraction of phase $0 \leq \pi \leq N_p$. We write $\mathbf{s} := (s_0, s_1, \dots, s_{N_p})$ and assume that $\sum_{\pi=0}^{N_p} s_\pi = 1$.

Laws will be postulated for the balance of the mass of each component, force balance for each phase, a single energy balance, and a single entropy imbalance. The mass balances involve the following quantities.

- $m_0 : \mathcal{T} \rightarrow [0, \infty)$ is the mass density of the solid phase.
- $m_c : \mathcal{T} \rightarrow [0, \infty)$ is the mass density of component c , for $1 \leq c \leq N_c$. The vector of fluid mass densities is denoted by $\mathbf{m} := (m_1, \dots, m_{N_c})$.
- $M_{c\pi} : \mathcal{T} \rightarrow [0, \infty)$ represents the mass density of component c in phase π for $1 \leq c \leq N_c$ and $1 \leq \pi \leq N_p$. The matrix of mass densities is denoted by $M = [M_{c\pi}]$. Clearly $m_c = \sum_{\pi=1}^{N_p} M_{c\pi}$.
- $\mathbf{h}_c : \mathcal{T} \rightarrow \mathcal{V}$ represents the mass flux of component c due to diffusion.
- $h_c : \mathcal{T} \rightarrow \mathbb{R}$ is the supply (sinks and sources) of component c .

Next we list the quantities appearing in the balances of forces. It is assumed that the solid phase is elastic, while the fluid phases are viscous. In the fluid phases, the viscous forces will be modeled as body forces, dependent on the velocity of the fluid relative to the other phases, rather than surface forces as is customary. In this way, the force balance for a fluid phase will take the form of a Darcy Law. This formulation of viscous forces in porous flow is motivated by results in homogenization theory [25]. Also, the phases will interact with each other through body forces, as is common in mixture theory.

- $\mathbf{T}_\pi : \mathcal{T} \rightarrow \text{Sym}^2$ is the elastic stress for phase $0 \leq \pi \leq N_p$. For the fluid phases, $\pi \neq 0$, we assume $\mathbf{T}_\pi = -s_\pi p_\pi \mathbf{1}$, where p_π is the pressure in the fluid phase. The term $s_\pi p_\pi$ is sometimes referred to as the partial pressure in phase π .
- $\mathbf{f}_{\pi\pi'} : \mathcal{T} \rightarrow \mathcal{V}$ with $\pi \neq \pi'$ is the force density that phase π' exerts on phase π . We assume that this can be additively decomposed into the sum of an elastic force $\mathbf{f}_{E\pi\pi'}$ and a viscous force $\mathbf{f}_{V\pi\pi'}$ so that $\mathbf{f}_{\pi\pi'} = \mathbf{f}_{E\pi\pi'} + \mathbf{f}_{V\pi\pi'}$.
- $\mathbf{b}_\pi : \mathcal{T} \rightarrow \mathcal{V}$ is the body force density acting on phase π per unit volume of phase π .

The quantities appearing in the balance of energy are the following.

- $e : \mathcal{T} \rightarrow \mathbb{R}$ is the total internal energy density of the entire continuum.
- $e_\pi : \mathcal{T} \rightarrow \mathbb{R}$ is the internal energy density of phase $0 \leq \pi \leq N_p$. The vector of internal energy densities is denoted by $\mathbf{e} = (e_0, \dots, e_{N_p})$.
- $\mathbf{q} : \mathcal{T} \rightarrow \mathcal{V}$ represents the heat flux.

²Here we assume that the stresses are symmetric. In more general mixture theory, it is possible for stresses to be nonsymmetric. In this case one must postulate a torque balance for each phase. However, if for each phase there is no external supply of torques, then torque balance implies that the stresses are symmetric. See Truesdell [29] for details.

- $\mu_c : \mathcal{T} \rightarrow \mathbb{R}$ is the chemical potential of component $1 \leq c \leq N_c$. The vector of chemical potentials is denoted by $\boldsymbol{\mu} := (\mu_1, \dots, \mu_{N_c})$.
- $r : \mathcal{T} \rightarrow \mathbb{R}$ is the supply of energy.

Finally, we list the quantities appearing in the entropy imbalance.

- $\eta : \mathcal{T} \rightarrow \mathbb{R}$ is the total entropy density of the entire continuum.
- $\eta_\pi : \mathcal{T} \rightarrow \mathbb{R}$ is the entropy density of phase $0 \leq \pi \leq N_p$.
- $\theta : \mathcal{T} \rightarrow (0, \infty)$ is the (absolute) temperature.
- $\psi_\pi = e_\pi - \theta \eta_\pi$ is the (Helmholtz) free energy of phase $0 \leq \pi \leq N_p$. The free energy is used in place of the entropy in the isothermal setting.

2.2 Balances for Inertialess Continua

All of the balances will first be formulated globally using a spatial region $V(t)$ that is connecting with the motion of the solid phase. We will assume that the fluid phases are transported by the porous solid so that their velocities take the form $\mathbf{v}_\pi = \mathbf{v}_0 + \mathbf{v}_\pi^r$ ($1 \leq \pi \leq N_p$), and that inertial effects are negligible.

1. *Mass balance:* The global form of mass balance for the solid states that

$$\frac{d}{dt} \int_{V(t)} m_0 \, dv = 0. \tag{3}$$

The local form of this balance is

$$\frac{\partial}{\partial t} m_0 + \operatorname{div}(m_0 \mathbf{v}_0) = 0. \tag{4}$$

For fluid component c , global mass balance reads

$$\frac{d}{dt} \int_{V(t)} m_c \, dv = \int_{V(t)} h_c \, dv - \int_{\partial V(t)} \left(\mathbf{h}_c + \sum_{\pi=1}^{N_p} M_{c\pi} \mathbf{v}_\pi^r \right) \cdot \mathbf{n} \, da, \tag{5}$$

where \mathbf{n} is the exterior unit normal. Localizing this balance yields

$$\frac{\partial}{\partial t} m_c + \operatorname{div} \left(m_c \mathbf{v}_0 + \mathbf{h}_c + \sum_{\pi=1}^{N_p} M_{c\pi} \mathbf{v}_\pi^r \right) = h_c. \tag{6}$$

2. *Force balance:* Assume for each subcollection $P \subset \{0, \dots, N_p\}$ of phases that

$$\sum_{\pi \in P} \left[\int_{V(t)} s_\pi \mathbf{b}_\pi \, dv + \int_{\partial V(t)} \mathbf{T}_\pi \mathbf{n} \, da + \sum_{\pi' \in P^c} \int_{V(t)} \mathbf{f}_{\pi\pi'} \, dv \right] = \mathbf{0}, \tag{7}$$

where P^c is the complement of P in $\{0, \dots, N_p\}$. The local form is

$$\sum_{\pi \in P} \left[s_\pi \mathbf{b}_\pi + \operatorname{div}(\mathbf{T}_\pi) + \sum_{\pi' \in P^c} \mathbf{f}_{\pi\pi'} \right] = \mathbf{0}. \tag{8}$$

3. *Energy balance:* The global form is

$$\begin{aligned} \frac{d}{dt} \int_{V(t)} e \, dv &= \int_{V(t)} r \, dv - \int_{\partial V(t)} \mathbf{q} \cdot \mathbf{n} \, da \\ &+ \sum_{c=1}^{N_c} \left[\int_{V(t)} \mu_c h_c \, dv - \int_{\partial V(t)} \mu_c \mathbf{h}_c \cdot \mathbf{n} \, da \right] \\ &+ \sum_{\pi=0}^{N_p} \left[\int_{V(t)} s_\pi \mathbf{b}_\pi \cdot \mathbf{v}_\pi \, dv + \int_{\partial V(t)} (\mathbf{T}_\pi \mathbf{v}_\pi - e_\pi \mathbf{v}_\pi^r) \cdot \mathbf{n} \, da \right]. \end{aligned} \tag{9}$$

The local form reads

$$\begin{aligned} \frac{\partial}{\partial t} e + \operatorname{div} \left[e \mathbf{v}_0 + \mathbf{q} + \sum_{c=1}^{N_c} \mu_c \mathbf{h}_c + \sum_{\pi=0}^{N_p} (e_\pi \mathbf{v}_\pi^r - \mathbf{T}_\pi \mathbf{v}_\pi) \right] \\ = r + \sum_{c=1}^{N_c} \mu_c h_c + \sum_{\pi=0}^{N_p} s_\pi \mathbf{b}_\pi \cdot \mathbf{v}_\pi. \end{aligned} \tag{10}$$

4. *Entropy imbalance:* The global form of entropy imbalance is

$$\frac{d}{dt} \int_{V(t)} \eta \, dv \geq \int_{V(t)} \frac{r}{\theta} \, dv - \int_{\partial V(t)} \frac{1}{\theta} \mathbf{q} \cdot \mathbf{n} \, da - \sum_{\pi=1}^{N_p} \int_{\partial V(t)} \eta_\pi \mathbf{v}_\pi^r \cdot \mathbf{n} \, da. \tag{11}$$

The local form is

$$\frac{\partial}{\partial t} \eta + \operatorname{div} \left(\eta \mathbf{v}_0 + \frac{\mathbf{q}}{\theta} + \sum_{\pi=1}^{N_p} \eta_\pi \mathbf{v}_\pi^r \right) \geq \frac{r}{\theta}. \tag{12}$$

2.3 Basic Consequences of the Balances

The aforementioned balances have a few consequences that will be useful in our analysis. First of all, using results in Noll’s theory of interactions [21], force balance yields the law of mutual action:

$$\mathbf{f}_{\pi\pi'} = -\mathbf{f}_{\pi'\pi} \quad \text{for all } \pi, \pi' \in \{0, \dots, N_p\}. \tag{13}$$

The energy balance can be rewritten in a different form which will be useful below. Eliminating the mass supplies h_c and the external body forces $s_\pi \mathbf{b}_\pi$ from the energy balance using (6) and (8), and using (13), the energy balance becomes (recall $\mathbf{T}_\pi = -s_\pi p_\pi \mathbf{1}$ for $\pi \neq 0$ and the notation (2))

$$\begin{aligned} \dot{e} - \dot{\mathbf{m}} \cdot \boldsymbol{\mu} &= r - \operatorname{div} \mathbf{q} - \sum_{c=1}^{N_c} \nabla \mu_c \cdot \mathbf{h}_c - \sum_{\pi=1}^{N_p} \left(\mathbf{v}_\pi^r \cdot \sum_{\substack{\pi'=0 \\ \pi' \neq \pi}}^{N_p} \mathbf{f}_{V\pi\pi'} \right) \\ &+ \nabla \mathbf{v}_0 \cdot \left[\mathbf{T}_0 + \left(\boldsymbol{\mu} \cdot \mathbf{m} - e - \sum_{\pi=1}^{N_p} s_\pi p_\pi \right) \mathbf{1} \right] \end{aligned}$$

$$\begin{aligned}
 & + \sum_{\pi=1}^{N_p} \mathbf{v}_{\pi}^r \cdot \left[\sum_{c=1}^{N_c} \mu_c \nabla M_{c\pi} - \nabla e_{\pi} - \sum_{\substack{\pi'=0 \\ \pi' \neq \pi}}^{N_p} \mathbf{f}_{E\pi\pi'} \right] \\
 & + \sum_{\pi=1}^{N_p} \operatorname{div}(\mathbf{v}_{\pi}^r) \left[\sum_{c=1}^{N_c} \mu_c M_{c\pi} - e_{\pi} - s_{\pi} p_{\pi} \right].
 \end{aligned} \tag{14}$$

This equation can be used to eliminate the radiation from the entropy imbalance to obtain

$$\begin{aligned}
 \theta \dot{\eta} - \dot{e} + \dot{\mathbf{m}} \cdot \boldsymbol{\mu} & \geq \frac{1}{\theta} \nabla \theta \cdot \mathbf{q} + \sum_{c=1}^{N_c} \nabla \mu_c \cdot \mathbf{h}_c + \sum_{\pi=1}^{N_p} \left(\mathbf{v}_{\pi}^r \cdot \sum_{\substack{\pi'=0 \\ \pi' \neq \pi}}^{N_p} \mathbf{f}_{V\pi\pi'} \right) \\
 & - \nabla \mathbf{v}_0 \cdot \left[\mathbf{T}_0 + \left(\boldsymbol{\mu} \cdot \mathbf{m} - e - \sum_{\pi=1}^{N_p} s_{\pi} p_{\pi} + \theta \eta \right) \mathbf{1} \right] \\
 & - \sum_{\pi=1}^{N_p} \mathbf{v}_{\pi}^r \cdot \left[\sum_{c=1}^{N_c} \mu_c \nabla M_{c\pi} - \nabla e_{\pi} - \sum_{\substack{\pi'=0 \\ \pi' \neq \pi}}^{N_p} \mathbf{f}_{E\pi\pi'} + \theta \nabla \eta_{\pi} \right] \\
 & - \sum_{\pi=1}^{N_p} \operatorname{div}(\mathbf{v}_{\pi}^r) \left[\sum_{c=1}^{N_c} \mu_c M_{c\pi} - e_{\pi} - s_{\pi} p_{\pi} + \theta \eta_{\pi} \right].
 \end{aligned} \tag{15}$$

This last relation is sometimes referred to as the reduced entropy inequality and does not involve any external influences. It is the form of the entropy imbalance that is most useful when applying the Coleman–Noll procedure.

3 Constitutive Relations

3.1 Local Thermodynamic Equilibrium

It is necessary to specify how the mass of each component m_c , total internal energy e , and entropy η , are partitioned among the various phases, and to determine how much of each phase is present at any particular point. Classical thermodynamics was developed to determine macroscopic properties of equilibrated systems from their constituents (energy, volume, mass). Porous flow models postulate a separation of time scales so that this theory can be used to model the state of the fluids at the pore scale. This is realized by assuming that at each point of the continuum the phases are formed by maximizing the total entropy subject to constraints on the total amount of internal energy and mass available, and the deformation \mathbf{F} . This characterizes equilibria of classical thermodynamic ensembles [28]. This assumption has been made in poroelasticity before [10], but a clear statement of the second law has not been formulated precisely in the multi-component multiphase fluid setting.

We assume that the entropy of a fluid phase, $\pi \neq 0$, is specified by the constitutive law

$$\eta_{\pi} = \hat{\eta}_{\pi}(e_{\pi}, s_{\pi}, \{M_{c\pi}\}_{c=1}^{N_c}), \tag{16}$$

and the entropy for the solid phase is determined by

$$\eta_0 = \hat{\eta}_0(e_0, s_0, \mathbf{F}). \tag{17}$$

The entropy for the entire continuum is determined by the local equilibrium assumption introduced below. The total internal energy at the pore scale is assumed to consist of the internal energy of the phases and the energy concentrated at the interfaces between the phases associated with surface tension and wetting. This additional contribution to the internal energy depends upon the phases present, and in the porous flow model is constitutively specified by a function of the form

$$e_I = \hat{e}_I(\mathbf{e}, \mathbf{s}, M, \mathbf{F}), \tag{18}$$

where, recall, $\mathbf{e} = (e_0, \dots, e_{N_p})$.

Assumption 1 (Local Equilibrium) The entropy η of the continuum is specified by

$$\begin{aligned} \eta &= \hat{\eta}(e, \mathbf{m}, \mathbf{F}) \\ &:= \sup_{(\mathbf{e}, \mathbf{s}, M)} \left\{ \hat{\eta}_0(e_0, s_0, \mathbf{F}) + \sum_{\pi=1}^{N_p} \hat{\eta}_\pi(e_\pi, s_\pi, \{M_{c\pi}\}_{c=1}^{N_c}) \mid \hat{e}_I(\mathbf{e}, \mathbf{s}, M, \mathbf{F}) \right. \\ &\quad \left. + \sum_{\pi=0}^{N_p} e_\pi = e, \sum_{\pi=0}^{N_p} s_\pi = 1, \sum_{\pi=1}^{N_p} M_{c\pi} = m_c \right\}. \end{aligned} \tag{19}$$

Moreover, for each $(e, \mathbf{m}, \mathbf{F})$ the maximum is attained at a unique point

$$\begin{aligned} (\mathbf{e}, \mathbf{s}, M) &= \arg \max_{(\mathbf{e}, \mathbf{s}, M)} \left\{ \hat{\eta}_0(e_0, s_0, \mathbf{F}) + \sum_{\pi=1}^{N_p} \hat{\eta}_\pi(e_\pi, s_\pi, \{M_{c\pi}\}_{c=1}^{N_c}) \mid \hat{e}_I(\mathbf{e}, \mathbf{s}, M, \mathbf{F}) \right. \\ &\quad \left. + \sum_{\pi=0}^{N_p} e_\pi = e, \sum_{\pi=0}^{N_p} s_\pi = 1, \sum_{\pi=1}^{N_p} M_{c\pi} = m_c \right\}. \end{aligned} \tag{20}$$

The supremum in these expressions is taken over $\mathbf{e} \in \mathbb{R}^{N_p+1}$, $\mathbf{s} \in [0, 1]^{N_p+1}$, and $M \in [0, \infty)^{N_c \times N_p}$.

By the uniqueness part of this assumption, we obtain constitutive laws for the energies, volume fractions, and mass densities:

$$(\mathbf{e}, \mathbf{s}, M) = (\hat{e}(e, \mathbf{m}, \mathbf{F}), \hat{\mathbf{s}}(e, \mathbf{m}, \mathbf{F}), \hat{M}(e, \mathbf{m}, \mathbf{F})).$$

Under the assumption that the entropy and interfacial energy functions are smooth, the local thermodynamic equilibrium assumption implies the existence of Lagrange multipliers $(\lambda^e, \lambda^s, \lambda^m) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{N_c}$ and KKT (Karush–Kuhn–Tucker) multipliers

$$(\lambda^0, \lambda^1, \Lambda) \in [0, \infty)^{1+N_\pi} \times [0, \infty)^{1+N_\pi} \times [0, \infty)^{N_\pi N_c}$$

such that

$$\frac{\partial \eta_\pi}{\partial e_\pi} = \lambda^e \left(1 + \frac{\partial e_I}{\partial e_\pi} \right), \quad \pi \in \{0, \dots, N_p\}, \tag{21}$$

$$\frac{\partial \eta_\pi}{\partial s_\pi} + \lambda_\pi^0 - \lambda_\pi^1 = \lambda^s + \lambda^e \frac{\partial e_I}{\partial s_\pi}, \quad \pi \in \{0, \dots, N_p\}, \tag{22}$$

$$\frac{\partial \eta_\pi}{\partial M_{c\pi}} + \Lambda_{c\pi} = \lambda_c^m + \lambda^e \frac{\partial e_I}{\partial M_{c\pi}}, \quad \pi \in \{1, \dots, N_p\}, \tag{23}$$

$$s_\pi \lambda_\pi^0 = 0, \quad (1 - s_\pi) \lambda_\pi^1 = 0, \quad \pi \in \{0, \dots, N_p\}, \tag{24}$$

$$M_{c\pi} \Lambda_{c\pi} = 0, \quad \pi \in \{1, \dots, N_p\}, \tag{25}$$

with $c \in \{1, \dots, N_c\}$. We shall see that the Lagrange multipliers λ^e , λ^s , and λ_c^m are related to the temperature θ , a pressure p , and the chemical potentials μ_c , respectively.

Let $(\bar{e}, \bar{\mathbf{m}}, \bar{\mathbf{F}})$ be a parameter dependent family of states at which the entropy η can be considered. By the assumption of local equilibrium, there corresponds a parameter dependent family $(\bar{\mathbf{e}}, \bar{\mathbf{s}}, \bar{\mathbf{M}})$ such that

$$\hat{\eta}(\bar{e}, \bar{\mathbf{m}}, \bar{\mathbf{F}}) = \hat{\eta}_0(\bar{e}_0, \bar{s}_0, \bar{\mathbf{F}}) + \sum_{\pi=1}^{N_p} \hat{\eta}_\pi(\bar{e}_\pi, \bar{s}_\pi, \{\bar{M}_{c\pi}\}_{c=1}^{N_c}).$$

Differentiating this equation with respect to the parameter, using the chain rule, and (21)–(25) yields

$$\eta' = \frac{\partial \eta}{\partial e} \bar{e}' + \frac{\partial \eta}{\partial \mathbf{m}} \cdot \bar{\mathbf{m}}' + \frac{\partial \eta}{\partial \mathbf{F}} \cdot \bar{\mathbf{F}}' = \lambda^e \bar{e}' + \lambda^m \cdot \bar{\mathbf{m}}' + \left(\frac{\partial \eta_0}{\partial \mathbf{F}} - \lambda^e \frac{\partial e_I}{\partial \mathbf{F}} \right) \cdot \bar{\mathbf{F}}', \tag{26}$$

where prime denotes the derivative with respect to the parameter. Since the process $(\bar{e}, \bar{\mathbf{m}}, \bar{\mathbf{F}})$ was arbitrary, it follows that

$$\frac{\partial \eta}{\partial e} = \lambda^e, \quad \frac{\partial \eta}{\partial \mathbf{m}} = \lambda^m, \quad \text{and} \quad \frac{\partial \eta}{\partial \mathbf{F}} = \frac{\partial \eta_0}{\partial \mathbf{F}} - \lambda^e \frac{\partial e_I}{\partial \mathbf{F}}. \tag{27}$$

3.2 Restrictions Due to the Second Law

Coleman and Noll’s interpretation [12] of the second law of thermodynamics is that the entropy imbalance (12) must hold for all thermodynamic processes. Moreover, they stipulated that external influences can be specified arbitrarily, so for any thermodynamic process one can choose h_c , \mathbf{b}_π , and r such that the balances of mass, force, and energy hold. Since the second law does not restrict the class of processes, it should be interpreted as a restriction on constitutive laws. Here, rather than specifying all of the constitutive laws and use the second law to find if and only if restrictions on them, we only find sufficient conditions for the second law to hold and use these to motivate additional constitutive laws that are consistent with the second law of thermodynamics.

Proposition 1 *Assume that local equilibrium holds so that the entropy η is determined by (19) and that the energies, volume fractions, and densities of the different phases are determined by (20). Under these constitutive assumptions, the entropy imbalance holds for all thermodynamics processes if*

1. *The entropy of fluid phase $\pi \neq 0$ is given by*

$$\eta_\pi = \frac{1}{\theta} \left(e_\pi + s_\pi p_\pi - \sum_{c=1}^{N_c} M_{c\pi} \mu_c \right). \tag{28}$$

2. The stress in the solid phase is given by

$$\mathbf{T}_0 = (e_0 - \theta \eta_0 + e_I) \mathbf{1} - \theta \frac{\partial \eta_0}{\partial \mathbf{F}} \mathbf{F}^\top + \frac{\partial e_I}{\partial \mathbf{F}} \mathbf{F}^\top. \tag{29}$$

3. The interfacial energy only depends upon the volume fractions and the deformation gradient; that is,

$$e_I = \hat{e}_I(\mathbf{s}, \mathbf{F}). \tag{30}$$

4. The temperature, phase pressures, and the chemical potential of the components are related to the Lagrange multipliers through

$$\theta = 1/\lambda^e, \quad p_\pi = p + \frac{\partial e_I}{\partial s_\pi}, \quad \mu_c = -\theta \lambda_c^m, \tag{31}$$

where $p := \lambda^s \theta$.

5. The diffusive mass fluxes, heat flux, and viscous forces satisfy

$$\sum_{c=1}^{N_c} \mathbf{h}_c \cdot \nabla \mu_c \leq 0, \quad \mathbf{q} \cdot \nabla \theta \leq 0, \quad \sum_{\pi=1}^{N_p} \left(\mathbf{v}_\pi^r \cdot \sum_{\substack{\pi'=0 \\ \pi' \neq \pi}}^{N_p} \mathbf{f}_{V\pi\pi'} \right) \leq 0. \tag{32}$$

6. The elastic forces between the phases satisfy

$$p_\pi \nabla s_\pi = \sum_{\substack{\pi'=0 \\ \pi' \neq \pi}}^{N_p} \mathbf{f}_{E\pi\pi'}, \quad \pi \neq 0. \tag{33}$$

Proof A chain rule calculation similar to (26) in the reduced entropy inequality (15) shows

$$\begin{aligned} 0 \leq & (\lambda^e \theta - 1) \dot{e} + (\boldsymbol{\mu} + \theta \boldsymbol{\lambda}^m) \cdot \dot{\mathbf{m}} \\ & - \frac{1}{\theta} \nabla \theta \cdot \mathbf{q} - \sum_{c=1}^{N_c} \nabla \mu_c \cdot \mathbf{h}_c - \sum_{\pi=1}^{N_p} \left(\mathbf{v}_\pi^r \cdot \sum_{\substack{\pi'=0 \\ \pi' \neq \pi}}^{N_p} \mathbf{f}_{V\pi\pi'} \right) \\ & + \nabla \mathbf{v}_0 \cdot \left[\mathbf{T}_0 - \left(\frac{\partial e_I}{\partial \mathbf{F}} - \theta \frac{\partial \eta_0}{\partial \mathbf{F}} \right) \mathbf{F}^\top + \left(\boldsymbol{\mu} \cdot \mathbf{m} - e - \sum_{\pi=1}^{N_p} s_\pi p_\pi + \theta \eta \right) \mathbf{1} \right] \\ & + \sum_{\pi=1}^{N_p} \mathbf{v}_\pi^r \cdot \left[\sum_{c=1}^{N_c} \mu_c \nabla M_{c\pi} - \nabla e_\pi - \sum_{\substack{\pi'=0 \\ \pi' \neq \pi}}^{N_p} \mathbf{f}_{E\pi\pi'} + \theta \nabla \eta_\pi \right] \\ & + \sum_{\pi=1}^{N_p} \operatorname{div}(\mathbf{v}_\pi^r) \left[\sum_{c=1}^{N_c} \mu_c M_{c\pi} - e_\pi - s_\pi p_\pi + \theta \eta_\pi \right]. \end{aligned}$$

We must use Items 1–6 to show that this inequality holds regardless of the state of the continuum. It follows from Items 1, 4, 5, and 6 that it suffices to show that

$$\mathbf{T}_0 = \left(\frac{\partial e_I}{\partial \mathbf{F}} - \theta \frac{\partial \eta_0}{\partial \mathbf{F}} \right) \mathbf{F}^\top - \left(\boldsymbol{\mu} \cdot \mathbf{m} - e - \sum_{\pi=1}^{N_p} s_\pi p_\pi + \theta \eta \right) \mathbf{1} \tag{34}$$

and

$$\sum_{c=1}^{N_c} \mu_c \nabla M_{c\pi} - \nabla e_\pi - p_\pi \nabla s_\pi + \theta \nabla \eta_\pi = \mathbf{0}. \tag{35}$$

To obtain the first of these, sum (28) from $\pi = 1$ to N_p and use the facts

$$\eta = \sum_{\pi=0}^{N_p} \eta_\pi \quad \text{and} \quad e = e_I + \sum_{\pi=0}^{N_p} e_\pi$$

to find that

$$e_0 - \theta \eta_0 + e_I = e - \theta \eta - \mathbf{m} \cdot \boldsymbol{\mu} + \sum_{\pi=1}^{N_p} s_\pi p_\pi.$$

Using this equation and Item 2 yields (34). To obtain (35), begin by taking the gradient of (16) and use (21)–(25) to find that

$$\theta \nabla \eta_\pi = \theta \lambda^e \left(1 + \frac{\partial e_I}{\partial e_\pi} \right) \nabla e_\pi + \left(\theta \lambda^s + \theta \lambda^e \frac{\partial e_I}{\partial s_\pi} \right) \nabla s_\pi + \sum_{c=1}^{N_c} \left(\theta \lambda_c^m + \theta \lambda^e \frac{\partial e_I}{\partial M_{c\pi}} \right) \nabla M_{c\pi}.$$

Combining this with Items 3 and 4 results in (35), which completes the proof. □

3.3 Comments and Consequences of the Restrictions

Item 1 of the Proposition 1 is consistent with the common assumption in classical thermodynamics that η_π is a homogeneous function of e_π , s_π , and $M_{c\pi}$; see, for example, [28].

Item 6 specifies the net elastic forces on a fluid phase π by the other phases. The net elastic force on the solid phase coming from the fluid phases can be computed from the fact that (13) implies

$$\mathbf{0} = \sum_{\pi=0}^{N_p} \sum_{\substack{\pi'=0 \\ \pi' \neq \pi}}^{N_p} \mathbf{f}_{E\pi\pi'} = \sum_{\pi'=1}^{N_p} \mathbf{f}_{E0\pi'} + \sum_{\pi=1}^{N_p} \sum_{\substack{\pi'=0 \\ \pi' \neq \pi}}^{N_p} \mathbf{f}_{E\pi\pi'}$$

and hence by (31)₂

$$\begin{aligned} \sum_{\pi'=1}^{N_p} \mathbf{f}_{E0\pi'} &= - \sum_{\pi=1}^{N_p} p_\pi \nabla s_\pi = - \sum_{\pi=1}^{N_p} \left(p + \frac{\partial e_I}{\partial s_\pi} \right) \nabla s_\pi \\ &= p \nabla s_0 - \nabla e_I + \frac{\partial e_I}{\partial s_0} \nabla s_0 + \frac{\partial e_I}{\partial \mathbf{F}} \nabla \mathbf{F}. \end{aligned} \tag{36}$$

This formula together (33) specifies net elastic force $\sum_{\substack{\pi'=0 \\ \pi' \neq \pi}}^{N_p} \mathbf{f}_{E\pi\pi'}$ on any phase π in terms of the given constitutive laws.

If (33) is used in the force balance for fluid phase π and the divergence of the stress is expanded, then one obtains

$$s_\pi \mathbf{b}_\pi - s_\pi \nabla p_\pi + \sum_{\substack{\pi'=0 \\ \pi' \neq \pi}}^{N_p} \mathbf{f}_{V\pi\pi'} = \mathbf{0}. \tag{37}$$

Thus, the net elastic force acting on the fluid phase from other phases cancels with part of the term associated with the divergence of the stress.

The results of Proposition 3.2 can also be used to rewrite the energy balance (14) in the form

$$\begin{aligned} \dot{e} - \boldsymbol{\mu} \cdot \dot{\mathbf{m}} = r - \operatorname{div} \mathbf{q} - \sum_{c=1}^{N_c} \nabla \mu_c \cdot \mathbf{h}_c - \sum_{\pi=1}^{N_p} \left(\mathbf{v}_\pi^r \cdot \sum_{\substack{\pi'=0 \\ \pi' \neq \pi}}^{N_p} \mathbf{f}_{V\pi\pi'} \right) \\ + \nabla \mathbf{v}_0 \cdot \left(\frac{\partial e_I}{\partial \mathbf{F}} \mathbf{F}^\top - \theta \frac{\partial \eta_0}{\partial \mathbf{F}} \mathbf{F}^\top - \theta \boldsymbol{\eta} \mathbf{1} \right) - \sum_{\pi=1}^{N_p} \theta \operatorname{div} (\eta_\pi \mathbf{v}_\pi^r). \end{aligned} \tag{38}$$

This form of the energy balance can be used to obtain a dissipation relation. Namely, using (27) and (31) we have

$$\theta \dot{\eta} = \dot{e} - \boldsymbol{\mu} \cdot \dot{\mathbf{m}} + \left(\frac{\partial e_I}{\partial \mathbf{F}} \mathbf{F}^\top - \theta \frac{\partial \eta_0}{\partial \mathbf{F}} \mathbf{F}^\top \right) \cdot \nabla \mathbf{v}_0.$$

Using this in (38) and then integrating the term involving the heat flux by parts, one obtains for any region $V(t)$ convecting with the solid phase that

$$\begin{aligned} -\frac{d}{dt} \int_{V(t)} \eta \, dv - \int_{V(t)} \frac{1}{\theta} \left\{ \frac{1}{\theta} \mathbf{q} \cdot \nabla \theta + \sum_{c=1}^{N_c} \nabla \mu_c \cdot \mathbf{h}_c + \sum_{\pi=1}^{N_p} \left(\mathbf{v}_\pi^r \cdot \sum_{\substack{\pi'=0 \\ \pi' \neq \pi}}^{N_p} \mathbf{f}_{V\pi\pi'} \right) \right\} dv \\ = - \int_{V(t)} \frac{r}{\theta} \, dv + \int_{\partial V(t)} \left(\frac{\mathbf{q}}{\theta} + \sum_{\pi=1}^{N_p} \eta_\pi \mathbf{v}_\pi \right) \cdot \mathbf{n} \, da. \end{aligned}$$

This equation can be used to establish bounds upon the gradients of the temperature, chemical potentials, and pressure. Such estimates are important for the development of stable numerical schemes.

3.4 Fluxes and Darcy’s Law

Prototypical constitutive laws for the mass and heat flux are provided by Fick and Fourier laws. Here, these take the form

$$\mathbf{h}_c = -\hat{\mathbf{K}}_c(\mathbf{e}, \mathbf{s}, M) \nabla \mu_c \quad \text{and} \quad \mathbf{q} = -\hat{\mathbf{K}}_\theta(\mathbf{e}, \mathbf{s}, M) \nabla \theta, \tag{39}$$

where it is assumed that $\hat{\mathbf{K}}_c(\mathbf{e}, \mathbf{s}, M)$ and $\hat{\mathbf{K}}_\theta(\mathbf{e}, \mathbf{s}, M)$ are positive semidefinite symmetric tensors, which ensures that (32)_{1,2} hold.

The specification of the viscous forces $\mathbf{f}_{V\pi\pi'}$ is related to Darcy’s law, which is fundamental to the theory of porous flow and is discussed in detail in all of the monographs on the subject [2, 9, 10, 25]. These forces should depend on the velocities of the phases π and π' and, possibly, the volume fractions. For this reason, we assume that

$$\mathbf{f}_{V\pi\pi'} = -\hat{\mathbf{A}}_{\pi\pi'}(\mathbf{s})(\mathbf{v}_\pi - \mathbf{v}_{\pi'}), \tag{40}$$

where the $\hat{\mathbf{A}}_{\pi\pi'}(\mathbf{s})$ are assumed to be positive semidefinite symmetric tensors and satisfy $\hat{\mathbf{A}}_{\pi\pi'}(\mathbf{s}) = \hat{\mathbf{A}}_{\pi'\pi}(\mathbf{s})$. These properties ensure that the inequality (32)₃ holds. Substituting (40) into the force balance for phase π (37) results in

$$s_\pi \mathbf{b}_\pi - s_\pi \nabla p_\pi - \sum_{\substack{\pi'=0 \\ \pi' \neq \pi}}^{N_p} \hat{\mathbf{A}}_{\pi\pi'}(\mathbf{s})(\mathbf{v}_\pi - \mathbf{v}_{\pi'}) = \mathbf{0}. \tag{41}$$

In the case of a single fluid phase, if $\hat{\mathbf{A}}_{10}(s_0, s_1)$ is invertible, the above equation can be written in the form

$$\mathbf{v}_1^r = s_1 \hat{\mathbf{A}}_{10}(s_0, s_1)^{-1}(\mathbf{b}_1 - \nabla p_1),$$

which is the traditional form of Darcy’s law. In this way one can see that Darcy’s law can be viewed as a consequence of force balance, in the absence of inertial forces, together with an appropriate constitutive law for the viscous forces. Moreover, the tensors $\mathbf{A}_{\pi\pi'}$ correspond to the (pseudo) inverses of the permeability tensors that appear in the engineering and experimental literature.

It is possible to formulate Darcy’s law as the Euler–Lagrange equation of a (convex) dissipation function. Set

$$I_s(\{\mathbf{v}_\pi^r\}_{\pi=1}^{N_p}) := \frac{1}{2} \|\{\mathbf{v}_\pi^r\}_{\pi=1}^{N_p}\|_A^2 - \sum_{\pi=1}^{N_p} s_\pi (\mathbf{b}_\pi - \nabla p_\pi) \cdot \mathbf{v}_\pi^r,$$

where $\|\cdot\|_A$ is the norm associated with the inner-product

$$(\{\mathbf{v}_\pi^r\}_{\pi=1}^{N_p}, \{\mathbf{w}_\pi^r\}_{\pi=1}^{N_p})_A := \sum_{\pi=1}^{N_p} \left(\sum_{\substack{\pi'=0 \\ \pi' < \pi}}^{N_p} \mathbf{A}_{\pi\pi'} (\mathbf{v}_\pi^r - \mathbf{v}_{\pi'}^r) \cdot (\mathbf{w}_\pi^r - \mathbf{w}_{\pi'}^r) \right). \tag{42}$$

The associated Euler–Lagrange equation is

$$(\{\mathbf{v}_\pi^r\}_{\pi=1}^{N_p}, \{\mathbf{w}_\pi^r\}_{\pi=1}^{N_p})_A = \sum_{\pi=1}^{N_p} s_\pi (\mathbf{b}_\pi - \nabla p_\pi) \cdot \mathbf{w}_\pi^r \quad \text{for all } \{\mathbf{w}_\pi^r\}_{\pi=1}^{N_p},$$

which is the variational formulation of Darcy’s law (41).

3.5 Mechanical and Isothermal Case

Often it is assumed that the temperature is constant or that the thermal energy in the solid phase dominates that of the fluid phases so that the temperature of the fluids become that of the solid much faster than the other processes taking place. In this case the energy balance

reduces to an equation for the temperature of the solid that is decoupled from the other balances. For example, the temperature could satisfy a classic heat equation

$$\frac{\partial}{\partial t}(c\theta) - \text{div}(\mathbf{K}_\theta \nabla \theta) = r,$$

where c and \mathbf{K}_θ are the specific heat and conductivity of the solid phase, respectively.

To specify a constant temperature it is preferable to use the Helmholtz free-energy rather than the entropy in the local thermodynamic equilibrium assumption. Letting $\psi_\pi = e_\pi - \theta \eta_\pi$ denote the free-energy density of phase π , the local equilibrium assumption requires that

$$\psi = \hat{\psi}(\theta, \mathbf{m}, \mathbf{F}) := \inf_{(s, M)} \left\{ \hat{\psi}_0(\theta, s_0, \mathbf{F}) + \sum_{\pi=1}^{N_p} \hat{\psi}_\pi(\theta, s_\pi, \{M_{c\pi}\}_{c=1}^{N_c}) + \hat{e}_I(\theta, \mathbf{s}, M, \mathbf{F}) \mid \sum_{\pi=0}^{N_p} s_\pi = 1, \sum_{\pi=1}^{N_p} M_{c\pi} = m_c \right\}. \tag{43}$$

Moreover, for each $(\theta, \mathbf{m}, \mathbf{F})$ the minimum is attained at a unique point

$$(\mathbf{s}, M) = \arg \min_{(s, M)} \left\{ \hat{\psi}_0(\theta, s_0, \mathbf{F}) + \sum_{\pi=1}^{N_p} \hat{\psi}_\pi(\theta, s_\pi, \{M_{c\pi}\}_{c=1}^{N_c}) + \hat{e}_I(\theta, \mathbf{s}, M, \mathbf{F}) \mid \sum_{\pi=0}^{N_p} s_\pi = 1, \sum_{\pi=1}^{N_p} M_{c\pi} = m_c \right\}. \tag{44}$$

The infimum in these expressions is taken over $\mathbf{s} \in [0, 1]^{N_p+1}$ and $M \in [0, \infty)^{N_c \times N_p}$.

As in Sect. 3.1, there are Lagrange multipliers (λ^s, λ^m) and KKT multipliers $(\lambda^0, \lambda^1, \Lambda)$ such that

$$\frac{\partial \psi_\pi}{\partial s_\pi} + \lambda_\pi^0 - \lambda_\pi^1 = \lambda^s - \frac{\partial e_I}{\partial s_\pi}, \quad \pi \in \{0, \dots, N_p\}, \tag{45}$$

$$\frac{\partial \psi_\pi}{\partial M_{c\pi}} + \Lambda_{c\pi} = \lambda_c^m - \frac{\partial e_I}{\partial M_{c\pi}}, \quad \pi \in \{1, \dots, N_p\}, \tag{46}$$

as well as (24) and (25) with $c \in \{1, \dots, N_c\}$.

Replacing the entropy with the free-energy in the reduced entropy inequality results in

$$\begin{aligned} -\dot{\psi} - \eta \dot{\theta} + \dot{\mathbf{m}} \cdot \boldsymbol{\mu} &\geq \frac{1}{\theta} \nabla \theta \cdot \mathbf{q} + \sum_{c=1}^{N_c} \nabla \mu_c \cdot \mathbf{h}_c + \sum_{\pi=1}^{N_p} \left(\mathbf{v}_\pi^r \cdot \sum_{\substack{\pi'=0 \\ \pi' \neq \pi}}^{N_p} \mathbf{f}_{V\pi\pi'} \right) \\ &- \nabla \mathbf{v}_0 \cdot \left[\mathbf{T}_0 + \left(\boldsymbol{\mu} \cdot \mathbf{m} - \psi - \sum_{\pi=1}^{N_p} s_\pi p_\pi \right) \mathbf{1} \right] \\ &- \sum_{\pi=1}^{N_p} \mathbf{v}_\pi^r \cdot \left[\sum_{c=1}^{N_c} \mu_c \nabla M_{c\pi} - \nabla \psi_\pi - \sum_{\substack{\pi'=0 \\ \pi' \neq \pi}}^{N_p} \mathbf{f}_{E\pi\pi'} - \eta_\pi \nabla \theta \right] \\ &- \sum_{\pi=1}^{N_p} \text{div}(\mathbf{v}_\pi^r) \left[\sum_{c=1}^{N_c} \mu_c M_{c\pi} - \psi_\pi - s_\pi p_\pi \right], \end{aligned}$$

which is sometimes referred to as the free-energy imbalance. The following result is analogous to Proposition 1, and is presented without proof.

Proposition 2 Assume that local equilibrium holds so that the Helmholtz free-energy ψ is determined by (43) and that the volume fractions and densities of the different phases are determined by (44). Under these constitutive assumptions, the free-energy imbalance holds for all thermodynamic processes if

1. The entropy of each phase is given by $\eta_\pi = -\frac{\partial \psi_\pi}{\partial \theta}$ for $0 \leq \pi \leq N_p$.
2. The free-energy of fluid phase $\pi \neq 0$ is given by

$$\psi_\pi = -s_\pi p_\pi + \sum_{c=1}^{N_c} M_{c\pi} \mu_c. \tag{47}$$

3. The stress in the solid phase is given by

$$\mathbf{T}_0 = (\psi_0 + e_I) \mathbf{1} + \frac{\partial \psi_0}{\partial \mathbf{F}} \mathbf{F}^\top + \frac{\partial e_I}{\partial \mathbf{F}} \mathbf{F}^\top. \tag{48}$$

4. The interfacial energy only depends upon the volume fractions and the deformation gradient; that is, $e_I = \hat{e}_I(\mathbf{s}, \mathbf{F})$.
5. The phase pressures and the chemical potential of the components are related to the Lagrange multipliers through

$$p_\pi = p + \frac{\partial e_I}{\partial s_\pi} \quad \text{and} \quad \mu_c = \lambda_c^m, \tag{49}$$

where $p := -\lambda^s$.

6. The diffusive mass fluxes, heat flux, and viscous forces satisfy

$$\sum_{c=1}^{N_c} \mathbf{h}_c \cdot \nabla \mu_c \leq 0, \quad \mathbf{q} \cdot \nabla \theta \leq 0, \quad \sum_{\pi=1}^{N_p} \left(\mathbf{v}_\pi^r \cdot \sum_{\substack{\pi'=0 \\ \pi' \neq \pi}}^{N_p} \mathbf{f}_{V\pi\pi'} \right) \leq 0. \tag{50}$$

7. The elastic forces between the phases satisfy

$$p_\pi \nabla s_\pi = \sum_{\substack{\pi'=0 \\ \pi' \neq \pi}}^{N_p} \mathbf{f}_{E\pi\pi'} \quad \pi \neq 0. \tag{51}$$

This result can be used to obtain an equation that describes the rate of change of the total Helmholtz free-energy. To see this, take the gradient of $\psi_\pi = \hat{\psi}_\pi(\theta, s_\pi, \{M_{c\pi}\}_{c=1}^{N_c})$ and use the chain rule to find

$$\nabla \psi_\pi = -\eta_\pi \nabla \theta - p_\pi \nabla s_\pi + \sum_{c=1}^{N_c} \mu_c \nabla M_{c\pi}.$$

Combining this with the gradient of (47) results in

$$\mathbf{0} = \eta_\pi \nabla \theta - s_\pi \nabla p_\pi + \sum_{c=1}^{N_c} M_{c\pi} \nabla \mu_c.$$

Now, using the chain rule to compute the time derivative of ψ , mass balance, and the previous equation, one can obtain

$$\begin{aligned} \frac{d}{dt} \int_{V(t)} \psi \, dv &= \int_{V(t)} \left(\mathbf{T}_0 - \sum_{\pi=1}^{N_p} s_\pi p_\pi \mathbf{1} \right) \cdot \nabla \mathbf{v}_0 + \sum_{\pi=1}^{N_p} s_\pi \nabla p_\pi \cdot \mathbf{v}_\pi \, dv \\ &\quad - \int_{V(t)} \left(\eta \dot{\theta} + \sum_{\pi=1}^{N_p} \eta_\pi \mathbf{v}_\pi^r \cdot \nabla \theta \right) \, dv + \sum_{c=1}^{N_c} \int_{V(t)} (\mu_c h_c + \nabla \mu_c \cdot \mathbf{h}_c) \, dv \\ &\quad - \int_{\partial V(t)} \mu_c \left(\mathbf{h}_c + \sum_{\pi=1}^{N_p} M_{c\pi} \mathbf{v}_\pi^r \right) \cdot \mathbf{n} \, da, \end{aligned}$$

where $V(t)$ is a region convecting with the motion of the solid. Integrating the term involving $\nabla \mathbf{v}_0$ by parts and then using the force balance (8), under isothermal conditions ($\dot{\theta} = 0$ and $\nabla \theta = \mathbf{0}$) the previous equation becomes

$$\begin{aligned} \frac{d}{dt} \int_{V(t)} \psi \, dv &- \int_{V(t)} \left\{ \sum_{\pi=1}^{N_p} \left(\sum_{\substack{\pi'=0 \\ \pi' \neq \pi}}^{N_p} \mathbf{f}_{V\pi\pi'} \cdot \mathbf{v}_\pi^r \right) + \sum_{c=1}^{N_c} \nabla \mu_c \cdot \mathbf{h}_c \right\} \, dv \\ &= \int_{V(t)} \left\{ \sum_{\pi=0}^{N_p} s_\pi \mathbf{b}_\pi \cdot \mathbf{v}_\pi + \sum_{c=1}^{N_c} \mu_c h_c \right\} \, dv \\ &\quad + \int_{\partial V(t)} \left\{ \left(\mathbf{T}_0 - \sum_{\pi=1}^{N_p} s_\pi p_\pi \mathbf{1} \right) \mathbf{v}_0 - \mu_c \left(\mathbf{h}_c + \sum_{\pi=1}^{N_p} M_{c\pi} \mathbf{v}_\pi^r \right) \right\} \cdot \mathbf{n} \, da. \end{aligned}$$

4 Connection with Other Theories

4.1 Biot Constants

One of the most dominant theories for poroelastic materials in the case of linear elastic response for the solid is Biot’s theory [3–6]. See the book by Cheng [10] for a recent overview of the subject or Sanchez-Palencia [25] for how the theory can be obtained via homogenization. Within this theory, only one fluid phase is considered and a global force balance is used. Thus, the stress in this theory is the total stress \mathbf{T} given as the sum the stresses in the solid and fluid phases:

$$\mathbf{T} = \mathbf{T}_0 + \mathbf{T}_1 = \mathbf{T}_0 - s_1 p_1 \mathbf{1}.$$

Constitutive laws for the total stress and the global pressure p are determined by Biot constants. Perhaps surprisingly, one of these constants appears in the constitutive laws for both the stress and the pressure. This fact has been reproduced using homogenization theory. Here we show that under certain simplifying assumptions, our theory reproduces these constitutive laws and the Biot constants. In the following calculations, assume that $0 < s_0 < 1$ so that we are not in a degenerate case. This implies that the KKT multipliers λ_0^1 and λ_0^1 are both zero by (24).

To obtain the constitutive laws appearing in Biot’s theory we make two simplifying assumptions:

- there is no interfacial energy; that is, $e_I = 0$,
- only small displacements and small changes in volume fraction occur.

In Biot’s theory of poroelasticity, there is no mention of any interfacial energy, so the first of these assumptions is reasonable. Moreover, Biot’s theory is linear, so the second assumption is also justified.

Notice that by (31)₂ the first assumption implies that $p_\pi = p$ for all π , so there is only one pressure. Since we will be working with small displacements, it makes sense to work with a referential description. For $J := \det \mathbf{F}$, set

$$e_{0R} := J e_0, \quad s_{0R} := J s_0, \quad p_R := J p, \quad \eta_{0R} := J \eta_0.$$

The constitutive law $\hat{\eta}_{0R}$ for η_{0R} is related to $\hat{\eta}_0$ by

$$\hat{\eta}_{0R}(J e_0, J s_0, \mathbf{F}) = J \hat{\eta}_0(e_0, s_0, \mathbf{F}).$$

Differentiating this equation with respect to \mathbf{F} and using the fact that $\frac{\partial J}{\partial \mathbf{F}} = J \mathbf{F}^{-\top}$ results in

$$\theta \frac{\partial \eta_{0R}}{\partial \mathbf{F}} + s_{0R} p_R \mathbf{F}^{-\top} = \left(\theta \frac{\partial \eta_0}{\partial \mathbf{F}} \mathbf{F}^\top - e_0 \mathbf{1} + \theta \eta_0 \mathbf{1} \right) J \mathbf{F}^\top,$$

and hence, from (29), the Piola stress $\mathbf{T}_{0R} = J \mathbf{T}_0 \mathbf{F}^{-\top}$ for the solid is given by

$$\mathbf{T}_{0R} = -\theta \frac{\partial \eta_{0R}}{\partial \mathbf{F}} - s_{0R} p_R \mathbf{F}^{-\top}.$$

Since the Piola stress for the fluid phase is given by $\mathbf{T}_{1R} = -s_1 p_R \mathbf{F}^{-\top}$, it follows that the total Piola stress is

$$\mathbf{T}_R = \theta \frac{\partial \eta_{0R}}{\partial \mathbf{F}} - p_R \mathbf{F}^{-\top}. \tag{52}$$

Based on the small displacements and small changes in volume fraction assumption, the parameter

$$\epsilon := \sqrt{|\nabla \mathbf{u}_0|^2 + |s_{0R} - s_{0R_0}|^2}$$

is small, where \mathbf{u}_0 is the displacement of the solid and s_{0R_0} is a given reference volume fraction for the solid from which we are considering small perturbations. Assuming that η_{0R} has a local maximum at $(e_{0R}, s_{0R}, \mathbf{F}) = (e_{0R_0}, s_{0R_0}, \mathbf{1})$, an asymptotic expansion in ϵ of the constitutive law for η_{0R} reads³

$$\eta_{0R} = \nabla \mathbf{u}_0 \cdot \mathbb{A} \nabla \mathbf{u}_0 + 2(s_{0R} - s_{0R_0}) \mathbf{A} \cdot \nabla \mathbf{u}_0 - d^2 (s_{0R} - s_{0R_0})^2 + o(\epsilon^2),$$

where

$$\begin{aligned} \mathbb{A} &:= \frac{1}{2} \frac{\partial^2 \hat{\eta}_{0R}}{\partial \mathbf{F}^2}(e_{0R_0}, s_{0R_0}, \mathbf{1}), \\ \mathbf{A} &:= \frac{1}{2} \frac{\partial^2 \hat{\eta}_{0R}}{\partial s_{0R} \partial \mathbf{F}}(e_{0R_0}, s_{0R_0}, \mathbf{1}), \end{aligned}$$

³One could also consider small changes in e_R , but here this is not done so we can compare the result with the traditional Biot theory, which is isothermal.

$$-a^2 := \frac{1}{2} \frac{\partial^2 \hat{\eta}_{0R}}{\partial s_{0R}^2} (e_{0R\circ}, s_{0R\circ}, \mathbf{1}).$$

Taking this expansion as a constitutive law for η_{0R} , from (22) and (31), we obtain

$$\frac{p_R}{\theta} = J(2\mathbf{A} \cdot \nabla \mathbf{u}_0 - 2a^2(s_{0R} - s_{0R\circ})) = 2\mathbf{A} \cdot \nabla \mathbf{u}_0 - 2a^2(s_{0R} - s_{0R\circ}) + o(\epsilon).$$

This motivates completing the square in the quadratic asymptotic expansion for η_{0R} in a particular way so as to obtain a term like the right-hand side of the previous equation. Namely, we can write

$$\eta_{0R} = \nabla \mathbf{u}_0 \cdot \tilde{\mathbb{A}} \nabla \mathbf{u}_0 - \frac{1}{a^2} (\mathbf{A} \cdot \nabla \mathbf{u}_0 - a^2(s_{0R} - s_{0R\circ}))^2,$$

where $\tilde{\mathbb{A}} \nabla \mathbf{u}_0 := \mathbb{A} \nabla \mathbf{u}_0 + \frac{1}{a^2} (\mathbf{A} \cdot \nabla \mathbf{u}_0) \mathbf{A}$. Now, if we set

$$\mathbb{C} := -2\theta \tilde{\mathbb{A}}, \quad \mathbf{H} := \frac{1}{a^2} \mathbf{A}, \quad h := -\frac{1}{2a^2\theta}$$

we can write

$$-\theta \eta_{0R} = \frac{1}{2} \nabla \mathbf{u}_0 \cdot \mathbb{C} \nabla \mathbf{u}_0 - \frac{1}{2h} (\mathbf{H} \cdot \nabla \mathbf{u}_0 - (s_{0R} - s_{0R\circ}))^2.$$

Using this in (52) and linearizing the term $p_R \mathbf{F}^{-\top}$ on the right-hand side yields

$$\mathbf{T}_R = \mathbb{C} \nabla \mathbf{u}_0 + p_R (\mathbf{H} - \mathbf{1}) \quad \text{and} \quad s_{0R} - s_{0R\circ} = \mathbf{H} \cdot \nabla \mathbf{u}_0 + h p_R. \tag{53}$$

The material constant \mathbf{H} appears in both the formula for the stress and for the pressure. This is consistent with what appears in [10] and [25]. These formulas differ slightly from those found in the literature since the measure of strain $s_{0R} - s_{0R\circ}$ used here differs from those used elsewhere. The fourth-order tensor \mathbb{C} has the usual major symmetry since it came from the tensor \mathbb{A} , which is a second derivative of the entropy with respect to \mathbf{F} . The minor symmetry of \mathbb{C} will follow from the principle of material frame-indifference.

4.2 Alternate Form of Mass Balance

In some formulations of poroelasticity involving only one fluid phase made up of one component, the mass balance for this component is replaced by an equation involving the pressure and the volume fraction of the phase. Here this equation is derived under certain assumptions and its relation with mass balance is examined.

The assumptions we will make in the context of the single fluid phase made up of a single component are the following:

- the fluid phase is incompressible,
- there is no interfacial energy,
- only small displacements and small changes in volume fraction can occur.

By the definition of incompressibility, the fluid phase can only undergo isochoric motions and thus

$$\frac{d}{dt} \int_{V_f(t)} s_1 dv = 0,$$

where $V_f(t)$ is any volume *convecting with the fluid phase*. Localizing this balance yields

$$\frac{\partial}{\partial t} s_1 + \operatorname{div}(s_1 \mathbf{v}_1) = 0. \tag{54}$$

Writing this in a reference configuration and using the assumption of small displacements results in

$$\frac{\partial}{\partial t} s_{1R} + \operatorname{div}(s_{1R} \mathbf{v}_1) = 0, \tag{55}$$

where $s_{1R} := J s_1$ is the volume fraction of the fluid per unit volume in the reference configuration. Using the fact that the volume fraction of the fluid and solid must add to one and that the reference volume fraction is constant, with the help of (53)₂ the previous equation can be written as

$$-\frac{\partial}{\partial t} (\mathbf{H} \cdot \nabla \mathbf{u}_0 + h p_R) + \operatorname{div}(s_{1R} \mathbf{v}_1) = 0, \tag{56}$$

which is sometimes referred to as the storage equation [30].

A relationship between (54), and hence (56), and mass balance becomes clear in the case where the fluid phase has no mass diffusion and no sources or sinks of mass, in which case mass balance (6) reads

$$\frac{\partial}{\partial t} m_1 + \operatorname{div}(m_1 \mathbf{v}_1) = 0.$$

If we let \bar{m}_1 denote the mass of the fluid per unit volume of the fluid so that $m_1 = s_1 \bar{m}_1$, then (54) and the previous equation imply that

$$\frac{\partial}{\partial t} \bar{m}_1 + \nabla \bar{m}_1 \cdot \mathbf{v}_1 = 0, \tag{57}$$

which says that \bar{m}_1 is constant on particle paths. This means that once the fluid velocity is known, \bar{m}_1 can be found by integrating along such a path. Some authors take the previous equation as the definition of incompressibility. When this is done, (57) together with mass balance yields (54). For this reason, (54), or (56) if there are small displacements and small changes in volume fraction, are sometimes referred to as an alternate form of mass balance. See, for example, the book by Bear [2].

4.3 Alternative Statements of the Force Balances

The force balance for the fluids used by the authors in [26] to develop a continuum description of porous flow in a rigid medium took the form

$$\mathbf{0} = \int_{V(t)} \sum_{\substack{\pi'=0 \\ \pi' \neq \pi}}^{N_p} \mathbf{f}_{V\pi\pi'} dv + \int_{V_\pi(t)} \mathbf{b}_\pi dv_\pi - \int_{\partial V_\pi(t)} p_\pi \mathbf{n} da_\pi,$$

where dv_π and da_π are the volume and area elements associated with the volume occupied by phase π . With the identification $dv_\pi = s_\pi dv$ the local form of this law becomes

$$s_\pi \mathbf{b}_\pi - s_\pi \nabla p_\pi + \sum_{\substack{\pi'=0 \\ \pi' \neq \pi}}^{N_p} \mathbf{f}_{V\pi\pi'} = \mathbf{0},$$

which is identical to Eq. (37) which resulted from the more general force balance postulated in Sect. 2 and subsequent restrictions due to the second law. A slightly simpler derivation of the model proposed here is obtained if this force balance is assumed a priori for the fluids, and the force balance for the solid takes the form

$$\int_{V(t)} (s_0 \mathbf{b}_0 + \mathbf{f}_E) dv + \int_{\partial V(t)} \mathbf{T}_0 \mathbf{n} da + \sum_{\pi=1}^{N_p} \int_{V(t)} \mathbf{f}_{V_0\pi} dv = \mathbf{0},$$

where \mathbf{f}_E is the force on the solid due to elastic stresses (pressures) in the fluid. The local form is

$$s_0 \mathbf{b}_0 + \text{div}(\mathbf{T}_0) + \mathbf{f}_E + \sum_{\pi=1}^{N_p} \mathbf{f}_{V_0\pi} = \mathbf{0},$$

and the formula for \mathbf{f}_E appearing on the right hand side of Eq. (36) follows directly from Coleman–Noll procedure.

5 Summary

In summary, the constitutive assumptions for the model are determined by

- the entropy η_π for the fluid phases (16) and the solid phase (17),
- the interfacial energy e_I specified by (30),
- local thermodynamics equilibrium, Assumption 1,
- the stress relation (29) for the solid phase,
- the Fick and Fourier laws (39),
- (33) and (36), which along with (13), determine $\sum_{\substack{\pi'=0 \\ \pi' \neq \pi}}^{N_p} \mathbf{f}_{E\pi\pi'}$,
- the formula (40) for $\mathbf{f}_{V_\pi\pi'}$.

Substituting these constitutive laws into the balances yields a system of partial differential equations that describe the poroelastic continuum. In this list of balances we use the total force balance for the system in place of the force balance for the solid phase. The reasoning for this will be explained at the end of this section.

The mass balance for component c is

$$\frac{\partial}{\partial t} m_c + \text{div} \left(m_c \mathbf{v}_0 - \mathbf{K}_c \nabla \mu_c + \sum_{\pi=1}^{N_p} M_{c\pi} \mathbf{v}_\pi^r \right) = h_c.$$

Force balance for fluid phase π is

$$s_\pi \mathbf{b}_\pi - s_\pi \nabla p_\pi - \sum_{\substack{\pi'=0 \\ \pi' \neq \pi}}^{N_p} \mathbf{A}_{\pi\pi'} \mathbf{v}_{\pi\pi'}^r = \mathbf{0},$$

and for the entire continuum is

$$\sum_{\pi=0}^{N_p} s_\pi \mathbf{b}_\pi + \text{div} \left[\left(\frac{\partial e_I}{\partial \mathbf{F}} - \theta \frac{\partial \eta_0}{\partial \mathbf{F}} \right) \mathbf{F}^\top + \left(e_0 - \theta \eta_0 - \sum_{\pi=1}^{N_p} s_\pi p_\pi \right) \mathbf{1} \right] = \mathbf{0}. \tag{58}$$

The balance of energy becomes

$$\begin{aligned} \dot{e} - \dot{\mathbf{m}} \cdot \boldsymbol{\mu} &= r + \operatorname{div}(\mathbf{K}_\theta \nabla \theta) + \sum_{c=1}^{N_c} \nabla \mu_c \cdot \mathbf{K}_c \nabla \mu_c + \sum_{\pi=1}^{N_p} \left(\mathbf{v}_\pi^r \cdot \sum_{\substack{\pi'=0 \\ \pi' \neq \pi}}^{N_p} \mathbf{A}_{\pi\pi'} (\mathbf{v}_\pi^r - \mathbf{v}_{\pi'}^r) \right) \\ &+ \nabla \mathbf{v}_0 \cdot \left[\left(\frac{\partial e_I}{\partial \mathbf{F}} - \theta \frac{\partial \eta_0}{\partial \mathbf{F}} \right) \mathbf{F}^\top - \theta \eta \mathbf{1} \right] - \sum_{\pi=1}^{N_p} \theta \operatorname{div}(\eta_\pi \mathbf{v}_\pi^r). \end{aligned}$$

The mass balance for the solid phase is not listed here since the solid mass does not appear in any of the other balances and is essentially decoupled from them. Also, the deformation gradient is determined from the kinematic equation $\dot{\mathbf{F}} = (\nabla \mathbf{v}_0) \mathbf{F}$.

The quantities to be solved for are the velocity \mathbf{v}_0 and deformation gradient \mathbf{F} of the solid, the relative velocities \mathbf{v}_π^r of the fluid phases $1 \leq \pi \leq N_p$, the masses of the components m_c , and the energy e . The other quantities are either determined by constitutive laws or are given a priori, such as the external influences h_c , \mathbf{b}_π , and r . Notice that the force balances for the fluid phases are algebraic in the velocities.

The force balance for just the solid phase reads

$$\begin{aligned} s_0 \mathbf{b}_0 + \operatorname{div} \left[\left(\frac{\partial e_I}{\partial \mathbf{F}} - \theta \frac{\partial \eta_0}{\partial \mathbf{F}} \right) \mathbf{F}^\top + (e_0 - \theta \eta_0) \mathbf{1} \right] + p \nabla s_0 \\ + \frac{\partial e_I}{\partial s_0} \nabla s_0 + \frac{\partial e_I}{\partial \mathbf{F}} \nabla \mathbf{F} + \sum_{\pi=1}^{N_p} \mathbf{A}_{0\pi} \mathbf{v}_\pi^r = \mathbf{0}. \end{aligned}$$

Since we are assuming force balance for all of the fluid phases, using a total force balance for the continuum or just using the force balance for the solid phase are equivalent. In the literature on poroelastic continuum, the total force balance (58) is used rather than the solid force balance. Comparing the two balances it is not hard to see why. The total force balance has a clear divergence structure and does not involve the relative velocities.

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

References

- Alt, H.W., DiBenedetto, E.: Nonsteady flow of water and oil through inhomogeneous porous media. *Ann. Sc. Norm. Super. Pisa, Cl. Sci.* **12**(3), 335–392 (1985). http://www.numdam.org/item?id=ASNSP_1985_4_12_3_335_0
- Bear, J.: *Dynamics of Fluids in Porous Media*. Dover, New York (1972)
- Biot, M.: General theory of three-dimensional consolidation. *J. Appl. Phys.* **12**, 155–164 (1941)
- Biot, M.: Theory of elasticity and consolidation for a porous anisotropic solid. *J. Appl. Phys.* **26**, 182–185 (1955)
- Biot, M.: General solutions of the equations of elasticity and consolidation for a porous material. *J. Appl. Mech.* **78**, 91–96 (1956)
- Biot, M., Willis, D.: The elastic coefficients of the theory of consolidation. *J. Appl. Mech.* **24**, 594–601 (1957)
- Chen, Z.: Degenerate two-phase incompressible flow. I. Existence, uniqueness and regularity of a weak solution. *J. Differ. Equ.* **171**(2), 203–232 (2001). <https://doi.org/10.1006/jdeq.2000.3848>
- Chen, Z., Ewing, R.E.: Degenerate two-phase incompressible flow. III. Sharp error estimates. *Numer. Math.* **90**(2), 215–240 (2001). <https://doi.org/10.1007/s002110100291>

9. Chen, Z., Huan, G., Ma, Y.: Computational Methods for Multiphase Flows in Porous Media. Computational Science & Engineering SIAM, Philadelphia (2006). <https://doi.org/10.1137/1.9780898718942>
10. Cheng, A.H.: Poroelasticity. Theory and Applications of Transport in Porous Media, vol. 27. Springer, Berlin (2016)
11. Cimmelli, V.A., Jou, D., Ruggeri, T., Ván, P.: Entropy principle and recent results in non-equilibrium theories. Entropy **16**(3), 1756–1807 (2014). <https://doi.org/10.3390/e16031756>
12. Coleman, B.D., Noll, W.: The thermodynamics of elastic materials with heat conduction and viscosity. Arch. Ration. Mech. Anal. **13**, 167–178 (1963)
13. Darcy, H.: Recherches expérimentales relatives au mouvement de l'eau dans les tuyaux. Mallet-Bachelier, Paris (1857)
14. DiBenedetto, E., Showalter, R.E.: Implicit degenerate evolution equations and applications. SIAM J. Math. Anal. **12**(5), 731–751 (1981). <https://doi.org/10.1137/0512062>
15. Dinariyev, O., Evseev, N.: Multiphase flow modeling with density functional method. Comput. Geosci. **20**(4), 835–856 (2016). <https://doi.org/10.1007/s10596-015-9527-2>
16. Ewing, R.: The Mathematics of Reservoir Simulation. Frontiers in Applied Mathematics. SIAM, Philadelphia (1983). https://books.google.com/books?id=k_4Y-RBOK9oC
17. Ganis, B., Kumar, K., Pencheva, G., Wheeler, M.F., Yotov, I.: A global Jacobian method for mortar discretizations of a fully implicit two-phase flow model. Multiscale Model. Simul. **12**(4), 1401–1423 (2014). <https://doi.org/10.1137/140952922>
18. Gibson, N.L., Medina, F.P., Peszynska, M., Showalter, R.E.: Evolution of phase transitions in methane hydrate. J. Math. Anal. Appl. **409**(2), 816–833 (2014). <https://doi.org/10.1016/j.jmaa.2013.07.023>
19. Kroener, D., Luckhaus, S.: Flow of oil and water in a porous medium. J. Differ. Equ. **55**(2), 276–288 (1984). [https://doi.org/10.1016/0022-0396\(84\)90084-6](https://doi.org/10.1016/0022-0396(84)90084-6)
20. Lauser, A., Hager, C., Helmig, R., Wohlmuth, B.: A new approach for phase transitions in miscible multi-phase flow in porous media. Adv. Water Resour. **34**, 957–966 (2011). <https://doi.org/10.1016/j.advwatres.2011.04.021>
21. Noll, W.: Lectures on the foundations of continuum mechanics and thermodynamics. Arch. Ration. Mech. Anal. **52**, 62–92 (1973)
22. Peszyńska, M., Lu, Q., Wheeler, M.F.: Coupling different numerical algorithms for two phase fluid flow. In: The Mathematics of Finite Elements and Applications, X, MAFELAP 1999, Uxbridge, pp. 205–214. Elsevier, Oxford (2000). <https://doi.org/10.1016/B978-008043568-8/50012-2>
23. Peszynska, M., Showalter, R.E., Webster, J.T.: Advection of methane in the hydrate zone: model, analysis and examples. Math. Methods Appl. Sci. **38**(18), 4613–4629 (2015). <https://doi.org/10.1002/mma.3401>
24. Rice, J.R., Cleary Michael, P.: Some basic stress diffusion solutions for fluid saturated elastic porous media with compressible constituents. Rev. Geophys. **14**(2), 227–241 (1976). <https://doi.org/10.1029/RG014i002p00227>
25. Sanchez-Palencia, E.: Nonhomogeneous Media and Vibration Theory. Lecture Notes in Physics, vol. 127. Springer, Berlin (1980)
26. Seguin, B., Walkington, N.J.: Multi-component multiphase flow. Arch. Ration. Mech. Anal. (2017, submitted)
27. Showalter, R.E.: Diffusion in poro-elastic media. J. Math. Anal. Appl. **251**(1), 310–340 (2000). <https://doi.org/10.1006/jmaa.2000.7048>
28. Swendsen, R.: An Introduction to Statistical Mechanics and Thermodynamics. Oxford Graduate Texts. Oxford University Press, Oxford (2012). <https://books.google.com/books?id=hbW609uEobsC>
29. Truesdell, C.: Rational Thermodynamics. McGraw-Hill, New York (1969). A course of lectures on selected topics, with an appendix on the symmetry of the heat-conduction tensor by C.C. Wang
30. Verruijt, A.: An Introduction to Soil Mechanics. Springer, Berlin (2010)
31. Visintin, A.: Models of Phase Transitions. Progress in Nonlinear Differential Equations and Their Applications, vol. 28. Birkhäuser, Boston (1996). <https://doi.org/10.1007/978-1-4612-4078-5>